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CONTINUOUS INFINITESIMAL GENERATORS OF A CLASS OF NONLINEAR EVOLUTION OPERATORS IN BANACH SPACES

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Abstract. A class of nonlinear evolution operators is introduced and a characterization of continuous infinitesimal generators of such evolution operators is given by applying the results on semigroups of Lipschitz operators.

Let X be a real Banach space with norm $\|\cdot\|$. Let Ω be a closed subset of $[0,\infty)\times X$ such that $\Omega(t)=\{x\in X;(t,x)\in\Omega\}\neq\emptyset$ for $t\in[0,\infty)$. Let A be a continuous mapping from Ω into X. Given $(\tau,x)\in\Omega$, we consider the following initial value problem:

$$(\mathrm{IVP};\tau,x)\qquad \begin{cases} u'(t)=A\big(t,u(t)\big) & \text{for}\quad \tau\le t<\infty,\\ u(\tau)=x. \end{cases}$$

Set $\Delta=\{(t,\tau);0\le\tau\le t<\infty\}$. Suppose that the problem (IVP; τ,x) has a unique (continuously differentiable) solution $u(\cdot)$ on $[\tau,\infty)$. Defining by $U(t,\tau)x=u(t)$, we have the following properties:

(E1) $U(\tau,\tau)x=x$ and $U(t,s)U(s,\tau)x=U(t,\tau)x$ for $(\tau,x)\in\Omega$ and $t,s\in[0,\infty)$ such that $t\ge s\ge\tau$.

(E2) For any $(\tau,x)\in\Omega$, $U(s,\tau)x$ converges to $U(t,\tau)x$ in X as $s\to t$ in $[\tau,\infty)$.

By a (nonlinear) *evolution operator* on Ω, we mean a family $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ of operators $U(t,\tau):\Omega(\tau)\to\Omega(t)$ satisfying (E1) and (E2). We consider the following additional condition on such a family $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ which ensures the continuous dependence of solutions $u(\cdot)$ on the initial data $(\tau,x)\in\Omega$:

(E3) For any $T>0$, there exists $M_T\in(0,\infty)$ such that

$$\|U(\tau+t,\tau)x-U(\sigma+t,\sigma)y\|\le M_T\big(|\tau-\sigma|+\|x-y\|\big)$$

for $(\tau,x),(\sigma,y)\in\Omega$ and $t\in[0,T]$.

The aim of this paper is to prove the following theorem, which provides a characterization of the continuous infinitesimal generator A such that the solution operator to (IVP; τ,x) becomes an evolution operator on Ω satisfying condition (E3). Our class of evolution operators is rather narrow but closely related to the ones discussed in Murakami [12], Martin [9], Lakshmikantham et al. [8] and Kato [4]. The theorem is proved by the use of the results for the autonomous case by Kobayashi-Tanaka [6].

Theorem 1. *There exists an evolution operator* $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ *on* Ω *such that* (E3) *is satisfied and that* $u(t)=U(t,\tau)x$ *is a unique solution to* (IVP; τ,x) *on* $[\tau,\infty)$ *for any* $(\tau,x)\in\Omega$ *if and only if the mapping* A *on* Ω *satisfies the following conditions* (Ω1) *and* (Ω2)*:*

(Ω1) *For any* $(\tau,x)\in\Omega$,

$$\liminf_{h\to+0} d(x+hA(\tau,x),\Omega(\tau+h))/h=0,$$

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where $d(x,S)=\inf_{y\in S}\|x-y\|$ *for* $x\in X$ *and* $S\subset X$.

(Ω2) *There exist a number* $\omega\in[0,\infty)$ *and* $V:(\boldsymbol{R}\times X)\times(\boldsymbol{R}\times X)\to[0,\infty)$, *which satisfies conditions* (V1) *and* (V2) *below, such that*

$$D_+V((\tau,x),(\sigma,y))(A(\tau,x),A(\sigma,y))\le\omega V((\tau,x),(\sigma,y)) \tag{1}$$

for $(\tau,x),(\sigma,y)\in\Omega$, *where*

$$\begin{aligned}&D_+V((\tau,x),(\sigma,y))(\xi,\eta)\\&=\liminf_{h\to+0}\big(V((\tau+h,x+h\xi),(\sigma+h,y+h\eta))-V((\tau,x),(\sigma,y))\big)/h\end{aligned}$$

for $(\tau,x),(\sigma,y)\in\boldsymbol{R}\times X$ *and* $(\xi,\eta)\in X\times X$.

(V1) *There exists* $L\in(0,\infty)$ *such that*

$$\begin{aligned}&|V((\tau,x),(\sigma,y))-V((\hat\tau,\hat x),(\hat\sigma,\hat y))|\\&\le L\big(|\tau-\hat\tau|+|\sigma-\hat\sigma|+\|x-\hat x\|+\|y-\hat y\|\big)\end{aligned}$$

for $(\tau,x),(\sigma,y),(\hat\tau,\hat x),(\hat\sigma,\hat y)\in\boldsymbol{R}\times X$.

(V2) *There exists* $M\in[1,\infty)$ *such that*

$$|\tau-\sigma|+\|x-y\|\le V((\tau,x),(\sigma,y))\le M\big(|\tau-\sigma|+\|x-y\|\big)$$

for $(\tau,x),(\sigma,y)\in\Omega$.

Moreover, in this case, we have

$$V\big((\tau+t,U(\tau+t,\tau)x),(\sigma+t,U(\sigma+t,\sigma)y)\big)\le e^{\omega t}V((\tau,x),(\sigma,y)) \tag{2}$$

and

$$\|U(\tau+t,\tau)x-U(\sigma+t,\sigma)y\|\le Me^{\omega t}\big(|\tau-\sigma|+\|x-y\|\big) \tag{3}$$

for $(\tau,x),(\sigma,y)\in\Omega$ *and* $t\in[0,\infty)$.

Proof. Let $\mathcal{X}$ be the real Banach space $\boldsymbol{R}\times X$ with norm $\|(t,x)\|_{\mathcal{X}}=|t|+\|x\|$ for $(t,x)\in\mathcal{X}$. We define $\mathcal{A}:\Omega\to\mathcal{X}$ by $\mathcal{A}(t,x)=(1,A(t,x))$ for $(t,x)\in\Omega$. Obviously, $\mathcal{A}$ is a continuous mapping on Ω into $\mathcal{X}$. We note that Ω is closed in $\mathcal{X}$. We note also that, for any $(\tau,x)\in\Omega$, $\boldsymbol{u}:[0,\infty)\to\boldsymbol{R}\times X$ is a solution to the initial value problem

$$\begin{cases}\boldsymbol{u}'(t)=\mathcal{A}\boldsymbol{u}(t) & \text{for}\quad 0\le t<\infty,\\ \boldsymbol{u}(0)=(\tau,x),\end{cases} \tag{4}$$

if and only if $\boldsymbol{u}(t)=(t+\tau,v(t+\tau))$ for $t\ge0$, where $v(t)$ is a solution to (IVP; τ,x). Indeed, let $\boldsymbol{u}(t)=(s(t),u(t))$ be a solution to (4). Then, $s'(t)=1$ and $s(t)=t+\tau$ since $s(0)=\tau$. Therefore,

$$u'(t)=A(s(t),u(t))=A(t+\tau,u(t))\quad\text{for}\quad t\ge0\quad\text{and}\quad u(0)=x.$$

Hence, $v(t)$ defined by $v(t)=u(t-\tau)$ for $t\in[\tau,\infty)$ is a solution to (IVP; τ,x) and $\boldsymbol{u}(t)=(t+\tau,v(t+\tau))$. Conversely, let $v(t)$ be a solution to (IVP; τ,x) and $\boldsymbol{u}(t)=(t+\tau,v(t+\tau))$. Then, $\boldsymbol{u}(0)=(\tau,v(\tau))=(\tau,x)$ and

$$\boldsymbol{u}'(t)=(1,v'(t+\tau))=(1,A(t+\tau,v(t+\tau)))=\mathcal{A}(t+\tau,v(t+\tau))=\mathcal{A}\boldsymbol{u}(t)$$

for $t\ge0$.

Suppose that there exists an evolution operator $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ on Ω such that (E3) is satisfied and that $v(t)=U(t,\tau)x$ is a unique solution to (IVP; τ,x) on $[\tau,\infty)$ for any $(\tau,x)\in$

Ω. Let $(\tau, x) \in \Omega$ and $v(t) = U(t,\tau)x$ for $t \geq \tau$. Then, since $v(\tau+h) = U(\tau+h,\tau)x \in \Omega(\tau+h)$ for $h > 0$, we have

$$\begin{aligned}&\limsup_{h\to+0} d(x + hA(\tau, x), \Omega(\tau + h))/h \\ &\leq \limsup_{h\to+0} \|x + hA(\tau, x) - v(\tau + h)\|/h = \|A(\tau, v(\tau)) - v'(\tau)\| = 0.\end{aligned}$$

Thus, (Ω1) is satisfied. We define $\mathcal{U}(t) : \Omega \to \Omega$ by

$$\mathcal{U}(t)(\tau, x) = (\tau + t, U(\tau + t, \tau)x)$$

for $(\tau, x) \in \Omega$ and $t \in [0, \infty)$. By (E1), we have $\mathcal{U}(0)(\tau, x) = (\tau, U(\tau, \tau)x) = (\tau, x)$ and

$$\begin{aligned}\mathcal{U}(t)\mathcal{U}(s)(\tau, x) &= \mathcal{U}(t)(\tau + s, U(\tau + s, \tau)x) \\ &= ((\tau + s) + t, U((\tau + s) + t, \tau + s)U(\tau + s, \tau)x) \\ &= (\tau + (s + t), U(\tau + (s + t), \tau)x) = \mathcal{U}(t + s)(\tau, x)\end{aligned}$$

for $s, t \in [0, \infty)$ and $(\tau, x) \in \Omega$. By (E2), $\mathcal{U}(s)(\tau, x) = (\tau + s, U(\tau + s, \tau)x) \to (\tau + t, U(\tau + t, \tau)x) = \mathcal{U}(t)(\tau, x)$ in $\boldsymbol{R} \times X$ as $s \to t$ in $[0, \infty)$. Hence the family $\{\mathcal{U}(t)\}_{t\in[0,\infty)}$ is a semigroup on Ω. Since, for any (τ, x), $\boldsymbol{u}(t) = \mathcal{U}(t)(\tau, x)$ is a solution to (4), the mapping $\mathcal{A}$ is the infinitesimal generator of the semigroup $\{\mathcal{U}(t)\}_{t\in[0,\infty)}$. Condition (E3) implies that, for any $T > 0$, there exists $M_T \in (0, \infty)$ such that

$$\begin{aligned}&\|\mathcal{U}(t)(\tau, x) - \mathcal{U}(t)(\sigma, y)\|_{\mathcal{X}} \\ &= |(\tau + t) - (\sigma + t)| + \|U(\tau + t, \tau)x - U(\sigma + t, \sigma)y\| \\ &\leq |\tau - \sigma| + M_T\big(|\tau - \sigma| + \|x - y\|\big) \leq (M_T + 1)\|(\tau, x) - (\sigma, y)\|_{\mathcal{X}}\end{aligned}$$

for $(\tau, x), (\sigma, y) \in \Omega$ and $t \in [0, T]$. Hence, it follows from [6, Theorem 4.2] that there exist a number $\omega \in [0, \infty)$ and $V : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfying conditions (V1) and (V2) such that

$$V(\mathcal{U}(t)(\tau, x), \mathcal{U}(t)(\sigma, y)) \leq e^{\omega t} V((\tau, x), (\sigma, y))$$

for $(\tau, x), (\sigma, y) \in \Omega$ and $t \in [0, \infty)$. Hence, by the definition of $\mathcal{U}(t)$, (2) holds for $(\tau, x), (\sigma, y) \in \Omega$ and $t \in [0, \infty)$. By (2) and (V2), (3) also holds for $(\tau, x), (\sigma, y) \in \Omega$ and $t \in [0, \infty)$. Since $\mathcal{A}$ is the infinitesimal generator of $\{\mathcal{U}(t)\}_{t\in[0,\infty)}$, [6, Theorem 4.2] implies that

$$\begin{aligned}\liminf_{h\to+0}\big(V((\tau, x) + h\mathcal{A}(\tau, x), (\sigma, y) + h\mathcal{A}(\sigma, y)) - V((\tau, x), (\sigma, y))\big)/h \\ \leq \omega V((\tau, x), (\sigma, y))\end{aligned} \tag{5}$$

for $(\tau, x), (\sigma, y) \in \Omega$. By the definition of $\mathcal{A}$, we have

$$\begin{aligned}&D_+V((\tau, x), (\sigma, y))(A(\tau, x), A(\sigma, y)) \\ &= \liminf_{h\to+0}\Big(V\big((\tau + h, x + hA(\tau, x)), (\sigma + h, y + hA(\sigma, y))\big) - V((\tau, x), (\sigma, y))\Big)/h \\ &= \liminf_{h\to+0}\big(V((\tau, x) + h\mathcal{A}(\tau, x), (\sigma, y) + h\mathcal{A}(\sigma, y)) - V((\tau, x), (\sigma, y))\big)/h\end{aligned} \tag{6}$$

for $(\tau, x), (\sigma, y) \in \Omega$. Hence, (1) holds for any $(\tau, x), (\sigma, y) \in \Omega$.

We suppose conversely that the mapping A satisfies conditions (Ω1) and (Ω2). Let $(\tau, x) \in \Omega$. Then, by (Ω1), there exist $h_n > 0$ and $x_n \in \Omega(\tau + h_n)$ such that $h_n \to 0$ and $\|x +$

$h_nA(\tau,x) - x_n\|/h_n \to 0$ as $n \to \infty$. We have

$$\begin{aligned} &\|(\tau,x) + h_n\mathcal{A}(\tau,x) - (\tau+h_n,x_n)\|_{\mathcal{X}}/h_n \\ &= \|(\tau,x) + h_n(1,A(\tau,x)) - (\tau+h_n,x_n)\|_{\mathcal{X}}/h_n \\ &= \|x + h_nA(\tau,x) - x_n\|/h_n \to 0 \end{aligned}$$

as $n \to \infty$. Since $(\tau+h_n,x_n) \in \Omega$, it follows that

$$\liminf_{h\to+0} d_{\mathcal{X}}((\tau,x) + h\mathcal{A}(\tau,x),\Omega)/h = 0,$$

where $d_{\mathcal{X}}((t,x),\mathcal{S}) = \inf_{(s,y)\in\mathcal{S}}\|(t,x)-(s,y)\|_{\mathcal{X}}$ for $(t,x) \in \mathcal{X}$ and $\mathcal{S} \subset \mathcal{X}$. By ($\Omega$2), there exist a number $\omega \in [0,\infty)$ and $V : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ satisfying (V1) and (V2) such that (1) holds true for any $(\tau,x),(\sigma,y) \in \Omega$. Using (6) again, we see from (1) that (5) holds true for any $(\tau,x),(\sigma,y) \in \Omega$. Therefore, [6, Theorem 4.2] implies that $\mathcal{A}$ is the infinitesimal generator of a semigroup $\{\mathcal{U}(t)\}_{t\in[0,\infty)}$ on Ω such that, for any $(\tau,x) \in \Omega$, $\boldsymbol{u}(t) = \mathcal{U}(t)(\tau,x)$ is a unique solution to the initial value problem (4) and

$$V(\mathcal{U}(t)(\tau,x),\mathcal{U}(t)(\sigma,y)) \le e^{\omega t}V((\tau,x),(\sigma,y)) \tag{7}$$

for $(\tau,x),(\sigma,y) \in \Omega$ and $t \in [0,\infty)$. Let $(\tau,x) \in \Omega$ and $\boldsymbol{u}(t) = \mathcal{U}(t)(\tau,x)$ for $t \in [0,\infty)$. Then we have $\boldsymbol{u}(t) = (t+\tau, v(t+\tau))$, where $v(t)$ is a solution to (IVP; τ,x). By virtue of the unicity of the solution $\boldsymbol{u}(t)$ to (4), the solution $v(t)$ is uniquely determined by (τ,x). Thus, we define $U(t,\tau)x \in X$ by $U(t,\tau)x = v(t)$ for $t \in [\tau,\infty)$. Since $\boldsymbol{u}(t-\tau) = (t,v(t)) = (t,U(t,\tau)x) \in \Omega$, we see that $U(t,\tau)x \in \Omega(t)$ for $t \in [\tau,\infty)$. Since $\{\mathcal{U}(t)\}_{t\in[0,\infty)}$ is a semigroup on Ω, we have

$$(t,U(t,\tau)x) = \mathcal{U}(t-\tau)(\tau,x) = \lim_{s\to t}\mathcal{U}(s-\tau)(\tau,x) = \lim_{s\to t}(s,U(s,\tau)x)$$

in $\boldsymbol{R} \times X$ and $U(t,\tau)x = \lim_{s\to t} U(s,\tau)x$ in X for $t \ge \tau$. Let $t \ge s \ge \tau$. Then,

$$\begin{aligned} &(t,U(t,\tau)x) = \mathcal{U}(t-\tau)(\tau,x) = \mathcal{U}(t-s)\mathcal{U}(s-\tau)(\tau,x) \\ &= \mathcal{U}(t-s)(s,U(s,\tau)x) = (t,U(t,s)U(s,\tau)x) \end{aligned}$$

and $U(t,\tau)x = U(t,s)U(s,\tau)x$. Thus $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ is an evolution operator on Ω. Moreover, (7) implies that

$$\begin{aligned} &\|U(\tau+t,\tau)x - U(\sigma+t,\sigma)y\| \le \|\mathcal{U}(t)(\tau,x) - \mathcal{U}(t)(\sigma,y)\|_{\mathcal{X}} \\ &\le V(\mathcal{U}(t)(\tau,x),\mathcal{U}(t)(\sigma,y)) \le e^{\omega t}V((\tau,x),(\sigma,y)) \\ &\le Me^{\omega t}\|(\tau,x)-(\sigma,y)\|_{\mathcal{X}} = Me^{\omega t}\big(|\tau-\sigma| + \|x-y\|\big) \end{aligned}$$

for $(\tau,x),(\sigma,y) \in \Omega$ and $t \in [0,\infty)$. Hence, condition (E3) is satisfied by $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$. □

Remark 1. The kinds of conditions (Ω1) and (Ω2) were found by Nagumo [13] and Okamura [14], respectively.

Remark 2. Our proof of Theorem 1 is suggested by Evans-Massey [3].

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