Strong Convergence Theorems for Fixed Points of Nonlinear Mappings of Nonexpansive Type

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Preface

The main purpose of this thesis is to present the theory of fixed point of nonlinear mappings in nonlinear functional analysis in a systematic way. In particular, we prove strong convergence theorems for fixed points problems of nonlinear mappings of nonexpansive type in Banach spaces.

Nonlinear functional analysis is an area of mathematics which has grown up greatly over the past few decades. It is significantly influenced by nonlinear problems posed in physics, sciences, engineering, and economics. Many problems in nonlinear functional analysis are related to finding fixed points of nonexpansive mappings. The theory of maximal monotone operators has emerged as an effective and powerful tool for studying a wide class of problems arising in various many fields. For example, many problems in convex programming, minimization problems and variational inequalities, can be formulated as finding zeros of maximal monotone operators. In 1976, Rockafellar [66] has set up a fundamental convergence analysis of an algorithm for finding a zero of a maximal monotone operator in a Hilbert space. The algorithm is called a proximal point algorithm. In this method, resolvents of the maximal monotone operator play a crucial role for finding a zero of a maximal monotone operator. A resolvent of a maximal monotone operator is a nonexpansive mapping which is an obvious generalization of a contraction mapping. Therefore finding zeros of maximal monotone operators is reduced to a fixed point problem for nonexpansive mappings, that is, the problem finding fixed points of nonexpansive mappings. The subdifferential of a proper, convex and lower semicontinuous functional is maximal monotone and the resolvents of a maximal monotone operator are everywhere defined nonexpansive mappings. Nonexpansive mappings also appear in applications as the transition operators for initial value problems of differential inclusions associated with accretive operators. Nonexpansive mappings are intimately connected with the monotonicity methods developed since the early 1960's, and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties. As a result of these, the study of fixed point theory for nonexpansive mappings has attracted the interest of numerous scientists and has become a flourishing area of research.

In fixed point theory, it is important to construct fixed points. Rockafellar [66] has posed an open question whether (or not) the proximal point method always converges strongly. This question was resolved in the negative later on. Naturally, the question arises whether the proximal point method can be modified, preferably in a simple way, so that strong convergence is guaranteed. Solodov and Svaiter [68] have proposed a new proximal type algorithm, which converges strongly, by combining proximal point iterations with certain computationally simple projection steps. This algorithm is called a hybrid method. Motivated by [68], Nakajo and Takahashi [51] have proved a strong convergence theorem for a nonexpansive mapping in a Hilbert space by using the hybrid method. Moreover, Takahashi, Takeuchi and Kubota [74] have introduced a new hybrid iterative scheme called a shrinking projection method for a nonexpansive mapping in a Hilbert space. It is an advantage of the hybrid method and the shrinking projection method that strong convergence of iterative sequences is guaranteed without any compact assumptions. These are now powerful methods, which play an important role in finding fixed points of nonlinear mappings in Banach spaces. From this background, many authors have studied iterative methods for finding a fixed point of nonlinear mappings of nonexpansive type in Banach spaces with tolerance requirements which are less restrictive and more constructive than in the classical setting. It is expected that the iterative methods for nonlinear mappings of nonexpansive type can be applied to finding a zero of maximal monotone operators in Banach spaces.

The aim of this thesis is to give new iterative methods for constructing fixed points of nonlinear mappings of nonexpansive type. With this in mind we have divided the thesis into three chapters. In Chapter 1, we explain certain notation, terminologies and basic results used throughout the thesis. In Chapter 2, we prove strong convergence theorems for finding a common element of the set of solutions for a generalized equilibrium problem and the set of common fixed points for countably infinite family of relatively nonexpansive (see Section 2.2) mappings by using the hybrid method in Banach spaces: In contrast to the case of Hilbert spaces, the resolvent of a maximal monotone operator is not generally a nonexpansive mapping in the case of Banach spaces. Recently, Matsushita and Takahashi [46] have introduced the class of relatively nonexpansive mappings in Banach spaces. The class includes all of resolvents of maximal monotone operators with zero points on a uniformly convex and uniformly smooth Banach space and all of nonexpansive mappings with fixed points in a Hilbert space. On the other hand, recent developments in fixed point theory reflect that algorithmic constructions for the approximation of fixed point problems are vigorously purpose and analyzed for various classes of mappings in different spaces. In the recent years, there are many researches concerning the problem of approximating a common fixed point of nonlinear mappings in various classes, by using W-mappings and convex combinations (see Section 2.2). Motivated by these concepts, we investigate the strong convergence theorems for finding a common element of a countably infinite family of relatively nonexpansive mappings and a generalized equilibrium problem by using W-mappings and convex combinations, respectively. It is expected that these results can be applied to generalized equilibrium problems with countably infinite constraints. In Section 2.2, we recall some basic notions and give the definition of W-mappings and convex combinations of mappings. We present and prove our main results which are strong convergence theorems of W-mappings and convex combinations in Section 2.3 and Section 2.4, respectively. In Chapter 3, we prove strong convergence theorems by using the shrinking projection method with respect to Bregman distances. In 1967, Bregman [12] has discovered an elegant and effective technique for the using of the so-called Bregman distance function (see Section 3.2) in the process of designing and analyzing feasibility and optimization algorithms. Many authors have studied iterative methods for approximating fixed points of nonexpansive mappings with respect to the Bregman distance. However, as far as we know, the cases where nonlinear mappings are not Lipschitz continuous with respect to the Bregman distance have not been studied yet. From this background, we introduce new classes of nonlinear mappings which are extensions of asymptotically quasi-nonexpansive mappings with respect to the Bregman distance in the intermediate sense (see Section 3.4). Motivated by the above results, we design new hybrid iterative schemes for finding fixed points of these mappings in reflexive Banach spaces. Our results are generalization of results by [74]. In Section 3.2, we present several preliminary definitions and results. In Section 3.3, we recall the notion of Mosco convergence and two kinds of projection with respect to the Bregman distance. One is the generalization of generalized projection and the other the sunny generalized nonexpansive retraction. In Section 3.4, we introduce new classes of mappings which are extensions of asymptotically quasi-nonexpansive mappings in the intermediate sense. We study the properties of the set of fixed points of these mappings. In Section 3.5, we prove new strong convergence theorems of the shrinking projection method for these mappings.

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Chapter 1

Preliminaries

In this chapter we explain certain notation, terminologies and elementary results used in this thesis.

Throughout this thesis, we denote by $\mathbf{N}$ and $\mathbf{R}$ the sets of all nonnegative integers and real numbers, respectively. Moreover, we assume that E is a real Banach space with the norm $\|\cdot\|$, E^* is the dual space of E and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*. We denote the strong convergence of a sequence $\{x_n\}$ to x by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

1.1 Lipschitzian and nonexpansive mappings

Let C be a nonempty subset of E, T a mapping of C into E and $k \in \mathbf{R}$. The mapping T is said to be *k-Lipschitz continuous* if

$$\|Tx - Ty\| \leq k\|x - y\|$$

for all $x, y \in C$. If $0 \leq k < 1$, the mapping T is called *contraction*. If $k = 1$, the mapping T is said to be *nonexpansive*. The mapping T is said to be *locally Lipschitz continuous* if, for any $x \in C$, there exist a neighbourhood U_x of x and a constant k such that $\|Ty - Tz\| \leq k\|y - z\|$ for all $y, z \in U_x$.

A point $p \in C$ is called a *fixed point* of T if $Tp = p$. We denote by $F(T)$ the set of fixed points of T.

1.2 Monotone operators

A set-valued operator $A \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be *maximal monotone* if $A = B$ for any monotone operator $B \subset E \times E^*$ such that $A \subset B$. Let $\alpha > 0$. An operator A of C into E^* is said to be *α-inverse strongly monotone* if

$$\langle x - y, Ax - Ay \rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$. If A is an α-inverse strongly monotone operator, then A is obviously $1/\alpha$-Lipschitzian.

1.3 Topological spaces

Let $d : E \times E \to [0, \infty)$ be a function. Recall that d is called a *metric* on E if the following properties hold:

(i) *identity of indiscernibles*: $d(x, y) = 0$ if and only if $x = y$ for some $x, y \in E$;

(ii) *symmetry*: $d(x, y) = d(y, x)$ for all $x, y \in E$;

(iii) *triangle inequality*: $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in E$.

A value of metric d at (x, y) is called the *distance between x and y*.

Let C be a subset of E. An element $x \in C$ is said to be an *interior point of C* if there exists $r > 0$ such that $\{y \in E : d(x, y) < r\} \subset E$. The subset C is said to be *open* if every point of C is an interior point of C. The subset C is said to be *closed* if $E \setminus C$ is open. The subset C is said to be *convex* if $tx + (1 - t)y \in C$ for all $x, y \in C$ and $t \in [0, 1]$. The subset C is said to be *bounded* if its diameter $\sup\{d(x, y) : x, y \in C\}$ is finite.

1.4 Convex functions and subdifferentials

Let $f : E \to (-\infty, +\infty]$ be a function. The *effective domain* of f is defined by

$$\text{dom} f := \{x \in E : f(x) < +\infty\}.$$

The function f is said to be *proper* if $\text{dom} f$ is nonempty. We denote by $\text{int}\,\text{dom} f$ the *interior* of the effective domain of f. We denote by $\text{ran} f$ the *range* of f.

The function f is said to be *bounded* if there exists $L > 0$ such that $|f(x)| \leq L < +\infty$ for all $x \in E$. The function f is said to be *locally bounded* if for each $x \in E$, there exist $L > 0$ and a neighborhood B_x of x such that $|f(y)| \leq L < +\infty$ for all $y \in B_x$. The function f is said to be *convex* on E if it satisfies

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in E$ and $\lambda \in [0, 1]$. The function f is said to be *lower semicontinuous* on E if

$$\liminf_{y \to x} f(y) \geq f(x)$$

for all $x \in E$. The function f is said to be *continuous at $x \in E$* if for every net $\{x_\alpha\}$ in E,

$$x_\alpha \to x \quad \text{implies} \quad f(x_\alpha) \to f(x).$$

The function f is said to be *continuous on E* if it is continuous at each point of E.

Proposition 1.4.1 ([6], Proposition 1.2, p. 6)**.** *Let $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function on E. Then f is continuous on* $\text{int}\,\text{dom} f$.

Proof. Let $x_0 \in \text{int}\,\text{dom} f$. Without loss of generality, we assume that $x_0 = 0$ and that $f(0) = 0$. Since the set $\{x \in E : f(x) > -\varepsilon\}$ is open it suffices to show that $\{x \in E : f(x) < \varepsilon\}$ is a neighborhood of the origin. We set $C = \{x \in E : f(x) \leq \varepsilon\} \cap \{x \in E : f(-x) \leq \varepsilon\}$. Clearly, C is a closed balanced set of E, that is, $\alpha x \in C$ for $|\alpha| \leq 1$ and $x \in C$. Moreover, C is absorbing, that is, for every $x \in E$ there exists $\alpha > 0$ such that $\alpha x \in C$, since the function $t \mapsto f(tx)$ is convex and finite in a neighborhood of the origin and therefore continuous. Since E is a Banach space, the preceding properties of C imply that it is a neighborhood of the origin, as claimed. $\square$

Proposition 1.4.2 ([11], Theorem 1.7, p. 66)**.** *Let $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function on E. Then f is locally Lipschitz continuous on* $\operatorname{int}\operatorname{dom}f$.

Proof. Assume that $x \in \operatorname{int}\operatorname{dom}f$. Define $E_n := \{x \in E : f(x) \leq n\}$ for all $n \in \mathbf{N}$. Then E_n are closed subsets of E since f is lower semicontinuous. Moreover, $\operatorname{int}\operatorname{dom}f \subset \bigcup_{n=1}^{\infty} E_n$. By the Baire category theorem, there exists $N \in \mathbf{N}$ such that $\operatorname{int}\operatorname{dom}f \cap \operatorname{int} E_N \neq \emptyset$. Assume that $y \in \operatorname{int}\operatorname{dom}f$ and $\delta > 0$ such that $B(y, \delta) \subset \operatorname{int}\operatorname{dom}f \cap \operatorname{int} E_N$, where $B(y, \delta) := \{z \in \operatorname{int}\operatorname{dom}f : \|z - y\| \leq \delta\}$. Put $\alpha > 0$ small enough and $z = (1+\alpha)x - \alpha y \in \operatorname{int}\operatorname{dom}f$. Since f is convex and $\operatorname{int}\operatorname{dom}f$ is a convex set, we have $[z, B(y, \delta)] \subset \operatorname{int}\operatorname{dom}f$, where $[z, B(y, \delta)]$ is a convex hull of $\{z\} \cup B(y, \delta)$. For any $u \in [z, B(y, \delta)]$, there exist $\lambda \in [0, 1]$ and $v \in B(y, \delta)$ such that $u = \lambda z + (1 - \lambda)v$. Then

$$f(u) \leq \lambda f(z) + (1 - \lambda)f(v) \leq \max\{f(z), n\}.$$

This implies that f is bounded above on $[z, B(y, \delta)]$. Hence $B(x, \alpha\delta/(1 + \alpha)) \subset [z, B(y, \delta)]$.

Since f is locally bounded, there exist $L_x > 0$ and $\delta > 0$ such that $|f| \leq L_x$ on $B(x, 2\delta) \subset \operatorname{int}\operatorname{dom}f$. Put $y, z \in B(x, \delta)$. Set $d := \|y - z\|$ and $u = z + \delta(z - y)/d$. We have $u \in B(x, 2\delta)$ since

$$\|u - x\| = \left\| z - x + \frac{\delta}{d}(z - y) \right\| \leq \|z - x\| + \delta \leq 2\delta.$$

Since $z = (\delta y + du)/(d + \delta)$ and f is convex, we have

$$f(z) \leq \frac{\delta}{d + \delta} f(y) + \frac{d}{d + \delta} f(u).$$

This implies

$$f(z) - f(y) \leq \frac{d}{d + \delta}(f(u) - f(y)) \leq \frac{d}{\delta}(f(u) - f(y)) \leq \frac{2L_x}{\delta}\|y - z\|.$$

Interchanging y and z, we obtain

$$f(y) - f(z) \leq \frac{2L_x}{\delta}\|y - z\|.$$

Therefore $|f(z) - f(y)| \leq L\|y - z\|$ for all $y, z \in B(x, \delta)$, where $L = \frac{2L_x}{\delta}$. □

The *Fenchel conjugate* function of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) := \sup\{\langle \xi, x\rangle - f(x) : x \in E\}.$$

Proposition 1.4.3 ([6], Proposition 1.3, p. 6)**.** *Let $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function on E. Then f^* is also proper, convex and lower semicontinuous on E^*.*

Proof. As supremum of a set of affine functions, f^* is convex and lower semicontinuous. Moreover, by Proposition 1.4.1, we see that $f^* \not\equiv \infty$. □

The function f is said to be *strongly coercive* (*cf.* [83]) if

$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty.$$

We know that f is strongly coercive if and only if f^* is bounded on bounded sets (see [8], Theorem 3.3, p. 10). The function f is said to be *cofinite* if $\mathrm{dom} f^* = E^*$. We know that a strongly coercive function f is cofinite. Moreover, if E is finite-dimensional, then f is cofinite if and only if it is strongly coercive (see [8], Theorem 3.4, p. 10).

Given a proper and convex function $f : E \to (-\infty, +\infty]$, the *subdifferential* of f is a mapping $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) := \{x^* \in E^* : f(y) \geq f(x) + \langle x^*, y - x \rangle, \ \forall y \in E\}$$

for all $x \in E$. In general, ∂f is a multivalued operator from E into E^* not always defined everywhere. If f is proper, convex and lower semicontinuous on E, then ∂f is a maximal monotone operator from E into E^* (see [7], Theorem 2.43, p. 88).

Proposition 1.4.4 ([7], Proposition 2.47, p. 91)**.** *Let $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then the following conditions are equivalent to each other:*

(i) $\mathrm{ran}\, \partial f = E^*$ *and* $\partial f^* = (\partial f)^{-1}$ *is bounded on bounded subsets of* E^**.*

(ii) f *is strongly coercive.*

Proof. (i)$\Rightarrow$(ii): Since f is bounded from below by an affine function, no loss of generality results in assuming that $f \geq 0$ on E. Let $r > 0$. Then, for every $z \in E^*$ with $\|z\| \leq r$, there exist $v \in \mathrm{dom}\, \partial f$ and $R > 0$ such that $z \in \partial f(v)$ and $\|v\| \leq R$. Since $f(u) - f(v) \geq \langle z, u - v \rangle$ for all $u \in E$, we have

$$\langle z, u \rangle \leq f(u) - f(v) + \langle z, v \rangle \leq f(u) + Rr$$

for all $u \in \mathrm{dom} f$ and $z \in E^*$ with $\|z\| \leq r$. Hence

$$f(u) + Rr \geq r\|u\|$$

or

$$\frac{f(u)}{\|u\|} \geq r - \frac{Rr}{\|u\|}$$

for all $u \in E$. This implies that f is strongly coercive.

(ii)$\Rightarrow$(i): Let $x_0 \in \mathrm{dom}\, \partial f$. By the definition of ∂f, we have $\langle \partial f(x), x - x_0 \rangle \geq f(x) - f(x_0)$ for all $x \in \mathrm{dom}\, \partial f$. Then ∂f is coercive, that is, for any $y \in \partial f(x)$,

$$\lim_{\|x\| \to \infty} \frac{\langle y, x - x_0 \rangle}{\|x\|} \geq \lim_{\|x\| \to \infty} \frac{f(x) - f(x_0)}{\|x\|} = +\infty.$$

Since ∂f is maximal monotone and coercive, we have $\mathrm{ran}\, \partial f = E^*$ (see [7], Corolally 1.143, p. 55). Moreover, it is readily seen that the operator $(\partial f)^{-1}$ is bounded on every bounded subsets of E^*. $\square$

Let $f : E \to (-\infty, +\infty]$ be a proper and convex function. Let D be a nonempty open convex subset of E. If $x \in D$, then, for each $y \in E$, the *right-hand directional derivative*

$$f^\circ(x, y) := \lim_{t \downarrow 0^+} \frac{f(x + ty) - f(x)}{t}.$$

exists and defines a sublinear functional on E. (see [53], Lemma 1.2, p. 2).

If f is finite at x, then the difference quotient $t \to t^{-1}\big(f(x+ty) - f(x)\big)$ is monotonically increasing on $(0, \infty)$ for every $y \in E$. Let $x \in \operatorname{int} \operatorname{dom} f$. For any $y \in E$, we define the *directional derivative of f at x in the direction y* by

$$f'(x, y) := \lim_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t}. \tag{1.4.1}$$

The function f is said to be *Gâteaux differentiable at x* if the limit (1.4.1) exists for each $y \in E$. It is immediate from this definition (requiring the existence of a two-sided limit) that f is Gâteaux differentiable at x if and only if $-f^\circ(x, -y) = f^\circ(x, y)$ for each $y \in E$. Since a sublinear functional g is linear if and only if $g(-x) = -g(x)$ for all x, this shows that f is Gâteaux differentiable at x if and only if $y \to f^\circ(x, y)$ is linear in y. In particular, if this is true, then $f'(x, \cdot)$ is a linear functional on E. In this case, we denote the *gradient* of f at x by $\nabla f(x) : E \to (-\infty, +\infty)$ defined by $\langle \nabla f(x), y \rangle = f^\circ(x, y)$ for every $y \in E$. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at each $x \in \operatorname{int} \operatorname{dom} f$. The function f is said to be *Fréchet differentiable at x* if the limit (1.4.1) is attained uniformly in $\|y\| = 1$. The function f is said to be *uniformly Fréchet differentiable on a subset C of E* if the limit (1.4.1) is attained uniformly for $x \in C$ and $\|y\| = 1$.

Proposition 1.4.5 ([4], Corollary 10, p. 150)**.** *Let f be a continuously Fréchet differentiable and convex functional on E. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse strongly monotone.*

Proposition 1.4.6 ([53], Corollary 1.7, p. 5)**.** *If a convex function $f : E \to \mathbf{R}$ is continuous at $x_0 \in \operatorname{dom} f$, then the right-hand derivative of f at x_0 is a continuous sublinear functional on E.*

Proof. Given $x_0 \in \operatorname{dom} f$, there exist a neighborfood B of x_0 and $M > 0$ such that, if $x \in E$, then $f(x_0 + tx) - f(x_0) \leq Mt\|x\|$ provided $t > 0$ is sufficiently small that $x_0 + tx \in B$. Thus, for any $x \in E$,

$$f^\circ(x_0, x) = \lim_{t \downarrow 0^+} \frac{f(x_0 + tx) - f(x_0)}{t} \leq M\|x\|.$$

This implies that $f^\circ(x_0, \cdot)$ is continuous. $\square$

Proposition 1.4.7 ([60], Proposition 2.1, p. 474)**.** *If a convex function $f : E \to \mathbf{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Proof. If this result is not true, there exist a positive number ε and bounded sequences $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ such that $\|x_n - y_n\| \to 0$ as $n \to \infty$ and

$$\langle \nabla f(x_n) - \nabla f(y_n), w_n \rangle \geq 2\varepsilon, \tag{1.4.2}$$

where $\{w_n\}_{n\in\mathbf{N}}$ is a sequence in E with $\|w_n\| = 1$ for $n \in \mathbf{N}$. Since f is unformly Fréchet differentiable, there exists a positive number δ such that

$$f(y_n + tw_n) - f(y_n) - t\langle \nabla f(y_n), w_n \rangle \leq \varepsilon t \tag{1.4.3}$$

for all $0 < t < \delta$ and $n \in \mathbf{N}$. Since f is convex, we have

$$\langle \nabla f(x_n), y_n + tw_n - x_n \rangle \leq f(y_n + tw_n) - f(x_n)$$

for all $n \in \mathbf{N}$. This implies

$$t\langle \nabla f(x_n), w_n \rangle \leq f(y_n + tw_n) + \langle \nabla f(x_n), x_n - y_n \rangle - f(x_n). \tag{1.4.4}$$

By (1.4.2), (1.4.3) and (1.4.4), we have

$$\begin{aligned} 2\varepsilon t &\leq t\langle \nabla f(x_n) - \nabla f(y_n), w_n \rangle \\ &\leq f(y_n + tw_n) - f(y_n) - t\langle \nabla f(y_n), w_n \rangle + \langle \nabla f(x_n), x_n - y_n \rangle + f(y_n) - f(x_n) \\ &\leq \varepsilon t + \langle \nabla f(x_n), x_n - y_n \rangle + f(y_n) - f(x_n). \end{aligned}$$

Since ∇f is bounded on bounded subsets of E (see [15], Proposition 1.1.11, p. 17), $\langle \nabla f(x_n), x_n - y_n \rangle \to 0$ as $n \to \infty$, while $f(y_n) - f(x_n) \to 0$ as $n \to \infty$ since f is uniformly continuous on bounded subsets of E (see [3], Theorem 1.8, p. 13). Therefore $2\varepsilon t \leq \varepsilon t$, which is a contradiction. □

A convex function $f : E \to \mathbf{R}$ is said to be *uniformly convex* if the function $\delta_f : [0, +\infty) \to [0, +\infty]$ defined by

$$\delta_f(t) := \inf\left\{ \frac{1}{2}f(x) + \frac{1}{2}f(y) - f\Big(\frac{x+y}{2}\Big) : \|y - x\| = t,\ x, y \in \mathrm{dom} f \right\}$$

is positive whenever $t > 0$. The function δ_f is called the *modulus of convexity* of f.

Proposition 1.4.8 ([83], Proposition 3.6.4)**.** *Let $f : E \to \mathbf{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertion are equivalent to each other:*

(i) *f is strongly coercive and uniformly convex on bounded subsets of E;*

(ii) *f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of* $\mathrm{dom} f^* = E^*$.

1.5 Geometry of Banach spaces

Let X be a nonempty set and Y a set. A mapping $T : X \to Y$ is said to be *surjective* (or *onto*) if for every $y \in Y$, there exists $x \in X$ such that $T(x) = y$.

A Banach space E is said to be *reflexive* if the natural mapping $E \to E^{**}$ is surjective. A Banach space E is said to have the *Kadec-Klee property* if, for every sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$. It is known that a uniformly convex Banach space has the Kadec-Klee property.

Let E be a Banach space. Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in S(E)$ with $x \neq y$. The Banach space E is said to be *uniformly convex* if, for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in S(E)$,

$$\|x - y\| \geq \varepsilon \quad \text{implies} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

It is well known that each uniformly convex Banach space is reflexive and strictly convex (see [72], Theorems 4.1.2 and 4.1.6, pp. 93–97).

A Banach space E is said to be *smooth* if there exists

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.5.1}$$

for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit (1.5.1) is attained uniformly for all $x, y \in S(E)$. It is well known that every uniformly smooth Banach space is reflexive with uniformly Gâteaux differentiable norm (see [72], Theorems 4.1.6 and 4.3.7, pp. 97 and 111).

A mapping J of E into 2^{E^*} defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$ is called the *normalized duality mapping*. By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in E$. If E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$ (see [72], Theorems 4.2.2 and 4.2.4 pp. 100–102). Moreover, the normalized duality mapping J has the following properties (see [72]).

Proposition 1.5.1 ([72], Theorem 4.2.2, p. 100)**.** *If E is reflexive Banach space, then the normalized duality mapping J of E is surjective.*

Proof. For each $f \in E^*$, by the Hahn-Banach theorem, there exists $u \in E^{**}$ such that $\langle f, u \rangle = \|f\|$ and $\|u\| = 1$. Putting $x = \|f\|u$, we have $\langle f, x \rangle = \langle f, \|f\|u \rangle = \|f\|^2 = \|x\|^2$. Since $E = E^{**}$, we have $f \in J(x)$. This implies that J is a mapping of E onto E^*. $\square$

Proposition 1.5.2 ([72], Theorem 4.2.2, p. 100)**.** *If E^* is strictly convex Banach space, then the normalized duality mapping J of E is single-valued.*

Proof. We know that $J(0) = \{0\}$. Let $x \neq 0$ and $f, g \in J(x)$. We have $\langle x, f \rangle = \|f\|^2 = \|x\|^2 = \|g\|^2 = \langle x, g \rangle$ and hence $2\|x\|^2 = \langle x, f + g \rangle \leq \|x\|\|f + g\|$. Thus $\|f\| + \|g\| = 2\|x\| \leq \|f + g\|$. This implies $\|f + g\| = \|f\| + \|g\|$. Since E^* is strictly convex, we have $g = \alpha f$ for some $\alpha \in \mathbf{R}$. We have $\langle x, f \rangle = \langle x, g \rangle = \langle x, \alpha f \rangle = \alpha \langle x, f \rangle$. This implies $\alpha = 1$ and hence $f = g$. $\square$

Proposition 1.5.3 ([72], Theorems 4.3.1 and 4.3.2, p. 107). *E is a smooth Banach space if and only if the normalized duality mapping J of E is single-valued.*

Proof. First we show the necessity. Since $J(0) = \{0\}$, we assume $x \neq 0$. Let $\|x\| = 1$. For $f \in J(x), y \in S(E)$ and $\lambda > 0$, we have

$$\frac{\langle y, f\rangle}{\|x\|} = \frac{\langle x + \lambda y, f\rangle - \|x\|^2}{\lambda\|x\|} \leq \frac{\|x\|\|x + \lambda y\| - \|x\|^2}{\lambda\|x\|} = \frac{\|x + \lambda y\| - \|x\|}{\lambda}.$$

Similarly, if $\lambda < 0$, we have $\langle y, f\rangle/\|x\| \geq (\|x + \lambda y\| - \|x\|)/\lambda$. By the smoothness of E,

$$\tau(x, y) := \lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

exists and $\langle y, f\rangle = \|x\|\tau(x, y)$. Then $J(x)$ is single-valued. If $x \neq 0$, then $J(x) = \|x\|J(x/\|x\|)$. Therefore $J(x)$ is single-valued for all $x \neq 0$.

Next we show that the sufficiency. Since J is single-valued, J is norm-to-weak* continuous (see [72], Lemma 4.3.3, p. 108). For $x, y \in S(E)$ and $\lambda > 0$,

$$\begin{aligned}
\frac{\langle y, J(x)\rangle}{\|x\|} &\leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \\
&\leq \frac{\|x + \lambda y\|^2 - \langle x, J(x + \lambda y)\rangle}{\lambda\|x + \lambda y\|} \\
&= \frac{\langle x + \lambda y, J(x + \lambda y)\rangle - \langle x, J(x + \lambda y)\rangle}{\lambda\|x + \lambda y\|} \\
&= \frac{\langle \lambda y, J(x + \lambda y)\rangle}{\lambda\|x + \lambda y\|} = \frac{\langle y, J(x + \lambda y)\rangle}{\|x + \lambda y\|}.
\end{aligned}$$

Similarly, if $\lambda < 0$, then

$$\frac{\langle y, J(x)\rangle}{\|x\|} \geq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \geq \frac{\langle y, J(x + \lambda y)\rangle}{\|x + \lambda y\|}.$$

Therefore, we have

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} = \frac{\langle y, J(x)\rangle}{\|x\|}.$$

This implies that E is smooth. □

Proposition 1.5.4 ([72], Theorem 4.3.4, p. 109). *If E^* is uniformly convex, then the normalized duality mapping J of E is uniformly norm-to-norm continuous on each bounded subset of E.*

Proof. Since E^* is uniformly convex, E^* is strictly convex. By Proposition 1.5.2, J is sigle-valued. Suppose that J is not uniformly continuous on some bounded set B of E. Then there exist $\varepsilon > 0$ and sequences $\{x_n\}, \{y_n\}$ in B such that $\|x_n - y_n\| < 1/n$ and $\|J(x_n) - J(y_n)\| \geq \varepsilon$. Suppose $x_n \to 0$ as $n \to \infty$. Then $y_n \to 0$ as $n \to \infty$. By $\|x_n\| = \|J(x_n)\|$ and $\|y_n\| = \|J(y_n)\|$, we have $J(x_n) \to 0$ and $J(y_n) \to 0$ as $n \to \infty$. This contradicts $\|J(x_n) - J(y_n)\| \geq \varepsilon$. Now let $x_n \not\to 0$ as $n \to \infty$. Then there exist $\alpha > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i}\| \geq \alpha$. Since $\|x_n\| - \|y_n\| \leq \|x_n - y_n\| < 1/n$, we can assume $\|y_{n_i}\| \geq \alpha/2$. Thus, without loss of generality, we assume that there exists $\beta > 0$ such that $\|x_n\| \geq \beta$ and $\|y_n\| \geq \beta$. Putting $u_n = x_n/\|x_n\|$ and $v_n = y_n/\|y_n\|$, we have $\|u_n\| = \|v_n\| = 1$ and

$$\begin{aligned}\|u_n - v_n\| &= \left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|} \right\| = \left\| \frac{\|y_n\|x_n - \|x_n\|y_n}{\|x_n\|\|y_n\|} \right\| \\ &\leq \frac{1}{\beta^2}\|(\|y_n\| - \|x_n\|)x_n + \|x_n\|(x_n - y_n)\| \\ &\leq \frac{1}{\beta^2}\left(\Big| \|y_n\| - \|x_n\| \Big| \|x_n\| + \|x_n\|\|x_n - y_n\| \right) \to 0\end{aligned}$$

as $n \to \infty$. Further, since

$$\begin{aligned}\langle u_n, J(u_n) + J(v_n)\rangle &= \|u_n\|^2 + \langle u_n - v_n, J(v_n)\rangle + \|v_n\|^2 \\ &\geq 2 - \|u_n - v_n\|\|J(v_n)\| = 2 - \|u_n - v_n\|,\end{aligned}$$

we have

$$\left\| \frac{J(u_n) + J(v_n)}{2} \right\| \geq \frac{1}{2}\langle u_n, J(u_n) + J(v_n)\rangle \geq 1 - \frac{\|u_n - v_n\|}{2}$$

and hence

$$\liminf_{n\to\infty} \left\| \frac{J(u_n) + J(v_n)}{2} \right\| \geq 1.$$

On the other hand, since

$$\limsup_{n\to\infty} \left\| \frac{J(u_n) + J(v_n)}{2} \right\| \leq \limsup_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2} \right) = 1,$$

we have

$$\lim_{n\to\infty} \left\| \frac{J(u_n) + J(v_n)}{2} \right\| = 1.$$

Since $\|J(u_n)\| = \|J(v_n)\| = 1$ and E^* is uniformly convex, we have $\|J(u_n) - J(v_n)\| \to 0$ as $n \to \infty$. We have

$$J(x_n) - J(y_n) = \|x_n\|(J(u_n) - J(v_n)) + (\|x_n\| - \|y_n\|)J(v_n).$$

This implies $\|J(x_n) - J(y_n)\| \to 0$ as $n \to \infty$, which contradicts $\|J(x_n) - J(y_n)\| \geq \varepsilon > 0$. □

Proposition 1.5.5 ([72], Theorem 4.3.7, p. 111)**.** *Let E be a Banach space. Then, E is uniformly smooth if and only if E^* is uniformly convex.*

Proof. First we show the necessity. Let $f, g \in S(E^*)$ and $\|f - g\| \geq \varepsilon > 0$. Since the norm of E is uniformly Fréchet differentiable, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |t| \leq \delta$ implies

$$\left| \frac{\|x + ty\| - \|x\|}{t} - \langle y, J(x)\rangle \right| < \frac{\varepsilon}{8}$$

for all $x, y \in S(E)$. Fix t with $0 < t < \delta$. Then we have

$$\|x + ty\| < \frac{1}{8}t\varepsilon + t\langle y, J(x)\rangle + 1$$

and

$$\|x - ty\| < \frac{1}{8}t\varepsilon - t\langle y, J(x)\rangle + 1.$$

Hence

$$\|x + ty\| + \|x - ty\| < 2 + \frac{1}{4}t\varepsilon$$

for all $x, y \in S(E)$. By $\|f - g\| \geq \varepsilon > 0$, there exists $y_0 \in S(E)$ such that $(f - g)(y_0) > \varepsilon/2$. Thus

$$\begin{aligned}
\|f + g\| &= \sup\{(f + g)(x) : x \in S(E)\} \\
&= \sup\{f(x + ty_0) + g(x - ty_0) - (f - g)(ty_0) : x \in S(E)\} \\
&< \sup\{\|x + ty_0\| + \|x - ty_0\| - \frac{1}{2}t\varepsilon : x \in S(E)\} \\
&\leq 2 + \frac{1}{4}t\varepsilon - \frac{1}{2}t\varepsilon = 2 - \frac{1}{4}t\varepsilon.
\end{aligned}$$

This implies that E^* is uniformly convex.

Next we show that the sufficiency. Let $x, y \in S(E)$. As in the proof of Proposition 1.5.3, if $\lambda > 0$, then

$$\frac{\langle y, J(x)\rangle}{\|x\|} \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq \frac{\langle y, J(x + \lambda y)\rangle}{\|x + \lambda y\|}$$

and if $\lambda < 0$, then

$$\frac{\langle y, J(x)\rangle}{\|x\|} \geq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \geq \frac{\langle y, J(x + \lambda y)\rangle}{\|x + \lambda y\|}.$$

By Proposition 1.5.4, E has a uniformly Fréchet differentiable norm. $\square$

A function $g : \mathbf{R} \to \mathbf{R}$ is said to be *strictly increasing* if $g(x_1) < g(x_2)$ for all $x_1, x_2 \in \mathbf{R}$ with $x_1 < x_2$. Let $\mathcal{G} = \{g : [0, \infty) \to [0, \infty) : g(0) = 0,\ g$ is continuous, strictly increasing and convex on $[0, \infty)\}$.

Proposition 1.5.6 ([81], Theorem 2, p. 1133)**.** *A Banach space E is uniformly convex if and only if, for every bounded subset B of E, there exists $g_B \in \mathcal{G}$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g_B(\|x - y\|)$$

for all $x, y \in B$ and $0 \leq \lambda \leq 1$.

Proposition 1.5.7 ([81], Theorem 2, p. 1133)**.** *Let $s > 0$. A Banach space E is uniformly convex if and only if there exists $g \in \mathcal{G}$ such that*

$$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, j\rangle + g(\|y\|)$$

for all $x, y \in \{z \in E : \|z\| \leq s\}$ and $j \in Jx$.

Chapter 2

Strong Convergence Theorems for Generalized Equilibrium Problems and Relatively Nonexpansive Mappings

2.1 Introduction

Let E be a real Banach space with the norm $\|\cdot\|$ and C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbf{R}$ be a bifunction and A a nonlinear operator of C into E^*. A *generalized equilibrium problem* is finding $u \in C$ such that

$$f(u, y) + \langle Au, y - u\rangle \geq 0 \tag{2.1.1}$$

for all $y \in C$. The set of solutions of (2.1.1) is denoted by EP, that is,

$$EP = \{u \in C : f(u, y) + \langle Au, y - u\rangle \geq 0, \ \forall y \in C\}.$$

If $A = 0$, then the problem (2.1.1) is equivalent to that of finding a point $u \in C$ such that

$$f(u, y) \geq 0 \tag{2.1.2}$$

for all $y \in C$, which is called an *equilibrium problem.* The set of solutions of (2.1.2) is denoted by $EP(f)$. If $f = 0$, then the problem (2.1.1) is equivalent to that of finding a point $u \in C$ such that

$$\langle Au, y - u\rangle \geq 0 \tag{2.1.3}$$

for all $y \in C$ which is called a *variational inequality.* The set of solutions of (2.1.3) is denoted by $VI(C, A)$. The problem (2.1.1) is very general in the sense that it includes optimization problems, variational inequalities, minimax problems and numerous problems in physics, economics and others. Some methods have been proposed for solving the generalized equilibrium problem, the equilibrium problem and the variational inequality in Hilbert spaces (see [69, 70]) and in Banach spaces (see [46, 76]).

In 2008, Takahashi and Takahashi [70] have proved a strong convergence theorem for finding an element of $F(S) \cap EP$ in a Hilbert space H, where S is a nonexpansive mapping of a nonempty closed convex subset $C \subset H$ into itself and A is an inverse strongly monotone operator of C into H. Recently, Chang, Lee and Chan [19] have considered iterative methods for finding an element of $F(S) \cap F(T) \cap EP$ in a certain Banach space E, where S and T are two relatively nonexpansive (see Section 2.2) mappings of a nonempty closed convex subset $C \subset E$ into itself and A is an inverse strongly monotone operator of C into E^*. On the other hand, Matsushita, Nakajo and Takahashi [44] have introduced iterative methods for finding an element of $\bigcap_{i=0}^{\infty} F(S_i)$, where S_i is a relatively nonexpansive mapping of C into itself for each $i \geq 0$.

In this chapter, motivated by Chang *et al.* [19] and Matsushita *et al.* [44], we introduce new iterative methods for finding an element of $\bigcap_{i=0}^{\infty} F(S_i) \cap EP$, where S_i is a relatively nonexpansive mapping of C into itself for each $i \geq 0$ and A is an inverse-strongly monotone operator of C into E^*.

2.2 Preliminaries

Throughout this chapter, we assume that $\mathcal{G} = \{g : [0, \infty) \to [0, \infty) : g(0) = 0,\ g$ is continuous, strictly increasing and convex on $[0, \infty)\}$.

Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty, closed and convex subset of E. Throughout this paper, we denote by ϕ *Lyapunov functional* $\phi : E \times E \to \mathbf{R}^+$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$ (see [1, 30, 59]). It is obvious that the following conditions:

(ϕ_1) $\phi(x, y) = 0$ if and only if $x = y$;

(ϕ_2) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$.

Proposition 2.2.1 ([30], Proposition 2, p. 940)**.** *Let E be a smooth and uniformly convex Banach space and $\{x_n\}, \{y_n\}$ two sequences of E. If $\phi(x_n, y_n) \to 0$ as $n \to \infty$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Proof. It follows from $\phi(x_n, y_n) \to 0$ as $n \to \infty$ that $\{\phi(x_n, y_n)\}$ is bounded. Then if one of the sequences $\{x_n\}$ and $\{y_n\}$ is bounded, so is the other because of (ϕ_2). By Proposition 1.5.7, there exists a function $g \in \mathcal{G}$ such that

$$\begin{aligned} g(\|x_n - y_n\|) &\leq \|y_n + (x_n - y_n)\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Jy_n\rangle \\ &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n, Jy_n\rangle + 2\|y_n\|^2 \\ &= \phi(x_n, y_n). \end{aligned}$$

It follows from $\phi(x_n, y_n) \to 0$ that $g(\|x_n - y_n\|) \to 0$. Therefore the properties of g yield that $x_n - y_n \to 0$ as $n \to \infty$. $\square$

Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a smooth Banach space. It is obvious from the definition of ϕ that $\phi(x_n, y_n) \to 0$ whenever $\|x_n - y_n\| \to 0$. By this fact and

Proposition 2.2.1, we see that if $\{x_n\}$ and $\{y_n\}$ are two bounded sequences in a uniformly smooth and uniformly convex Banach space, then

$$\|x_n - y_n\| \to 0 \quad \Leftrightarrow \quad \|Jx_n - Jy_n\| \to 0 \quad \Leftrightarrow \quad \phi(x_n, y_n) \to 0.$$

Proposition 2.2.2 ([30], Proposition 3, p. 940)**.** *Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty, closed and convex subset of E and $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$.*

Proof. Note that E is reflexive and that $\|y_n\| \to \infty$ as $n \to \infty$ implies $\phi(y_n, x) \to \infty$ as $n \to \infty$. We see that there exists $x_0 \in C$ such that $\phi(x_0, x) = \inf\{\phi(y, x) : y \in C\}$. Since E is strictly convex, the function $\|\cdot\|^2$ is strictly convex, that is, $\|\lambda x_1 + (1-\lambda)x_2\|^2 < \lambda\|x_1\|^2 + (1-\lambda)\|x_2\|^2$ for all $x_1, x_2 \in E$ with $x_1 \neq x_2$ and $\lambda \in (0, 1)$. Then the function $\phi(\cdot, y)$ is also strictly convex. Therefore $x_0 \in C$ is unique. □

Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty, closed and convex subset of E. Following Alber [1], the *generalized projection* Π_C of E onto C is defined by

$$\Pi_C x = \arg\min_{y \in C} \phi(y, x)$$

for $x \in E$. We have the following results for generalized projections.

Proposition 2.2.3 ([30], Proposition 4, p. 941)**.** *Let E be a smooth Banach space, C a nonempty and convex subset of E, $x \in E$ and $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if $\langle y - x_0, Jx_0 - Jx \rangle \geq 0$ for all $y \in C$.*

Proof. Suppose that $x_0 = \Pi_C x$. Let $y \in C$ and $\lambda \in (0.1)$. It follows from $\phi(x_0, x) \leq \phi((1-\lambda)x_0 + \lambda y, x)$ that

$$\begin{aligned}
0 &\leq \|(1-\lambda)x_0 + \lambda y\|^2 - 2\langle (1-\lambda)x_0 + \lambda y, Jx \rangle + \|x\|^2 - \|x_0\|^2 + 2\langle x_0, Jx \rangle - \|x\|^2 \\
&= \|(1-\lambda)x_0 + \lambda y\|^2 - \|x_0\|^2 - 2\lambda\langle y - x_0, Jx \rangle \\
&\leq 2\lambda\langle y - x_0, J((1-\lambda)x_0 + \lambda y) \rangle - 2\lambda\langle y - x_0, Jx \rangle \\
&= 2\lambda\langle y - x_0, J((1-\lambda)x_0 + \lambda y) - Jx \rangle.
\end{aligned}$$

Letting $\lambda \downarrow 0$, we obtain $\langle y - x_0, Jx_0 - Jx \rangle \geq 0$ since J is norm-to-weak* continuous.

Suppose that $\langle y - x_0, Jx_0 - Jx \rangle \geq 0$ for all $y \in C$. For any $y \in C$, we have

$$\begin{aligned}
\phi(y, x) - \phi(x_0, x) &= \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 - \|x_0\|^2 + 2\langle x_0, Jx \rangle - \|x\|^2 \\
&= \|y\|^2 - \|x_0\|^2 - 2\langle y - x_0, Jx \rangle \\
&\geq 2\langle y - x_0, Jx_0 \rangle - 2\langle y - x_0, Jx \rangle \\
&= 2\langle y - x_0, Jx_0 - Jx \rangle \geq 0.
\end{aligned}$$

This implies $x_0 = \arg\min_{y \in C} \phi(y, x)$. □

Proposition 2.2.4 ([30], Proposition 5, p. 941)**.** *Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty, closed and convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$$

for all $y \in C$.

Proof. By Proposition 2.2.3, we have

$$\begin{aligned}
&\phi(y,x)-\phi(\Pi_C x,x)-\phi(y,\Pi_C x)\\
&=\|y\|^2-2\langle y,Jx\rangle+\|x\|^2-\|\Pi_C x\|^2+2\langle \Pi_C x,Jx\rangle-\|x\|^2\\
&\quad -\|y\|^2+2\langle y,J\Pi_C x\rangle-\|\Pi_C x\|^2\\
&=-2\langle y,Jx\rangle+2\langle \Pi_C x,Jx\rangle+2\langle y,J\Pi_C x\rangle-2\|\Pi_C x\|^2\\
&=2\langle y-\Pi_C x,J\Pi_C x-Jx\rangle\ge 0
\end{aligned}$$

for all $y\in C$. □

A point $p\in C$ is called an *asymptotic fixed point* of T (*cf.* [58]) if there exists a sequence $\{x_n\}\subset C$ such that $x_n \rightharpoonup p$ and $\|x_n-Tx_n\|\to 0$. We denote by $\hat{F}(T)$ the set of all asymptotic fixed points of T. A mapping T is said to be *relatively nonexpansive* (*cf.* [45, 46]) if $\hat{F}(T)=F(T)\neq\emptyset$ and $\phi(u,Tx)\le\phi(u,x)$ for all $u\in F(T)$ and $x\in C$.

Proposition 2.2.5 ([46], Proposition 2.4, p. 260)**.** *Let E be a smooth and strictly convex Banach space, C a nonempty, closed and convex subset of E and T a relatively nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Proof. First we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence of $F(T)$ such that $x_n\to\hat{x}\in C$. By the definition of T, $\phi(x_n,T\hat{x})\le\phi(x_n,\hat{x})$ for each $n\in\mathbf{N}$. This implies

$$\phi(\hat{x},T\hat{x})=\lim_{n\to\infty}\phi(x_n,T\hat{x})\le\lim_{n\to\infty}(x_n,\hat{x})=\phi(\hat{x},\hat{x})=0.$$

This implies $\hat{x}=T\hat{x}$. Hence $\hat{x}\in F(T)$.

Next we show that $F(T)$ is convex. For $x,y\in F(T)$ and $t\in(0,1)$, put $z=tx+(1-t)y$. It is sufficient to show that $Tz=z$. Indeed, we have

$$\begin{aligned}
\phi(z,Tz)&=\|z\|^2-2\langle z,JTz\rangle+\|Tz\|^2\\
&=\|z\|^2-2t\langle x,JTz\rangle-2(1-t)\langle y,JTz\rangle+\|Tz\|^2\\
&=\|z\|^2+t\phi(x,Tz)+(1-t)\phi(y,Tz)-t\|x\|^2-(1-t)\|y\|^2\\
&\le\|z\|^2+t\phi(x,z)+(1-t)\phi(y,z)-t\|x\|^2-(1-t)\|y\|^2\\
&=\|z\|^2-2\langle tx+(1-t)y,JTz\rangle+\|z\|^2\\
&=\|z\|^2-2\langle z,JTz\rangle+\|z\|^2=0.
\end{aligned}$$

This implies $z=Tz$. □

Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty, closed and convex subset of E, $\{S_i\}_{i=0}^{\infty}$ a family of mappings of C into inself and $\{\beta_{n,i}:0\le i\le n\}_{n=0}^{\infty}\subset[0,1]$ a sequence of real numbers. For any $n\ge 0$, let us define a mapping W_n of C into itself as follows:

$$\begin{aligned}
&U_{n,n+1}=I,\\
&U_{n,n}=\Pi_C J^{-1}\big(\beta_{n,n}J(S_nU_{n,n+1})+(1-\beta_{n,n})J\big),
\end{aligned}$$

$$
\begin{aligned}
&U_{n,n-1} = \Pi_C J^{-1}\big(\beta_{n,n-1} J(S_{n-1}U_{n,n}) + (1-\beta_{n,n-1})J\big),\\
&\vdots\\
&U_{n,i} = \Pi_C J^{-1}\big(\beta_{n,i} J(S_i U_{n,i+1}) + (1-\beta_{n,i})J\big),\\
&\vdots\\
&U_{n,1} = \Pi_C J^{-1}\big(\beta_{n,1} J(S_1 U_{n,2}) + (1-\beta_{n,1})J\big),\\
&W_n = U_{n,0} = J^{-1}\big(\beta_{n,0} J(S_0 U_{n,1}) + (1-\beta_{n,0})J\big),
\end{aligned} \tag{2.2.1}
$$

where I is the identity mapping on C. Such a mapping W_n is called a *W-mapping* generated by $\{S_i\}_{i=0}^n$ and $\{\beta_{n,i}\}_{i=0}^n$. We have the following result for the W-mappings.

Proposition 2.2.6 ([44], Proposition 2.5, p. 1468)**.** *Let E be a uniformly smooth and strictly convex Banach space, C a nonempty, closed and convex subset of E and $\{S_i\}_{i=0}^n$ a family of relatively nonexpansive mappings of C into itself such that $\bigcap_{i=0}^n F(S_i) \neq \emptyset$. Let $\{\beta_{n,i}\}_{i=0}^n$ be a sequence of real numbers such that $0 < \beta_{n,0} \leq 1$ and $0 < \beta_{n,i} < 1$ for every $1 \leq i \leq n$. Let $\{U_{n,i}\}_{i=0}^{n+1}$ be a sequence defined by (2.2.1) and W_n the W-mapping generated by $\{S_i\}_{i=0}^n$ and $\{\beta_{n,i}\}_{i=0}^n$. Then the following hold:*

(i) *$F(W_n) = \bigcap_{i=0}^n F(S_i)$;*

(ii) *for every $0 \leq i \leq n$, $x \in C$ and $z \in F(W_n)$, $\phi(z, U_{n,i}x) \leq \phi(z, x)$ and $\phi(z, S_i U_{n,i+1}x) \leq \phi(z, x)$.*

Proof. (i): It is obvious that $\bigcap_{i=1}^n F(S_i) \subset F(W_n)$. Suppose that $u \in \bigcap_{i=1}^n F(S_i)$ and $z \in F(W_n)$. By Propositions 1.5.6 and 2.2.4, we have

$$
\begin{aligned}
\phi(u,z) &= \phi(u, W_n z)\\
&= \phi(u, J^{-1}(\beta_{n,0} J(S_0 U_{n,1} z) + (1-\beta_{n,0}) Jz))\\
&= \|u\|^2 - 2\langle u, \beta_{n,0} J(S_0 U_{n,1} z) + (1-\beta_{n,0}) Jz\rangle + \|\beta_{n,0} J(S_0 U_{n,1} z) + (1-\beta_{n,0}) Jz\|^2\\
&\leq \|u\|^2 - 2\beta_{n,0}\langle u, J(S_0 U_{n,1} z)\rangle - 2(1-\beta_{n,0})\langle u, Jz\rangle + \beta_{n,0}\|S_0 U_{n,1} z\|^2\\
&\quad + (1-\beta_{n,0})\|z\|^2 - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} z) - Jz\|)\\
&= \beta_{n,0}\phi(u, S_0 U_{n,1} z) + (1-\beta_{n,0})\phi(u,z) - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} z) - Jz\|)\\
&\leq \beta_{n,0}\phi(u, U_{n,1} z) + (1-\beta_{n,0})\phi(u,z) - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} z) - Jz\|)\\
&\leq \beta_{n,0}\phi(u, J^{-1}(\beta_{n,1} J(S_1 U_{n,2} z) + (1-\beta_{n,1}) Jz)) + (1-\beta_{n,0})\phi(u,z)\\
&\quad - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} z) - Jz\|)\\
&\leq \beta_{n,0}\{\beta_{n,1}\phi(u, U_{n,2} z) + (1-\beta_{n,1})\phi(u,z) - \beta_{n,1}(1-\beta_{n,1}) g(\|J(S_1 U_{n,2} z) - Jz\|)\}\\
&\quad + (1-\beta_{n,0})\phi(u,z) - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} z) - Jz\|)\\
&\leq \cdots\\
&\leq \phi(u,z) - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} z) - Jz\|)\\
&\quad - \beta_{n,0}\beta_{n,1}(1-\beta_{n,1}) g(\|J(S_1 U_{n,2} z) - Jz\|) - \cdots\\
&\quad - \beta_{n,0}\beta_{n,1}\cdots\beta_{n,n}(1-\beta_{n,n}) g(\|J(S_n U_{n,n+1} z) - Jz\|)
\end{aligned}
$$

for some $g \in \mathcal{G}$. This implies

$$
g(\|J(S_1 U_{n,2} z) - Jz\|)) = \cdots = g(\|J(S_n z) - Jz\|) = 0.
$$

Hence $S_k z = z$ and $U_{n,k}z = z$ for $k = 1, 2, \ldots, n$. If $\beta_{n,0} < 1$, then $S_0 z = z$ since $\|J(S_0U_{n,1}z) - Jz\| = 0$. On the other hand, if $\beta_{n,0} = 1$, then $S_0 z = z$ since $z = W_n z = S_0U_{n,1}z$. Therefore $z \in \bigcap_{i=1}^{n} F(S_i)$, that is, $F(W_n) \subset \bigcap_{i=1}^{n} F(S_i)$.

(ii): Suppose that $k = 0, 1, \ldots, n$, $x \in C$ and $z \in F(W_n)$. As in the proof of (i),

$$\begin{aligned}
\phi(z, U_{n,k}x) &\leq \beta_{n,k}\phi(z, S_kU_{n,k+1}x) + (1 - \beta_{n,k})\phi(z, x) \\
&\leq \beta_{n,k}\phi(z, U_{n,k+1}x) + (1 - \beta_{n,k})\phi(z, x) \\
&\leq \beta_{n,k}\{\beta_{n,k+1}\phi(z, U_{n,k+2}x) + (1 - \beta_{n,k+1})\phi(z, x)\} + (1 - \beta)\phi(z, x) \\
&\leq \cdots \leq \phi(z, x).
\end{aligned}$$

This implies $\phi(z, S_kU_{n,k+1}x) \leq \phi(z, U_{n,k+1}x) \leq \phi(z, x)$ for every $k = 0, 1, \ldots, n$. □

Let E be a smooth and uniformly convex, C a nonempty, closed and convex subset of E, $\{S_i\}_{i=0}^{\infty}$ a family of relatively nonexpansive mappings of C into itself and $\{\lambda_{n,i} : 0 \leq i \leq n\}_{n=0}^{\infty} \subset [0, 1]$ a sequence of real numbers. For any $n \geq 0$, let V_n be a mapping of C into E defined by

$$V_n = J^{-1} \sum_{i=0}^{n} \lambda_{n,i} JS_i. \tag{2.2.2}$$

We have the following result for convex combinations of relatively nonexpansive mappings.

Proposition 2.2.7 ([44], Proposition 2.6, p. 1469)**.** *Let E be a smooth and uniformly convex Banach space, C a nonempty, closed and convex subset of E and $\{S_i\}_{i=0}^{\infty}$ a family of relatively nonexpansive mappings of C into itself such that $\bigcap_{i=0}^{\infty} F(S_i) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=0}^{n} \subset [0, 1]$ such that $\sum_{i=0}^{n} \lambda_{n,i} = 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \lambda_{n,i} > 0$ for each $i \geq 0$. Let V_n be a mapping of C into E defined by* (2.2.2). *Then the following hold:*

(i) $\bigcap_{n=0}^{\infty} F(V_n) = \bigcap_{i=0}^{\infty} F(S_i)$;

(ii) *for every $n \geq 0$, $x \in C$ and $z \in \bigcap_{i=0}^{\infty} F(S_i)$, $\phi(z, V_n x) \leq \phi(z, x)$.*

Proof. (i): It is obvious that $\bigcap_{i=0}^{\infty} F(S_i) \subset \bigcap_{n=0}^{\infty} F(V_n)$. Suppose that $u \in \bigcap_{i=0}^{\infty} F(S_i)$ and $z \in \bigcap_{n=0}^{\infty} F(V_n)$. For large enough $n \in \mathbf{N}$ and $1 \leq l < m \leq n$, by Proposition 1.5.6, we have

$$\begin{aligned}
&\phi(u, z) \\
&= \phi(u, V_n z) \\
&= \|u\|^2 - 2\left\langle u, \sum_{i=0}^{n} \lambda_{n,i} J(S_i x_n) \right\rangle + \left\| \sum_{i=0}^{n} \lambda_{n,i} J(S_i z) \right\|^2 \\
&= \|u\|^2 - 2\left\langle u, \sum_{i=0}^{n} \lambda_{n,i} J(S_i x_n) \right\rangle \\
&\quad + \left\| (\lambda_{n,l} + \lambda_{n,m}) \frac{\lambda_{n,l} J(S_l z) + \lambda_{n,m} J(S_m z)}{\lambda_{n,l} + \lambda_{n,m}} + (1 - (\lambda_{n,l} + \lambda_{n,m})) \frac{\sum_{\substack{i=0,1,\ldots,m \\ i \neq l,m}} \lambda_{n,i} J(S_i z)}{1 - (\lambda_{n,l} + \lambda_{n,m})} \right\|^2 \\
&\leq \|u\|^2 - 2\left\langle u, \sum_{i=0}^{n} \lambda_{n,i} J(S_i z) \right\rangle + (\lambda_{n,l} + \lambda_{n,m}) \left\| \frac{\lambda_{n,l} J(S_l z) + \lambda_{n,m} J(S_m z)}{\lambda_{n,l} + \lambda_{n,m}} \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ (1-(\lambda_{n,l}+\lambda_{n,m}))\left\| \frac{\sum_{\substack{i=0,1,\dots,n\\ i\neq l}} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\|^2 \\
&\le \|u\|^2 - 2\left\langle u, \sum_{i=0}^{n} \lambda_{n,i} J(S_i z) \right\rangle + (\lambda_{n,l}+\lambda_{n,m}) \left\{ \frac{\lambda_{n,l}}{\lambda_{n,l}+\lambda_{n,m}} \|J(S_l z)\|^2 \right. \\
&\left. + \frac{\lambda_{n,m}}{\lambda_{n,l}+\lambda_{n,m}} \|J(S_m z)\|^2 - \frac{\lambda_{n,l}}{\lambda_{n,l}+\lambda_{n,m}} \cdot \frac{\lambda_{n,m}}{\lambda_{n,l}+\lambda_{n,m}} g(\|J(S_l z) - J(S_m z)\|) \right\} \\
&+ \sum_{\substack{i=0,1,\dots,n\\ i\neq l}} \lambda_{n,i} \|J(S_i z)\|^2 \\
&= \sum_{i=1}^{n} \lambda_{n,i} \{\|u\|^2 - 2\langle u, J(S_i z)\rangle + \|J(S_i z)\|^2\} - \frac{\lambda_{n,l}\lambda_{n,m}}{\lambda_{n,l}+\lambda_{n,m}} g(\|J(S_l z) - J(S_m z)\|) \\
&= \sum_{i=1}^{n} \lambda_{n,i} \phi(u, S_i z) - \frac{\lambda_{n,l}\lambda_{n,m}}{\lambda_{n,l}+\lambda_{n,m}} g(\|J(S_l z) - J(S_m z)\|) \\
&\le \sum_{i=1}^{n} \lambda_{n,i} \phi(u, z) - \frac{\lambda_{n,l}\lambda_{n,m}}{\lambda_{n,l}+\lambda_{n,m}} g(\|J(S_l z) - J(S_m z)\|) \\
&= \phi(u, z) - \frac{\lambda_{n,l}\lambda_{n,m}}{\lambda_{n,l}+\lambda_{n,m}} g(\|J(S_l z) - J(S_m z)\|)
\end{aligned}
$$

for some $g \in \mathcal{G}$. We have $g(\|J(S_l z) - J(S_m z)\|) = 0$ since $\lambda_{n,l}, \lambda_{n,m} > 0$ for large enough $n \in \mathbf{N}$. This implies $J(S_l z) = J(S_m z)$, that is, $S_l z = S_m z$ for every $l, m \in \mathbf{N}$ with $l \neq m$. Therefore $z \in \bigcap_{i=0}^{\infty} F(S_i)$.

(ii): Suppose that $n \in \mathbf{N}$, $x \in C$ and $z \in \bigcap_{i=0}^{\infty} F(S_i)$. As in the proof of (i), we have

$$
\begin{aligned}
\phi(z, V_n x) &= \|z\|^2 - 2\left\langle z, \sum_{i=1}^{n} \lambda_{n,i} J(S_i x) \right\rangle + \left\| \sum_{i=1}^{n} \lambda_{n,i} J(S_i x) \right\|^2 \\
&\le \|z\|^2 - 2\sum_{i=1}^{n} \lambda_{n,i} \langle z, J(S_i x)\rangle + \sum_{i=1}^{n} \lambda_{n,i} \|J(S_i x)\|^2 \\
&= \sum_{i=1}^{n} \lambda_{n,i} \phi(z, S_i x) \le \phi(z, x).
\end{aligned}
$$

$\square$

Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty, closed and convex subset of E. For solving the equilibrium problem, let us assume that a bifunction $f : C \times C \to \mathbf{R}$ satisfies the following conditions:

(A_1) $f(x, x) = 0$ for all $x \in C$;
(A_2) f is monotone, that is, $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;
(A_3) f is *upper-hemicontinuous*, that is, $\limsup_{t\downarrow 0} f(x + t(z - x), y) \le f(x, y)$ for all $x, y, z \in C$;
(A_4) the function $y \mapsto f(x, y)$ is convex and lower semicontinuous for all $x \in C$.

Proposition 2.2.8 ([10]). *Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty, closed and convex subset of E. Let $f : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (A_1)–(A_4). Let $r > 0$ and $x \in E$. Then there exists $u \in C$ such that*

$$f(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0$$

for all $y \in C$.

Proposition 2.2.9 ([76], Lemma 2.8, p. 47). *Let E be a uniformly smooth and strictly convex Banach space, C a nonempty, closed and convex subset of E and $f : C \times C \to \mathbf{R}$ a bifunction satisfying (A_1)–(A_4). For $r > 0$ and $x \in E$, define a mapping T_r of E into C as follows:*

$$T_r(x) = \{u \in C : f(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \ \forall y \in C\} \tag{2.2.3}$$

for all $x \in E$. Then the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is a firmly nonexpansive-type mapping (cf. [37]), that is,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$$

for all $x, y \in E$;

(iii) *$F(T_r) = \hat{F}(T_r) = EP(f)$;*

(iv) *$EP(f)$ is a closed and convex subset of C.*

Proof. (i): For $x \in E$ and $r > 0$, let $z_1, z_2 \in T_r x$. Then

$$f(z_1, z_2) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jx\rangle \geq 0$$

and

$$f(z_2, z_1) + \frac{1}{r}\langle z_1 - z_2, Jz_2 - Jx\rangle \geq 0.$$

Adding the two inequalities above, we have

$$f(z_1, z_2) + f(z_2, z_1) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq 0.$$

By (A_2) and $r > 0$, we have $\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq 0$. Since E is strictly convex, we have $z_1 = z_2$.

(ii): For $x, y \in E$, we have

$$f(T_r x, T_r y) + \frac{1}{r}\langle T_r y - T_r x, JT_r x - Jx\rangle \geq 0$$

and

$$f(T_r y, T_r x) + \frac{1}{r}\langle T_r x - T_r y, JT_r y - Jy\rangle \geq 0.$$

Adding the two inequalities above, we have

$$f(T_r x, T_r y) + f(T_r y, T_r x) + \frac{1}{r}\langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy\rangle \geq 0.$$

By (A_2) and $r > 0$, we have

$$\langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy \rangle \geq 0.$$

Therefore, for any $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle.$$

(iii): We have the following:

$$\begin{aligned} u \in F(T_r) &\Leftrightarrow u = T_r u \\ &\Leftrightarrow f(u, y) + \frac{1}{r}\langle y - u, Ju - Ju \rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow f(u, y) \geq 0, \quad \forall y \in C \\ &\Leftrightarrow u \in EP(f). \end{aligned}$$

Next we show that $\hat{F}(T_r) = EP(f)$. Let $p \in \hat{F}(T_r)$. Thus there exists $z_n \in E$ such that $z_n \rightharpoonup p$ and $z_n - T_r z_n \to 0$ as $n \to \infty$. Moreover, we have $T_r z_n \rightharpoonup p$. Hence $p \in C$. Since J is uniformly continuous on bounded sets, we have

$$\lim_{n \to \infty} \frac{1}{r} \|Jz_n - JT_r z_n\| = 0.$$

By the definition of T_r, we have

$$f(T_r z_n, y) + \frac{1}{r}\langle y - T_r z_n, JT_r z_n - Jz_n \rangle \geq 0.$$

Since

$$\frac{1}{r}\langle y - T_r z_n, JT_r z_n - Jz_n \rangle \geq -f(T_r z_n, y) \geq f(y, T_r z_n)$$

and f is lower semicontinuous and convex in the second variable, we have

$$0 \geq \liminf_{n \to \infty} f(y, T_r z_n) \geq f(y, p).$$

Hence $f(y, p) \leq 0$ for all $y \in C$. Let $y \in C$ and set $x_t = ty + (1 - t)p$ for $t \in (0, 1]$. Thus we have

$$0 = f(x_t, x_t) \leq tf(x_t, y) + (1 - t)f(x_t, p) \leq tf(x_t, y).$$

Dividing by t, we obtain $f(x_t, y) \geq 0$. Letting $t \downarrow 0$, by (A_3), we have $f(p, y) \geq 0$ for all $y \in C$. Therefore $p \in EP(f)$.

(iv): By (ii), we have

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle \tag{2.2.4}$$

for all $x, y \in E$. Moreover, we obtain

$$\begin{aligned} &\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \\ &= 2\|T_r x\|^2 - 2\langle T_r x, JT_r y \rangle - 2\langle T_r y, JT_r x \rangle + 2\|T_r y\|^2 \end{aligned}$$

$$
\begin{aligned}
&= 2\langle T_r x, JT_r x - JT_r y\rangle + 2\langle T_r y, JT_r y - JT_r x\rangle \\
&= 2\langle T_r x - T_r y, JT_r x - JT_r y\rangle
\end{aligned} \tag{2.2.5}
$$

and

$$
\begin{aligned}
&\phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y) \\
&= \|T_r x\|^2 - 2\langle T_r x, Jy\rangle + \|y\|^2 + \|T_r y\|^2 - 2\langle T_r y, Jx\rangle + \|x\|^2 \\
&\quad - \|T_r x\|^2 + 2\langle T_r x, Jx\rangle - \|x\|^2 - \|T_r y\|^2 + 2\langle T_r y, Jy\rangle - \|y\|^2 \\
&= 2\langle T_r x, Jx - Jy\rangle - 2\langle T_r y, Jx - Jy\rangle \\
&= 2\langle T_r x - T_r y, Jx - Jy\rangle.
\end{aligned} \tag{2.2.6}
$$

By (2.2.4), (2.2.5) and (2.2.6), we have

$$
\begin{aligned}
\phi(T_r x, T_r y) + \phi(T_r y, T_r x) &\le \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y) \\
&\le \phi(T_r x, y) + \phi(T_r y, x)
\end{aligned} \tag{2.2.7}
$$

for all $x, y \in C \subset E$. Taking $y = p \in F(T_r)$, we obtain $\phi(p, T_r x) \le \phi(p, x)$. By (iii), T_r is relatively nonexpansive on C. By Proposition 2.2.5, $EP(f) = F(T_r)$ is a closed and convex subset of C. □

Proposition 2.2.10 ([76], Lemma 2.9, p. 50). *Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty, closed and convex subset of E, $f : C \times C \to \mathbf{R}$ a bifunction satisfying (A_1)–(A_4) and $r > 0$. Let T_r be the mapping defined by* (2.2.3)*. Then*

$$\phi(p, T_r x) + \phi(T_r x, x) \le \phi(p, x)$$

for all $p \in F(T_r)$ and $x \in E$.

Proof. By (2.2.7), we have

$$\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \le \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y)$$

for all $x, y \in E$. Letting $y = q \in F(T_r)$, we have $\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x)$. □

For solving the generalized equilibrium problem, let us assume that a nonlinear operator A of C into E^* is an α-inverse strongly monotone and a bifunction $f : C \times C \to \mathbf{R}$ satisfies the conditions (A_1)–(A_4).

Proposition 2.2.11 ([19], Lemma 2.5, p. 2262). *Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty, closed and convex subset of E and A an α-inverse strongly monotone operator of C into E^*. Let $f : C \times C \to \mathbf{R}$ be a bifunction satisfying (A_1)–(A_4) and $g : C \times C \to \mathbf{R}$ a bifunction defined by*

$$g(x, y) = f(x, y) + \langle Ax, y - x\rangle$$

for all $x, y \in C$. Let $r > 0$ and $x \in E$. Then g satisfies (A_1)–(A_4) and there exists $u \in C$ such that

$$g(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \ge 0$$

for all $y \in C$.

Proof. By the definition of g, (A_1) is satisfied. Since f satisfies (A_3) and $A : C \to E^*$ is α-inverse strongly monotone, we have

$$\begin{aligned} g(x,y) + g(y,x) &= f(x,y) + f(y,x) + \langle Ax, y-x\rangle + \langle Ay, x-y\rangle \\ &\le 0 + \langle Ax - Ay, y - x\rangle \\ &\le -\alpha\|Ax - Ay\|^2 \le 0. \end{aligned}$$

This implies that g satisfies (A_2). Since A is α-inverse strongly monotone, it is easy to see that A is $1/\alpha$-Lipschitzian continuous. Again, since f satisfies (A_3), we have

$$\begin{aligned} &\limsup_{t\downarrow 0} g(x + t(z-x), y) \\ &= \limsup_{t\downarrow 0}\{f(x+t(z-x),y) + \langle A(x+t(z-x)), y-(x+t(z-x))\rangle\} \\ &\le f(x,y) + \lim_{t\downarrow 0}\{\langle A(x+t(z-x)), y-(x+t(z-x))\rangle\} \\ &= f(x,y) + \langle A(x), y-x\rangle = g(x,y). \end{aligned}$$

This implies that g satisfies (A_3). By assumption, the function $y \mapsto f(x,y)$ is convex and lower semicontinuous. Since the function $y \mapsto \langle Ay - x\rangle$ is convex and continuous, the function $y \mapsto g(x,y)$ is convex and lower semicontinuous, that is, g satisfies (A_4). By Proposition 2.2.8, the conclusion of Proposition 2.2.11 is obtained. □

Proposition 2.2.12 ([19], Lemma 2.6, p. 2263)**.** *Let E be a uniformly smooth and strictly convex Banach space, C a nonempty, closed and convex subset of E, A an α-inverse strongly monotone operator of C into E^* and $f : C \times C \to \mathbf{R}$ a bifunction satisfying (A_1)–(A_4). For any $r > 0$ and $x \in E$, define a mapping K_r of E into C as follows:*

$$K_r(x) = \{u \in C : f(u,y) + \langle Au, y-u\rangle + \frac{1}{r}\langle y-u, Ju - Jx\rangle \ge 0,\ \forall y \in C\}$$

for all $x \in E$. Then the following hold:

(i) *K_r is single-valued;*

(ii) *K_r is a firmly nonexpansive-type mapping, that is,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \le \langle K_r x - K_r y, Jx - Jy\rangle$$

for all $x, y \in E$;

(iii) *$F(K_r) = \hat{F}(K_r) = EP$;*

(iv) *EP is a closed and convex subset of C;*

(v) *$\phi(p, K_r x) + \phi(K_r x, x) \le \phi(p, x)$ for all $p \in F(K_r)$.*

Moreover, the mapping K_r is relatively nonexpansive.

Proof. Putting $g(x,y) = f(x,y) + \langle Ax, y-x\rangle$ for all $x, y \in C$. By Proposition 2.2.11, the function $g : C \times C \to \mathbf{R}$ satisfies the conditions (A_1)–(A_4). We rewrite the mapping $K_r : E \to C$ as

$$K_r(x) = \{u \in C : g(u,y) + \frac{1}{r}\langle y-u, Ju-Jx\rangle \ge 0,\ \forall y \in C\}.$$

By Propositions 2.2.9 and 2.2.10, the conclusion of Proposition 2.2.12 is obtained. □

2.3 Strong convergence theorems of W-mappings

In this section, we prove strong convergence theorems of W-mappings for finding a common element of the set of solutions for a generalized equilibrium problem and the set of common fixed points of infinite relatively nonexpansive mappings in a Banach space.

Theorem 2.3.1 ([77], Theorem 3.1, p. 287)**.** *Let E be a uniformly smooth and uniformly convex Banach space and C a nonempty, closed and convex subset of E. Let $f : C \times C \to \mathbf{R}$ be a bifunction satisfying (A_1)–(A_4) and $\{S_i\}_{i=0}^{\infty}$ an infinite family of relatively nonexpansive mappings of C into itself such that $F := \bigcap_{i=0}^{\infty} F(S_i) \cap EP(f) \neq \emptyset$. Let $\{\beta_{n,i}\}_{i=0}^{n} \subset (0,1)$ be a sequence of real numbers such that $\liminf_{n\to\infty} \beta_{n,i}(1-\beta_{n,i}) > 0$ and W_n the W-mapping generated by $\{S_i\}_{i=0}^{n}$ and $\{\beta_{n,i}\}_{i=0}^{n}$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ y_n = W_n x_n, \\ u_n \in T_{\gamma_n} y_n, \text{ that is, } f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \text{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases} \tag{2.3.1}$$

for $n \geq 0$, where $\Pi_{C_n\cap Q_n}$ is the generalized projection of E onto $C_n \cap Q_n$ and $\{\gamma_n\} \subset [r, \infty)$ for some $r > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. We divide the proof into six steps.

Step 1. We prove that $C_n \cap Q_n \subset C$ is closed and convex for all $n \geq 0$. In fact, it is obvious that C_n is closed and Q_n is closed and convex for all $n \geq 0$. It follows that C_n is convex for all $n \geq 0$ since $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2.$$

Thus $C_n \cap Q_n$ is closed and convex for all $n \geq 0$.

Step 2. We prove that $F \subset C_n \cap Q_n$ for all $n \geq 0$. Let $u_n = T_{\gamma_n} y_n$ for all $n \geq 0$ and $u \in F$. By Proposition 2.2.6 (i), we have $u \in F(W_n)$ for all $n \geq 0$. By Proposition 2.2.9, we obtain T_{γ_n} is relatively nonexpansive. Since S_i is also relatively nonexpansive for all $n \geq 0$, by Proposition 2.2.6 (ii), we have

$$\begin{aligned} \phi(u, u_n) &= \phi(u, T_{\gamma_n} y_n) \leq \phi(u, y_n) = \phi(u, W_n x_n) \\ &= \phi\Big(u, J^{-1}\big(\beta_{n,0} J(S_0 U_{n,1} x_n) + (1-\beta_{n,0}) Jx_n\big)\Big) \\ &= \|u\|^2 - 2\langle u, \beta_{n,0} J(S_0 U_{n,1} x_n) + (1-\beta_{n,0}) Jx_n\rangle \\ &\quad + \|\beta_{n,0} J(S_0 U_{n,1} x_n) + (1-\beta_{n,0}) Jx_n\|^2 \\ &\leq \|u\|^2 - 2\beta_{n,0}\langle u, J(S_0 U_{n,1} x_n)\rangle - 2(1-\beta_{n,0})\langle u, Jx_n\rangle \\ &\quad + \beta_{n,0}\|S_0 U_{n,1} x_n\|^2 + (1-\beta_{n,0})\|x_n\|^2 \\ &= \beta_{n,0}\phi(u, S_0 U_{n,1} x_n) + (1-\beta_{n,0})\phi(u, x_n) \end{aligned}$$

$$\leq \beta_{n,0}\phi(u,x_n) + (1-\beta_{n,0})\phi(u,x_n) = \phi(u,x_n). \tag{2.3.2}$$

This implies $u \in C_n$ and so $F \subset C_n$ for all $n \geq 0$. By induction, now we prove that $F \subset C_n \cap Q_n$ for all $n \geq 0$. In fact, since $Q_0 = C$, we have $F \subset C_0 \cap Q_0$. Suppose that $F \subset C_k \cap Q_k$ for some $k \geq 0$. Then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \Pi_{C_k \cap Q_k} x_0$. By the definition of x_{k+1}, we have

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1} \rangle \geq 0 \tag{2.3.3}$$

for all $z \in C_k \cap Q_k$. Since $F \subset C_k \cap Q_k$, we obtain (2.3.3) for all $z \in F$. This implies $z \in Q_{k+1}$, and so $F \subset Q_{k+1}$. Therefore $F \subset C_n \cap Q_n$ for all $n \geq 0$.

Step 3. We prove that $\{x_n\}$ is bounded. By the definition of Q_n and Proposition 2.2.3, we have $x_n = \Pi_{Q_n} x_0$ for all $n \geq 0$. Hence, by Proposition 2.2.4,

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \leq \phi(u, x_0)$$

for all $u \in F \subset Q_n$ and $n \geq 0$. This implies that $\{\phi(x_n, x_0)\}$ is bounded, and so $\{x_n\}$ and $\{u_n\}$ are bounded in C.

Step 4. We prove that $\|x_n - u_n\| \to 0$ and $\|Jx_n - Ju_n\| \to 0$. Since $x_n = \Pi_{Q_n} x_0$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$, we have $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$ for all $n \geq 0$. This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing, and so there exists the limit $\lim_{n\to\infty} \phi(x_n, x_0)$. By Proposition 2.2.4, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{Q_n} x_0) &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0)\end{aligned}$$

for all $n \geq 0$. This implies

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.3.4}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, by the definition of C_n, we obtain

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \tag{2.3.5}$$

Since E is smooth and uniformly convex, by (2.3.4), (2.3.5) and Proposition 2.2.1, we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.3.6}$$

Since J is uniformly continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{2.3.7}$$

Step 5. We prove that $\omega(\{x_n\}) \subset F$, where $\omega(\{x_n\})$ is the set consisting all of the weak limits points of $\{x_n\}$. In fact, for any $p \in \omega(\{x_n\})$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup p$. We shall prove that $p \in \bigcap_{i=0}^{\infty} F(S_i)$. We have

$$\phi(u, x_n) - \phi(u, u_n) = \|x_n\|^2 - \|u_n\|^2 + 2\langle u, Ju_n - Jx_n \rangle$$

$$\leq \big|\|x_n\| - \|u_n\|\big|(\|x_n\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_n\|$$
$$\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_n\| \tag{2.3.8}$$

for all $n \geq 0$. By (2.3.6) and (2.3.7), we obtain

$$\lim_{n\to\infty}\big(\phi(u, x_n) - \phi(u, u_n)\big) = 0. \tag{2.3.9}$$

By Proposition 2.2.6 (ii), we have

$$\phi(u, U_{n,i}x_n) \leq \phi(u, x_n) \quad \text{and} \quad \phi(u, S_iU_{n,i+1}x_n) \leq \phi(u, U_{n,i+1}x_n) \leq \phi(u, x_n)$$

for each $0 \leq i \leq n$. Thus $\{S_iU_{n,i+1}x_n\}_{n\geq i}$ and $\{U_{n,i}x_n\}_{n\geq i}$ are bounded sequences in C for all $i \geq 0$. By Propositions 1.5.6, 2.2.4 and 2.2.6 (ii), we have

$$\begin{aligned}
\phi(u, U_{n,i}x_n) &\leq \phi\Big(u, J^{-1}\big(\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\big)\Big) \\
&\quad - \phi\Big(U_{n,i}x_n, J^{-1}\big(\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\big)\Big) \\
&= \|u\|^2 - 2\langle u, \beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\rangle \\
&\quad + \|\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\|^2 \\
&\quad - \phi\Big(U_{n,i}x_n, J^{-1}\big(\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\big)\Big) \\
&= \|u\|^2 - 2\langle u, \beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\rangle \\
&\quad + \beta_{n,i}\|S_iU_{n,i+1}x_n\|^2 + (1-\beta_{n,i})\|x_n\|^2 \\
&\quad - \beta_{n,i}(1-\beta_{n,i})g(\|J(S_iU_{n,i+1}x_n) - Jx_n\|) \\
&\quad - \phi\Big(U_{n,i}x_n, J^{-1}\big(\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\big)\Big) \\
&= \beta_{n,i}\phi(u, S_iU_{n,i+1}x_n) + (1-\beta_{n,i})\phi(u, x_n) \\
&\quad - \beta_{n,i}(1-\beta_{n,i})g(\|J(S_iU_{n,i+1}x_n) - Jx_n\|) \\
&\quad - \phi\Big(U_{n,i}x_n, J^{-1}\big(\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\big)\Big) \\
&\leq \beta_{n,i}\phi(u, U_{n,i+1}x_n) + (1-\beta_{n,i})\phi(u, x_n) \\
&\quad - \beta_{n,i}(1-\beta_{n,i})g(\|J(S_iU_{n,i+1}x_n) - Jx_n\|) \\
&\quad - \phi\Big(U_{n,i}x_n, J^{-1}\big(\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\big)\Big)
\end{aligned}$$

for some $g \in \mathcal{G}$ and for all $1 \leq i \leq n$. This implies

$$\begin{aligned}
&\phi(u, u_n) \leq \phi(u, y_n) = \phi(u, W_nx_n) = \phi(u, U_{n,0}x_n) \\
&= \|u\|^2 - 2\beta_{n,0}\langle u, J(S_0U_{n,1}x_n)\rangle - 2(1-\beta_{n,0})\langle u, Jx_n\rangle \\
&\quad + \|\beta_{n,0}J(S_0U_{n,1}x_n) + (1-\beta_{n,0})Jx_n\|^2 \\
&\leq \beta_{n,0}\phi(u, U_{n,1}x_n) + (1-\beta_{n,0})\phi(u, x_n) \\
&\quad - \beta_{n,0}(1-\beta_{n,0})g(\|J(S_0U_{n,1}x_n) - Jx_n\|) \\
&\leq \beta_{n,0}\Big\{\beta_{n,1}\phi(u, U_{n,2}x_n) + (1-\beta_{n,1})\phi(u, x_n)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \beta_{n,1}(1-\beta_{n,1})g(\|J(S_1U_{n,2}x_n) - Jx_n\|) \\
&\quad - \phi\Big(U_{n,1}x_n, J^{-1}\big(\beta_{n,1}J(S_1U_{n,2}x_n) + (1-\beta_{n,1})Jx_n\big)\Big)\Big\} \\
&\quad + (1-\beta_{n,0})\phi(u,x_n) - \beta_{n,0}(1-\beta_{n,0})g(\|J(S_0U_{n,1}x_n) - Jx_n\|) \\
&\le \cdots \\
&\le \phi(u,x_n) - \beta_{n,0}(1-\beta_{n,0})g(\|J(S_0U_{n,1}x_n) - Jx_n\|) \\
&\quad - \beta_{n,0}\beta_{n,1}(1-\beta_{n,1})g(\|J(S_1U_{n,2}x_n) - Jx_n\|) - \cdots \\
&\quad - \beta_{n,0}\beta_{n,1}\cdots\beta_{n,n}(1-\beta_{n,n})g(\|J(S_nU_{n,n+1}x_n) - Jx_n\|) \\
&\quad - \beta_{n,0}\phi\Big(U_{n,1}x_n, J^{-1}\big(\beta_{n,1}J(S_1U_{n,2}x_n) + \big(1-\beta_{n,1})Jx_n\big)\Big) - \cdots \\
&\quad - \beta_{n,0}\beta_{n,1}\cdots\beta_{n,n-1} \\
&\qquad \times \phi\Big(U_{n,n}x_n, J^{-1}\big(\beta_{n,n}J(S_nU_{n,n+1}x_n) + \big(1-\beta_{n,n})Jx_n\big)\Big)
\end{aligned} \tag{2.3.10}
$$

for all $n \ge 0$. By (2.3.9), (2.3.10) and the definition of $\beta_{n,i}$, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} g(\|J(S_iU_{n,i+1}x_n) - Jx_n\|) = 0, \\
&\lim_{n\to\infty} \phi\Big(U_{n,i+1}x_n, J^{-1}\big(\beta_{n,i+1}J(S_{i+1}U_{n,i+2}x_n) + (1-\beta_{n,i+1})Jx_n\big)\Big) = 0
\end{aligned}
$$

for all $i \ge 0$. By the definition of g and Proposition 2.2.1, we have

$$
\lim_{n\to\infty} \|J(S_iU_{n,i+1}x_n) - Jx_n\| = 0, \tag{2.3.11}
$$

$$
\lim_{n\to\infty} \big\|U_{n,i+1}x_n - J^{-1}\big(\beta_{n,i+1}J(S_{i+1}U_{n,i+2}x_n) + (1-\beta_{n,i+1})Jx_n\big)\big\| = 0. \tag{2.3.12}
$$

By (2.3.11), we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \|\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n - Jx_n\| \\
&= \lim_{n\to\infty} \beta_{n,i}\|J(S_iU_{n,i+1}x_n) - Jx_n\| = 0.
\end{aligned} \tag{2.3.13}
$$

Since J^{-1} is also norm-to-norm continuous on bounded sets, by (2.3.11) and (2.3.13), we have

$$
\lim_{n\to\infty} \|S_iU_{n,i+1}x_n - x_n\| = 0, \tag{2.3.14}
$$

$$
\lim_{n\to\infty} \big\|J^{-1}\big(\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\big) - x_n\big\| = 0 \tag{2.3.15}
$$

for all $i \ge 0$. By (2.3.12) and (2.3.15), we obtain

$$
\lim_{n\to\infty} \|U_{n,i+1}x_n - x_n\| = 0 \tag{2.3.16}
$$

for all $i \ge 0$. Since $x_{n_k} \rightharpoonup p$, we have $U_{n_k,i+1}x_{n_k} \rightharpoonup p$ for all $i \ge 0$. By (2.3.14) and (2.3.16), we obtain

$$
\lim_{n\to\infty} \|S_iU_{n,i+1}x_n - U_{n,i+1}x_n\| = 0
$$

for each $i \geq 0$. Since $U_{n_k,i+1}x_{n_k} \rightharpoonup p$ and S_i is relatively nonexpansive, we have $p \in \hat{F}(S_i) = F(S_i)$ for all $i \geq 0$. Hence $p \in \bigcap_{i=0}^{\infty} F(S_i)$. Now we shall prove that $p \in EP(f)$. By (2.3.2), (2.3.9) and Proposition 2.2.10, we have

$$\begin{aligned} \phi(u_n, y_n) = \phi(T_{\gamma_n} y_n, y_n) &\leq \phi(u, y_n) - \phi(u, T_{\gamma_n} y_n) \\ &\leq \phi(u, x_n) - \phi(u, u_n) \to 0 \end{aligned}$$

as $n \to \infty$. By Proposition 2.2.1, we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{2.3.17}$$

Since $x_{n_k} \rightharpoonup p$, by (2.3.6) and (2.3.17), we have $u_{n_k} \rightharpoonup p$ and $y_{n_k} \rightharpoonup p$. Since J is uniformly continuous on any bounded set of E, by (2.3.17), we have $\|Ju_n - Jy_n\| \to 0$ as $n \to \infty$. By the assumption that $\gamma_n \geq r$, we have

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \|Ju_n - Jy_n\| = 0. \tag{2.3.18}$$

Since $u_n = T_{\gamma_n} y_n$, we obtain

$$f(u_n, y) + \frac{1}{\gamma_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0 \tag{2.3.19}$$

for all $y \in C$. Replacing n by n_k in (2.3.19), by (A_2), we have

$$\frac{1}{\gamma_{n_k}} \langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k} \rangle \geq -f(u_{n_k}, y) \geq f(y, u_{n_k}) \tag{2.3.20}$$

for all $y \in C$. Since $y \mapsto f(x, y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in (2.3.20), by (2.3.18) and (A_4), we have $f(y, p) \leq 0$ for all $y \in C$. For $t \in (0, 1]$ and $y \in C$, letting $y_t = ty + (1 - t)p$, then $y_t \in C$ and $f(y_t, p) \leq 0$. By (A_1) and (A_4), we obtain

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, p) \leq tf(y_t, y).$$

Dividing by t, we have $f(y_t, y) \geq 0$ for all $y \in C$. Letting $t \downarrow 0$, by (A_3), we obtain $f(p, y) \geq 0$. Therefore $p \in EP(f)$, and so $p \in F$. This implies $\omega(\{x_n\}) \subset F$.

Step 6. We prove that $\omega(\{x_n\})$ is a singleton and $x_n \to \Pi_F x_0$. Let $w = \Pi_F x_0$. Since $w \in F \subset C_n \cap Q_n$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$, we have $\phi(x_{n+1}, x_0) \leq \phi(w, x_0)$ for all $n \geq 0$. Since the norm is weakly lower semicontinuous, this implies

$$\begin{aligned} \phi(p, x_0) &= \|p\|^2 - 2\langle p, Jx_0 \rangle + \|x_0\|^2 \\ &\leq \liminf_{k\to\infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0 \rangle + \|x_0\|^2) \\ &= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\ &\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0). \end{aligned} \tag{2.3.21}$$

By the definition of w and (2.3.21), we have $p = w$. This implies that $\omega(\{x_n\})$ is a singleton and $\phi(x_{n_k}, x_0) \to \phi(w, x_0)$. Hence

$$\begin{aligned} 0 &= \lim_{k\to\infty} \big(\phi(x_{n_k}, x_0) - \phi(w, x_0)\big) \\ &= \lim_{k\to\infty} (\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0\rangle) \\ &= \lim_{k\to\infty} \|x_{n_k}\|^2 - \|w\|^2, \end{aligned}$$

that is,

$$\lim_{k\to\infty} \|x_{n_k}\|^2 = \|w\|^2. \tag{2.3.22}$$

Since E is uniformly convex, it has the Kadec-Klee property. By (2.3.22) and $x_{n_k} \rightharpoonup w$, we have $x_{n_k} \to w = \Pi_F x_0$. Since $\omega(x_n)$ is a singleton, we have $x_n \to \Pi_F x_0$. □

The following theorems can be obtained by Theorem 2.3.1.

Theorem 2.3.2 ([77], Theorem 3.3, p. 293)**.** *Let E be a uniformly smooth and uniformly convex Banach space and C a nonempty, closed and convex subset of E. Let A be an α-inverse strongly monotone operator of C into E^*, $f : C \times C \to \mathbf{R}$ a bifunction satisfying (A_1)–(A_4) and $\{S_i\}_{i=0}^{\infty}$ an infinite family of relatively nonexpansive mappings of C into itself such that $F := \bigcap_{i=0}^{\infty} F(S_i) \cap EP \neq \emptyset$. Let $\{\beta_{n,i}\}_{i=0}^{n}$ be a sequence of real numbers such that $\liminf_{n\to\infty} \beta_{n,i}(1 - \beta_{n,i}) > 0$ and W_n the W-mapping generated by $\{S_i\}_{i=0}^{n}$ and $\{\beta_{n,i}\}_{i=0}^{n}$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ y_n = W_n x_n, \\ u_n \in K_{\gamma_n} y_n, \text{ that is,} \\ \quad f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \text{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases} \tag{2.3.23}$$

for $n \geq 0$, where $\Pi_{C_n \cap Q_n}$ is the generalized projection of E onto $C_n \cap Q_n$ and $\{\gamma_n\} \subset [r, \infty)$ for some $r > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. Let $g(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n\rangle$. By Propositions 2.2.11 and 2.2.12, (2.3.23) is equivalent to (2.3.1) in Theorem 2.3.1. Therefore the conclusion of Theorem 2.3.2 can be deduced from Theorem 2.3.1. □

Corollary 2.3.3 ([76], Theorem 3.1, p. 50)**.** *Let E be a uniformly smooth and uniformly convex Banach space and C a nonempty, closed and convex subset of E. Let $f : C \times C \to \mathbf{R}$ a bifunction satisfying (A_1)–(A_4) and S a relatively nonexpansive mapping from C into itself such that $F := F(S) \cap EP(f) \neq \emptyset$. Let $\{\alpha_n\} \subset [0, 1]$ be a sequence of real numbers such* *that* $\lim_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ y_n = J^{-1}(\alpha_n JSx_n + (1-\alpha_n)Jx_n), \\ u_n \in C \textit{ such that } f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \textit{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n\cap Q_n} x_0 \end{cases} \tag{2.3.24}$$

for $n \geq 0$, *where* $\Pi_{C_n\cap Q_n}$ *is the generalized projection of* E *onto* $C_n \cap Q_n$ *and* $\{\gamma_n\} \subset [r, \infty)$ *for some* $r > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* Π_F *is the generalized projection of* E *onto* F.

Proof. Let $S_n = S$, $\beta_{n,0} = \alpha_n$ and $\{\beta_{n,i}\}_{i=1}^n = \{0\}$ for all $n \geq 0$ in Theorem 2.3.1. This shows that (2.3.1) is equivalent to (2.3.24). Therefore the conclusion of Theorem 2.3.3 can be deduced from Theorem 2.3.1. □

2.4 Strong convergence theorems of convex combinations

In this section, we prove strong convergence theorems of convex combinations for finding a common element of the set of solutions for a generalized equilibrium problem and the set of common fixed points of infinite relatively nonexpansive mappings in a Banach space.

Theorem 2.4.1 ([77], Theorem 4.1, p. 294)**.** *Let* E *be a uniformly smooth and uniformly convex Banach space and* C *a nonempty, closed and convex subset of* E. *Let* $f : C \times C \to \mathbf{R}$ *be a bifunction satisfying* (A_1)–(A_4) *and* $\{S_i\}_{i=0}^\infty$ *an infinite family of relatively nonexpansive mappings of* C *into itself such that* $F := \bigcap_{i=0}^\infty F(S_i) \cap EP(f) \neq \emptyset$. *Let* $\{\lambda_{n,i}\}_{i=0}^n \subset [0, 1)$ *be a sequence of real numbers such that* $\sum_{i=0}^n \lambda_{n,i} = 1$ *for all* $n \geq 0$ *and* $\lim_{n\to\infty} \lambda_{n,i} > 0$ *for each* $i \geq 0$, *and* V_n *the mapping defined by* (2.2.2). *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ y_n = V_n x_n, \\ u_n \in T_{\gamma_n} y_n, \textit{ that is, } f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \textit{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n\cap Q_n} x_0 \end{cases} \tag{2.4.1}$$

for $n \geq 0$, *where* $\Pi_{C_n\cap Q_n}$ *is the generalized projection of* E *onto* $C_n \cap Q_n$ *and* $\{\gamma_n\} \subset [r, \infty)$ *for some* $r > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* Π_F *is the generalized projection of* E *onto* F.

Proof. We divide the proof into six steps.

Step 1. We prove that $C_n \cap Q_n \subset C$ is closed and convex for all $n \geq 0$. In fact, it is obvious that C_n is closed and Q_n is closed and convex for all $n \geq 0$. It follows that C_n is convex for all $n \geq 0$ since $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2.$$

Thus $C_n \cap Q_n$ is closed and convex for all $n \geq 0$.

Step 2. We prove that $F \subset C_n \cap Q_n$ for all $n \geq 0$. Let $u_n = T_{\gamma_n} y_n$ for all $n \geq 0$ and $u \in F$. By Propositions 2.2.7 (i) and 2.2.9 (iii), we have $u \in \bigcap_{n=0}^{\infty} F(V_n) \cap F(T_{\gamma_n})$. By Proposition 2.2.9, we obtain T_{γ_n} is relatively nonexpansive. By Proposition 2.2.7 (ii), we have

$$\phi(u, u_n) = \phi(u, T_{\gamma_n} y_n) \leq \phi(u, y_n) = \phi(u, V_n x_n) \leq \phi(u, x_n). \tag{2.4.2}$$

This implies $u \in C_n$ and so $F \subset C_n$ for all $n \geq 0$. By induction, now we prove that $F \subset C_n \cap Q_n$ for all $n \geq 0$. In fact, since $Q_0 = C$, we have $F \subset C_0 \cap Q_0$. Suppose that $F \subset C_k \cap Q_k$ for some $k \geq 0$. Then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \Pi_{C_k \cap Q_k} x_0$. By the definition of x_{k+1}, we have

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1}\rangle \geq 0 \tag{2.4.3}$$

for all $z \in C_k \cap Q_k$. Since $F \subset C_k \cap Q_k$, we have (2.4.3) for all $z \in F$. This implies $z \in Q_{k+1}$, and so $F \subset Q_{k+1}$. Therefore $F \subset C_n \cap Q_n$ for all $n \geq 0$.

Step 3. We prove that $\{x_n\}$ is bounded. By the definition of Q_n, we have $x_n = \Pi_{Q_n} x_0$ for all $n \geq 0$. Hence, by Proposition 2.2.4,

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \leq \phi(u, x_0)$$

for all $u \in F \subset Q_n$ and $n \geq 0$. This implies that $\{\phi(x_n, x_0)\}$ is bounded, and so $\{x_n\}$ and $\{u_n\}$ are bounded in C. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $x_n = \Pi_{Q_n} x_0$, we have $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$ for all $n \geq 0$. This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing. Hence there exists the limit $\lim_{n\to\infty} \phi(x_n, x_0)$. By Proposition 2.2.4, we have

$$\begin{aligned} \phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{Q_n} x_0) &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \end{aligned}$$

for all $n \geq 0$. This implies

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.4.4}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, by the definition of C_n, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \tag{2.4.5}$$

Since E is smooth and uniformly convex, by (2.4.4), (2.4.5) and Proposition 2.2.1, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.4.6}$$

Since J is uniformly norm-to-norm continuous on bounded subsets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{2.4.7}$$

Step 4. We prove that $\|S_l x_n - x_n\| \to 0$ for all $l \geq 0$. By the definition of $\lambda_{n,i}$, we have $1 - \lambda_{n,l} = \sum_{\substack{i=0,1,\ldots,n \\ i\neq l}} \lambda_{n,i}$. For large enough $n \geq 0$ and $0 \leq l \leq n$, Proposition 1.5.6 implies

$$\begin{aligned}
&\phi(u, u_n) \leq \phi(u, y_n) = \phi(u, V_n x_n) \\
&= \|u\|^2 - 2\sum_{i=0}^{n} \lambda_{n,i}\langle u, J(S_i x_n)\rangle + \left\| J^{-1} \sum_{i=0}^{n} \lambda_{n,i} J(S_i x_n) \right\|^2 \\
&= \|u\|^2 - 2\sum_{i=0}^{n} \lambda_{n,i}\langle u, J(S_i x_n)\rangle \\
&\quad + \left\| \lambda_{n,l} J(S_l x_n) + (1-\lambda_{n,l}) \frac{\sum_{\substack{i=0,1,\ldots,n \\ i\neq l}} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\|^2 \\
&\leq \|u\|^2 - 2\sum_{i=0}^{n} \lambda_{n,i}\langle u, J(S_i x_n)\rangle + \lambda_{n,l}\|S_l x_n\|^2 \\
&\quad + (1-\lambda_{n,l}) \left\| \frac{\sum_{\substack{i=0,1,\ldots,n \\ i\neq l}} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\|^2 \\
&\quad - \lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{\substack{i=0,1,\ldots \\ n, i\neq l}} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\| \right) \\
&= \|u\|^2 - 2\sum_{i=0}^{n} \lambda_{n,i}\langle u, J(S_i x_n)\rangle + \sum_{i=0}^{n} \lambda_{n,i}\|S_i x_n\|^2 \\
&\quad - \lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{\substack{i=0,1,\ldots,n \\ i\neq l}} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\| \right) \\
&= \phi(u, S_i x_n) - \lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{\substack{i=0,1,\ldots,n \\ i\neq l}} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\| \right) \\
&\leq \phi(u, x_n) - \lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{\substack{i=0,1,\ldots,n \\ i\neq l}} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\| \right)
\end{aligned}$$

for some $g \in \mathcal{G}$. Thus

$$\begin{aligned}
&\lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{\substack{i=0,1,\ldots \\ n, i\neq l}} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\| \right) \\
&\leq \phi(u, x_n) - \phi(u, u_n)
\end{aligned}$$

$$= \|x_n\|^2 - \|u_n\|^2 + 2\langle u, Ju_n - Jx_n\rangle$$
$$\leq 2\|u\| \cdot \|Ju_n - Jx_n\| + (\|x_n\| + \|u_n\|)\|x_n - u_n\|.$$

This implies, together with (2.4.6) and (2.4.7),

$$\lim_{n\to\infty} \left\| J(S_l x_n) - \frac{\sum_{\substack{i=0,1,\ldots,n \\ i\neq l}} \lambda_{n,i} J(S_i x_n)}{1 - \lambda_{n,l}} \right\| = 0 \tag{2.4.8}$$

for all $l \geq 0$. By (2.4.2), (2.4.6), (2.4.7) and Proposition 2.2.10, we have

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(T_{\gamma_n} y_n, y_n) \\
&\leq \phi(u, y_n) - \phi(u, T_{\gamma_n} y_n) \\
&\leq \phi(u, x_n) - \phi(u, u_n) \\
&\leq \|x_n - u_n\|(\|x_n\| + \|y_n\|) + 2\|u\|\|Ju_n - Jx_n\| \to 0.
\end{aligned}$$

This implies

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{2.4.9}$$

By (2.4.6) and (2.4.9), we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| \leq \lim_{n\to\infty} \{\|x_n - u_n\| + \|u_n - y_n\|\} = 0.$$

Since J is uniformly norm-to-norm continuous on bounded subsets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{2.4.10}$$

Since

$$\begin{aligned}
\|Jx_n - J(S_l x_n)\| &\leq \|Jx_n - J(V_n x_n)\| + \|J(S_l x_n) - J(V_n x_n)\| \\
&= \|Jx_n - Jy_n\| + \left\| J(S_l x_n) - \sum_{i=0}^{n} \lambda_{n,i} J(S_i x_n) \right\| \\
&= \|Jx_n - Jy_n\| \\
&\quad + (1 - \lambda_{n,l}) \left\| J(S_l x_n) - \frac{\sum_{\substack{i=0,1,\ldots,n \\ i\neq l}} \lambda_{n,i} J(S_i x_n)}{1 - \lambda_{n,l}} \right\|
\end{aligned}$$

for large enough $n \geq 0$, by (2.4.8) and (2.4.10), we obtain $\|Jx_n - J(S_l x_n)\| \to 0$ as $n \to \infty$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets, we have

$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0 \tag{2.4.11}$$

for all $l \geq 0$.

Step 5. We prove that $\omega(\{x_n\}) \subset F$, where $\omega(\{x_n\})$ is the set consisting all of the weak limits points of $\{x_n\}$. In fact, for any $p \in \omega(\{x_n\})$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup p$. Since S_i is relatively nonexpansive, (2.4.11) implies $p \in \bigcap_{i=0}^{\infty} \hat{F}(S_i) =$

$\bigcap_{i=0}^{\infty} F(S_i)$. Now we prove that $p \in EP(f)$. Since $x_{n_k} \rightharpoonup p$, by (2.4.6) and (2.4.9), we have $u_{n_k} \rightharpoonup p$ and $y_{n_k} \rightharpoonup p$. Since J is uniformly continuous on any bounded set of E, by (2.4.9), we have $\|Ju_n - Jy_n\| \to 0$. By the assumption that $\gamma_n > r$, we have

$$\lim_{n\to\infty} \frac{1}{\gamma_n}\|Ju_n - Jy_n\| = 0. \tag{2.4.12}$$

Since $u_n = T_{\gamma_n} y_n$, we obtain

$$f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \tag{2.4.13}$$

for all $y \in C$. Replacing n by n_k in (2.4.13), by (A_2), we have

$$\frac{1}{\gamma_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k}\rangle \geq -f(u_{n_k}, y) \geq f(y, u_{n_k}) \tag{2.4.14}$$

for all $y \in C$. Since $y \mapsto f(x, y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in (2.4.14), by (2.4.12) and (A_4), we obtain $f(y, p) \leq 0$ for all $y \in C$. For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)p$. Thus we have $y_t \in C$ and $f(y_t, p) \leq 0$. By (A_1) and (A_4), we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

Dividing by t, we obtain $f(y_t, y) \geq 0$ for all $y \in C$. Letting $t \downarrow 0$, by (A_3), we have $f(p, y) \geq 0$ for all $y \in C$. Therefore $p \in EP(f)$, and so $p \in F$. This implies $\omega(\{x_n\}) \subset F$.

Step 6. We prove that $\omega(\{x_n\})$ is a singleton and $x_n \to \Pi_F x_0$. Let $w = \Pi_F x_0$. Since $w \in F \subset C_n \cap Q_n$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$, we have $\phi(x_{n+1}, x_0) \leq \phi(w, x_0)$ for all $n \geq 0$. Since the norm is weakly lower semicontinuous, this implies

$$\begin{aligned} \phi(p, x_0) &= \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 \\ &\leq \liminf_{k\to\infty}(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2) \\ &= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\ &\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0). \end{aligned} \tag{2.4.15}$$

By the definition of w and (2.4.15), we have $p = w$. This implies that $\omega(\{x_n\})$ is a singleton and $\phi(x_{n_k}, x_0) \to \phi(w, x_0)$. Therefore

$$\begin{aligned} 0 &= \lim_{k\to\infty} \big(\phi(x_{n_k}, x_0) - \phi(w, x_0)\big) \\ &= \lim_{k\to\infty} (\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0\rangle) \\ &= \lim_{k\to\infty} \|x_{n_k}\|^2 - \|w\|^2, \end{aligned}$$

that is,

$$\lim_{k\to\infty} \|x_{n_k}\|^2 = \|w\|^2. \tag{2.4.16}$$

Since E is uniformly convex, it has the Kadec-Klee property. By (2.4.16) and $x_{n_k} \rightharpoonup w$, we have $x_{n_k} \to w = \Pi_F x_0$. Since $\omega(x_n)$ is a singleton, we have $x_n \to \Pi_F x_0$. $\square$

The following theorem can be obtained by Theorem 2.4.1.

Theorem 2.4.2 ([77], Theorem 4.2, p. 298)**.** *Let E be a uniformly smooth and uniformly convex Banach space and C a nonempty closed convex subset of E. Let A be an α-inverse strongly monotone operator of C into E^*, $f : C \times C \to \mathbf{R}$ a bifunction satisfying* (A_1)–(A_4) *and $\{S_i\}_{i=0}^{\infty}$ an infinite family of relatively nonexpansive mappings of C into itself such that $F := \bigcap_{i=0}^{\infty} F(S_i) \cap EP \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=0}^{n} \subset [0,1)$ be a sequence of real numbers such that $\sum_{i=0}^{n} \lambda_{n,i} = 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \lambda_{n,i} > 0$ for each $i \geq 0$, and V_n the mapping defined by* (2.2.2)*. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ y_n = V_n x_n, \\ u_n \in K_{\gamma_n} y_n, \ \textit{that is}, \\ \quad f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \ \textit{for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases} \tag{2.4.17}$$

for $n \geq 0$, where $\Pi_{C_n \cap Q_n}$ is the generalized projection of E onto $C_n \cap Q_n$ and $\{\gamma_n\} \subset [r, \infty)$ for some $r > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. Let $g(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n\rangle$. By Propositions 2.2.11 and 2.2.12, (2.4.17) is equivalent to (2.4.1) in Theorem 2.4.1. Therefore the conclusion of Theorem 2.4.2 can be deduced from Theorem 2.4.1. $\square$

Chapter 3

Shrinking Projection Methods with Respect to Bregman Distances

3.1 Introduction

Let E be a smooth, strictly convex and reflexive real Banach space with the norm $\|\cdot\|$, C a nonempty, closed and convex subset of E and T a nonlinear mapping from C into itself. For an arbitrary point $x \in E$, consider the set $\{z \in C : \|x - z\| = \min_{y \in C} \|x - y\|\}$. We know that this set is always a singleton. Let P_C be a mapping of E onto C defined by

$$P_C x = \arg\min_{y \in C} \|x - y\|.$$

Such a mapping P_C is called the *metric projection*. Takahashi, Takeuchi and Kubota [74] have introduced a new hybrid iterative scheme called a *shrinking projection method* for nonexpansive mappings in Hilbert spaces. They proved that a sequence generated by the shrinking projection method converges strongly to a fixed point of a nonexpansive mapping. It is an advantage of projection methods that strong convergence of iterative sequences is guaranteed without any compact assumptions.

The mapping T is said to be *asymptotically nonexpansive* (*cf.* [24]) if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \mathbf{N}$. Schu [67] has considered a *modified Mann iteration* for asymptotically nonexpansive mappings. Inchan [29] has introduced a modified Mann iteration for asymptotically nonexpansive mappings by the shrinking projection method.

The mapping T is said to be *asymptotically nonexpansive in the intermediate sense* (*cf.* [13, 32]) if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y \in C} \big(\|T^n x - T^n y\| - \|x - y\|\big) \leq 0. \tag{3.1.1}$$

If $F(T) \neq \emptyset$ and (3.1.1) holds for all $p \in F(T)$ and $x \in C$, that is,

$$\limsup_{n\to\infty} \sup_{p \in F(T), x \in C} \big(\|p - T^n x\| - \|p - x\|\big) \leq 0, \tag{3.1.2}$$

then T is said to be *asymptotically quasi-nonexpansive in the intermediate sense.* It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense may not be Lipschitz continuous. Motivated by Takahashi *et al.* [74] and Schu [67], many authors have studied iterative methods for approximating fixed points of asymptotically quasi-nonexpansive mappings in the intermediate sense (see [26, 27, 56]). However, as far as we know, it has not been studied yet for the cases of asymptotically quasi-nonexpansive with respect to the Bregman distance in the intermediate sense.

On the other hand, we know two kinds of mappings in Banach spaces which generalized the metric projections in Hilbert spaces. Let $\langle \cdot, \cdot \rangle$ be the pairing between E and the dual space of E and J the normalized duality mapping of E. Let $\phi : E \times E \to \mathbf{R}^+$ be the *Lyapunov functional* (*cf.* [1]) defined by $\phi(x, y) := \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for all $x, y \in E$. The first kind is the projection introduced by [1]: For an arbitrary point $x \in E$, consider the set $\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\}$. It is known that this set is always a singleton (see [1]). Let Π_C be a mapping of E onto C defined by

$$\Pi_C x = \arg\min_{y \in C} \phi(y, x).$$

Such a mapping Π_C is called the *generalized projection.* The other is the projection found in [28]: The mapping T is said to be *generalized nonexpansive* (*cf.* [28]) if $F(T) \neq \emptyset$ and $\phi(Tx, p) \leq \phi(x, p)$ for all $x \in C$ and $p \in F(T)$. Given two nonempty subsets $K \subset C \subset E$, an operator $R : C \to K$ is called a *retraction* of C onto K if $Rx = x$ for each $x \in K$. A retraction $R : C \to K$ is said to be *sunny* (*cf.* [25, 57]) if $R\big(Rx + t(x - Rx)\big) = Rx$ for each $x \in C$ and any $t \geq 0$, whenever $Rx + t(x - Rx) \in C$. A nonempty subset C of E is said to be a *sunny generalized nonexpansive retract* of E if there exists a sunny generalized nonexpansive retraction of E onto C. We know that a sunny generalized nonexpansive retraction R_C from E onto C is uniquely determined (see [28]). We know also that $z = R_C x$ for all $x \in E$ if and only if $\phi(x, z) = \min_{y \in C} \phi(x, y)$ (see [36]). By these facts, R_C is characterized by

$$R_C x = \arg\min_{y \in C} \phi(x, y)$$

for $x \in E$. The projections Π_C and R_C are generalization of the metric projection in Hilbert spaces. In connection with the Bregman distance (see Section 3.2), there exist projections which are generalizations of the projections Π_C and R_C, respectively (see Section 3.3). Therefore we can construct hybrid iterative schemes with respect to Bregman distances, which are generalizations of schemes for the generalized projection and the sunny generalized nonexpansive retraction.

In this chapter, we introduce new classes of nonlinear mappings, that is, asymptotically quasi-nonexpansive mappings with respect to the Bregman distance in the intermediate sense. Motivated by the above results, we design new hybrid iterative schemes using the shrinking projection method with respect to Bregman distances for finding fixed points of the mappings in reflexive Banach spaces.

3.2 Preliminaries

Throughout this chapter, we assume that E is a real reflexive Banach space. A function $f : E \to (-\infty, +\infty]$ is said to be *admissible* if f is proper, convex and lower semicontinuous on E and Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. Under these conditions we know that ∂f is single-valued and $\partial f = \nabla f$ (see [15], Proposition 1.1.10, p. 13). An admissible function $f : E \to (-\infty, +\infty]$ is called *Legendre* (*cf.* [8]) if it satisfies additionally the following two conditions:

(L_1) $\operatorname{int}\operatorname{dom} f \neq \emptyset$ and ∂f is single-valued on its domain;

(L_2) $\operatorname{int}\operatorname{dom} f^* \neq \emptyset$ and subdifferential ∂f^* is single-valued on its domain.

Let f be a Legendre function on E. Since E is reflexive, we always have $\nabla f = (\nabla f^*)^{-1}$. When this fact is combined with conditions (L_1) and (L_2), we obtain the following equalities:

$$\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^* \quad \text{and} \quad \operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}\operatorname{dom} f.$$

It follows that f is Legendre if and only if f^* is Legendre (see [8], Corollary 5.5, p. 634).

Example 3.2.1. The following functions are Legendre on $E = \mathbf{R}^n$: Let $x \in \mathbf{R}^n$.

(i) Halved energy: $f(x) = \|x\|^2/2 = \frac{1}{2}\sum_{j=1}^n x_j^2$.

(ii) Boltzmann-Shannon entropy: $f(x) = \begin{cases} \sum_{j=1}^n (x_j \ln(x_j) - x_j), & x \geq 0; \\ +\infty, & \text{otherwise.} \end{cases}$

(iii) Burg entropy: $f(x) = \begin{cases} -\sum_{j=1}^n \ln(x_j), & x > 0; \\ +\infty, & \text{otherwise.} \end{cases}$

Note that $\operatorname{int}\operatorname{dom} f = \mathbf{R}^n$ in (i), whereas $\operatorname{int}\operatorname{dom} f = \{x \in \mathbf{R}^n : x_j > 0,\ j = 1, \ldots, n\}$ in (ii) and (iii).

Let $f : E \to (-\infty, +\infty]$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. A bifunction $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ given by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called a *Bregman distance with respect to f* (*cf.* [12, 18]). In general, the Bregman distance is not a metric since it is not symmetric and does not satisfy the triangle inequality. However, it has the following important property, which is called the *three point identity* (*cf.* [20]): for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle. \tag{3.2.1}$$

Example 3.2.2. The Bregman distances corresponding to the Legendre functions of Example 3.2.1 are as follows: Let $x, y \in \mathbf{R}^n$.

(i) Euclidean distance: $D_f(y, x) = \|y - x\|^2/2$.

(ii) Kullback-Leibler divergence: $D_f(y, x) = \sum_{j=1}^n (y_j \ln(y_j/x_j) - y_j + x_j)$.

(iii) Itakura-Saito divergence: $D_f(y, x) = \sum_{j=1}^n (\ln(x_j/y_j) + y_j/x_j - 1)$.

For a Legendre function $f : E \to (-\infty, +\infty]$, we associate a bifunction $W^f : \operatorname{dom} f^* \times \operatorname{dom} f \to [0, +\infty)$ defined by

$$W^f(\xi, x) := f(x) - \langle \xi, x \rangle + f^*(\xi)$$

for $(\xi, x) \in \operatorname{dom} f^* \times \operatorname{dom} f$.

Proposition 3.2.3 ([42], Proposition 1, p. 1047). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Then the following hold:*

(i) *The function $W^f(\cdot, x)$ is convex for all $x \in \mathrm{dom} f$;*

(ii) $W^f(\nabla f(x), y) = D_f(y, x)$ *for all $x \in \mathrm{int}\,\mathrm{dom} f$ and $y \in \mathrm{dom} f$.*

Proof. (i) By Proposition 1.4.3, f^* is convex. Therefore $W^f(\cdot, x)$ is convex.

(ii)For $x \in \mathrm{int}\,\mathrm{dom} f$ and $y \in \mathrm{dom} f$, we have $f(x) + f^*(\nabla f(x)) = \langle \nabla f(x), x\rangle$. Therefore

$$\begin{aligned} W^f(\nabla f(x), y) &= f(y) - \langle \nabla f(x), y\rangle + f^*(\nabla f(x)) \\ &= f(y) - \langle \nabla f(x), y\rangle + \langle \nabla f(x), x\rangle - f(x) \\ &= f(y) - f(x) - \langle \nabla f(x), y - x\rangle \\ &= D_f(y, x). \end{aligned}$$

□

Proposition 3.2.4 ([42], Proposition 10, p. 1052). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of* $\mathrm{dom} f^* = E^*$. *Let $x \in \mathrm{dom} f$. If the sequence $\{D_f(x, x_n)\}_{n\in\mathbf{N}}$ is bounded, then the sequence $\{x_n\}_{n\in\mathbf{N}}$ is also bounded.*

Proof. Since $\{D_f(x, x_n)\}_{n\in\mathbf{N}}$ is bounded, there exists $M > 0$ such that $D_f(x, x_n) < M$ for all $n \in \mathbf{N}$. By the definition of W_f, we have

$$f(x) - \langle \nabla f(x_n), x\rangle + f^*(\nabla f(x_n)) = W^f(\nabla f(x_n), x) = D_f(x, x_n) < M.$$

This implies that the sequence $\{\nabla f(x_n)\}_{n\in\mathbf{N}}$ is contained in the sub-level set $\{y \in \mathrm{ran}\nabla f = \mathrm{dom}\nabla f^* = \mathrm{int}\,\mathrm{dom} f^* : \psi(y) \le M - f(x)\}$ of the function $\psi = f^* - \langle \cdot, x\rangle$. By Proposition 1.4.3, the function f^* is proper and lower semicontinuous. By the Moreau-Rockafellar theorem ([65], Theorem 7A (a), p. 60), the function ψ is coercive, that is, $\lim_{\|x\|\to\infty} \psi(x) = +\infty$. Consequently, all sub-level sets of ψ are bounded. Hence $\{\nabla f(x_n)\}_{n\in\mathbf{N}}$ is bounded. By our hypothesis, ∇f^* is bounded on bounded subsets of E^*. Therefore the sequence $\{x_n\}_{n\in\mathbf{N}} = \{\nabla f^*(\nabla f(x_n))\}_{n\in\mathbf{N}}$ is bounded. □

Let $f : E \to (-\infty, +\infty]$ be a convex function on E which is Gâteaux differentiable on $\mathrm{int}\,\mathrm{dom} f$. A *modulus of total convexity of f at* $x \in \mathrm{dom} f$ is a function $v_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \mathrm{dom} f,\ \|y - x\| = t\}.$$

The function f is said to be *totally convex at* $x \in \mathrm{int}\,\mathrm{dom} f$ (*cf.* [14]) if $v_f(x, t)$ is positive for all $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point of $\mathrm{int}\,\mathrm{dom} f$. A *modulus of total convexity of f* on nonempty bounded subset $B \subset E$ is a function $v_f(B, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap \mathrm{int}\,\mathrm{dom} f\}$$

for $t \in (0, \infty)$. The function f is said to be *totally convex on bounded sets* if, for any nonempty bounded set $B \subset E$, $v_f(B, t)$ is positive for all $t > 0$.

Proposition 3.2.5 ([14], Proposition 2.4, p. 26)**.** *Let $f : E \to (-\infty, +\infty]$ be an admissible function and $x \in \operatorname{int}\operatorname{dom} f$.*

(i) *If $c \in [1, \infty)$ and $t \geq 0$, then $v_f(x, ct) \geq cv_f(x, t)$.*

(ii) *The function $v_f(x, \cdot)$ is nondecreasing. It is strictly increasing if and only if f is totally convex at x.*

Proof. (i) If $c = 1$, $t = 0$ or $v_f(x, ct) = +\infty$, then the result is obvious. Otherwise, let ε be a positive real number. By the definition of V_f, there exists a point $u \in \operatorname{dom} f$ such that $\|u - x\| = ct$ and

$$v_f(x, ct) + \varepsilon > D_f(u, x) = f(u) - f(x) - \langle \nabla f(x), u - x \rangle. \tag{3.2.2}$$

For every $\alpha \in (0, 1)$, denote $u_\alpha = \alpha u + (1 - \alpha)x$. Let $\beta = 1/c$. Then we have $\|u_\beta - x\| = \beta\|u - x\| = t$. Note that, for any $\alpha \in (0, 1)$,

$$\frac{\alpha}{\beta}u_\beta + (1 - \frac{\alpha}{\beta})x = \frac{\alpha}{\beta}(\beta u + (1 - \beta)x) + (1 - \frac{\alpha}{\beta})x = u_\alpha. \tag{3.2.3}$$

The function $t \mapsto (f(x + t(u - x)) - f(x))/t$ from $\mathbf{R} \setminus \{0\}$ into $(-\infty, \infty]$ is nondecreasing on $(0, 1)$. By (3.2.2), we have

$$v_f(x, ct) + \varepsilon > f(u) - f(x) - \frac{f(x + \alpha(u - x)) - f(x)}{\alpha} \langle \nabla f(x), u - x \rangle$$

for all $\alpha \in (0, 1)$. By (3.2.3), we have

$$\begin{aligned}
v_f(x, ct) + \varepsilon &> \frac{1}{\alpha}\Big\{\alpha f(u) + (1 - \alpha)f(x) - f(x + \alpha(u - x))\Big\} \\
&= \frac{1}{\alpha}\Big\{\alpha f(u) + (1 - \alpha)f(x) - \frac{\alpha}{\beta}f(u_\beta) - \Big(1 - \frac{\alpha}{\beta}\Big)f(x)\Big\} \\
&\quad + \frac{1}{\alpha}\Big\{\frac{\alpha}{\beta}f(u_\beta) + \Big(1 - \frac{\alpha}{\beta}\Big)f(x) - f(u_\alpha)\Big\} \\
&= \frac{1}{\beta}\Big\{\beta f(u) + (1 - \beta)f(x) - f(u_\beta)\Big\} \\
&\quad + \frac{1}{\alpha}\Big\{\frac{\alpha}{\beta}f(u_\beta) + \Big(1 - \frac{\alpha}{\beta}\Big)f(x) - f\Big(\frac{\alpha}{\beta}u_\beta + \Big(1 - \frac{\alpha}{\beta}\Big)x\Big)\Big\}.
\end{aligned}$$

The first term of the last sum is nonnegative since f is convex. Thus

$$\begin{aligned}
v_f(x, ct) + \varepsilon &> \frac{1}{\alpha}\Big\{\frac{\alpha}{\beta}f(u_\beta) + \Big(1 - \frac{\alpha}{\beta}\Big)f(x) - f\Big(\frac{\alpha}{\beta}u_\beta + \Big(1 - \frac{\alpha}{\beta}\Big)x\Big)\Big\} \\
&= \frac{1}{\beta}\Big\{f(u_\beta) - f(x) - \frac{\beta}{\alpha}\Big(f\Big(x + \frac{\alpha}{\beta}(u_\beta - x)\Big) - f(x)\Big)\Big\}.
\end{aligned}$$

Letting $\alpha \to 0$, we have $v_f(x, ct) + \varepsilon > cD_f(u_\beta, x) \geq cv_f(x, t)$. Since ε is an arbitrary positive real number, this proves (i).

(ii) Suppose that $0 < s < t$. By (i), we have

$$v_f(x, t) \geq \frac{t}{s}v_f(x, s) \geq v_f(x, s). \tag{3.2.4}$$

Thus the function $v_f(x,\cdot)$ is nondecreasing. If f is totally convex, then the last inequality in (3.2.4) is strict. This implies that $v_f(x,\cdot)$ is strictly increasing on $[0,\tau_f(x))$, where $\tau_f(x)\in(0,+\infty]$. The converse is obvious. $\square$

We remark in passing that $f:E\to(-\infty,+\infty]$ is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [16, 17]).

Proposition 3.2.6 ([16], Proposition 4.2, p. 16)**.** *Let $f:E\to(-\infty,+\infty]$ be a proper and convex function whose domain contains at least two different points. If f is lower semicontinuous, then f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets.*

Proof. Suppose that f is uniformly convex on bounded sets. Take B a bounded set such that $B\cap \mathrm{dom}f\neq\emptyset$. Denote by C the closed convex hull of B and $K:=\{x\in E: d(x,C)\leq 1\}$, where $d(x,C):=\inf\{\|x-y\|: y\in C\}$ for $x\in E$ and $C\subset E$. Obviously, K is closed, convex and bounded and C is a subset of the interior of K. Let ι_K be the indicator function of K and define $g=f+\iota_K$. The function g is uniformly convex since f is uniformly convex on bounded sets. Consider $\mathcal{F}:=\{\psi:[0,+\infty)\to[0,+\infty]: \psi$ is convex and lower semicontinuous, $\mathrm{int}\,\mathrm{dom}\psi\neq\emptyset,\ \psi(0)=0$ and $\psi(t)>0$ for $t>0\}$. Hence there exists a function $\psi\in\mathcal{F}$ such that

$$g(y)-g(x)\geq g^{\circ}(x,y-x)+\psi(\|y-x\|)$$

for all $x,y\in\mathrm{dom}g$ (see [82], Theorem 2.2, p. 353). This implies

$$f(y)-f(x)\geq g^{\circ}(x,y-x)+\psi(\|y-x\|)$$

for all $x\in C\cap\mathrm{dom}f$ and $y\in K\cap\mathrm{dom}f$. By definition of g, we have $g^{\circ}(x,y-x)=f^{\circ}(x,y-x)$ whenever $x\in C\cap\mathrm{dom}f$ and $\in K\cap\mathrm{dom}f$. Thus

$$f(y)-f(x)\geq f^{\circ}(x,y-x)+\psi(\|y-x\|).$$

If $\|y-x\|=t\in(0,1]$, then

$$f(y)-f(x)\geq f^{\circ}(x,y-x)+\psi(t).$$

Thus $v_f(B,t)\geq v_f(C,t)\geq\psi(t)>0$ for all $t\in(0,1]$. Since $v_f(B,\cdot)$ is nondecreasing, $v_f(B,t)>0$ for all $t>0$, that is, f is totally convex on bounded sets.

Conversely, assume that f is totally convex on bounded sets. Then

$$f(y)-f(x)\geq f^{\circ}(x,y-x)+v_f(C,t)$$

for all $x\in C\cap\mathrm{dom}f$ and $\in K\cap\mathrm{dom}f$ with $\|y-x\|=t$. Thus

$$h(y)-h(x)\geq h^{\circ}(x,y-x)+v_f(C,\|y-x\|)\geq h^{\circ}(x,y-x)+\overline{co}\,v_f(C,\|y-x\|),$$

whenever $x,y\in\mathrm{dom}h$, where $\overline{co}\,v_f$ is the closed convex hull of v_f. The functional $\overline{co}f(C,\cdot)$ is convex, lower semicontinuous and positive on $(0+\infty)$ (see [82], Proposition A.5, p. 372). This implies that h is uniformly convex (see [82], Theorem 2.2, p. 353), that is, $\mu_h(E,t)>0$ for all $t>0$. Therefore $\mu_f(B,t)\geq\mu_f(C,t)\geq\mu_h(E,t)>0$ for all $t>0$. $\square$

Proposition 3.2.7 ([61], Lemma 3.1, p. 31). *Let $f : E \to \mathbf{R}$ be a totally convex function and $x \in \operatorname{int} \operatorname{dom} f$. If the sequence $\{D_f(x_n, x)\}_{n\in\mathbf{N}}$ is bounded, then the sequence $\{x_n\}_{n\in\mathbf{N}}$ is also bounded.*

Proof. Since $\{D_f(x_n, x)\}_{n\in\mathbf{N}}$ is bounded, there exists $M > 0$ such that $D_f(x_n, x) < M$ for all $n \in \mathbf{N}$. By the definition of the modulus of totally convexity at x, we have

$$0 < v_f(x, \|x_n - x\|) \leq D_f(x_n, x) < M. \tag{3.2.5}$$

By Proposition 3.2.5 (ii), the function $v_f(x, \cdot)$ is strictly increasing on $(0, \infty)$. This implies $v_f(x, 1) > 0$. Suppose by way of contradiction that $\{x_n\}_{n\in\mathbf{N}}$ is not bounded. Then there exists a subsequence $\{n(k)\}_{k\in\mathbf{N}}$ of positive real numbers such that $\|x_{n(k)}\| \to +\infty$ as $k \to \infty$. Consequently, $\|x_{n(k)} - x\| \to +\infty$ as $k \to \infty$. This shows that the sequence $\{v_f(x, \|x_n - x\|)\}_{n\in\mathbf{N}}$ is not bounded. Indeed, there exists some $k_0 > 0$ such that $\|x_{n(k)} - x\| > 1$ for all $k > k_0$. By Proposition 3.2.5 (i), we have

$$\lim_{k\to\infty} v_f(x, \|x_{n(k)} - x\|) \geq \lim_{k\to\infty} \|x_{n(k)} - x\| v_f(x, 1) = +\infty$$

since $v_f(x, 1) > 0$. This contradicts (3.2.5). Therefore $\{x_n\}_{n\in\mathbf{N}}$ is bounded. □

A function $f : E \to (-\infty, +\infty]$ is said to be *sequentially consistent* (*cf.* [17]) if

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \text{implies} \quad \lim_{n\to\infty} \|y_n - x_n\| = 0$$

for any two sequences $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ in $\operatorname{int} \operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded.

Proposition 3.2.8 ([15], Lemma 2.1.2, p. 67). *A function $f : E \to (-\infty, +\infty]$ is totally convex on bounded subsets of E if and only if it is sequentially consistent.*

Proof. Assume that f is totally convex. Suppose by way of contradiction that there exist two sequences $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ contained in $\operatorname{int} \operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded, $D_f(y_n, x_n) \to 0$ as $n \to \infty$ and $\{\|y_n - x_n\|\}_{n\in\mathbf{N}}$ does not converge to zero. This implies that there exist a positive number M and subsequences $\{x_{n(k)}\}_{k\in\mathbf{N}}$ and $\{y_{n(k)}\}_{k\in\mathbf{N}}$ of $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$, respectively, such that $M \leq \|y_{n(k)} - x_{n(k)}\|$ for all $k \in \mathbf{N}$. The set B of all x_n is bounded. Thus, for all $k \in \mathbf{N}$, we have

$$D_f(y_{n(k)}, x_{n(k)}) \geq v_f(x_{n(k)}, \|y_{n(k)} - x_{n(k)}\|) \geq v_f(x_{n(k)}, M) \geq \inf_{x\in B} v_f(x, M).$$

This implies $\inf_{x\in B} v_f(x, M) = 0$, which contradicts our assumption.

Assume that f is sequentially consistent. Suppose by way of contradiction that there exists a nonempty bounded subset $B \subseteq \operatorname{int} \operatorname{dom} f$ such that $\inf_{x\in B} v_f(x, t) = 0$ for some positive real number t. Then there exists a sequence $\{x_n\}_{n\in\mathbf{N}}$ contained in B such that, for each positive integer n,

$$\frac{1}{n} > v_f(x_n, t) = \inf\{D_f(y, x_n) : y \in \operatorname{dom} f,\ \|y - x_n\| = t\}.$$

Then there exists a sequence $\{y_n\}_{n\in\mathbf{N}} \subseteq B$ such that, for each positive integer n, one has $\|y_n - x_n\| = t$ and $D_f(y_n, x_n) < 1/n$. The sequence $\{x_n\}_{n\in\mathbf{N}}$ is bounded since it is contained in B. Moreover, we have $D_f(y_n, x_n) \to 0$ as $n \to \infty$. Therefore

$$0 < t = \lim_{n\to\infty} \|y_n - x_n\| = 0,$$

which contradicts our assumption. □

3.3 Bregman projections

The concept of Bregman projection was first used by Bregman [12], while the terminology is due to Censor and Lent [18]. It has been shown that this generalized projection is a good replacement for the metric projection in optimization methods and in algorithms for solving convex feasibility problems. Let $f : E \to (-\infty, +\infty]$ be an admissible function.

3.3.1 The left Bregman projection

Given nonempty, closed and convex subset C of $\mathrm{dom} f$, the *left Bregman projection* proj_C^f *with respect to* f (*cf.* [12, 18]) from $\mathrm{int}\,\mathrm{dom} f$ onto C is defined by

$$\mathrm{proj}_C^f(x) := \arg\min_{y \in C} D_f(y, x) = \{z \in C : D_f(z, x) \le D_f(y, x),\ \forall y \in C\}$$

for $x \in \mathrm{int}\,\mathrm{dom} f$. If a Banach space E is reflexive and a function f is admissible, strongly coercive and totally convex, then there exists a unique minimizer of the function $D_f(\cdot, x)$ in C (see [2, 15]).

Proposition 3.3.1 ([2], Corollary 2.1, p. 38)**.** *Let* $f : E \to \mathbf{R}$ *is a strongly coercive and strictly convex function and* C *a nonempty, closed and convex subset of* $\mathrm{dom} f$*. Then* $\mathrm{proj}_C^f(x)$ *exists uniquely for all* $x \in \mathrm{int}\,\mathrm{dom} f$*.*

Proof. Denote $D_f(C, x) := \inf\{D_f(y, x) : y \in C\}$. By Proposition 1.4.1, the function f is continuous on $\mathrm{int}\,\mathrm{dom} f$. By Proposition 1.4.6, $f^\circ(x, \cdot)$ is continuous for each $x \in \mathrm{int}\,\mathrm{dom} f$. Consequently, for each $x \in \mathrm{int}\,\mathrm{dom} f$, the function $D_f(\cdot, x)$ is also continuous. Clearly, $D_f(C, x)$ is finite and there exists a sequence $\{x_n\}_{n \in \mathbf{N}}$ in C such that

$$\lim_{n \to \infty} D_f(x_n, x) = D_f(C, x).$$

Since f is strongly coercive, the sequence $\{x_n\}_{n \in \mathbf{N}}$ is bounded. Suppose by way of contradiction that $\{x_n\}_{n \in \mathbf{N}}$ is unbounded. This implies that there exists a subsequence $\{x_{n(k)}\}_{k \in \mathbf{N}}$ such that

$$\lim_{k \to \infty} \frac{f(x_{n(k)})}{\|x_{n(k)}\|} = +\infty. \tag{3.3.1}$$

Observe that

$$\begin{aligned} D_f(x_{n(k)}, x) &= f(x_{n(k)}) - f(x) - \langle \nabla f(x), x_{n(k)} - x \rangle \\ &= f(x_{n(k)}) - f^\circ(x, x_{n(k)}) - (f(x) - f^\circ(x, x)) \\ &\ge f(x_{n(k)}) - \|x_{n(k)}\| \|f^\circ(x, \cdot)\|_* - (f(x) - f^\circ(x, x)) \\ &= \|x_{n(k)}\| \Big(\frac{f(x_{n(k)})}{\|x_{n(k)}\|} - \|f^\circ(x, \cdot)\|_* \Big) - (f(x) - f^\circ(x, x)). \end{aligned}$$

By (3.3.1), we have

$$D_f(C, x) = \lim_{k \to \infty} D_f(x_{n(k)}, x) = +\infty,$$

which is a contradiction.

Since E is reflexive, the bounded sequence $\{x_n\}_{n\in\mathbf{N}}$ has a weakly convergent subsequence $\{x_{n(k)}\}_{k\in\mathbf{N}}$. Let x^* be the weak limit of $\{x_{n(k)}\}_{k\in\mathbf{N}}$. Since C is closed and convex, it is weakly closed. Hence $x^* \in C$. The epigraph of the convex and lower semicontinuous function $D_f(\cdot, x)$ is closed and convex in $E \times \mathbf{R}$. Thus it is also weakly closed in $E \times \mathbf{R}$. By consequence, the weak limit of the sequence $\{(x_{n(k)}, D_f(x_{n(k)}, x))\}_{k\in\mathbf{N}}$ belongs to the epigraph of $D_f(\cdot, x)$, that is, $D_f(x^*, x) \le D_f(C, x)$. Since $x^* \in C$, the proof is complete.

□

The left Bregman projection with respect to totally convex functions has the following variational characterization.

Proposition 3.3.2 ([17], Corollary 4.4, p. 23)**.** *Let $f : E \to (-\infty, +\infty]$ be a totally convex function. Let C be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$ *and $x \in \operatorname{int}\operatorname{dom} f$. If $\hat{x} \in C$, then the following statements are equivalent:*

(i) *The vector $\hat{x}$ is the left Bregman projection of x onto C with respect to f;*

(ii) *The vector $\hat{x}$ is the unique solution of the variational inequality;*

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \ge 0 \quad \textit{for all} \quad y \in C; \tag{3.3.2}$$

(iii) *The vector $\hat{x}$ is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \le D_f(y, x) \quad \textit{for all} \quad y \in C.$$

Proof. Suppose that (i) holds. Then $D_f(\hat{x}, x) \le D_f(w, x)$ for all $w \in C$. In particular, this holds for $w = (1-t)\hat{x} + ty$ for all $y \in C$ and $t \in [0, 1]$. Hence we obtain

$$\begin{aligned} 0 &\ge D_f(\hat{x}, x) - D_f(w, x) \\ &= f(\hat{x}) - f(w) - \langle \nabla f(x), w - \hat{x})\rangle \\ &\ge \langle \nabla f(w) - \nabla f(x), \hat{x} - w)\rangle \\ &= \langle \nabla f((1-t)\hat{x} + ty)) - \nabla f(x), t(\hat{x} - y)\rangle \end{aligned}$$

Letting here $t \to 0^+$, we have (3.3.2).

Suppose that (ii) holds. Then, for any $y \in C$, we have

$$D_f(y, x) - D_f(\hat{x}, x) = f(y) - f(\hat{x}) - \langle \nabla f(x), y - \hat{x}\rangle \ge \langle \nabla f(x) - \nabla f(\hat{x}), \hat{x} - y\rangle \ge 0.$$

This implies that $\hat{x}$ minimizes $D_f(\cdot, x)$ over C, that is, $\hat{x} = \operatorname{proj}_C^f(x)$.

To show that (ii) and (iii) are equivalent, it is sufficient to observe that

$$D_f(y, \hat{x}) + D_f(\hat{x}, x) - D_f(y, x) = \langle \nabla f(x) - \nabla f(\hat{x}), y - \hat{x}\rangle$$

for all $y \in C$. □

Remark. Let $f(x) = \|x\|^2/2$ for $x \in E$.

(i) If E is a Hilbert space, then the left Bregman projection proj_C^f is reduced to the metric projection P_C.

(ii) If E is a smooth, strictly convex and reflexive Banach space, then the left Bregman projection proj_C^f is reduced to the generalized projection Π_C.

Let $\{C_n\}_{n\in\mathbf{N}}$ be a sequence of subsets of E. We denote by s-Li$_n C_n$ the set of limit points of $\{C_n\}$, that is, $x \in$ s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \in C_n$ for each $n \in \mathbf{N}$ and $x_n \to x$ as $n \to \infty$. Similarly, we denote by w-Ls$_n C_n$ the set of weak cluster points of $\{C_n\}$; $y \in$ w-Ls$_n C_n$ if and only if there exists $\{y_{n_i}\} \subset E$ such that $y_{n_i} \in C_{n_i}$ for each $i \in \mathbf{N}$ and $y_{n_i} \rightharpoonup y$ as $i \to \infty$. Using these definitions, we define *Mosco convergence* (*cf.* [48]) of $\{C_n\}$. If C_0 satisfies

$$\underset{n}{\text{s-Li}}\, C_n = C_0 = \underset{n}{\text{w-Ls}}\, C_n,$$

then we say that $\{C_n\}$ is a Mosco convergent sequence to C_0. In this case, we denote it by

$$C_0 = \text{M-}\lim_n C_n.$$

Proposition 3.3.3 ([64], Theorem 4.5, p. 12)**.** *Let $f : E \to (-\infty, +\infty]$ be a totally convex function which is Fréchet differentiable on* $\operatorname{int}\operatorname{dom} f$*. Let $\{C_n\}_{n\in\mathbf{N}}$ be a sequence of nonempty, closed and convex subsets of* $\operatorname{int}\operatorname{dom} f$ *and C_0 a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$*. Then the following statements are equivalent:*

(i) *The sequence $\{C_n\}$ converges in the sense of Mosco to C_0;*

(ii) $\lim_{n\to\infty} \operatorname{proj}^f_{C_n}(x) = \operatorname{proj}^f_{C_0}(x)$ *for all $x \in \operatorname{int}\operatorname{dom} f$.*

Proof. (i)⇒(ii): Fix $x \in \operatorname{int}\operatorname{dom} f$ and denote $x_0 := \operatorname{proj}^f_C(x)$ and $x_n := \operatorname{proj}^f_{C_n}(x)$. Let $u \in C_0$ and $u_n \in C_n$ such that $u_n \to u$ as $n \to \infty$. Then, for any $n \in \mathbf{N}$, $D_f(u_n, x_n) + D_f(x_n, x) \le D_f(u_n, x)$. Since the sequence $\{D_f(u_n, x)\}_{n\in\mathbf{N}}$ converges to $D_f(u, x)$, it is bounded and the sequence $\{D_f(x_n, x)\}_{n\in\mathbf{N}}$ is also bounded. Note that $D_f(x_n, x) \ge v_f(x, \|x_n - x\|)$ for all $n \in \mathbf{N}$. By the strict monotonicity of $v_f(x, \cdot)$, this yields the boundedness of the sequence $\{x_n\}_{n\in\mathbf{N}}$. Hence there exists some subsequence $\{x_{n(j)}\}_{j\in\mathbf{N}}$ which converges weakly to some $y \in E$. By the definition of w-Ls$_n C_n$, we have $y \in C_0$. Since f is convex and lower semicontinuous, it is weakly lower semicontinuous. By consequence,

$$D_f(y, x) \le \liminf_{j\to\infty} D_f(x_n, x) \le \lim_{j\to\infty} D_f(u_n, x) = D_f(u, x).$$

Since u is arbitrarily chosen in C_0, we have $y = \operatorname{proj}^f_C(x)$. As this weak cluster point is unique, we obtain that the entire sequence $\{x_n\}_{n\in\mathbf{N}}$ converges weakly to x_0. Hence

$$D_f(x_0, x) \le \liminf_{n\to\infty} D_f(x_n, x) \le \limsup_{n\to\infty} D_f(x_n, x) \le D_f(u, x)$$

for all $u \in C_0$. In particular, this holds for x_0. Therefore the following limit exists and $D_f(x_n, x) \to D_f(x_0, x)$ as $n \to \infty$. Note that

$$D_f(x_n, x) - D_f(x_0, x) - D_f(x_n, x_0) = \langle \nabla f(x) - \nabla f(x_0), x_0 - x_n \rangle.$$

Letting $n \to \infty$, we obtain $D_f(x_n, x_0) \to 0$ as $n \to \infty$. By Proposition 3.2.8, we have $x_n \to x_0$ as $n \to \infty$.

(ii)⇒(i): Clearly, $x \in$ s-Li$_n C_n$: if $x \in C_0$, there exists $\{\operatorname{proj}^f_{C_n}(x)\}_{n\in\mathbf{N}}$ such that $\operatorname{proj}^f_{C_n}(x) \in C_n$ for all $n \in \mathbf{N}$, and $\operatorname{proj}^f_{C_n}(x) \to \operatorname{proj}^f_{C_0}(x) = x$ as $n \to \infty$. It remains to prove $y \in$ w-Ls$_n C_n$. Let $\{x_i\}_{i\in\mathbf{N}}$ with $x_i \in C_i$ such that it converges weakly to some $x \in E$. If $y_0 := \operatorname{proj}^f_{C_0}(x)$ and $y_i := \operatorname{proj}^f_{C_i}(x)$, then the hypothesis yields $y_i \to y_0$ as $i \to \infty$. By Proposition 3.3.2, we have $\langle \nabla f(y_i) - \nabla f(x), x_i - y_i \rangle \ge 0$. Letting $i \to \infty$, we obtain $\langle \nabla f(y_0) - \nabla f(x), x - y_0 \rangle \ge 0$. Since f is strictly convex and consequently ∇f is strictly monotone, we have $x = y_0 \in C_0$. □

3.3.2 The right Bregman projection

Given a nonempty, closed and convex subset C of $\operatorname{int}\operatorname{dom} f$, the *right Bregman projection* $\overrightarrow{\operatorname{proj}}_C^f$ *with respect to* f (*cf.* [9, 43]) from $\operatorname{int}\operatorname{dom} f$ onto C is defined by

$$\overrightarrow{\operatorname{proj}}_C^f(x) := \arg\min_{y\in C} D_f(x,y) = \{z \in C : D_f(x,z) \le D_f(x,y),\ \forall y \in C\}$$

for $x \in \operatorname{int}\operatorname{dom} f$. Since D_f is not convex in the second variable, it is not clear a priori that the right Bregman projection is well defined. However, Bauschke, Wang, Ye and Yuan [9] and Martín-Márquez, Reich and Sabach [43] have proved

$$\overrightarrow{\operatorname{proj}}_C^f = \nabla f^* \circ \operatorname{proj}_{\nabla f(C)}^{f^*} \circ \nabla f \tag{3.3.3}$$

and established several other properties of $\overrightarrow{\operatorname{proj}}_C^f$. The right Bregman projection with respect to totally convex functions has the following variational characterization.

Proposition 3.3.4 ([43], Proposition 4.11, p. 5459)**.** *Let $f : E \to \mathbf{R}$ be a function such that f^* is admissble and totally convex. Let C be a nonempty subset of* $\operatorname{int}\operatorname{dom} f$ *such that $\nabla f(C)$ is closed and convex. Let $x \in \operatorname{int}\operatorname{dom} f$. If $\hat{x} \in C$, then the following conditions are equivalent to each other:*

(i) *The vector $\hat{x}$ is the right Bregman projection of x onto C with respect to f;*

(ii) *The vector $\hat{x}$ is the unique solution z of the variational inequality*

$$\langle \nabla f(z) - \nabla f(y), x - z\rangle \ge 0 \quad \textit{for all} \quad y \in C;$$

(iii) *The vector $\hat{x}$ is the unique solution z of the inequality*

$$D_f(z,y) + D_f(x,z) \le D_f(x,y) \quad \textit{for all} \quad y \in C.$$

Proof. Since $\nabla f(C)$ is closed and convex, the left Bregman projection onto $\nabla f(C)$ with respect to the totally convex function f^* is well defined and characterized in Proposition 3.3.2. It is clear from (3.3.3) that (i) is equivalent to the fact that the vector $\nabla f(\hat{x})$ is the left Bregman projection of $\nabla f(x)$ onto $\nabla f(C)$ with respect to f^*. By Proposition 3.3.2 (ii), (i) is equivalent to $\hat{x}$ being the unique solution z of the inequality

$$\langle \nabla f^*(\nabla f(x)) - \nabla f^*(\nabla f(z)), \nabla f(z) - \xi\rangle \ge 0$$

for all $\xi \in \nabla f(C)$. This is equivalent to $\langle \nabla f(z) - \nabla f(y), x - z\rangle \ge 0$ for all $y \in C$. Using the three point identity (3.2.1), we can also prove that (ii) is equivalent to (iii). □

Remark. Let $f(x) = \|x\|^2/2$ for $x \in E$.

(i) If E is a Hilbert space, then the right Bregman projection $\overrightarrow{\operatorname{proj}}_C^f$ is reduced to the metric projection P_C.

(ii) If E is a smooth, strictly convex and relative Banach space, then the right Bregman projection $\overrightarrow{\operatorname{proj}}_C^f$ is reduced to the sunny generalized nonexpansive retraction R_C.

3.4 Bregman asymptotically quasi-nonexpansive in the intermediate sense

In this section, we introduce and consider the new nonlinear mappings with respect to Bregman distances based on asymptotically quasi-nonexpansive mappings in the intermediate sense. Let $f : E \to (-\infty, +\infty]$ be an admissible function.

3.4.1 Left Bregman nonexpansive mappings

Let C be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$ and T a mapping from C into $\operatorname{int}\operatorname{dom} f$. The mapping T is said to be *left Bregman quasi-nonexpansive* with respect to $F(T)$ (*cf.* [62]) if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x) \quad \text{for all} \quad p \in F(T),\ x \in C.$$

The mapping T is said to be *left Bregman asymptotically quasi-nonexpansive* (*cf.* [80]) if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for every $n \in \mathbf{N}$,

$$D_f(p, T^n x) \leq k_n D_f(p, x) \quad \text{for all} \quad p \in F(T),\ x \in C.$$

Every Bregman quasi-nonexpansive mapping is Bregman asymptotically quasi-nonexpansive with $k_n = 1$.

We introduce a new class of mappings: the mapping T is said to be *left Bregman asymptotically quasi-nonexpansive in the intermediate sense* if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in F(T),\ x\in C} \big(D_f(p, T^n x) - D_f(p, x)\big) \leq 0. \tag{3.4.1}$$

Put

$$\xi_n = \max\left\{0, \sup_{p\in F(T),\ x\in C} \big(D_f(p, T^n x) - D_f(p, x)\big)\right\}.$$

The inequality (3.4.1) implies $\lim_{n\to\infty} \xi_n = 0$. Then (3.4.1) is reduced to the following inequality

$$D_f(p, T^n x) \leq D_f(p, x) + \xi_n \tag{3.4.2}$$

for all $p \in F(T)$ and $x \in C$, where $\{\xi_n\}$ is a sequence such that $\xi_n \to 0$ as $n \to \infty$. Left Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense are not Lipschitz continuous in general.

Example 3.4.1. Assume that $E = \mathbf{R}$, $C = [1/2, 3/2]$ and $T : C \to C$ defined by

$$Tx = \begin{cases} 1, & x \in [\frac{1}{2}, 1], \\ 1 - \sqrt{\frac{x-1}{2}}, & x \in (1, \frac{3}{2}]. \end{cases} \tag{3.4.3}$$

Note that $F(T) = \{1\}$ and $T^n x = 1$ for all $x \in C$ and $n \geq 2$. If $f : \mathbf{R} \to (-\infty, +\infty]$ is a Legendre function, then T is left Bregman asymptotically quasi-nonexpansive in the intermediate sense since

$$\limsup_{n\to\infty} \sup_{x\in C} \big(D_f(1, T^n x) - D_f(1, x)\big) \leq \limsup_{n\to\infty} \sup_{x\in C} D_f(1, T^n x) = 0.$$

However, T is not Lipschitzian with respect to Bregman distances in Example 3.2.2. Indeed, suppose that there exists $L > 0$ such that $D_f(Ty, Tx) \le LD_f(y, x)$ for all $x, y \in C$. By Taylor's theorem, there exists $t \in (0, 1)$ such that

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x\rangle = \frac{1}{2}\nabla^2 f(x + t(y - x))(y - x)^2. \tag{3.4.4}$$

(i) Let $f(x) = \|x\|^2/2$ on $\mathrm{dom} f = \mathbf{R}$ and $D_f(y, x) = \|y - x\|^2/2$ for all $x, y \in \mathbf{R}$. Put $y = 1$ and $x = 1 + 1/2(L + 1)$. Since $Tx = 1 - 1/2\sqrt{L + 1}$, we have

$$\frac{1}{8(L+1)} = \frac{1}{2}\left\|\frac{-1}{2\sqrt{L+1}}\right\|^2 = \frac{1}{2}\|Ty - Tx\|^2 \le \frac{L}{2}\|y - x\|^2 = \frac{L}{8(L+1)^2}.$$

This implies $L + 1 \le L$, which is a contradiction.

(ii) Let $f(x) = x\ln(x) - x$ on $\mathrm{dom} f = [0, +\infty)$ and $D_f(y, x) = y\ln(y/x) - y + x$ for all $x \in (0, +\infty)$ and $y \in [0, +\infty)$. Note that $\nabla^2 f(x) = 1/x$. Put $x = 1$. By (3.4.4), we have

$$D_f(y, 1) = \frac{(y-1)^2}{2(1 + t(y - 1))} \le \frac{(y-1)^2}{2} \quad \text{for } y \ge 1$$

and

$$D_f(y, 1) = \frac{(y-1)^2}{2(1 + t(y - 1))} \ge \frac{(y-1)^2}{2} \quad \text{for } 0 < y \le 1.$$

If $y = 1 + 1/2(L + 1)$, we have

$$\frac{1}{8(L+1)} = \frac{1}{2}\left(\frac{-1}{2\sqrt{L+1}}\right)^2 \le D_f(Ty, 1) \le LD_f(y, 1) \le \frac{L}{2}\left(\frac{1}{2(L+1)}\right)^2 = \frac{L}{8(L+1)^2}.$$

This implies $L + 1 \le L$, which is a contradiction.

(iii) Let $f(x) = -\ln(x)$ on $\mathrm{dom} f = (0, +\infty)$ and $D_f(y, x) = \ln(x/y) + y/x - 1$ for all $x, y \in (0, +\infty)$. Note that $\nabla^2 f(x) = 1/x^2$. Put $y = 1$. By (3.4.4), we have

$$D_f(1, x) = \frac{(1-x)^2}{2(x + t(1 - x))^2} \le \frac{(1-x)^2}{2} \quad \text{for } x \ge 1$$

and

$$D_f(1, x) = \frac{(1-x)^2}{2(x + t(1 - x))^2} \ge \frac{(1-x)^2}{2} \quad \text{for } 0 < x \le 1.$$

If $x = 1 + 1/2(L + 1)$, we have

$$\frac{1}{8(L+1)} = \frac{1}{2}\left(\frac{1}{2\sqrt{L+1}}\right)^2 \le D_f(1, Tx) \le LD_f(1, x) \le \frac{L}{2}\left(\frac{-1}{2(L+1)}\right)^2 = \frac{L}{8(L+1)^2}.$$

This implies $L + 1 \le L$, which is a contradiction.

Remark. Let $f(x) = \|x\|^2/2$ for $x \in E$.

(i) If E is a Hilbert space, then left Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense is reduced to asymptotically quasi-nonexpansive mappings in the intermediate sense (3.1.2).

(ii) If E is a smooth, strictly convex and relatively Banach space, then left Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense is reduced to asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense (*cf.* [55]), that is, $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in F(T),\ x\in C} \big(\phi(p, T^n x) - \phi(p, x)\big) \leq 0.$$

Theorem 3.4.2. *Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is totally convex on bounded subsets of E. Suppose that ∇f^* is bounded on bounded subsets of* $\mathrm{dom} f^* = E^*$*. Let C be a nonempty, closed and convex subset of* $\mathrm{int}\,\mathrm{dom} f$*. Let $T : C \to C$ be a closed and left Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense. Then $F(T)$ is closed and convex.*

Proof. Since T is closed, we can easily conclude that $F(T)$ is closed. Now we show the convexness of $F(T)$. Let $p_1, p_2 \in F(T)$ and $p = tp_1 + (1-t)p_2$, where $t \in (0, 1)$. We prove that $p \in F(T)$. By (3.4.2), we have

$$D_f(p_i, T^n p) \leq D_f(p_i, p) + \xi_n \tag{3.4.5}$$

for $i = 1, 2$. By the three point identity (3.2.1), we know that

$$D_f(x, y) = D_f(x, z) + D_f(z, y) + \langle \nabla f(z) - \nabla f(y), x - z\rangle$$

for $x \in \mathrm{dom} f$ and $y, z \in \mathrm{int}\,\mathrm{dom} f$. This implies

$$D_f(p_i, T^n p) = D_f(p_i, p) + D_f(p, T^n p) + \langle \nabla f(p) - \nabla f(T^n p), p_i - p\rangle \tag{3.4.6}$$

for $i = 1, 2$. Combining (3.4.5) and (3.4.6) yields that

$$\begin{aligned} D_f(p, T^n p) &= D_f(p_i, T^n p) - D_f(p_i, p) - \langle \nabla f(p) - \nabla f(T^n p), p_i - p\rangle \\ &\leq \xi_n - \langle \nabla f(p) - \nabla f(T^n p), p_i - p\rangle \end{aligned} \tag{3.4.7}$$

for $i = 1, 2$. Multiplying t and $1-t$ on the both sides of (3.4.7) with $i = 1$ and $i = 2$, respectively, yields that

$$\lim_{n\to\infty} D_f(p, T^n p) \leq \lim_{n\to\infty} (\xi_n - \langle \nabla f(p) - \nabla f(T^n p), tp_1 + (1-t)p_2 - p\rangle) = 0.$$

This implies that $\{D_f(p, T^n p)\}_{n\in\mathbf{N}}$ is bounded. By Propositions 3.2.4 and 3.2.8, we see that the sequence $\{T^n p\}_{n\in\mathbf{N}}$ is bounded and $\|p - T^n p\| \to 0$ as $n \to \infty$. By the closedness of T, we have

$$p = \lim_{n\to\infty} T^{n+1} p = T\left(\lim_{n\to\infty} T^n p\right) = Tp$$

and hence $p \in F(T)$. Therefore $F(T)$ is convex. □

Theorem 3.4.3. *Let $f : E \to (-\infty, +\infty]$ be a Legendre and strongly coercive function which is totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of* $\mathrm{int}\,\mathrm{dom} f$ *and $T : C \to C$ a closed and left Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense. Then there exists a unique left Bregman projection from* $\mathrm{int}\,\mathrm{dom} f$ *onto $F(T)$.*

Proof. By Proposition 1.4.4, ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$ since f is Legendre and strongly coercive. By Proposition 3.3.1 and Theorem 3.4.2, there exists a unique minimizer of $D_f(\cdot, x)$ in $F(T)$. □

Theorem 3.4.2 can be reduced to the following results.

Corollary 3.4.4 ([80], Lemma 1, p. 3)**.** *Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$ *and $T : C \to C$ a closed and left Bergman asymptotically quasi-nonexpansive mapping with the sequence $\{k_n\}_{n\in\mathbf{N}} \subset [1, +\infty)$ such that $k_n \to 1$ as $n \to \infty$. Then $F(T)$ is closed and convex.*

Corollary 3.4.5 ([63], Lemma 15.5, p. 307)**.** *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$ *and $T : C \to C$ a left Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

3.4.2 Right Bregman nonexpansive mappings

Let C be a nonempty subset of $\operatorname{dom} f$ and T a mapping from C into $\operatorname{int}\operatorname{dom} f$. The mapping T is said to be *right Bregman quasi-nonexpansive* with respect to $F(T)$ (*cf.* [43]) if $F(T) \neq \emptyset$ and

$$D_f(Tx, p) \leq D_f(x, p) \quad \text{for all} \quad p \in F(T),\ x \in C.$$

Recall that the mapping T is said to be *right Bregman firmly quasi-nonexpansive* with respect to $F(T)$ (*cf.* [43]) if $F(T) \neq \emptyset$ and

$$\langle \nabla f(p) - \nabla f(Tx), Tx - x\rangle \geq 0$$

for all $p \in F(T)$ and $x \in C$, or equivalently,

$$D_f(Tx, p) + D_f(x, Tx) \leq D_f(x, p) \quad \text{for all} \quad p \in F(T),\ x \in C. \tag{3.4.8}$$

Given two nonempty subsets $K \subset C \subset \operatorname{int}\operatorname{dom} f$, the subset K is said to be a *sunny right Bregman quasi-nonexpansive retract of* C if there exists a sunny right Bregman quasi-nonexpansive retraction of C onto K.

Proposition 3.4.6 ([43], Propotition 4.1, p. 5456)**.** *Let $f : E \to (-\infty, +\infty]$ be a totally convex function and $K \subset C \subset$* $\operatorname{int}\operatorname{dom} f$ *two nonempty subsets. If C is convex and R is a retraction of C onto K, then R is sunny and right Bregman quasi-nonexpansive if and only if it is right Bregman firmly quasi-nonexpansive.*

Proof. First we assume that R is sunny right Bregman quasi-nonexpansive. Let $x \in C$ and $p \in K = F(R)$. Denote $x_t = Rx + t(x - Rx)$ for each $t \in [0, 1]$. Since R is a retraction and right Bregman quasi-nonexpansive, we have $D_f(Rx, p) = D_f(Rx_t, p) \leq D_f(x_t, p)$. Thus $Rx = \operatorname{proj}^f_{[x,Rx]}(p)$, where $[x, Rx] := \{tx + (1 - t)Rx : t \in [0, 1]\}$. Using Proposition 3.3.2, we have

$$\langle \nabla f(p) - \nabla f(Rx), Rx - x_t\rangle \geq 0$$

for each $t \in [0,1]$. Setting $t = 1$, we have $\langle \nabla f(p) - \nabla f(Rx), Rx - x\rangle \geq 0$ for all $x \in C$ and $p \in F(R)$, that is, R is right Bregman firmly quasi-nonexpansive.

Conversely, suppose that R is right Bregman firmly quasi-nonexpansive. By the three point identity (3.2.1), we have

$$\begin{aligned} D_f(x,p) &= D_f(x,Rx) + D_f(Rx,p) + \langle \nabla f(Rx) - \nabla f(p), x - Rx\rangle \\ &\geq D_f(x,Rx) + D_f(Rx,p) \geq D_f(Rx,p). \end{aligned}$$

for all $x \in C$ and $p \in K = F(R)$. This means that R is right Bregman quasi-nonexpansive. Now we prove that R is sunny. To this end, for any $x \in C$ and $t > 0$, set $x_t = Rx + t(x - Rx)$. By (3.4.8), we have

$$\langle \nabla f(Rx) - \nabla f(Rx_t), Rx_t - x_t\rangle \geq 0 \tag{3.4.9}$$

and $\langle \nabla f(Rx_t) - \nabla f(Rx), Rx - x\rangle \geq 0$. Since $x_t - Rx = t(x - Rx)$, we have

$$0 \leq t\langle \nabla f(Rx) - \nabla f(Rx_t), x - Rx\rangle = \langle \nabla f(Rx) - \nabla f(Rx_t), x_t - Rx\rangle. \tag{3.4.10}$$

Combining (3.4.9) and (3.4.10), we have $\langle \nabla f(Rx) - \nabla f(Rx_t), Rx_t - Rx\rangle \geq 0$. This implies $\langle \nabla f(Rx) - \nabla f(Rx_t), Rx_t - Rx\rangle = 0$. Since f is totally convex, it is strictly convex, and hence ∇f is strictly monotone. Therefore $Rx_t = Rx$, that is, R is sunny, as claimed. □

Proposition 3.4.7 ([43], Corollary 4.2, p. 5457)**.** *Let $f : E \to (-\infty, +\infty]$ be a totally convex function and $K \subset C \subset \operatorname{int}\operatorname{dom} f$ two nonempty subsets. If K is a sunny right Bregman quasi-nonexpansive retract of C, then the sunny right Bregman quasi-nonexpansive retraction of C onto K is uniquely defined.*

Proof. Assuming that there exist two sunny right Bregman quasi-nonexpansive retractions R and S of C onto K, we know by Proposition 3.4.6 that both these operators are right Bregman firmly quasi-nonexpansive. Hence, for any $x \in C$, we have $\langle \nabla f(Rx) - \nabla f(Sx), x - Rx\rangle \geq 0$ and $\langle \nabla f(Sx) - \nabla f(Rx), x - Sx\rangle \geq 0$ since $Rx, Sx \in K$. Thus $\langle \nabla f(Sx) - \nabla f(Rx), Sx - Rx\rangle \leq 0$. This implies $Sx = Rx$ since ∇f is strictly monotone. □

Proposition 3.4.8 ([43], Proposition 4.4, p. 5457)**.** *Let $f : E \to \mathbf{R}$ be a Legendre function. Assume that f and f^* are totally convex. Let K^* be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f^*$. Then the operator R defined by $R = \nabla f^* \circ \operatorname{proj}_{K^*}^{f^*} \circ \nabla f$ is a sunny right Bregman quasi-nonexpansive retraction of $\operatorname{int}\operatorname{dom} f$ onto $\nabla f^*(K^*)$.*

Proof. For any $x \in \nabla f^*(K^*)$, we have $\operatorname{proj}_{K^*}^{f^*}(\nabla f(x))$ since $\nabla f(x) \in K^*$. This implies

$$Rx = (\nabla f^* \circ \operatorname{proj}_{K^*}^{f^*} \circ \nabla f)(x) = \nabla f^*(\nabla f(x)) = x$$

for all $x \in \nabla f^*(K^*)$. Thus R is onto $\nabla f^*(K^*)$ and $Rx = x$ for all $x \in \nabla f^*(K^*)$, that is, R is a retraction of $\operatorname{int}\operatorname{dom} f$ onto $\nabla f^*(K^*)$. This implies $F(R) = \nabla f^*(K^*)$. By Proposition 3.3.2, we have

$$D_{f^*}(\xi, \operatorname{proj}_{K^*}^{f^*}(\eta)) + D_{f^*}(\operatorname{proj}_{K^*}^{f^*}(\eta), \eta) \leq D_{f^*}(\xi, \eta)$$

for all $\eta \in \operatorname{int}\operatorname{dom} f^*$ and $\xi \in K^*$. Thus

$$D_{f^*}\big(\nabla f(y), \operatorname{proj}_{K^*}^{f^*}(\nabla f(x))\big) + D_{f^*}(\operatorname{proj}_{K^*}^{f^*}(\nabla f(x)), \nabla f(x)) \leq D_{f^*}(\nabla f(y), \nabla f(x)) \tag{3.4.11}$$

for all $x \in \operatorname{int}\operatorname{dom} f$ and $y \in \nabla f^*(K^*)$. Since $(\nabla f)^{-1} = \nabla f^*$, it is easy to check that $D_f^*(\nabla f(y), \nabla f(x)) = D_f(x, y)$. Hence, by (3.4.11), we have

$$D_f(\nabla f^* \circ \operatorname{proj}_{K^*}^{f^*}(\nabla f(x)), y) + D_f\big(x, \nabla \circ \operatorname{proj}_{K^*}^{f^*}(\nabla f(x))\big) \le D_f(x, y).$$

This implies $D_f(Rx, y) + D_f(x, Rx) \le D_f(x, y)$ for all $x \in \operatorname{int}\operatorname{dom} f$ and $y \in \nabla f^*(K^*)$. In other words, R is right Bregman firmly quasi-nonexpansive. By Proposition 3.4.6, R is a sunny right Bregman quasi-nonexpansive retraction of $\operatorname{int}\operatorname{dom} f$ onto $\nabla f^*(C^*)$. □

We also know that the unique sunny right Bregman quasi-nonexpansive retraction of E onto C is given by the right Bregman projection defined by (3.3.3):

Proposition 3.4.9 ([43], Corollary 4.6, p. 5458)**.** *Let $f : E \to \mathbf{R}$ be a Legendre, cofinite and totally convex function, and assume that f^* is totally convex. Let C be a nonempty subset of* $\operatorname{int}\operatorname{dom} f$*. If $\nabla f(C)$ is closed and convex, then the right Bregman projection* (3.3.3) *is the unique sunny right Bregman quasi-nonexpansive retraction of* $\operatorname{int}\operatorname{dom} f$ *onto C.*

Proof. Since f is Legendre, we have $\operatorname{ran}\nabla f = \operatorname{int}\operatorname{dom} f^*$. By Proposition 3.4.8, $R = \nabla f^* \circ \operatorname{proj}_{\nabla f(C)}^{f^*} \circ \nabla f$ is a sunny right Bregman quasi-nonexpansive retraction of $\operatorname{int}\operatorname{dom} f$ onto $C = \nabla f^*(\nabla f(C))$. Thus C is a sunny right Bregman quasi-nonexpansive retract of $\operatorname{int}\operatorname{dom} f$. By Proposition 3.4.7, the unique sunny right Bregman quasi-nonexpansive retraction of $\operatorname{int}\operatorname{dom} f$ onto C is given by the conjugate operator $\nabla f^* \circ \operatorname{proj}_{\nabla f(C)}^{f^*} \circ \nabla f$, which is the right Bregman projection by (3.3.3). □

Let C be a nonempty subset of $\operatorname{dom} f$ and T a mapping from C into $\operatorname{int}\operatorname{dom} f$. We introduce a new class of mappings: the mapping T is said to be *right Bregman asymptotically quasi-nonexpansive in the intermediate sense* if $F(T) \ne \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in F(T),\ x\in C} \big(D_f(T^n x, p) - D_f(x, p)\big) \le 0. \tag{3.4.12}$$

Put

$$\eta_n = \max\left\{0, \sup_{p\in F(T),\ x\in C} \big(D_f(T^n x, p) - D_f(x, p)\big)\right\}.$$

The inequality (3.4.12) implies $\lim_{n\to\infty} \eta_n = 0$. Then (3.4.12) is reduced to the following:

$$D_f(T^n x, p) \le D_f(x, p) + \eta_n \tag{3.4.13}$$

for all $p \in F(T)$ and $x \in C$, where $\{\eta_n\}$ is a sequence such that $\eta_n \to 0$ as $n \to \infty$. Right Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense are not Lipschitz continuous in general.

Example 3.4.10. Assume that $E = \mathbf{R}$, $C = [1/2, 3/2]$ and $T : C \to C$ defined by (3.4.3). If $f : \mathbf{R} \to (-\infty, +\infty]$ is a Legendre function, then T is right Bregman asymptotically quasi-nonexpansive in the intermediate sense since

$$\limsup_{n\to\infty} \sup_{x\in C}\big(D_f(T^n x, 1) - D_f(x, 1)\big) \le \limsup_{n\to\infty} \sup_{x\in C} D_f(T^n x, 1) = 0.$$

By Example 3.4.1, we know that T is not Lipschitzian with respect to Bregman distances in Example 3.2.2.

Remark. Let $f(x) = \|x\|^2/2$ for $x \in E$. If E is a Hilbert space, then right Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense is reduced to asymptotically quasi-nonexpansive mappings in the intermediate sense (3.1.2).

Theorem 3.4.11. *Let $f : E \to (-\infty, +\infty]$ be a Legendre and strongly coercive function which is totally convex on bounded subsets of E. Let $T : \operatorname{int}\operatorname{dom}f \to \operatorname{int}\operatorname{dom}f$ be a closed and right Bregman asymptotically quasi-nonexpansive mapping in the intermediate sense. Then $\nabla f\big(F(T)\big)$ is closed and convex subset of E^*.*

Proof. First we show that $\nabla f\big(F(T)\big)$ is convex. Let $p_1, p_2 \in F(T)$ and $p = \nabla f^*\big(t\nabla f(p_1) + (1-t)\nabla f(p_2)\big)$, where $t \in (0,1)$. We prove that $p \in F(T)$. By the definition of Bregman distance, we have

$$\begin{aligned}
D_f(T^n p, p) &= f(T^n p) - f(p) - \langle \nabla f(p), T^n p - p\rangle \\
&= t\{f(T^n p) - f(p_1) - \langle \nabla f(p_1), T^n p - p_1\rangle\} \\
&\quad + (1-t)\{f(T^n p) - f(p_2) - \langle \nabla f(p_2), T^n p - p_2\rangle\} \\
&\quad - f(p) + tf(p_1) + (1-t)f(p_2) \\
&\quad + \langle \nabla f(p), p\rangle - t\langle \nabla f(p_1), p_1\rangle - (1-t)\langle \nabla f(p_2), p_2\rangle \\
&= tD_f(T^n p, p_1) + (1-t)D_f(T^n p, p_2) - f(p) + \langle \nabla f(p), p\rangle \\
&\quad + t(f(p_1) - \langle \nabla f(p_1), p_1\rangle) + (1-t)(f(p_2) - \langle \nabla f(p_2), p_2\rangle). \qquad (3.4.14)
\end{aligned}$$

It is known that $f(x) + f^*\big(\nabla f(x)\big) = \langle \nabla f(x), x\rangle$ for all $x \in E$. By (3.4.14), we have

$$\begin{aligned}
D_f(T^n p, p) &= tD_f(T^n p, p_1) + (1-t)D_f(T^n p, p_2) \\
&\quad + f^*\big(\nabla f(p)\big) - tf^*\big(\nabla f(p)\big) - (1-t)f^*\big(\nabla f(p)\big). \qquad (3.4.15)
\end{aligned}$$

By (3.4.13), we have $D_f(p_i, T^n p) \le D_f(p_i, p) + \eta_n$ for $i = 1, 2$. By (3.4.15), we have

$$\begin{aligned}
D_f(T^n p, p) &\le tD_f(p, p_1) + (1-t)D_f(p, p_2) + \eta_n \\
&\quad + f^*\big(\nabla f(p)\big) - tf^*\big(\nabla f(p)\big) - (1-t)f^*\big(\nabla f(p)\big) \\
&= f(p) - \langle \nabla f(p), p\rangle + f^*\big(\nabla f(p)\big) + \eta_n = \eta_n.
\end{aligned}$$

This implies

$$\lim_{n\to\infty} D_f(T^n p, p) = \lim_{n\to\infty} \eta_n = 0.$$

By Proposition 3.2.8, we have $\|T^n p - p\| \to 0$ as $n \to \infty$. By the closedness of T, we have

$$p = \lim_{n\to\infty} T^{n+1} p = T \lim_{n\to\infty} T^n p = Tp$$

and hence $p \in F(T)$.

Next we prove that $\nabla f\big(F(T)\big)$ is closed. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in $F(T)$ such that $\nabla f(x_n) \to x^* \in E^*$ as $n \to \infty$. Since f is strongly coercive, we have $\operatorname{ran}\nabla f = E^*$. Hence there exists $x \in E$ such that $x^* = \nabla f(x)$. It is sufficient to prove that $x \in F(T)$. Since $\{x_n\} \subset F(T)$ and T is right Bregman asymptotically quasi-nonexpansive in the intermediate sense, we have

$$D_f(T^n x, x_n) \le D_f(x, x_n) + \eta_n = f(x) + f^*\big(\nabla f(x_n)\big) - \langle \nabla f(x_n), x\rangle + \eta_n.$$

By assumption, f^* is continuous and $\nabla f(x_n) \to \nabla f(x)$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} D_f(T^n x, x_n) \leq f(x) + \lim_{n\to\infty} \big(f^*\big(\nabla f(x_n)\big) - \langle \nabla f(x_n), x\rangle + \eta_n\big) = 0.$$

On the other hand,

$$\begin{aligned} D_f(T^n x, x) &= D_f(T^n x, x_n) + f(x_n) + \langle \nabla f(x_n), T^n x - x_n\rangle - f(x) - \langle \nabla f(x), T^n x - x\rangle \\ &= D_f(T^n x, x_n) - f^*\big(\nabla f(x_n)\big) + f^*\big(\nabla f(x)\big) + \langle \nabla f(x_n) - \nabla f(x), T^n x\rangle. \end{aligned}$$

Hence $D_f(T^n x, x) \to 0$ as $n \to \infty$. By Proposition 3.2.8, we have $\|T^n x - x\| \to 0$ as $n \to \infty$. By the closedness of T, we have $x = Tx$ and hence $x \in F(T)$. □

Theorem 3.4.12. *Let $f : E \to (-\infty, +\infty]$ be a Legendre and strongly coercive function which is totally convex on bounded subsets of E such that f^* is totally convex. If $T : \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ is a closed and right Bregman asymptotically quasi-nonexpansive mapping in the intermediate sense, then there exists a unique sunny right Bregman quasi-nonexpansive retraction of* $\operatorname{int}\operatorname{dom} f$ *onto $F(T)$, which is the right Bregman projection onto $F(T)$.*

Proof. By the assumption of f and T, it follows from Theorem 3.4.11 that $\nabla f\big(F(T)\big)$ is closed and convex in E^*. Proposition 3.4.9 ensures that the right Bregman projection $\overrightarrow{\operatorname{proj}}^f_{F(T)}$ is the unique sunny right Bregman quasi-nonexpansive retraction of $\operatorname{int}\operatorname{dom} f$ onto $F(T)$. □

When a mapping T is right Bregman quasi-nonexpansive, Theorems 3.4.11 and 3.4.12 can be reduced to the following results.

Corollary 3.4.13 ([43], Proposition 3.3, p. 5454)**.** *Let $f : E \to (-\infty, +\infty]$ be a Legendre and cofinite function and $T : \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ a right Bregman quasi-nonexpansive mapping. Then $\nabla f\big(F(T)\big)$ is closed and convex subset of E^*.*

Corollary 3.4.14 ([43], Proposition 3.4, p. 5454)**.** *Let $f : E \to (-\infty, +\infty]$ be a Legendre function and C a nonempty subset of* $\operatorname{int}\operatorname{dom} f$ *such that $\nabla f(C)$ is closed and convex. If $T : C \to \operatorname{int}\operatorname{dom} f$ a right Bregman quasi-nonexpansive mapping, then $\nabla f\big(F(T)\big)$ is closed and convex subset of E^*.*

Corollary 3.4.15 ([43], Corollary 4.7, p. 5458)**.** *Let $f : E \to \mathbf{R}$ be a Legendre and cofinite and totally convex function. Assume that f^* is totally convex. If $T : \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ is a right Bregman quasi-nonexpansive mapping, then there exists a unique sunny R-BQNE retraction of* $\operatorname{int}\operatorname{dom} f$ *onto $F(T)$, which is the right Bregman projection onto $F(T)$.*

3.5 Strong convergence theorems of Bregman projections

In this section, we prove strong convergence theorems for finding a fixed point of a Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense by the shirinking projection method.

Let C be a nonempty, closed and convex subset of E and T a mapping from C into itself. The mapping T is said to be *asymptotically regular* if, for any $x \in C$,

$$\lim_{n\to\infty} \left\| T^{n+1}x - T^n x \right\| = 0.$$

3.5.1 The Shrinking projection method with left Bregman projections

Theorem 3.5.1. *Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is bounded, strongly coercive, uniformly Fréchet differentiable and totally convex on bounded subsets on E. Let C be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$. *Let $T : C \to C$ be a closed and left Bregman asymptotically quasi-nonexpansive mapping in the intermediate sense. Suppose that T is asymptotically regular on C and $F(T)$ is bounded. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence generated by*

$$\begin{cases} x_0 \in \operatorname{int}\operatorname{dom} f, \ \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \operatorname{proj}^f_{C_1} x_0, \\ y_n = \nabla f^*\big(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T^n x_n)\big), \\ C_{n+1} = \{z \in C_n : D_f(z, y_n) \le D_f(z, x_n) + \xi_n\}, \\ x_{n+1} = \operatorname{proj}^f_{C_{n+1}} x_0, \ n \in \mathbf{N}, \end{cases}$$

where $\operatorname{proj}^f_{C_n}$ *is the left Bregman projection from* $\operatorname{int}\operatorname{dom} f$ *onto C_n,*

$$\xi_n := \max\left\{0, \sup_{p\in F(T),\ x\in C} \big(D_f(p, T^n x) - D_f(p, x)\big)\right\}$$

and $0 \le \alpha_n \le a < 1$ for all $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ converges strongly to $\operatorname{proj}^f_{F(T)} x_0$, *where* $\operatorname{proj}^f_{F(T)}$ *is the left Bregman projection from* $\operatorname{int}\operatorname{dom} f$ *onto $F(T)$.*

Proof. We divide the proof into six steps.

Step 1. We show that C_n is closed and convex for all $n \in \mathbf{N}$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \in \mathbf{N}$. We see that, for $z \in C_m$, $D_f(z, y_m) \le D_f(z, x_m) + \xi_m$ is equivalent to

$$\langle \nabla f(x_m) - \nabla f(y_m), z\rangle \le f(y_m) - f(x_m) - \langle \nabla f(y_m), y_m\rangle + \langle \nabla f(x_m), x_m\rangle + \xi_m. \quad (3.5.1)$$

Now we prove that C_{m+1} is closed. Let $z_i \in C_{m+1}$ such that $z_i \to z$ as $i \to \infty$. By (3.5.1), we have

$$\langle \nabla f(x_m) - \nabla f(y_m), z_i\rangle \le f(y_m) - f(x_m) - \langle \nabla f(y_m), y_m\rangle + \langle \nabla f(x_m), x_m\rangle + \xi_m.$$

This implies

$$\langle \nabla f(x_m) - \nabla f(y_m), z\rangle = \lim_{i\to\infty} \langle \nabla f(x_m) - \nabla f(y_m), z_i\rangle$$

$$\leq f(y_m) - f(x_m) - \langle \nabla f(y_m), y_m \rangle + \langle \nabla f(x_m), x_m \rangle + \xi_m$$

and hence $z \in C_{m+1}$. Thus C_n is closed for all $n \in \mathbf{N}$. Next we prove that C_{m+1} is convex. Let $x, y \in C_{m+1}$ and $z = tx + (1-t)y$, where $t \in (0,1)$. By (3.5.1), we have

$$\begin{aligned}
&\langle \nabla f(x_m) - \nabla f(y_m), z \rangle \\
&= t\langle \nabla f(x_m) - \nabla f(y_m), x \rangle + (1-t)\langle \nabla f(x_m) - \nabla f(y_m), y \rangle \\
&\leq (t + 1 - t)(f(y_m) - f(x_m) - \langle \nabla f(y_m), y_m \rangle + \langle \nabla f(x_m), x_m \rangle + \xi_m) \\
&= f(y_m) - f(x_m) - \langle \nabla f(y_m), y_m \rangle + \langle \nabla f(x_m), x_m \rangle + \xi_m
\end{aligned}$$

and hence $z \in C_{m+1}$. Thus C_n is convex for all $n \in \mathbf{N}$. Therefore C_n is closed and convex, and this shows that $\mathrm{proj}^f_{C_n} x_0$ is well-defined for all $n \in \mathbf{N}$.

Step 2. We show that $F(T) \subset C_n$ for all $n \in \mathbf{N}$. Lat $p \in F(T)$. It is obvious that $F(T) \subset C_1 = C$. Suppose that $F(T) \subset C_m$ for some $m \in \mathbf{N}$. By Proposition 3.2.3, we have

$$\begin{aligned}
D_f(p, y_m) &= D_f\Big(p, \nabla f^*\big(\alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(T^m x_m)\big)\Big) \\
&= W^f(\alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(T^m x_m), p) \\
&\leq \alpha_m W^f(\nabla f(x_m), p) + (1-\alpha_m) W^f(\nabla f(T^m x_m), p) \\
&= \alpha_m D_f(p, x_m) + (1-\alpha_m) D_f(p, T^m x_m) \\
&\leq \alpha_m D_f(p, x_m) + (1-\alpha_m)(D_f(p, x_m) + \xi_m) \\
&\leq D_f(p, x_m) + \xi_m.
\end{aligned} \tag{3.5.2}$$

This implies $p \in C_{m+1}$. Therefore $F(T) \subset C_n$ for all $n \in \mathbf{N}$. Since $F(T)$ is nonempty, C_n is nonempty, closed and convex subset of $\mathrm{int}\,\mathrm{dom} f$.

Step 3. Put $C_0 = \bigcap_{n=1}^\infty C_n$. We show that $\{x_n\}_{n\in\mathbf{N}}$ converges to $\mathrm{proj}^f_{C_0}(x)$ as $n \to \infty$. By the construction of C_n, the sequence $\{C_n\}_{n\in\mathbf{N}}$ is nonincreasing of nonempty, closed and convex subsets of E. It follows that

$$\emptyset \neq F(T) \subset M\text{-}\lim_n C_n = \bigcap_{n=1}^\infty C_n = C_0.$$

By Proposition 3.3.3, $\{x_n\}_{n\in\mathbf{N}} = \{\mathrm{proj}^f_{C_n}(x_0)\}_{n\in\mathbf{N}}$ converges strongly to $\mathrm{proj}^f_{C_0}(x_0)$ as $n \to \infty$. To complete the proof, it is sufficient to show that $\mathrm{proj}^f_{C_0} = \mathrm{proj}^f_{F(T)}$.

Step 4. We show that $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ are bounded. Let $p \in F(T)$. By Proposition 3.3.2 (iii), we have

$$D_f(p, x_n) = D_f(p, \mathrm{proj}^f_{C_n} x_0) \leq D_f(p, x_0) - D_f(\mathrm{proj}^f_{C_n} x_0, x_0) \leq D_f(p, x_0).$$

This implies that $\{D_f(p, x_n)\}_{n\in\mathbf{N}}$ is bounded. By Proposition 1.4.4, ∇f^* is bounded on bounded subsets of $\mathrm{int}\,\mathrm{dom} f^*$ since f is Legendre and strongly coercive. By Proposition 3.2.4, the sequence $\{x_n\}_{n\in\mathbf{N}}$ is bounded. Moreover, by (3.5.2) and Proposition 3.2.4, the sequences $\{D_f(p, y_n)\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ are also bounded.

Step 5. We show that $\mathrm{proj}^f_{C_0}(x_0) \in F(T)$. Since $x_n = \mathrm{proj}^f_{C_n} x_0 \in C_n$ and $x_{n+1} = \mathrm{proj}^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have $D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0)$ for all $n \in \mathbf{N}$. This implies that $\{D_f(x_n, x_0)\}_{n\in\mathbf{N}}$ is nondecreasing and the limit of $D_f(x_n, x_0)$ as $n \to \infty$ exists. By Proposition 3.3.2 (iii), we have

$$D_f(x_{n+1}, x_n) = D_f(x_{n+1}, \mathrm{proj}^f_{C_n} x_0) \leq D_f(x_{n+1}, x_0) - D_f(\mathrm{proj}^f_{C_n} x_0, x_0) \leq D_f(x_{n+1}, x_0)$$

for all $n \in \mathbf{N}$. This implies

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.5.3}$$

By Proposition 3.2.8, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5.4}$$

By Proposition 1.4.7, we have

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0. \tag{3.5.5}$$

Since $x_{n+1} \in C_{n+1}$, we have $D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, x_n) + \xi_n$ for all $n \in \mathbf{N}$. By (3.5.3), we have $D_f(x_{n+1}, y_n) \to 0$ as $n \to \infty$. By Proposition 3.2.8, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. By Proposition 1.4.7, we have

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(y_n)\| = 0. \tag{3.5.6}$$

By the definition of y_n, we have

$$\|\nabla f(T^n x_n) - \nabla f(x_{n+1})\| \leq \frac{1}{1-\alpha_n}\|\nabla f(x_{n+1}) - \nabla f(y_n)\| + \frac{\alpha_n}{1-\alpha_n}\|\nabla f(x_{n+1}) - \nabla f(x_n)\|.$$

By (3.5.5), (3.5.6) and the definition of α_n, we have $\|\nabla f(T^n x_n) - \nabla f(x_{n+1})\| \to 0$ as $n \to \infty$. By Propositions 1.4.8 and 3.2.6, ∇f^* is uniformly continuous on bounded subsets of E^* and hence $\|T^n x_n - x_{n+1}\| = 0$ as $n \to \infty$. This implies $T^n x_n \to \mathrm{proj}^f_{C_0}(x_0)$ as $n \to \infty$. We have

$$\lim_{n\to\infty} \left\| T^{n+1} x_n - \mathrm{proj}^f_{C_0}(x_0) \right\| = \lim_{n\to\infty} \left(\left\| T^{n+1} x_n - T^n x_n \right\| + \left\| T^n x_n - \mathrm{proj}^f_{C_0}(x_0) \right\| \right) = 0.$$

This implies $TT^n x_n - \mathrm{proj}^f_{C_0}(x_0) \to 0$ as $n \to \infty$. By the closedness of T, we have $T(\mathrm{proj}^f_{C_0}(x_0)) = \mathrm{proj}^f_{C_0}(x_0)$. Therefore $\mathrm{proj}^f_{C_0}(x_0) \in F(T)$.

Step 6. We show that $\mathrm{proj}^f_{C_0}(x_0) \to \mathrm{proj}^f_{F(T)}(x_0)$ as $n \to \infty$. Put $z_0 = \mathrm{proj}^f_{F(T)}(x_0)$. Since $z_0 \in F(T) \subset C_n$ and $x_n = \mathrm{proj}^f_{C_n}(x_0)$, we have $D_f(x_n, x_0) \leq D_f(z_0, x_0)$ for all $n \in \mathbf{N}$. We have

$$\begin{aligned} D_f(\mathrm{proj}^f_{C_0}(x_0), x_0) &= f(\mathrm{proj}^f_{C_0}(x_0)) - f(x_0) - \langle \nabla f(x_0), \mathrm{proj}^f_{C_0}(x_0) - x_0 \rangle \\ &= \lim_{n\to\infty} (f(x_n) - f(x_0) - \langle \nabla f(x_0), x_n - x_0 \rangle) \\ &= \lim_{n\to\infty} D_f(x_n, x_0) \leq D_f(z_0, x_0). \end{aligned}$$

Therefore $z_0 = \mathrm{proj}^f_{C_0}(x_0)$ and hence $\{x_n\}$ converges strongly to z_0. □

If $f(x) = \|x\|^2/2$ for $x \in E$, then Theorem 3.5.1 is reduced to the following theorems.

Corollary 3.5.2 ([26], Theorem 2.1, p. 6)**.** *Let E be a reflaxive, strictly convex and smooth Banach space such that both of E and E^* have the Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that T is asymptotically regular on C and closed, and $F(T) \neq \emptyset$ is bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E, \ \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ n \in \mathbf{N}, \end{cases} \tag{3.5.7}$$

where

$$\xi_n := \max\left\{0, \sup_{p \in F(T), \ x \in C} \big(\phi(p, T^n x) - \phi(p, x)\big)\right\},$$

Π_{C_n} is the generalized projection from E onto C_n and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$. Then $\{x_n\}_{n \in \mathbf{N}}$ converges strongly to $\Pi_{F(T)} x_1$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. Using the technique used in the proof of Theorem 3.5.1 with $f(x) = \|x\|^2/2$ for $x \in E$, we have the sequence $\{x_n\}$ generated by (3.5.7) which converges strongly to $\Pi_{F(T)} x_1$. $\square$

Corollary 3.5.3 ([54], Theorem 2.1, p. 854)**.** *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty, closed and convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n \to \infty} k_n = 1$. Assume that T is asymptotically regular on C and $F(T) \neq \emptyset$ is bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E, \ \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + (k_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ n \in \mathbf{N}, \end{cases} \tag{3.5.8}$$

where $M_n := \sup_{p \in F(T)} \phi(p, x_n)$, Π_{C_n} is the generalized projection from E onto C_n and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$. Then $\{x_n\}_{n \in \mathbf{N}}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from E onto $F(T)$.

Proof. By the definition of T, we obtain $\phi(p, T^n x) - \phi(p, x) \leq (k_n - 1)\phi(p, x)$. Hence

$$\xi_n \leq \sup_{p \in F(T)} (k_n - 1)\phi(p, x_n) = (k_n - 1) M_n.$$

Therefore the iteration (3.5.1) is reduced to (3.5.8). $\square$

Corollary 3.5.4 ([74], Theorem 4.1, p. 283)**.** *Let H be a Hilbert space and C a nonempty, closed and convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and $x_0 \in H$. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence generated by*

$$\begin{cases} x_0 \in H, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N}, \end{cases} \tag{3.5.9}$$

where P_{C_n} is the metric projection from H onto C_n and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Proof. By the definition of T, we obtain $\|p - T^n x\|^2 - \|p - x\|^2 \leq 0$ and hence $\xi_n = 0$. Therefore the iteration (3.5.1) is reduced to (3.5.9). □

3.5.2 The Shrinking projection method with right Bregman projections

Theorem 3.5.5. *Let $f : E \to (-\infty, +\infty]$ be a Legendre and strongly coersive function which is totally convex on bounded subsets on E. Assume that f^* is admissible, totally convex and Fréchet differentiable on $\operatorname{int}\operatorname{dom} f^*$. Let C be a nonempty subset of $\operatorname{int}\operatorname{dom} f$ such that $\nabla f(C)$ is closed and convex. Let $T : C \to C$ be a closed and right Bregman asymptotically quasi-nonexpansive mapping in the intermediate sense. Suppose that T is asymptotically regular on C and $F(T)$ is bounded. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in C generated by*

$$\begin{cases} x_0 \in \operatorname{int}\operatorname{dom} f, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \overrightarrow{\operatorname{proj}}^f_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : D_f(y_n, z) \leq D_f(x_n, z) + \eta_n\}, \\ x_{n+1} = \overrightarrow{\operatorname{proj}}^f_{C_{n+1}}x_0, \ n \in \mathbf{N}, \end{cases} \tag{3.5.10}$$

where $\overrightarrow{\operatorname{proj}}^f_{C_n}$ is the right Bregman projection from $\operatorname{int}\operatorname{dom} f$ onto C_n,

$$\eta_n = \max\left\{0, \sup_{p\in F(T),\ x\in C} \big(D_f(T^n x, p) - D_f(x, p)\big)\right\}$$

and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ converges strongly to $\overrightarrow{\operatorname{proj}}^f_{F(T)}x_0$, where $\operatorname{proj}^f_{F(T)}$ is the right Bregman projection from $\operatorname{int}\operatorname{dom} f$ onto $F(T)$.

Proof. We divide the proof into six steps.

Step 1. We show that $\nabla f(C_n)$ is closed and convex for all $n \in \mathbf{N}$. It is obvious that $\nabla f(C_1) = \nabla f(C)$ is closed and convex. Suppose that $\nabla f(C_k)$ is closed and convex for $k \in \mathbf{N}$. We see that, for $z \in C_k$, $D_f(y_k, z) \leq D_f(x_k, z) + \eta_k$ is equivalent to

$$\langle \nabla f(z), x_k - y_k \rangle \leq f(x_k) - f(y_k) + \eta_k. \tag{3.5.11}$$

First we prove that $\nabla f(C_{k+1})$ is closed. Let $\{z_i\}_{i \in \mathbf{N}} \subset C_{k+1}$ with $\nabla f(z_i) \to z^*$ as $i \to \infty$. Since f is strongly coercive, we have $\operatorname{ran}\nabla f = E^*$. Hence there exists $z \in E$ such that $z^* = \nabla f(z)$. It is sufficient to prove that $z \in C_{k+1}$. By (3.5.11), we have

$$\langle \nabla f(z), x_k - y_k \rangle = \lim_{i \to \infty} \langle \nabla f(z_i), x_k - y_k \rangle \leq f(x_k) - f(y_k) + \eta_k$$

and hence $z \in C_{k+1}$. Thus $\nabla f(C_n)$ is closed for all $n \in \mathbf{N}$. Next we prove that $\nabla f(C_{k+1})$ is convex. Let $x, y \in C_{k+1}$ and $t \in (0,1)$. Define $z = \nabla f^*(t\nabla f(x) + (1-t)\nabla f(y))$. We prove that $z \in C_{k+1}$. By (3.5.11), we have

$$\begin{aligned}
\langle \nabla f(z), x_k - y_k \rangle &= \langle t\nabla f(x) + (1-t)\nabla f(y), x_k - y_k \rangle \\
&= t\langle \nabla f(x), x_k - y_k \rangle + (1-t)\langle \nabla f(y), x_k - y_k \rangle \\
&\leq f(x_k) - f(y_k) + \eta_k
\end{aligned}$$

and hence $z \in C_{k+1}$. Thus $\nabla f(C_n)$ is convex for all $n \in \mathbf{N}$. Therefore $\nabla f(C_n)$ is closed and convex. By Proposition 3.4.9, there exists a unique sunny right Bregman quasi-nonexpansive retraction of E onto C_n which is $\overrightarrow{\operatorname{proj}}^f_{C_n}$. Hence $\{x_n\}$ is well-defined.

Step 2. We show that $F(T) \subset C_n$ for all $n \in \mathbf{N}$. It is obvious that $F(T) \subset C_1 = C$. Suppose that $F(T) \subset C_k$ for $k \in \mathbf{N}$. Since f is convex, the function $D_f(\cdot, x)$ is also convex for all $x \in \operatorname{int}\operatorname{dom} f$. For any $p \in F(T)$, we have

$$\begin{aligned}
D_f(y_k, p) &= D_f(\alpha_k x_k + (1-\alpha_k)T^n x_k, p) \\
&\leq \alpha_k D_f(x_k, p) + (1-\alpha_n) D_f(T^n x_k, p) \\
&\leq \alpha_k D_f(x_k, p) + (1-\alpha_n)(D_f(x_k, p) + \eta_k) \\
&= D_f(x_k, p) + \eta_k.
\end{aligned} \tag{3.5.12}$$

This implies $p \in C_{k+1}$. Therefore $F(T) \subset C_n$ for all $n \in \mathbf{N}$. Since $F(T)$ is nonempty, C_n is nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$.

Step 3. Put $C_0^* = \bigcap_{n=1}^{\infty} \nabla f(C_n)$. We show that $\{x_n\}$ converges to $\nabla f^*(\operatorname{proj}^{f^*}_{C_0^*} \nabla f(x))$ as $n \to \infty$. Since $\{\nabla f(C_n)\}$ is a nonincreasing sequence with respect to inclusion of nonempty, closed and convex subsets of E^*, we have

$$\emptyset \neq \nabla f(F(T)) \subset M\text{-}\lim_{n \to \infty} \nabla f(C_n) = \bigcap_{n=1}^{\infty} \nabla f(C_n).$$

By Proposition 3.3.3, $\{\operatorname{proj}^{f^*}_{\nabla f(C_n)} \nabla f(x)\}$ converges strongly to $x^* = \operatorname{proj}^{f^*}_{C_0^*} \nabla f(x)$ as $n \to \infty$. Since E^* has a Fréchet differential norm, $(\nabla f)^{-1} = \nabla f^*$ is continuous. We have

$$x_n = \overrightarrow{\operatorname{proj}}^f_{C_n}(x) = \nabla f^* \circ \operatorname{proj}^{f^*}_{\nabla f(C_n)} \circ \nabla f(x) \to \nabla f^*(x^*)$$

as $n \to \infty$. To complete the proof, it is sufficient to show that $\nabla f^*(x^*) = \overrightarrow{\mathrm{proj}}^f_{F(T)}$.

Step 4. We show that $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ are bounded. Let $p \in F(T)$. By Proposition 3.3.4 (iii), we have

$$D_f(x_n, p) = D_f(\overrightarrow{\mathrm{proj}}^f_{C_n}(x_0), p) \le D_f(x_0, p) - D_f(x_0, \overrightarrow{\mathrm{proj}}^f_{C_n}(x_0)) \le D_f(x_0, p).$$

This implies that $\{D_f(x_n, p)\}_{n\in\mathbf{N}}$ is bounded. By Proposition 3.2.7, the sequence $\{x_n\}_{n\in\mathbf{N}}$ is bounded. Moreover, by (3.5.12) and Proposition 3.2.7, the sequences $\{D_f(y_n, p)\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ are also bounded.

Step 5. We show that $\nabla f^*(x^*) \in F(T)$. Since $x_n = \overrightarrow{\mathrm{proj}}^f_{C_n}(x_0)$ and $x_{n+1} = \overrightarrow{\mathrm{proj}}^f_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have $D_f(x_0, x_n) \le D_f(x_0, x_{n+1})$. This implies that $\{D_f(x_0, x_n)\}_{n\in\mathbf{N}}$ is nondecreasing and the limit of $D_f(x_0, x_n)$ as $n \to \infty$ exists. By Proposition 3.3.4 (iii), we have

$$\begin{aligned} D_f(x_n, x_{n+1}) &= D_f(\overrightarrow{\mathrm{proj}}^f_{C_n}(x_0), x_{n+1}) \\ &\le D_f(x_0, x_{n+1}) - D_f(x_0, \overrightarrow{\mathrm{proj}}^f_{C_n}(x_0)) \\ &\le D_f(x_0, x_{n+1}) \end{aligned}$$

for all $n \in \mathbf{N}$. This implies

$$\lim_{n\to\infty} D_f(x_n, x_{n+1}) = 0. \tag{3.5.13}$$

By Proposition 3.2.8, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.5.14}$$

Since $x_{n+1} \in C_{n+1}$, by (3.5.13), we have

$$\lim_{n\to\infty} D_f(y_n, x_{n+1}) \le \lim_{n\to\infty} (D_f(x_n, x_{n+1}) + \eta_n) = 0.$$

By Proposition 3.2.8, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.5.15}$$

By the definition of y_n, we have

$$\|T^n x_n - x_{n+1}\| \le \frac{1}{1-\alpha_n}\|x_{n+1} - y_n\| + \frac{\alpha_n}{1-\alpha_n}\|x_{n+1} - x_n\|.$$

By (3.5.14), (3.5.15) and the definition of α_n, we have $\|T^n x_n - x_{n+1}\| \to 0$ as $n \to \infty$. This implies

$$\lim_{n\to\infty} T^n x_n = \nabla f^*(x^*) = \nabla f^* \circ \mathrm{proj}^{f^*}_{C_0^*} \circ \nabla f(x_0).$$

We have

$$\lim_{n\to\infty} \left\|T^{n+1}x_n - \nabla f^*(x^*)\right\| = \lim_{n\to\infty} \Big(\left\|T^{n+1}x_n - T^n x_n\right\| + \|T^n x_n - \nabla f^*(x^*)\|\Big) = 0.$$

This implies $TT^n x_n - \nabla f^*(x^*) \to 0$ as $n \to \infty$. By the closedness of T, we have $T(\nabla f^*(x^*)) = \nabla f^*(x^*)$. Therefore $\nabla f^*(x^*) \in F(T)$.

Step 6. We show that $\overrightarrow{\mathrm{proj}}^f_{F(T)}(x_0) = \nabla f^*(x^*)$ as $n \to \infty$. Put $z_0^* = \overrightarrow{\mathrm{proj}}^f_{F(T)}(x_0)$. Since $z_0^* \in F(T) \subset C_n$ and $x_n = \overrightarrow{\mathrm{proj}}^f_{C_n}(x_0)$, we have $D_f(x_0, x_n) \le D_f(x_0, z_0^*)$ for all $n \in \mathbf{N}$. We have

$$\begin{aligned} D_f(x_0, \nabla f^*(x^*)) &= f(x_0) - f(\nabla f^*(x^*)) - \langle x^*, x_0 - \nabla f^*(x^*)\rangle \\ &= \lim_{n\to\infty} (f(x_0) - f(x_n) - \langle \nabla f(x_n), x_0 - x_n\rangle) \\ &= \lim_{n\to\infty} D_f(x_0, x_n) \le D_f(x_0, z_0^*). \end{aligned}$$

Therefore $z_0^* = \nabla f^*(x^*)$ and hence $\{x_n\}$ converges strongly to z_0^*. □

If $f(x) = \|x\|^2/2$ for $x \in E$, then Theorem 3.5.5 is reduced to the following theorem.

Corollary 3.5.6 ([75], Theorem 5.1, p. 973)**.** *Let E be a uniformly convex Banach space which has a Fréchet differential norm. Let $T : E \to E$ be a generalized nonexpansive mapping. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence generated by:*

$$\begin{cases} x_1 = x \in E \text{ chosen arbitrarily}, \\ C_1 = E, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x, \ n \in \mathbf{N}, \end{cases} \tag{3.5.16}$$

where $R_{C_{n+1}}$ is sunny generalized nonexpansive retraction of E onto C_{n+1} and $0 \le \alpha_n \le a < 1$ for all $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ converges strongly to $R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. By the definition of T, we obtain $\phi(T^n x, p) - \phi(x, p) \le 0$ and hence $\eta_n = 0$. Therefore the iteration (3.5.10) is reduced to (3.5.16). □

Afterword

The author is aiming to apply the research results in the body of this thesis to various fields. It is important to study some problems concerning nonlinear functional analysis and convex analysis by using fixed point theory. For example, fixed point theorems and strongly convergence theorems are used for the study of existence and approximation of solutions to evolution equations. In connection with these, we provide the following research result on evolution equations.

We introduce a class of nonlinear evolution operators and give a characterization of continuous infinitesimal generators of such evolution operators by applying the results on semigroups of Lipschitz operators. The following content provides a characterization of the continuous infinitesimal generator such that the solution operator to the initial valued problem associated with the generator becomes an evolution operator whose solutions depend continuously on the initial data.

Let X be a real Banach space with the norm $\|\cdot\|$. Let Ω be a closed subset of $[0,\infty)\times X$ such that $\Omega(t)=\{x\in X : (t,x)\in\Omega\}\neq\emptyset$ for $t\in[0,\infty)$. Let A be a continuous mapping from Ω into X. Given $(\tau,x)\in\Omega$, we consider the following initial value problem:

$$(\mathrm{IVP};\tau,x)\qquad \begin{cases} u'(t)=A\big(t,u(t)\big) & \text{for}\quad \tau\le t<\infty,\\ u(\tau)=x. \end{cases}$$

Set $\Delta=\{(t,\tau) : 0\le\tau\le t<\infty\}$. Suppose that the problem $(\mathrm{IVP};\tau,x)$ has a unique (continuously differentiable) solution $u(\cdot)$ on $[\tau,\infty)$. Defining by $U(t,\tau)x=u(t)$, we have the following properties:

(E1) $U(\tau,\tau)x=x$ and $U(t,s)U(s,\tau)x=U(t,\tau)x$ for $(\tau,x)\in\Omega$ and $t,s\in[0,\infty)$ such that $t\ge s\ge\tau$.

(E2) For any $(\tau,x)\in\Omega$, $U(s,\tau)x$ converges to $U(t,\tau)x$ in X as $s\to t$ in $[\tau,\infty)$.

By a (nonlinear) *evolution operator* on Ω, we mean a family $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ of operators $U(t,\tau):\Omega(\tau)\to\Omega(t)$ satisfying (E1) and (E2). We consider the following additional condition on such a family $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ which ensures the continuous dependence of solutions $u(\cdot)$ on the initial data $(\tau,x)\in\Omega$:

(E3) For any $T>0$, there exists $M_T\in(0,\infty)$ such that

$$\|U(\tau+t,\tau)x-U(\sigma+t,\sigma)y\|\le M_T\big(|\tau-\sigma|+\|x-y\|\big)$$

for $(\tau,x),(\sigma,y)\in\Omega$ and $t\in[0,T]$.

Theorem A ([78])**.** There exists an evolution operator $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ on Ω such that (E3) is satisfied and that $u(t) = U(t,\tau)x$ is a unique solution to (IVP; τ, x) on $[\tau,\infty)$ for any $(\tau, x) \in \Omega$ if and only if the mapping A on Ω satisfies the following conditions (Ω1) and (Ω2):

(Ω1) For any $(\tau, x) \in \Omega$,

$$\liminf_{h\to+0} d(x + hA(\tau, x), \Omega(\tau + h))/h = 0,$$

where $d(x, S) = \inf_{y\in S} \|x - y\|$ for $x \in X$ and $S \subset X$.

(Ω2) There exist a number $\omega \in [0,\infty)$ and $V : (\mathbf{R}\times X)\times(\mathbf{R}\times X) \to [0,\infty)$, which satisfies conditions (V1) and (V2) below, such that

$$D_+V((\tau, x), (\sigma, y))(A(\tau, x), A(\sigma, y)) \le \omega V((\tau, x), (\sigma, y)) \tag{3.5.17}$$

for $(\tau, x), (\sigma, y) \in \Omega$, where

$$\begin{aligned}&D_+V((\tau, x), (\sigma, y))(\xi, \eta)\\ &= \liminf_{h\to+0}\big(V((\tau + h, x + h\xi), (\sigma + h, y + h\eta)) - V((\tau, x), (\sigma, y))\big)/h\end{aligned}$$

for $(\tau, x), (\sigma, y) \in \mathbf{R} \times X$ and $(\xi, \eta) \in X \times X$.

(V1) There exists $L \in (0,\infty)$ such that

$$\begin{aligned}&|V((\tau, x), (\sigma, y)) - V((\hat\tau, \hat x), (\hat\sigma, \hat y))|\\ &\le L\big(|\tau - \hat\tau| + |\sigma - \hat\sigma| + \|x - \hat x\| + \|y - \hat y\|\big)\end{aligned}$$

for $(\tau, x), (\sigma, y), (\hat\tau, \hat x), (\hat\sigma, \hat y) \in \mathbf{R} \times X$.

(V2) There exists $M \in [1,\infty)$ such that

$$|\tau - \sigma| + \|x - y\| \le V((\tau, x), (\sigma, y)) \le M\big(|\tau - \sigma| + \|x - y\|\big)$$

for $(\tau, x), (\sigma, y) \in \Omega$.

Moreover, in this case, we have

$$V\big((\tau + t, U(\tau + t, \tau)x), (\sigma + t, U(\sigma + t, \sigma)y)\big) \le e^{\omega t}V((\tau, x), (\sigma, y))$$

and

$$\|U(\tau + t, \tau)x - U(\sigma + t, \sigma)y\| \le Me^{\omega t}\big(|\tau - \sigma| + \|x - y\|\big)$$

for $(\tau, x), (\sigma, y) \in \Omega$ and $t \in [0,\infty)$.

Remark. The kinds of conditions (Ω1) and (Ω2) were found by Nagumo [50] and Okamura [52], respectively. Our class of evolution operators is rather narrow but closely related to the ones discussed in Murakami [49], Martin [40], Lakshmikantham, Mitchell and Mitchell [39] and Kato [31].

Theorem A is proved by the use of the results for the autonomous case by Kobayashi and Tanaka [34]. Our proof of Theorem A is suggested by Evans and Massey [23].

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