Doctoral thesis

A Hydrologic-based Lumped Methodology for the Internal Dose Calculation

by

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Contents

1 General introduction

1.1 Research background

1.1.1 Fukushima Daiichi nuclear disaster in March 2011

The Great East Japan Earthquake attacked the north-east area in Japan on March 11, 2011. The system of electrical facilities to control Fukushima Daiichi nuclear power station was completely destroyed by the following tsunamis. An amount of radioactive isotopes had leaked and been diffused from the damaged reactor containment vessels in the vicinity of the plant, deposition of radioactive cesium was observed on March 2011 [1]. On May 12, 2011, immediately after the accident, the Japanese government issued a mandatory evacuation order for those living within a 20 km radius of the nuclear plant and indoor shelter and voluntary evacuation instruction residents of a 20 to 30 km area. After December 15, 2011, cold shutdown of the reactor and little possibility of radioactive leakage were recognized, reconsideration of evacuation order was started. At the present (December 2013), the area radioactive isotopes precipitated are separated “ Special Decontamination Area ” , [2,3].

Japan Atomic Energy Relations Organization briefly summarizes the sequence of the evacuation order until the present (December 2013) from after the accident [4].

Food containing radioactive isotopes has become an issue of great interest to the Japanese public, and has caused excessive anxiety because of a shortage of fundamental knowledge concerning radioactivity. It is indicated that there are serious gaps regarding grasping of actual damage between specialists with general public [5].

1.1.2 Previous research: Standard method in radiology

Concentrations of radioactivity in the human body and internal exposure have been researched extensively. Previous research of radiology, MIRD method [6–18] etc., divided the human body into 20 organs, and energy absorbed in the organs is calculated using a computational phantom that imitates the body, including a Monte Carlo simulation. Existing method [19] accepted as a consensus opinion for the radioactivity measurement employs 10 elliptical containers of polyethylene filled with water. The details of industrially accepted standard phantom is described in the report of American national standards institute [20].

The International Commission on Radiological Protection (ICRP) has contributed development of radiology since 1950. The history of ICRP is retroactive to International X-ray and Radium Protection Committee (IXRPC) founded in 1928. Recommendations and guidance of ICRP has been very influential in radiation protection.

ICRP also employs computational phantoms to analyze the pharmacokinetics of various nuclides in the human body, as described in ICRP Publ. 110 [21]. Recent researches [22,23] calculated the effective dose from ingestion using a computational phantom including directly obtained raw data from CT scans and MRI imaging. Methods proposed previously calculate internal exposure by dividing the human body into its constituent tissues. While computational simulation is a standard method for calculating exposure doses among radiology specialists, these methods, although exact, are too difficult for non-specialists to use because of the complex computer simulations involved. We suggest that a new methodology is needed because many people in both Japan and abroad are experiencing anxiety related to the health effects of radiation exposure.

1.2 Objectives

In this article, we aimed to understand the behavior and influences of radioactivity in the human body, using an approach different than the existing methods that is readily understandable. Hydrological runoff analysis developed in hydrology is used to describe the conservation of radioactive atoms in the human body. Runoff analysis is classified into two methods, that is, lumped hydrological method and distributed hydrological method. While lumped hydrological method is simplified runoff characteristics by parameters averaging a whole basin, distributed hydrological method divides a basin into each sub basins of which parameters are given individually.

Surely, there is no doubt that distributed model is helpful to know exact runoff characteristics. However, there are difficulties in that it requires complicated analysis. Lumped hydrological model is also practically useful to understand general runoff characteristics owing to the simplicity. Appropriately simplified lumping method can provide sufficiently accurate runoff characteristics in convenient way without the need for sophisticated distributed model. Previous researches in hydrology have made great efforts to study appropriate lumping of distributed model without spoiling the essence of complex natural phenomena [24–29]. The aim of this study is to apply the knowledge studied in hydrology that finds out the core of natural phenomena to radiology.

Previous methods that divide the human body into its constituent organs may be considered to correspond to a distributed model in hydrology. We propose a method that regards the human body as a single large tissue in this thesis. The methods proposed here correspond to a hydrological lumped model. A method that describes the storage of radioactivity in the human body using a hydrological runoff analysis is proposed.

Runoff analysis is useful for describing the conservation of a physical quantity of material. The storage of radioactive atoms in the human body is affected by both physical and biological factors; i.e., radioactive decay and excretion, respectively. Thus, it is reasonable to apply runoff analysis in this study. The human body is regarded as a vessel. The storage of a number of radioactive atoms in the human body can be expressed by a continuity equation, described as a function of time. Physical quantities related to radioactivity can be derived from the storage of a number of radioactive atoms in the body.

Concretely, calculated physical quantities are the number of radioactive atoms in the human body, intensity of radioactivity in the human body, the cumulative number of radioactive decay, and the effective dose. The intensity of radioactivity is linear with respect to the number of radioactive atoms. The cumulative number of radioactive decay is obtained from a time integral of the intensity of radioactivity in the human body. The accuracy of the analytic solution given by the present method in calculating the intensity of radioactivity in the human body is shown by comparison of the measured value by a whole body counter. Additionally, a method of calculating absorbed energy per one radioactive decay of individual nuclides is shown from decay schemes. The effective dose is calculated from the cumulative number of radioactive decay and energy absorbed per one radioactive decay. The equal accuracy of the present method in terms of calculating energy absorbed per one radioactive decay is demonstrated by comparison with previous research [6, 30].

The accuracy of the present method in calculating the effective dose is shown by comparison of the committed effective dose coefficient with that determined in previous research. The total effective dose delivered over a lifetime is termed the committed effective dose. In the case of adults, time until death is assumed to be 50 years. The committed effective dose coefficient is defined as committed effective dose per 1 Bq. A method to calculate committed effective dose coefficient is proposed, using the effective dose expressed as a function of time. ICRP calculated values of the coefficient [32]. Time series of committed effective dose coefficients calculated is shown using previously implemented ICRP software [33, 34]. The software is available for anyone from website of ICRP [35]. In the present study, we provide a method to calculate the coefficient. The accuracy of our method was examined by comparison with values determined in previous research by ICRP. Differences in time series of effective dose by age of cesium are shown, based on effective dose described as a function of time. Eq.ual accuracy of the present method to calculate the effective dose of cesium is demonstrated by comparison with results of previous research by ICRP [32]. In this study, we show that runoff analysis is useful in calculating physical quantities related to radioactivity. Regarding the human body as a single large tissue gives essentially equivalent results to previous research in which the human body was divided into its organs.

1.3 Structure of this article

The structure of this article is described in the followings. It will be useful guidance to read this article.

The purpose of chapter 2 is to calculate the number of radioactive atoms in the human body, radioactivity in the human body, and cumulative number of radioactive decay. The solution of first-order linear differential equation is required to calculate radioactivity in the human body. Appendix B shows the solution methods of first-order linear differential equation. Variation of constants and Laplace transform are introduced as examples of solution methods of first-order linear differential equation. While we accept that radioactive atoms decrease exponentially due to radioactive decay in the body of this article, appendix A provides the mathematical theory of the exponential decrease.

In chapter 3, we discuss the effective dose caused by internal exposure based on the cumulative number of radioactive decay calculated in chapter 2. The main subject of this article is β ray (electron) and γ ray (light, photon, electromagnetic wave), we require elementary knowledge of electromagnetism and quantum mechanics to compute the effective dose.

The energy of β ray is much large, therefore, the velocity of β ray is close to speed of light according to special relativity. The energy of β ray derives from the mass difference between a neutron with proton, mass-energy equivalence established in special relativity is also important. Appendix F provides knowledge of vector calculus to study electromagnetism and special relativity. We can study electromagnetism and special relativity in appendix G.

The transition probability of β decay and the energy spectrum of β ray can be discussed by the Fermi's golden rule derived from perturbation of the Hamiltonian in the Schrödinger equation, which is the governing equation of non-relativistic quantum mechanics. In appendix H, we review the fundamental ideas of quantum mechanics to discuss transition probability of β decay. Appendix I shows the elementary knowledge of nuclear physics for understanding of the energy spectrum of β ray.

In chapter 4, the radioactivity in the soils is computed as with the human body discussed in chapter 2. We obtain the effective dose caused by the intake of radioactivity on agricultural products derived from fallout by using the calculation method shown in chapter 3.

In chapter 5, we introduce stochastic calculus to discuss the uncertainty and randomness of the intake of radioactivity. Appendix E provides the mathematical footings of stochastic calculus and statistical mechanics regarding Brownian motion which is origin of stochastic calculus. Diffusion process of Brownian particles is described by the diffusion equation in global, appendix C is discussed the solution method of the diffusion equation by using the Fourier transform. In appendix E, we study the solution of advection-diffusion equation for discussion of the Fokker-Planck equation which is one of the formalisms of stochastic calculus.

Appendix J comprehensively discusses the conservation law which is the most universal idea in nature and also used as a fundamental idea in this article. While appendix J is an additional content in comparison with other sections, the universal law lying to natural science including the subjects of this article must be greatly useful.

Chapter 6 concludes the present study, and the benefits of adopting a hydrological lumped model related to radioactive substances in the human body are summarized.

1.4 Notation and convention

Mathematical or physical notation and convention in this thesis is summarized as follows.

- The natural logarithm is written as $\ln x$, i.e. $\ln e = 1$.
- Derived functions are described as $\dfrac{df(x)}{dx}$ and $\dfrac{\partial f(t,x)}{\partial x}$ for clear writing of the variable. Notations $f'(x)$ and f_x are not used as much as possible.
- According to convention in radiology, various derived units, for example, Bq, Gy, and Sv, are used. The international system of units is not necessarily used.
- $N(\mu, \sigma^2)$ stands for a normal distribution with average μ and variance σ^2.
- A vector is described in bold type, for example, as $\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}$. The description $\mathbf{x}$ often is used to emphasize that it notes space components. We use the description by index as x_i (covariant vector) and x^i (contravariant vector) appropriately.
- O is Landau notation. For example, $e^x = 1 + x + \dfrac{1}{2}x^2 + O(x^3)$ means that constant multiple of $|x^3|$ is sufficiently small.
- Mathematical symbols of set of numbers, for example, natural numbers set $\mathbb{N}$, complex numbers set $\mathbb{C}$, are used as necessary.
- ψ^* stands for the conjugate complex of ψ, that is, $\psi = a + ib \Rightarrow \psi^* = a - ib$.
- $|\psi\rangle\dagger$ stands for the Hermitian conjugate of $|\psi\rangle$, that is, $|\psi\rangle\dagger = \langle\psi|$.
- Natural units system is used for convenience. Velocity of light in vacuum c and reduced Planck constant $\hbar$ are omitted 1 as $c = \hbar = 1$. In this system,

$$[\text{energy}] = [\text{mass}] = [\text{time}]^{-1} = [\text{length}]^{-1}. \tag{1.1}$$

- $\delta(x)$ stands for the Dirac delta function which satisfies the followings:

$$\delta(x) = \begin{cases} \infty & (x = 0) \\ 0 & (x \neq 0) \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{1.2}$$

- δ^i_j stands for the Kronecker delta defined as:

$$\delta^i_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}, \tag{1.3}$$

- $\langle A \rangle$ stands for the average of A, where A is a random variable or a set.

- $N(\mu, \sigma^2)$ stands for the normal distribution of which average and variance are μ and σ^2 respectively.

2 Intensity of radioactivity in the human body and cumulative number of radioactive decay

2.1 Introduction

First in this chapter, we compute the number of radioactive atoms in the human body based on lumped hydrologic method. The relationship between the number of radioactive atoms with intensity of radioactivity is discussed. Subsequently, cumulative number of radioactive decay is associated with intensity of radioactivity in the human body. The fundamental equation is the continuity equation which describes the conservation of the number of atoms. Our mathematical requirement is only the first order linear ordinary differential equation. There is no mathematical difficulty. We can get analytic solution readily. This simplicity is the merit of our hydrologic based methodology.

It should be emphasized that the parameters used in this section are only (physical, biological, effective) half-lives of radionuclide. All they are well-known experimentally. There is no arbitrary parameters which are introduced to fit our results to the existing knowledge.

2.2 Theory and Method

2.2.1 Decrease of the number of radioactive atoms

Radioactive isotopes decay to stable isotopes with emission of radiation. It is known experimentally and theoretically that radioactive atoms decrease exponentially. For time t, a function of the number of radioactive atoms $N(t)$ is described as:

$$N(t) = N_0 e^{-\alpha_p t} \text{ (atoms)} \tag{2.1}$$

under the initial condition $N(0) = N_0$. α_p is the physical attenuation constant, it only depends on kinds of nuclides. Later, the relationship with the attenuation constant and the half-life is discussed. The number of radioactive decay events between change in time dt, i.e. runoff emerged from radioactive decay $q_p(t)$ is described as:

$$\frac{dN(t)}{dt} = -\alpha_p N(t) = -q_p(t) \text{ (atoms/unit time)}. \tag{2.2}$$

As shown in Eq. (2.2), physical runoff $q_p(t)$ is linear with respect to the number of radioactive atoms $N(t)$. A function of intensity of radioactivity $h(t)$ is the number of radioactive decay events per second:

$$h(t) = \alpha_p N(t) \text{ (Bq)}, \tag{2.3}$$

becquerel (Bq) is the number of radioactive decay events per second; unit of intensity of radioactivity.

2.2.2 Half-life

Half-life is the period of time it takes for the amount of a substance undergoing radioactive decay to decrease by half. From Eq. (2.1), half-life T_p is defined as:

$$\frac{1}{2}N_0 = N_0 e^{-\alpha_p T_p} \Rightarrow T_p = \frac{\ln 2}{\alpha_p}. \tag{2.4}$$

Up to here, physical decrease emerged from radioactive decay is only discussed. The period of time required to decrease the material by half due to radioactive decay is defined as the physical half-life T_p.

The major subject of this study is internal exposure caused by ingestion of radioactive isotopes from the viewpoint of radiological protection, thus, decrease of radioactive isotopes in the human body caused by biological effects, metabolism and excretion, has to be considered. It is known that the number of radioactive atoms in the human body decreases exponentially due to biological effects. The period of time required to decrease the material by half due to metabolism and excretion is defined as the biological half-life, T_b. As with physical half-life shown in Eq. (2.4), biological half-life satisfies $T_b = \ln 2/\alpha_b$, runoff emerged from metabolism and excretion is described as $q_b(t)$.

Effective runoff of radioactive isotopes in the human body $q_e(t)$ is sum of physical runoff $q_p(t)$ and biological runoff $q_b(t)$. Then, $N(t)$ is redefined as a function of the number of radioactive atoms in the human body. Runoff of radioactive isotopes from the human body between change in time dt is described as:

$$\begin{aligned}\frac{dN(t)}{dt} &= -q_p(t) - q_b(t) \\ &= -\alpha_p N(t) - \alpha_b N(t) \\ &= -(\alpha_p + \alpha_b)N(t) = -\alpha_e N(t)\end{aligned} \tag{2.5}$$

Here,

$$\alpha_e = \alpha_p + \alpha_b \tag{2.6}$$

is substituted. Eqs. (2.2) and (2.5) are equivalent analytically, because the difference of both is only the coefficients. They show the number of radioactive atoms in the human body also decreases exponentially. The effective half-life, T_e, is the period of time required to decrease the material by half in the human body, due to both physical and biological effects. As with physical half-life and biological half-life, the effective half-life is satisfies $T_e = \ln 2/\alpha_e$. From Eq. (2.6), relationship between T_e, T_p, and T_b is:

$$\frac{1}{T_e} = \frac{1}{T_p} + \frac{1}{T_b} \Rightarrow T_e = \frac{T_p T_b}{T_p + T_b}. \tag{2.7}$$

The parameters used in this study to calculate the number of radioactive atoms in the human body, intensity of radioactivity in the human body and cumulative number of radioactive decay are only these three types of half-lives. Table 2.1 shows the three types of half-lives with regard to representative nuclides, from Tuszynski and Dixon [38].

From Eqs. (2.3) and (2.5), the number of radioactive decay events between change in time dt is intensity of radioactivity in the human body as:

$$h(t) = -\alpha_p \frac{dN(t)}{dt} = -\alpha_p \alpha_e N(t) \text{ (Bq/body)} \tag{2.8}$$

Eq. (2.7) and Table 2.1 suggest that in cases one of the half-lives is markedly shorter than the other, it effectively becomes dominant. For example, T_p and T_b of ^{137}Cs are 30 years and 70 days (in the case of adult), respectively. T_e of ^{137}Cs is calculated as:

$$T_e = \frac{30(\text{years}) \times 70(\text{days})}{30(\text{years}) + 70(\text{days})} = 69.6(\text{days}) \simeq T_b. \tag{2.9}$$

Table 2.1: Physical half-life, biological half-life, and effective half-life of representative nuclides, taken from Tuszynski and Dixon [38].

Nuclide	T_p	T_b	T_e
^{3}H	12.3 (years)	12 (days)	12 (days)
^{14}C	5781 (years)	40 (days)	40 (days)
^{32}P	14.3 (days)	1155 (days)	14.1 (days)
^{40}K	1.25×10^9 (years)	30 (days)	30 (days)
^{59}Fe	45 (days)	600 (days)	42 (days)
^{60}Co	1930 (days)	10 (days)	10 (days)
^{65}Zn	244 (days)	933 (days)	193 (years)
^{90}Sr	28.9 (years)	50 (years)	18.3 (years)
^{131}I	8.04 (days)	138 (days)	7.6 (days)
^{137}Cs	30 (years)	70 (days)	70 (days)

Although T_p is 30 years (Table 2.1), T_e of ^{137}Cs almost equals to $T_b = 70$ (days). The result suggests that T_p has little effect on calculation of T_e of ^{137}Cs, T_b is important to calculation of radiation dose the human body exposed.

On the other hand, one of radioactive isotope of iodine ^{131}I is its $T_p = 8.04$(days), $T_b = 138$(days). T_e of ^{137}Cs is calculated 7.6 (days), $T_e \simeq T_p$. In contrast of ^{137}Cs, in the case of ^{131}I, the physical half-life T_p is dominant for calculation of the effective half-life T_e. Although that fact gives very important result for calculation of the exposure, it is described later in detail.

2.2.3 The continuity equation regarding the number of radioactive atoms in the human body

The conservation relationship between the number of radioactive atoms in the human body $N(t)$ with the effective runoff of radioactive atoms from the human body $q_{ea}(t)$ and ingestion amount per unit time of a number of radioactive atoms $r_a(t)$ is described for a single vessel in Fig 2.1. Outflow of radioactive atoms is defined as $q_a(t)$. In this study, physical outflow of radioactive atoms $q_{pa}(t)$ emerges from radioactive decay and biological outflow of radioactive atoms $q_{ba}(t)$ emerges from excretion; they are considered as factors of runoff. Effective outflow of radioactive atoms $q_{ea}(t)$ is the sum of $q_{pa}(t)$ and $q_{ba}(t)$ as:

$$q_{ea}(t) = q_{pa}(t) + q_{ba}(t) = (\alpha_p + \alpha_b)N(t) = \alpha_e N(t). \tag{2.10}$$

From Fig 2.1, the relationship between $N(t)$ and ingestion flux of the number of radioactive atoms $r_a(t)$ and $q_{ea}(t)$ is described as:

$$\frac{dN(t)}{dt} = r_a(t) - q_{ea}(t). \tag{2.11}$$

From Eq. (2.10), Eq. (2.11) is rewritten as:

$$\frac{dN(t)}{dt} = r_a(t) - \alpha_e N(t), \tag{2.12}$$

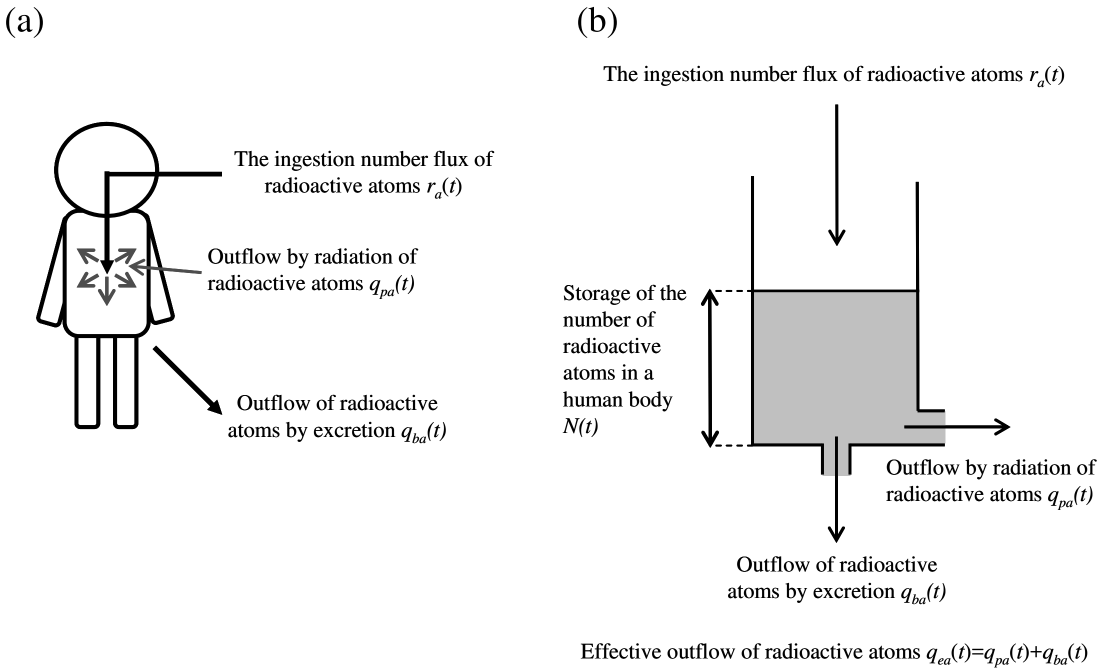

Fig 2.1: The relationship between the number ingestion flux of radioactive atoms $r_a(t)$, outflow by radiation of radioactive atoms $q_{pa}(t)$, outflow by excretion $q_{ba}(t)$, total effective outflow $q_{ea}(t)$ and storage of the number of radioactive atoms in a human body $N(t)$, which (a) is described schematically, (b) is described as a single vessel.

which is a continuity equation regarding the number of radioactive atoms in the human body. Eq. (2.12) is first order linear ordinary differential equation, the general solution can be obtained by basic analysis.

2.2.4 Relationship between the number of radioactive atoms and intensity of radioactivity

The number of radioactive atoms is discussed above. So, intensity of radioactivity (the number of radioactive decay events per unit time) is discussed because the main subject of this study is the influence of radiation on the human body. As shown in Eq. (2.3), intensity of radioactivity is α_p times the number of radioactive atoms. The relational expressions

$$r_r(t) = \alpha_p r_a(t), \tag{2.13}$$

and

$$\begin{align} q_{er}(t) &= \alpha_p q_{ea}(t) \tag{2.14} \\ &= \alpha_p \alpha_e N(t) = \alpha_e h(t) \tag{2.15} \end{align}$$

are satisfied, where $r_r(t)$ is the ingestion flux of intensity of radioactivity, and $q_{er}(t)$ is effective outflow of intensity of radioactivity, respectively. Multiplying by α_p on both sides of Eq. (2.12),

$$\alpha_p \frac{dN(t)}{dt} = \alpha_p r_a(t) - \alpha_p q_{ea}(t) \tag{2.16}$$

is obtained. From the above relational expressions, Eqs. (2.3), (2.13) and (2.14), Eq. (2.16) can be rewritten as:

$$\frac{dh(t)}{dt} = r_r(t) - q_{er}(t), \tag{2.17}$$

where $h(t)$ is reinterpreted as intensity of radioactivity in the human body. Eq. (2.17) can be rewritten further as:

$$\frac{dh(t)}{dt} = r_r(t) - \alpha_e h(t), \tag{2.18}$$

from Eq. (2.15).

Eq. (2.18) is analytically equivalent to Eq. (2.12), the same discussion applies to the number of radioactive atoms and the intensity of radioactivity. Ingestion flux of the number of radioactive atoms $r_a(t)$ is assumed, for simplicity, to be:

$$r_a(t) = \begin{cases} r_{a0} & (t < t_m) \\ 0 & (t \geq t_m) \end{cases}, \tag{2.19}$$

where t_m and r_{a0} are interpreted as the ingestion period and the mean ingestion flux of the number of radioactive atoms, respectively. $r_r(t)$ is linear with respect to $r_a(t)$, such that

$$r_r(t) = \alpha_p r_a(t) = \begin{cases} r_{r0} & (t < t_m) \\ 0 & (t \geq t_m) \end{cases} \tag{2.20}$$

is satisfied.

2.2.5 Cumulative number of radioactive decay

cumulative number of radioactive decay $I(t)$ is defined by time integral of h(t) as:

$$I(t) = \int_0^t h(\tau) d\tau \text{ (events/body)}. \tag{2.21}$$

$I(t)$ is the total number of radioactive decay events in period of time t, a non-dimensional and accumulative quantity.

2.3 Results and Discussion

2.3.1 The analytical solutions of the number of radioactive atoms in the human body and intensity of radioactivity

As mentioned in section 2.2.4, the equation of continuity of radioactive atoms in the human body (2.12) is a first-order linear ordinary differential equation; thus, the general solution can be obtained by basic analysis[1]. The particular solution of $N(t)$ is described as:

$$N(t) = \int_0^t e^{-\alpha_e(t-\tau)} r_a(\tau) d\tau \tag{2.22}$$

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[1] Concrete method to solve the differential equation is described in Appendix B.

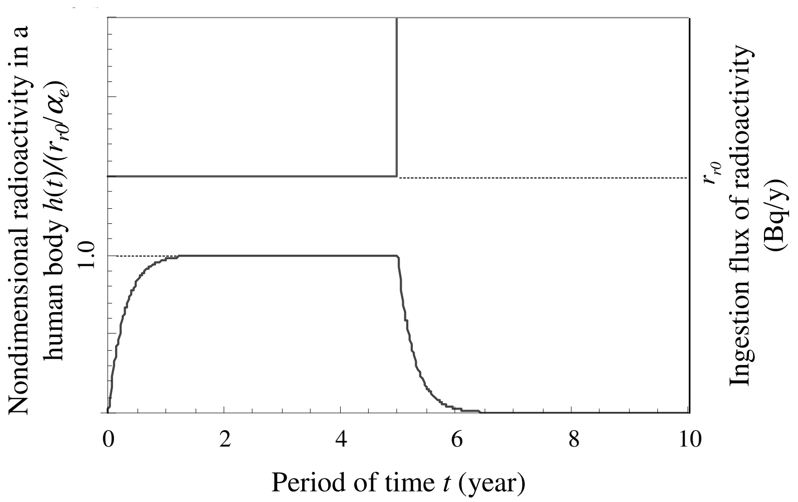

Fig 2.2: Time series of ingestion flux of radioactivity $r_r(t)$ and non-dimensional radiation intensity in the human body $h(t)/(r_{r0}/\alpha_e)$ in the case of ^{137}Cs.

under the initial condition that $N(0) = 0$. If ingestion flux of radioactive atoms $r_r(t)$ is a step function as Eq. (2.19), $N(t)$ is described as:

$$N(t) = \begin{cases} \dfrac{r_{a0}}{\alpha_e}(1 - e^{-\alpha_e t}) & (t < t_m) \\ \dfrac{r_{a0}}{\alpha_e}(-1 + e^{\alpha_e t_m})e^{-\alpha_e t} & (t \geq t_m) \end{cases}. \tag{2.23}$$

Multiplying by α_p on both sides of Eq. (2.22),

$$\alpha_p N(t) = \alpha_p \int_0^t e^{-\alpha_e(t-\tau)} r_a(\tau) d\tau \tag{2.24}$$

is obtained. From relationship Eqs. (2.3) and (2.13), Eq. (2.24) can be rewritten as:

$$h(t) = \alpha_p N(t) = \int_0^t e^{-\alpha_e(t-\tau)} r_r(\tau) d\tau. \tag{2.25}$$

As with the number of radioactive atoms $N(t)$, if ingestion flux of radioactive atoms $r_r(t)$ is a step function as Eq. (2.20), $h(t)$ is described as:

$$h(t) = \begin{cases} \dfrac{r_{r0}}{\alpha_e}(1 - e^{-\alpha_e t}) & (t < t_m) \\ \dfrac{r_{r0}}{\alpha_e}(-1 + e^{\alpha_e t_m})e^{-\alpha_e t} & (t \geq t_m) \end{cases}. \tag{2.26}$$

The time series of the ingestion flux of radioactivity and the non-dimensional intensity of radioactivity in the human body is shown in Fig 2.2; ^{137}Cs is used as the example. The ingestion period t_m is considered to be 5 years.

As shown in Fig 2.2, the intensity of radioactivity is saturated within approximately 1 year. After the end of ingestion, radioactivity is removed from the body within 1 year. ^{137}Cs is not required for the human body; thus, T_b is much shorter than T_p. So, T_b becomes

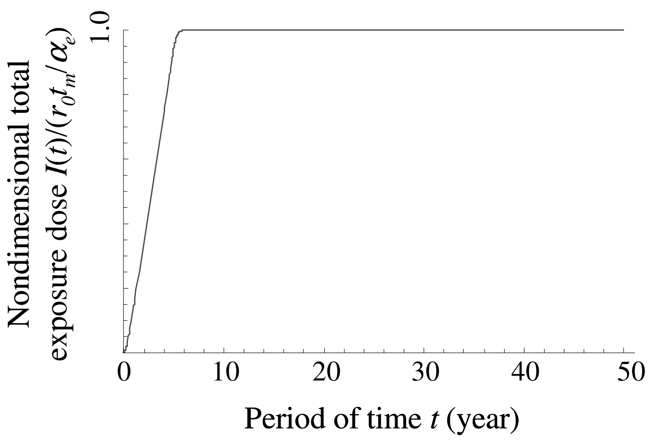

Fig 2.3: Time series of non-dimensional cumulative number of radioactive decay $I(t)/(r_{r0}t_m/\alpha_e)$ in the case of ^{137}Cs.

dominant, effectively, and T_e is nearly equal to T_b, as calculated in Eq. (2.9), although T_p is very long. Although ^{137}Cs is used as the example, it should be noted that these values of time do not have qualitative meaning, the time to be saturated and to be removed depends on nuclides.

For ^{137}Cs,

$$\frac{\lim_{(t_m,t)\to(\infty,1\text{ year})} h(t)}{\lim_{(t_m,t)\to(\infty,\infty)} h(t)} = \simeq 0.97 \tag{2.27}$$

and

$$h(1\text{ year}) = 0.97\frac{r_{r0}}{\alpha_e} \simeq \frac{r_{r0}}{\alpha_e} \tag{2.28}$$

are satisfied, we can say that the intensity of radioactivity is almost saturated to 1.

2.3.2 The analytical solution of cumulative number of radioactive decay

As shown in Eq. (2.21), the cumulative number of radioactive decay $I(t)$ is provided by a time integral of $h(t)$. $I(t)$ is described as:

$$I(t) = \begin{cases} \dfrac{r_0}{\alpha_e^2}(\alpha_e t - 1 + e^{-\alpha_e t}) & (t < t_m) \\ \dfrac{r_0}{\alpha_e^2}[\alpha_e t_m + e^{-\alpha_e t}(1 - e^{\alpha_e t_m})] & (t \geq t_m) \end{cases}. \tag{2.29}$$

Fig 2.3 shows the time series of non-dimensional cumulative number of radioactive decay under the same conditions with Fig 2.2. After the end of ingestion, $I(t)$ remains constant. In the case of T_e is sufficiently shorter than t_m, i.e. $T_e \ll t_m$,

$$I(t_m) \simeq \lim_{t\to\infty} I(t) = \frac{r_{r0}t_m}{\alpha_e}. \tag{2.30}$$

The result suggests that decreasing the ingestion period and the ingestion flux of radioactivity makes decrease the cumulative number of radioactive decay. The approximation of Eq. (2.30) is used hereafter as necessary.

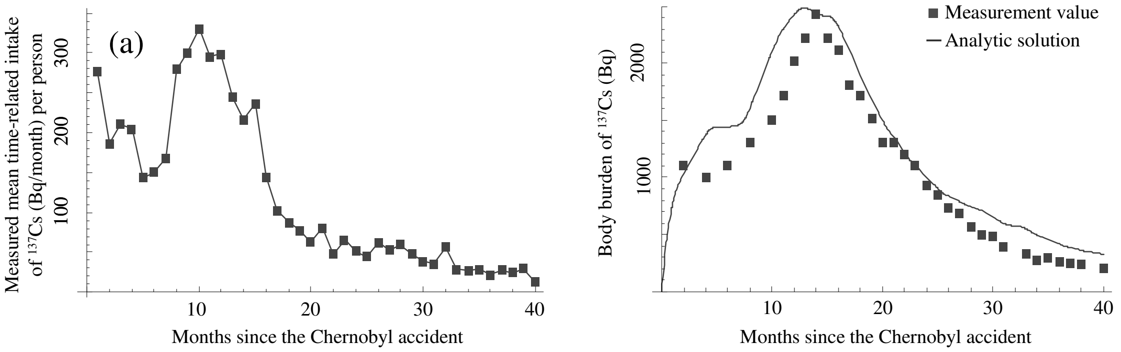

Fig 2.4: (a) Mean time-related intake of ^{137}Cs per person in the Federal Republic of Germany from W. Rühm et. al [39]. (b) Typical ^{137}Cs body burden of a male. (Plots are measurement values by whole-body counter quote from W. Rühm et. al [39]. The solid line is analytic solution given from interpolation of mean time-related intake, Eqs. (2.31) and (2.25))

Table 2.2: Age and gender-dependent food consumption rates (cr) in grams per day per person: data are valid for former Federal Republic of Germany in 1983 (Deutsche Gesellschaft für Ernährung 1989) [40]

Age (year)	4-6	7-9	10-12	13-14	15-18	19-35	36-50	51-65	>66
Male	1104	1269	1473	1661	2338	2444	2811	2616	2295
Female	973	1049	1212	1354	1951	1937	2076	1881	1657

2.3.3 Comparison of the analytical solution with measurement values regarding the intensity of radioactivity in the human body

After the Chernobyl disaster, time series of mean intake of ^{137}Cs and intensity of radioactivity in the human body and intensity of radioactivity in the human body were measured [39]. A sample of intensity of radioactivity in the human body is 41 men and women whose age is various as from 5 old to 61 old. The measurements were performed by using a whole-body counter (WBC). Fig 2.4 (a) is mean time-related intake of ^{137}Cs per person in the Federal Republic of Germany (read-out from Rühm et al. [39] Fig. 2).

Rühm et al gives the formula to estimate the individual intake of radioactivity of ^{137}Cs $r_{\rm ind}(t)$ (Bq mo^{-1}) from mean intake as:

$$r_{\rm ind}(t) = b_{\rm ind} \times (cr_{\rm age}/2000) \times r_{\rm ref}(t), \tag{2.31}$$

where $b_{\rm ind}$ is a factor to account for potential differences between individuals and mean food consumption, $cr_{\rm age}$ is age-dependent and sex-dependent food consumption rate (g d^{-1}) (Table 2.2 [40]), $r_{\rm ref}(t)$ is mean intake of radioactivity of ^{137}Cs (Bq mo^{-1}) based on a mean food consumption rate of 2,000 (g d^{-1}) (Fig. 2.4 (a)), respectively (see Rühm et al. [39] Eq.(1)).

From and Eqs. (2.18) and (2.31), intensity of radioactivity in the human body can be calculated. Plots in Fig 2.4 (b) are measurement value of ^{137}Cs body burden of a male after the Chernobyl accident by using a whole-body counter (read-out from Rühm et al. [39] Fig. 1). The solid line in Fig 2.4 (b) is analytic solution calculated by interpolation of mean

Table 2.3: Parameters of ^{137}Cs retention equation from the model adopted by ICRP Publ.67 [42].

Range of body weight (kg)	6	9.8	19	32	55	70
Age	3 months	1 year	5 years	10 years	15 years	Adult
Coefficient of short-term biological half-life a_1	-	-	0.45	0.30	0.13	0.10
Short biological half-life T_1 (days)	-	-	9.1	5.8	2.2	2
Coefficient of long-term biological half-life a_2	1	1	0.55	0.70	0.87	0.90
Long biological half-life T_2 (days)	16	13	30	50	93	110

time-related intake (Fig 2.4 (a)), Eqs. (2.18) and (2.31). Plots and solid line in Fig 2.4 (b) are very similar. Rühm et al. use exact ICRP model (see section XX and Fig XX) to estimate biological half-life, the present method provides almost the same result more readily. The accuracy of the present method to calculate radioactivity in the human body is demonstrated. In addition, the similar investigation of internal exposures has been carried out since October 2011 in Fukushima Prefecture [41]

2.3.4 Differences in the intensity of radioactivity in the human body and cumulative number of radioactive decay by age

The intensity of radioactivity and the cumulative number of radioactive decay differ by age because metabolic activity depends on age. Table 2.3 shows differences in parameters of the ^{137}Cs retention equation by age, as set out by ICRP [42].

More information of on the parameters is explained. Most orally ingested radioactive cesium is absorbed from the gastrointestinal tract immediately, and enters the blood. Then, a certain percentage will be excreted from the body relatively quickly, moving directly from the blood to the kidneys and then the urine. The remaining cesium accumulates in muscles temporarily, and is subsequently excreted via the urine.

Then, a_1 and a_2 are the ratios of short-term and long-term excretion (such that $a_1 + a_2 = 1$). T_1 is and T_2 are the two different biological half-lives. The separation of ingested intensity of radioactivity in the human body by short-term excretion and long-term excretion is shown in Fig 2.5.

From Table 2.3 and Fig 2.5, the intensity of radioactivity of ^{137}Cs in the human body $h(t)$, Eq. (2.25), is rewritten as:

$$\begin{aligned} h(t) =& h_1(t) + h_2(t) \\ =& a_1 \int_0^t e^{-\alpha_{e1}(t-\tau)} r_r(\tau) d\tau + a_2 \int_0^t e^{-\alpha_{e2}(t-\tau)} r_r(\tau) d\tau \end{aligned} \tag{2.32}$$

more exactly, where $\frac{1}{T_{e1}} = \frac{1}{T_p} + \frac{1}{T_1}$, $\frac{1}{T_{e2}} = \frac{1}{T_p} + \frac{1}{T_2}$, $\alpha_{e1} = \frac{\ln 2}{T_{e1}}$, and $\alpha_{e2} = \frac{\ln 2}{T_{e2}}$, respectively. If $r_r(t)$ is set as Eq. (2.20), then:

$$h(t) = \begin{cases} \frac{a_1 r_{r0}}{\alpha_{e1}}(1 - e^{-\alpha_{e1} t}) + \frac{a_2 r_{r0}}{\alpha_{e2}}(1 - e^{-\alpha_{e2} t}) & (t < t_m) \\ \frac{a_1 r_{r0}}{\alpha_{e1}}(-1 + e^{\alpha_{e1} t_m}) e^{-\alpha_{e1} t} + \frac{a_2 r_{r0}}{\alpha_{e2}}(-1 + e^{\alpha_{e2} t_m}) e^{-\alpha_{e2} t} & (t \geq t_m) \end{cases}. \tag{2.33}$$

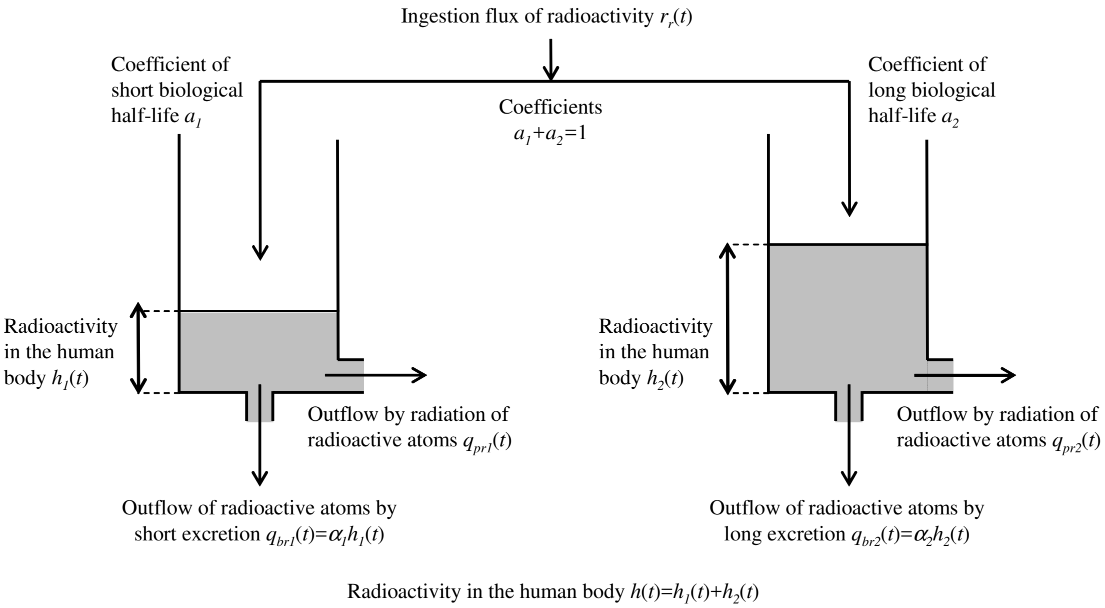

Fig 2.5: The separation of ingested intensity of radioactivity in the human body by short-term excretion and long-term excretion.

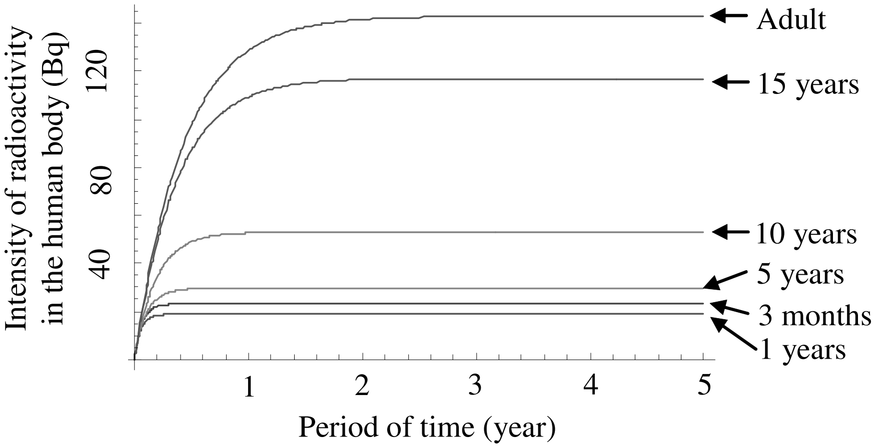

Fig 2.6: Time series of intensity of radioactivity in the human body of ^{137}Cs according to age when radioactivity of 1 Bq is ingested.

Fig 2.6 is a time series of intensity of radioactivity in the human body of ^{137}Cs per unit intake when food containing 1 Bq is ingested, according to age. Fig 2.6 shows that intensity of radioactivity in the human body of children, 3 months to 10 years old, is smaller than that of adults, because metabolism of children is more active than adults, consequently the biological half-life is shorter as shown in Table 2.3.

Eq. (2.33) gives more exact cumulative number of radioactive decay as a function of time

$I(t)$ of ^{137}Cs, based on the foregoing discussion, Eqs. (2.21) and (2.29), so that:

$$I(t) = \begin{cases} \dfrac{a_1 r_{r0}}{\alpha_{e1}^2}(\alpha_{e1}t - 1 + e^{-\alpha_{e1}t}) + \dfrac{a_2 r_{r0}}{\alpha_{e2}^2}(\alpha_{e2}t - 1 + e^{-\alpha_{e2}t}) & (t < t_m) \\ \dfrac{a_1 r_{r0}}{\alpha_{e1}^2}[\alpha_{e1}t_m + e^{-\alpha_{e1}t}(-1 + e^{-\alpha_{e1}t_m})] + \dfrac{a_2 r_{r0}}{\alpha_{e2}^2}[\alpha_{e2}t_m + e^{-\alpha_{e2}t}(-1 + e^{-\alpha_{e2}t_m})] & (t \geq t_m) \end{cases} \tag{2.34}$$

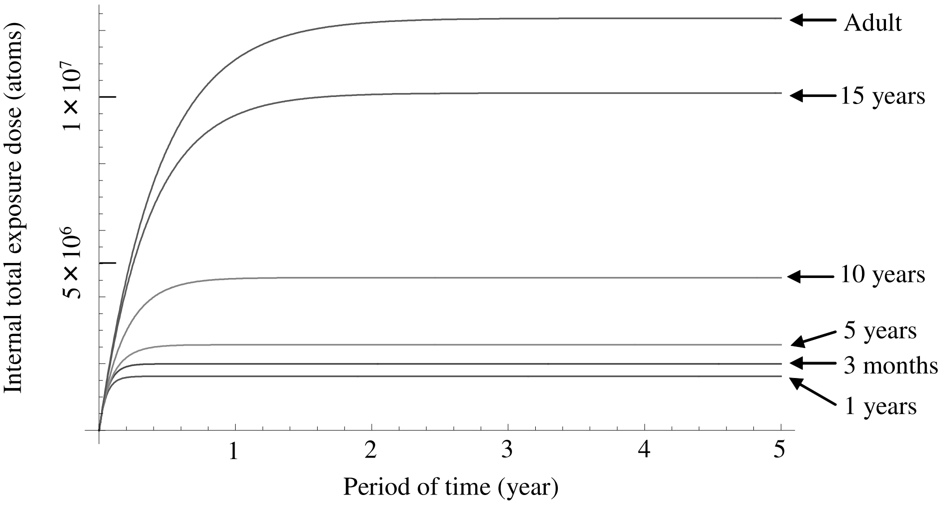

Fig 2.7: Time series of cumulative number of radioactive decay of ^{137}Cs according to age when radioactivity of 1 Bq is ingested.

Fig 2.7 is a time series of cumulative number of radioactive decay of ^{137}Cs per unit intake when food containing 1 Bq is ingested, according to age. As with Fig 2.6, Fig 2.7 shows that cumulative number of radioactive decay of children is smaller than adults. Demonstration of the accuracy of the present method is performed in detail later in section 3.3.3 by comparison of previous research by using ICRP model.

2.4 Conclusion

In this chapter, a readily-understandable methodology for calculating the number of radioactive atoms in the human body, intensity of radioactivity in the human body, cumulative number of radioactive decay is proposed based on knowledge developed in hydrology. Lumping of distributed model has been important issue in hydrology, great efforts have been carried out. This thesis attempts to expand the knowledge developed in hydrology to radiology. The accuracy of the hydrological-based method proposed in this thesis is demonstrated by comparison of intensity of radioactivity in the human body of analytic solution calculated by the present method with measurement by whole-body counter. The results shown in this chapter are summarized as follows.

1. A hydrological lumped model is useful for describing the number of radioactive atoms stored in the human body. A continuity equation can be applied to the number of radioactive atoms in the human body. The number of radioactive atoms in the human body, intensity of radioactivity in the human body, and cumulative number of radioactive decay can be described as functions of time rationally by hydrologically familiar method.

2. The effective half-life of a radioactive substance in a human body is derived from the physical and biological half-lives. If one of the half-lives is markedly longer than the other, it nearly equals the effective half-life. For example, the biological half-life of ^{137}Cs is much shorter than the physical half-life; thus, the biological half-life is dominant in the derivation of effective half-life.
3. In the case of ^{137}Cs, intensity of radioactivity in the human body saturates within approximately 1 year. After the end of ingestion, radioactivity is removed from the body within 1 year. Thus, decreasing the ingestion period and the ingestion flux of radioactivity causes decreases in cumulative number of radioactive decay and effective dose.
4. The accuracy of the present method was demonstrated by comparison with previous research. Measurement value and analytic solution obtained by the present method are very similar regarding intensity of radioactivity in the human body as shown in Fig.2.4 (b).

In the next chapter, the hydrological-based methodology is expanded to calculating of the absorbed energy into the human body and the effective dose caused by ingestion of radioactive isotopes based on the results of this chapter.

3 Effective dose calculation and its accuracy

3.1 Introduction

In the chapter 2, the number of radioactive atoms and intensity of radioactivity were discussed. In the following this chapter, absorbed energy per radioactive decay event and effective dose are estimated. Effective dose (Sv, sievert) is quantity which represents the stochastic biological effects. In this study, beta decay and gamma decay are focused on, because these two types of decay were observed in the vicinity of the nuclear plant after the disaster. In addition, radioactive cesium which evenly distributes whole the body is a major subject (nuclides concentrate on specific tissues or organs, such as iodine, are discussed later in detail). These suggest that the absorbed dose $D_{\mathrm{T,R}}$ (Gy, gray), the equivalent dose H_{T} (Sv), and the effective dose H_{eff} (Sv) equal each other within the scope of this study because the radiation weighting factors of beta ray and gamma ray are both 1, and concentration on specific tissues or organs (tissue weighting factor) can be neglected. Going forward, the two are regarded as equivalent without further comment. The only exception which concentration on specific organ (thyroid gland) is considered is radioactive iodine, a separate supplement is mentioned.

3.2 Theory and Methods

3.2.1 Beta decay

Fermi studied the theory of beta decay[2], and, in outline, it can be described as follows.

When β decay occurs, emission of an electron (e^-) and an electron antineutrino ($\bar{\nu_e}$) is observed per radioactive decay, at the same time converting a neutron (n) to a proton (p). The process is:

$$n \to p + e^- + \bar{\nu_e}. \tag{3.1}$$

In the case of γ decay occurs subsequently, the process

$$p \to p + \gamma \tag{3.2}$$

occurs in the emitted proton (proton-γ reaction).

In this study, the kinetic energy of an electron (β ray) is a subject of discussion from the viewpoint of radiological protection. The energy of the neutrino is negligible because its mean free path is much long and it is electrically neutral, and it has only tiny mass [43].

In β decay, the kinetic energy of an electron has an energy spectrum (Fig 3.1), is not expected deterministically; however, its maximum kinetic energy $K_e^{\max}$ can be determined for each nuclide. The rest mass energy difference between the parent nucleus and the daughter nucleus determines $K_e^{\max}$ as:

$$K_e^{\max} = [M(A,Z) - M(A,Z+1)]c^2, \tag{3.3}$$

where $M(A,Z)$ is mass of the nucleus, A is the sum number of protons and neutrons, i.e. mass number, and Z is the number of protons in the parent nuclide, i.e. atomic number of the parent nuclide. c is the velocity of light in a vacuum. Mass-energy equivalence is established on special relativity [3]. The mean kinetic energy of electrons $\langle K_e \rangle$ is expected

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[2] See Appendix I in detail.

[3] See Appendix G.

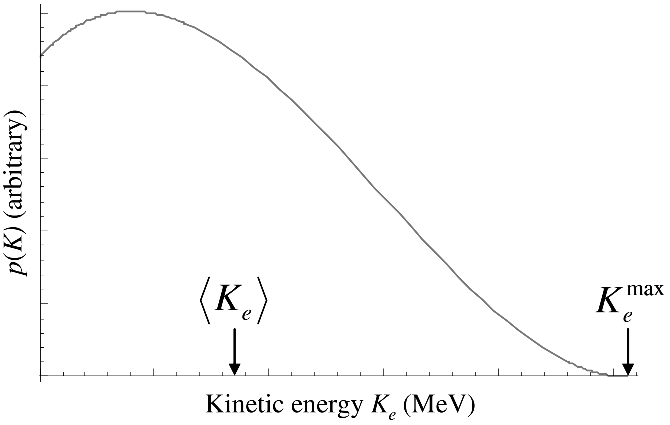

Fig 3.1: A schematic diagram of energy spectrum of β^- decay.

Table 3.1: $\langle K_e \rangle$-$K_e^{\max}$ ratio ξ for 8 nuclides [44].

Nuclide	^{14}C	^{32}P	^{40}K	^{59}Fe	^{60}Co	^{90}Sr	^{131}I	^{137}Cs
$\langle K_e \rangle$-$K_e^{\max}$ ratio ξ	0.32	0.41	0.43	0.32	0.30	0.36	0.32	0.34

roughly, the values of ratio ξ regarding 8 representative nuclides are summarized in Table 3.1 quote from Japan Atomic Energy Agency (JAEA) [44].

In this thesis, ξ is approximately to be as:

$$\xi = \frac{\langle K_e \rangle}{K_e^{\max}} \approx \frac{1}{3} \tag{3.4}$$

This result is known, theoretically and experimentally. In fact, ICRP gives a more precise mean kinetic energy of electrons [45], as $\langle K_e \rangle = 0.33 K_e^{\max} f(1 - \frac{\sqrt{Z}}{50})(1 + \frac{K_e^{\max}}{4})$. Experimentally known results [44] are shown in Table 3.1. However, Eq. (3.4) was used in this study for simplicity.

3.2.2 Absorbed energy per radioactive decay event

The decay scheme is explained to calculate the energy absorbed per radioactive decay event. Fig.3.2 shows the radioactive decay scheme of ^{137}Cs [46]. Thus, 94.6% of ^{137}Cs decays into a metastable $^{137\mathrm{m}}$Ba, with electron emission. The kinetic energy of the electron is maximally 0.5120 MeV. Subsequently, 85.1% of $^{137\mathrm{m}}$Ba becomes stable ^{137}Ba with γ ray emission (half-life of $^{137\mathrm{m}}$Ba is 2.56 min). The other 14.9% of $^{137\mathrm{m}}$Ba occurs internal conversion. The energy of the gamma ray is 0.6617 MeV. The other 5.4% of ^{137}Cs decays directly into ^{137}Ba with electron emission; the energy of the electron is maximally 1.174 MeV. The mean kinetic energy of the electrons and the absorptivity of the beta ray are required to calculate the energy absorbed into the human body per one radioactive decay event $\langle K_e \rangle$. As mentioned above, the mean kinetic energy of electrons is approximately 1/3 of their maximum kinetic energy. The absorptivity of γ ray into the human body is important for calculating the effective dose.

In previous research by the Committee on the Medical Internal Radiation Dose (MIRD), absorptivity was assumed to be 40% [11]. Also, ICRP assumed the absorptivity to be 60% in

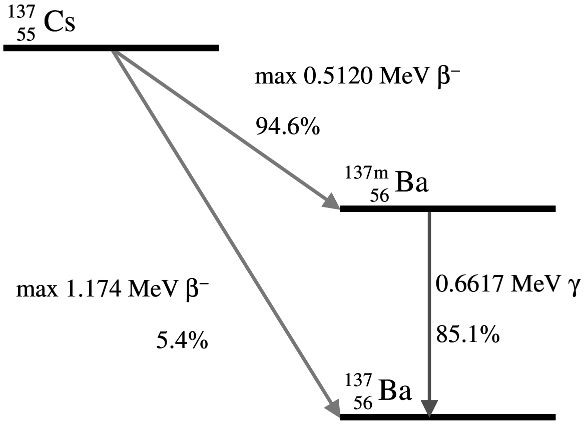

Fig 3.2: The radioactive decay scheme of ^{137}Cs, where the ordinate axis is energy, and the abscissa is the proton number. The arrows indicate the emitted particles. For the γ rays (vertical arrows), the γ energy are given; for the β decay (oblique arrow), the maximum beta energy.

ICRP Publ. 103 [47]. In this thesis, the absorptivity of γ ray ζ was assumed to be $\zeta = 1/2$, a value intermediate between the previous ICRP and MIRD values. $\langle K_e \rangle$ of ^{137}Cs is calculated concretely as an example. ^{137}Cs decays as Fig.3.2. $\langle K_e \rangle$ of ^{137}Cs is calculated as:

$$\begin{aligned}\langle E \rangle &= 0.946 \times \frac{1}{3} \times 0.512 \text{ (MeV) } (\mathrm{e}^-) \\ &+ 0.054 \times \frac{1}{3} \times 1.174 \text{ (MeV) } (\mathrm{e}^-) \\ &+ 0.851 \times \frac{1}{2} \times 0.6617 \text{ (MeV) } (\gamma) \\ &= 0.464. \text{ (MeV)}\end{aligned} \tag{3.5}$$

In the case of more the number of decay events are existing, $\langle K_e \rangle$ can be generally described as follows

$$\langle E \rangle = \sum_i \xi \mu_i K_{e,i}^{\max} + \sum_j \zeta \nu_j e_{\gamma,j}, \tag{3.6}$$

where i, j, μ, ν, and e_γ are the number of events of beta decay, the number of events of gamma decay, the probability of beta decay, the emission ratio of gamma decay, and the emitted energy of the gamma ray, respectively. Eqs. (3.5) and (3.6) show the simplicity of the present method. Absorbed energy per radioactive decay event can be calculated in the same way for other nuclides. The accuracy was investigated by comparison with read-out value from previous research [6]. As described below, the energy absorbed per one radioactive decay event is required to calculate the effective dose.

3.2.3 Absorbed dose, equivalent dose and effective dose

The absorbed dose $D_{\mathrm{T,R}}(t)$ is proportional to cumulative number of radioactive decay $I(t)$ as:

$$D_{\mathrm{T,R}}(t) = \frac{I(t) \times \langle E \rangle \text{ (MeV)} \times 1.6 \times 10^{-13} \text{ (J/MeV)}}{\text{weight (kg)}} \text{ (Gy, J/kg)}, \tag{3.7}$$

where 1.6×10^{13}(J/MeV) is the unit conversion from MeV to J. The weight of the human body is assumed to be 70 kg, supposedly an average adult male.

However, we cannot discuss health effects by radiation only the energy but the type of radiation. Particles which have large energy is more harmful than particles which have small energy to human's gene (recall photoelectric effect). Equivalent dose is a dose quantity taking into account the type of radiation. Weighting the absorbed dose $D_{\mathrm{T,R}}(t)$ by radiation weighting factor w_{R} and summing up, equivalent dose $H_{\mathrm{T}}(t)$ is obtained as:

$$H_{\mathrm{T}}(t) = \sum_{\mathrm{R}} w_{\mathrm{R}} D_{\mathrm{T,R}}(t) \ (\mathrm{Sv, J/kg}). \tag{3.8}$$

Radiation weighting factor w_{R} depends on the type of radiation. For example, w_{R} of α ray is $w_{\mathrm{R}} = 20$ because of mass and charge of α ray (helium nucleus). In case of β and γ ray, $w_{\mathrm{R}} = 1$, that is, $D_{\mathrm{T,R}}(t) = H_{\mathrm{T}}(t)$. The main subject of this article is nuclides of which decay mode is β and γ decay such as radiocesium, In that reason, we can regard equivalent dose is equal to absorbed dose in the scope of this article.

Equivalent dose is, however, unsatisfactory to discuss health effects by radiation. Some elements are required for human body metabolism, therefore they are concentrated in specific organ or tissue. For example, calcium is concentrated in bone, because compound of calcium is the main component of bone. Iodine is concentrated in thyroid to synthesize hormones. In that case, we should consider the dose weighting by each organs or tissues from the viewpoints of biochemistry.

Effective dose $H_{\mathrm{eff}}(t)$ is a dose quantity taking into account tissue weighting defined as:

$$H_{\mathrm{eff}}(t) = \sum_{\mathrm{T}} w_{\mathrm{T}} H_{\mathrm{T}} \ (\mathrm{Sv, J/kg}).,\tag{3.9}$$

where w_{T} is issue weighting factors. Of course, the effective dose is also proportional to cumulative number of radioactive decay $I(t)$.

Again, within the major scope of this thesis, regarding radioactive isotopes as cesium which are distributed evenly whole the body, absorbed dose, the equivalent dose, and the effective dose equal each other,

$$H_{\mathrm{eff}}(t) = \frac{I(t) \times \langle E \rangle \ (\mathrm{MeV}) \times 1.6 \times 10^{-13} \ (\mathrm{J/MeV})}{\mathrm{weight} \ (\mathrm{kg})} \ (\mathrm{Sv, J/kg}) \tag{3.10}$$

is satisfied as a general rule.

Eq.(3.10) is satisfactory regarding radioisotopes which are distributed whole the body (e.g. radiocesium). However, Eq.(3.10) is unsatisfactory for elements which is required for the human body (e.g. iron, zinc, iodine etc.) These elements should be considered their biochemical property, it will be discussed later in detail.

3.2.4 Committed effective dose

As mentioned in section 1.2, the total effective dose delivered over a lifetime is termed the committed effective dose (CED). In the case of adults, time until death is assumed to be 50 years. In previous research, the committed effective dose coefficient was used to calculate the CED. Values of the coefficient are calculated by ICRP [32, 35]. In this thesis, the effective dose $H_{\mathrm{eff}}(t)$ is described as a function of time, as Eq. (3.10).

The committed effective dose coefficient, C_{CED}, can be described as:

$$C_{\mathrm{CED}} = \frac{H_{\mathrm{eff}}(50\mathrm{y})}{r_{\mathrm{tot}}} \tag{3.11}$$

Table 3.2: The table which compares energy absorbed per one radioactive decay event calculated by the present method *versus* ICRP Publ.2.

Nuclide	Present method (MeV)	ICRP (MeV)
^{14}C	0.052	0.052
^{22}Na	1.2	1.6
^{24}Na	2.5	2.5
^{32}P	0.57	0.70
^{35}S	0.058	0.056
^{42}K	1.3	1.7
^{47}Sc	0.28	0.22
^{59}Fe	0.72	0.82
^{60}Co	1.5	1.6
^{131}I	0.38	0.44
^{137}Cs	0.47	0.52

3.3 Results and Discussion

3.3.1 Comparison of absorbed energy per radioactive decay event with previous research

The accuracy of the present method for calculating energy absorbed per radioactive decay event was investigated by comparison with previous research. Fig.3.3 compares values of energy absorbed per radioactive decay event by the present method with the energy absorbed as determined according to MIRD method [6] related to individual nuclides. Both results are similar; that is, the present method, despite its simplicity, shows accuracy about equal to that of the previously proposed complex method.

3.3.2 Calculation of effective dose

The value of the committed effective dose coefficient is calculated analytically. From Eqs. (2.30) and (3.10),

$$\begin{aligned} H_{\text{eff}}(50\text{y}) &= I(50\text{y}) \times \frac{\langle E\rangle(\text{MeV}) \times 1.6 \times 10^{-13}(\text{J/MeV})}{70(\text{kg})} \\ &\simeq \frac{r_{r0}t_m}{\alpha_e} \times \frac{\langle E\rangle(\text{MeV}) \times 1.6 \times 10^{-13}(\text{J/MeV})}{70(\text{kg})} \\ &= C_{\text{CED}} \times r_{r0}t_m \ (\text{Sv}) \end{aligned} \tag{3.12}$$

In Eq. (3.12), C_{CED} is exactly the committed effective dose coefficient. Eq. (3.12) can be rewritten as:

$$C_{\text{CED}} = \frac{H_{\text{eff}}(50\text{y})}{r_{r0}t_m} \ (\text{Sv/Bq}) \tag{3.13}$$

3.3.3 Differences in committed effective dose by age

The effective dose coefficient differs by age because metabolic activity depends on age. Again, Table 2.3 shows differences in parameters, such as the biological half-life, of the ^{137}Cs reten-

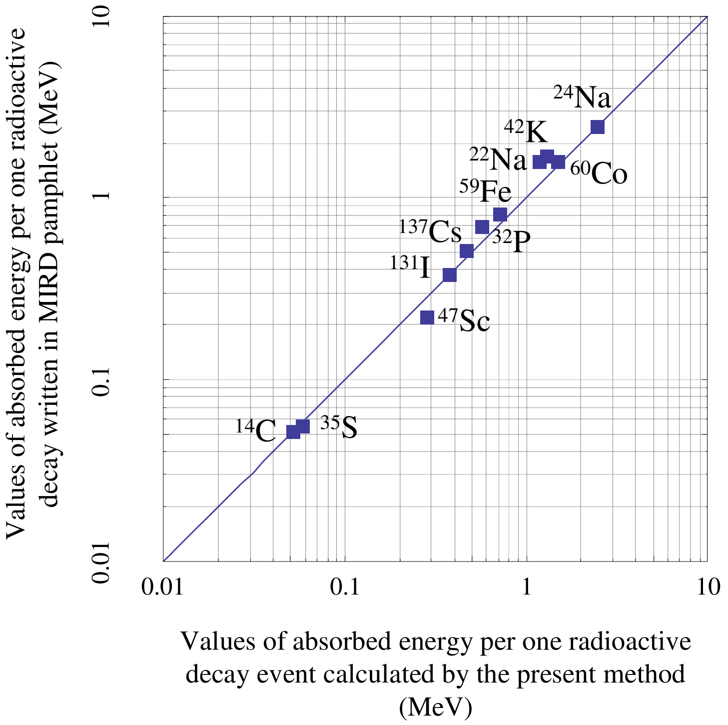

Fig 3.3: The graph which compares energy absorbed per one radioactive decay event calculated by the present method *versus* ICRP Publ.2.

tion equation by age, as set out by ICRP [42]. Eqs. (3.10) and (2.34) give effective dose as a function of time $H_{\text{eff}}(t)$. Difference of weight by age is shown in Table 2.3. Fig.3.4 shows a time series of effective dose of ^{137}Cs per unit intake when food containing 1 Bq/d is ingested during 1 day, according to age. A similar time series using a pharmacokinetic model related to cesium is shown by previous research [34].

Its accuracy is comparable with previous research using a complicated pharmacokinetic model. Plots in Fig.3.4 are the time series of effective calculated ICRP [Kurihara, ICRP-database], the solid lines in Fig.3.4 are the effective dose calculated by analytic solutions, Eqs. (3.10) and (2.34). Plots and solid line in Fig.3.4 are very similar, the accuracy of the present method to calculate effective dose is demonstrated. The saturated values of effective dose by age are similar to the committed effective dose coefficients calculated by ICRP [32] as shown in Table 3.3. These results further demonstrate the accuracy of the method proposed in this study.

Table 3.3: Comparison of committed effective dose (CED) of ^{137}Cs by age determined by ICRP Publ.72 with by the present method.

Age	3 months	1 year	5 years	10 years	15 years	Adult
CED calculated by ICRP (Sv/Bq)	2.1×10^{-8}	1.2×10^{-8}	9.6×10^{-9}	1.0×10^{-8}	1.3×10^{-8}	1.3×10^{-8}
CED by the present method (Sv/Bq)	2.5×10^{-8}	1.2×10^{-8}	1.0×10^{-8}	1.1×10^{-8}	1.4×10^{-8}	1.3×10^{-8}

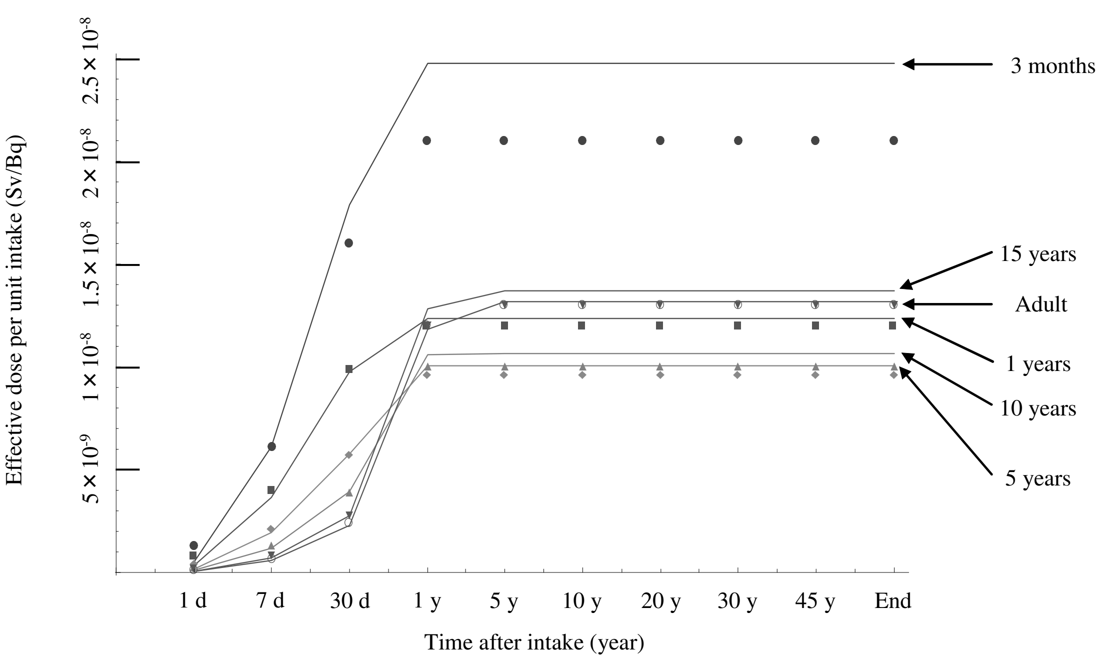

Fig 3.4: Time series of effective ^{137}Cs dose according to age when food contains 1 Bq/year is ingested for 1 year. Plots are calculated values by ICRP [35]. The solid line is analytic solution given from Eqs. (2.34) and (3.10))

3.3.4 Comparison of committed effective dose with previous research

Fig.3.5(a) compares values of committed effective dose coefficient calculated by the present method with those determined according to ICRP [32]. Eq. (3.13) was used to calculate the coefficient. Values of the coefficient are similar to those of the ICRP for five nuclides, ^{14}C, ^{32}P, ^{40}K, ^{60}Co, and ^{137}Cs. Indeed, the present method shows accuracy equal to that of previous research in terms of calculating these coefficients. The present method, which regards the human body as a single large tissue, shows accuracy equal to those of previous methods for non-essential nuclides that distribute throughout the whole body approximately evenly, such as cesium, phosphorus, and potassium.

However, for four nuclides, the calculated values are inconsistent. For ^{59}Fe, ^{65}Zn, and ^{90}Sr, the values calculated are larger, and for ^{131}I, the value is smaller than the ICRP results. These elements are required by the human body. Further, it is known that these elements concentrate in specific parts of the human body.

Regarding the inconsistencies for nuclides required by the human body, it is known in biochemistry and nutritional science that part of the ingested material is absorbed, and the rest is excreted promptly, as shown in Fig. 2.5. For these nuclides, such as iron and zinc, bioactivity affects to be the amount of the element constant. Short-term excretion ratio a_1 is much larger than long-term excretion ratio a_2. Thus, it is important to consider absorption by the digestive system when calculating the effective dose. The chemical properties of radioactive isotopes and stable isotopes are almost identical in the human body, so knowledge from biochemistry and nutritional science can be applied directly. Specifically, iron and zinc are minerals required by the human body, and their absorptivity is detailed in the biochemical and nutritional sciences literature. Fixing an appropriate value for the absorptivity of iron

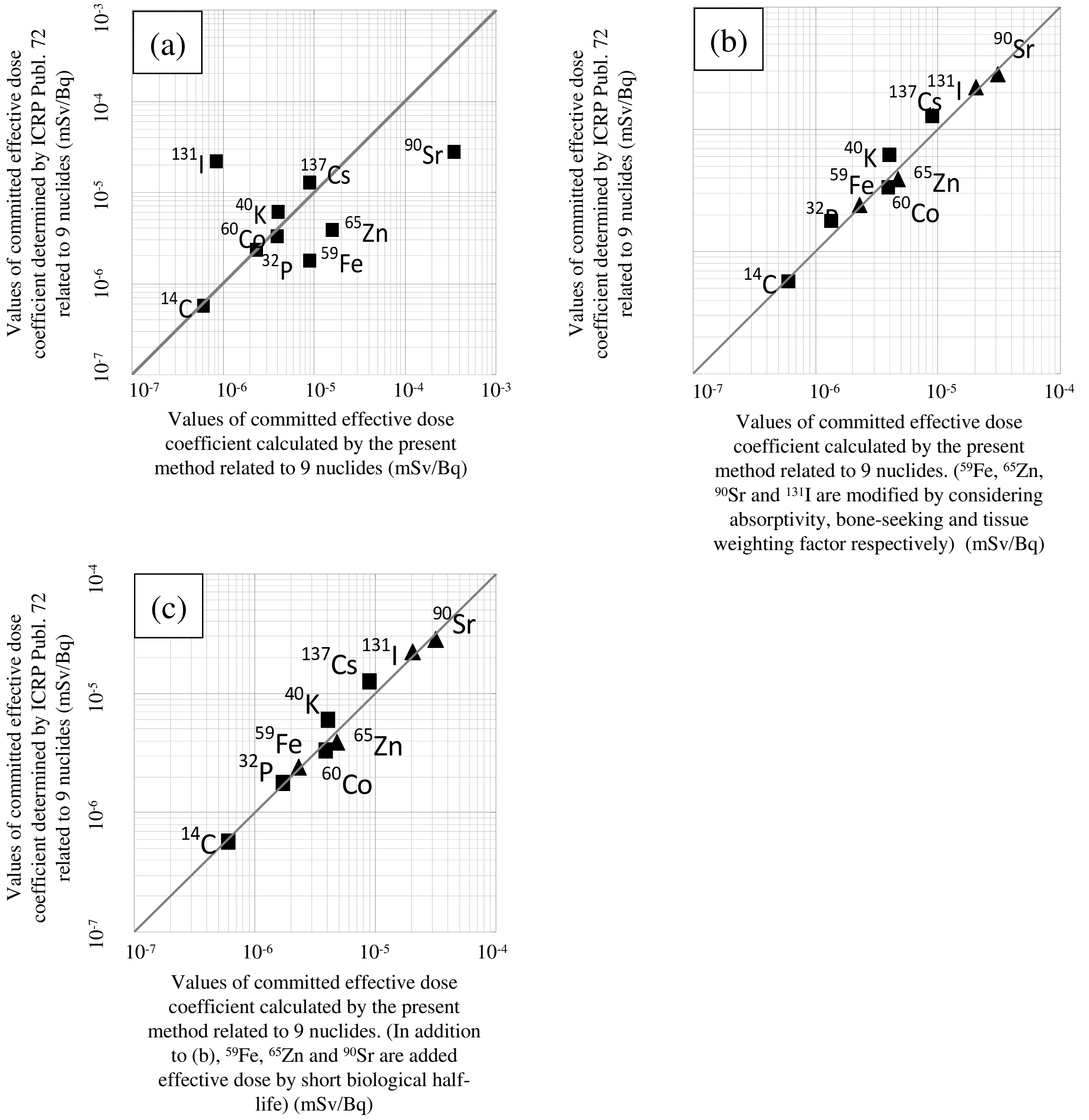

Fig 3.5: Comparison of the committed effective dose coefficient from previous research (ICRP) with (a) the present method (b) the present method, in which ^{59}Fe, ^{65}Zn, ^{90}Sr, and ^{131}I values are modified, (c) the present method in which ^{131}I is modified as in (b), and ^{59}Fe, ^{65}Zn, and ^{90}Sr are further modified by adding back the effective dose due to the short biological half-life.

is difficult generally because it depends on the ingested amounts of other nutrients, which can increase or decrease it.

In this study, the absorptivity of iron was assumed to be 15%, the value used by the Food and Agriculture Organization and the World Health Organization report [48]. Thus, 15% of iron ingested was considered when calculating the effective dose, and the rest was ignored.

The absorptivity of zinc depends on the amount ingested. Again, setting an appropriate value for zinc is difficult generally because the absorptivity of zinc depends on the amount ingested. In this study, absorptivity of zinc from intestinal tract was assumed to be 30%, as reported by previous researches [49–55]. As with iron, 30% of the zinc ingested was considered when calculating the effective dose and the rest was ignored. In the cases of iron and zinc, more plausible coefficients are obtained by multiplying by the absorptivity. Ministry of Health, Labour and Welfare Japan (MHLW) summarize above results as a survey [56].

Next, strontium is discussed. The human body confuses strontium with calcium because strontium and calcium are homologous elements. It is known that strontium is a bone seeker, a nuclide that tends to concentrate in bone. The ratio of strontium concentrated on bone by ingestion is 0.09 [57]. As with iron and zinc, 9% of the strontium ingested was considered in calculating the effective dose and the rest was ignored. A better-matched coefficient was obtained by multiplying by the bone-seeking ratio.

Finally, iodine is considered. It is known that iodine tends to accumulate in the thyroid gland. ICRP recommends using a tissue weighting factor of 0.04 to convert equivalent dose to effective dose related to the thyroid gland [47]. This means that, for example, exposure to 1 mSv for the whole body is equivalent to exposure to 25 mSv for the thyroid gland. The equivalent dose calculated by the present method uses exposure to the whole body because it regards a human body as a single tank. Thus, multiplying by 25 to calculate the equivalent dose provides appropriate in estimating the effective dose due to iodine, which concentrates in the thyroid gland. Fig.3.5(b) compares the coefficients by the present method with those determined according to ICRP Publ. 72 [32], with ^{59}Fe, ^{65}Zn, ^{90}Sr, and ^{131}I values modified as discussed above. The comparability of the ^{59}Fe, ^{65}Zn, ^{90}Sr, and ^{131}I values is better than in Fig.3.5(a). These results suggest that the modifications above are appropriate. Thus, the present methodology, corresponding to a hydrological lumped model, shows about equal accuracy to research in ICRP Publ.72 [32]. Vales for nuclides required by the human body, such as iron, zinc, strontium, and iodine, were modified by taking into account their biochemical properties. Table 3.4 summarizes the considered factors and the corrected values for the four nuclides.

Table 3.4: Modified factors and multiplied values relating to 4 nuclides.

Nuclide	Modified factor	Multiplied value
^{59}Fe	Absorptivity from a digestive system	0.15
^{65}Zn	Absorptivity from a digestive system	0.30
^{90}Sr	Ratio of bone-seeking	0.09
^{131}I	Tissue weighting factor of a thyroid	0.04^{-1}

The effective dose calculated will be more accurate if the neglected short biological half-life is added back. Eq. (2.34), which calculates a more accurate cumulative number of radioactive decay, gives a more accurate effective dose. The short biological half-life is set to 2 days. Fig.3.5(c) compares the coefficients calculated by the present method with those determined according to ICRP Publ. 72 [32]. ^{131}I is modified as in Fig.3.5(b), and ^{59}Fe, ^{65}Zn, and ^{90}Sr are modified by adding exposure due to the short biological half-life. The difference in effective dose made by adding the short biological half-life is quite small; it must, however, give a more accurate effective dose. Table 3.5 shows effective dose ratio caused by a long biological half-life versus that caused by a short biological half-life of ^{59}Fe, ^{65}Zn, and ^{90}Sr. It can be seen that the effective dose caused by a short biological half-life is much smaller than that caused by a long biological half-life.

Table 3.5: Effective dose ratio: long biological half-life *versus* short biological half-life for ^{59}Fe, ^{65}Zn and ^{90}Sr.

Nuclide	Effective dose ratio: long biological half-life *versus* short biological half-life
^{59}Fe	0.27
^{65}Zn	2.4×10^{-2}
^{90}Sr	3.0×10^{-3}

3.3.5 Examples calculating the committed effective dose

Three examples are provided, which illustrate how to calculate the effective dose when a human ingests food containing radioactive ^{137}Cs.

Exercise 1:

In meat, a concentration of 100 Bq/kg of ^{137}Cs has been measured. What effective dose does a person receive on eating 1 kg/d of this meat during 50 years? 100 Bq/kg is the standard limit in general foods for radioactive cesium, determined by the MHLW-Japan at the present (October 2013), enforced from April 1 2012 [29].

Answer 1:

Ingested radioactivity is 100(Bq/kg) × 1(kg/d) × 365(d/y) × 50(y) = 1.83×10^6 (Bq). From Eq. (2.30), cumulative number of radioactive decay during 50 years is:

$$I(50\mathrm{y}) \simeq \frac{r_{r0}t_m}{\alpha_e} = \frac{100(\mathrm{Bq/kg}) \times 1(\mathrm{kg/d}) \times 365(\mathrm{d/y}) \times 50(\mathrm{y})}{\ln 2/69.6(\mathrm{d}) \times 24(\mathrm{h/d}) \times 60^2(\mathrm{s/h})} = 1.6 \times 10^{13}. \tag{3.14}$$

From Eqs. (3.10) and (3.14), the committed effective dose is calculated as:

$$H_{\mathrm{eff}}(50\mathrm{y}) = \frac{I(50\mathrm{y}) \times 0.464(\mathrm{MeV}) \times 1.6 \times 10^{-13}(\mathrm{J/MeV})}{70(\mathrm{kg})} \simeq 17\mathrm{mSv}. \tag{3.15}$$

If the average energy absorbed into the human body per one radioactive decay event of ^{137}Cs calculated according to the ICRP, 0.521 MeV and Eq. (3.10) are used, the committed effective dose is calculated as 18 mSv. If the committed effective dose coefficient of ^{137}Cs calculated according to the ICRP, 0.013 μSv/Bq is used, and the value of the committed effective dose is calculated as: 0.013(μSv/Bq) × 1.83 × 100(Bq) = 23 (mSv). These results are similar, again indicating the accuracy of the present method. Despite its simplicity, the present method, using a hydrological lumped model, shows sufficiently high accuracy. For reference, the United Nations Scientific Committee on the Effects of Atomic Radiation says that total exposure dose due to natural radiation (world average) is 2.4 mSv/year [59]. UNSCEAR also noted that a dose rate less than 100 mSv/y indicates a linear no-threshold model [60]. ICRP Publ. 103 notes that exposure less than 100 mSv/year does not provide any clear and significant indications of cancer risk, medically and statistically [47]. The result shown in Eq. (3.15) suggests that the new criterion for radioactive cesium set by MHLW-Japan is excessive.

Exercise 2:

In meat, a concentration of 50 Bq/kg of ^{137}Cs and 50 Bq/kg of ^{134}Cs have been measured. What effective dose does a person receive on eating 1 kg/d of this meat during 50 years? Intensity of radioactivity of ^{137}Cs and ^{134}Cs was almost the same in immediately after the accident [61].

Answer 2:

In the same way as exercise 1, cumulative number of radioactive decay of ^{137}Cs during 50 years $I_{137}(50\text{y})$ is:

$$I_{137}(50\text{y}) \simeq \frac{r_{r0}t_m}{\alpha_e} = \frac{100(\text{Bq/kg}) \times 1(\text{kg/d}) \times 365(\text{d/y}) \times 50(\text{y})}{\ln 2/69.6(\text{d}) \times 24(\text{h/d}) \times 60^2(\text{s/h})} = 7.9 \times 10^{12}, \tag{3.16}$$

where α_{e137} is an effective attenuation constant of ^{137}Cs, relationship with effective half-life of ^{137}Cs, T_{e137}, is $\alpha_{e137} = \ln 2/T_{e137}$.

Physical half-life of ^{134}Cs is 2 years, biological half-life of ^{134}Cs is the same as ^{137}Cs. Effective half-life of ^{134}Cs is calculated as 63.9 days from Eq. (2.7). Therefore, cumulative number of radioactive decay of ^{134}Cs during 50 years $I_{134}(50\text{y})$ is:

$$I_{134}(50\text{y}) \simeq \frac{r_{r0}t_m}{\alpha_e} = \frac{100(\text{Bq/kg}) \times 1(\text{kg/d}) \times 365(\text{d/y}) \times 50(\text{y})}{\ln 2/63.9(\text{d}) \times 24(\text{h/d}) \times 60^2(\text{s/h})} = 7.3 \times 10^{12}, \tag{3.17}$$

where α_{e134} is an effective attenuation constant of ^{134}Cs, relationship with effective half-life of ^{134}Cs, T_{e134}, is $\alpha_{e134} = \ln 2/T_{e134}$.

The average energy absorbed into the human body per one radioactive decay event is calculated as 0.924 MeV. The committed effective dose is calculated as:

$$\begin{aligned} H_{\text{eff}}(50\text{y}) &= \frac{I_{137}(50\text{y}) \times 0.464(\text{MeV}) \times 1.6 \times 10^{-13}(\text{J/MeV})}{70(\text{kg})} \\ &\quad + \frac{I_{134}(50\text{y}) \times 0.9240(\text{MeV}) \times 1.6 \times 10^{-13}(\text{J/MeV})}{70(\text{kg})} \\ &\simeq 24\text{mSv} \end{aligned} \tag{3.18}$$

If the average energy absorbed into the human body per one radioactive decay event of ^{134}Cs calculated according to the ICRP, 1.09 MeV and Eq. (3.10) are used, the committed effective dose is calculated as 27 mSv. When the committed effective dose coefficient of ^{134}Cs calculated according to ICRP, 0.019 μSv/Bq is used, the value of the committed effective dose is calculated as: $0.5 \times 0.013(\mu\text{Sv/Bq}) \times 100(\text{Bq/kg}) \times 1(\text{kg}) \times 365(\text{d}) \times 50(\text{y}) + 0.5 \times 0.019(\mu\text{Sv/Bq}) \times 100(\text{Bq/kg}) \times 1(\text{kg}) \times 365(\text{d}) \times 50(\text{y}) = 29\text{mSv}$. As in Exercise 1, almost identical results were obtained.

Exercise 3:

In water, a concentration of 60000 Bq/L of ^{3}H has been measured. What effective dose does a person receive on drinking 2 L/d (required water consumption for adult male) of this water over 50 years? 60000 Bq/L is concentration limit specified by laws and regulations [62].

Answer 3:

Physical half-life of ^{3}H is 12 years, biological half-life of ^{3}H is 12 days and effective half-life of ^{3}H is 12 days as shown in Table 2.1, in the case of tritiated water. cumulative number of radioactive decay of ^{3}H during 50 years $I(50\mathrm{y})$ is:

$$I(50\mathrm{y}) \simeq \frac{r_{r0} t_m}{\alpha_e} = \frac{60000(\mathrm{Bq/L}) \times 2(\mathrm{L/d}) \times 365(\mathrm{d/y}) \times 50(\mathrm{y})}{\ln 2/12(\mathrm{d}) \times 24(\mathrm{h/d}) \times 60^2(\mathrm{s/h})} = 6.6 \times 10^{13}. \tag{3.19}$$

The average energy absorbed into the human body per one radioactive decay event is calculated as 6.2 keV from Eq. (3.4). The committed effective dose is calculated as:

$$H_{\mathrm{eff}}(50\mathrm{y}) = \frac{I(50\mathrm{y}) \times 6.2(\mathrm{keV}) \times 1.6 \times 10^{-16}(\mathrm{J/keV})}{70(\mathrm{kg})} \simeq 46\mathrm{mSv} \tag{3.20}$$

When the average energy absorbed into the human body per one radioactive decay event calculated according to the ICRP, 5.8 kev and Eq. (3.10) are used, the value of committed effective dose is calculated as 43 mSv. When the committed effective dose coefficient of ^{3}H calculated according to ICRP, 0.018 nSv/Bq is used, the value of the committed effective dose is calculated as: $0.018(\mathrm{nSv/Bq}) \times 60000(\mathrm{Bq/L}) \times 2(\mathrm{L/d}) \times 365(\mathrm{d/y}) \times 50(\mathrm{y}) = 40$ mSv. Again, as in Exercises 1 and 2, almost identical results were obtained. The effective dose per a year is 0.93 mSv, which is much lower than total exposure dose due to natural radiation.

3.4 Conclusion

In this chapter, we calculate the effective dose caused by the intake of radioisotope. The method to calculate the absorbed energy into the human body per one radioactive decay event is proposed. The effective dose is computed from the absorbed energy per radioactive decay and cumulative number of radioactive decay. It is demonstrated that the method to calculate the effective dose proposed in this article has comparable accuracy with previous study in radiology. The results of this chapter are summarized in the followings.

1. The absorbed energy into the human body per one radioactive decay event is estimated from the decay scheme considering the energy spectrum of β ray and absorptivity of γ ray. The average energy of β ray is estimated to one third of its maximum energy. Absorptivity of γ ray is estimated to be a half because the energy order of γ ray is 0.01-0.1 MeV. Reading out the decay schemes of each nuclides, we compute the absorbed energy per one radioactive decay.

2. Multiplying the absorbed energy per one radioactive decay by accumulative number of radioactive decay, we obtain total energy absorbed into the body. The absorbed dose is computed by dividing body weight into the total energy. The weight is set to be 70 kg assuming typical adult male.

3. In case of radiocesium, main subject of this article, absorbed dose, equivalent dose, and effective dose are equal to each other. This is because cesium is not required human body and does not concentrate in a particular organ. Comparison of committed effective dose with existing study shows that this method also applies to other nuclides.

4. Elements which human body requires, for example iron, zinc, strontium (congener of calcium), and iodine, should be considered biochemical properties to compute the effective dose. For iron and zinc, the absorptivity from digestive organs is considered. Strontium is modified taking into account the property as a bone-seeker. For iodine, we should consider concentration in thyroid and its tissue weighting factor. These modification give more consistent values of effective dose with existing study.

5. The values of committed effective dose are calculated assuming the case that people take radiocesium on food and water supply referring the standard limit for radiocesium in Japan in 2015. The standard limit of radiocesium yields only the dose much less than natural radiation.

In this chapter, we studied readily-understandable effective dose calculation method. As more practical problem, we will discuss radioactivity on agricultural products derived from fallout and the effective dose in next chapter.

4 The effective dose calculation caused by radioactivity on agricultural products derived from radioactive fallout

4.1 Introduction

In this chapter, we aim to expand the scope of the research from the human body to the soils. We model the transfer of radiocesium from the radioactive fallout into the soils to the intake of radioactivity on agricultural products. Subsequently, we calculate the effective dose by applying the methods we have been studied up to previous section. We show that our hydrologic-based method can be straightforwardly applied to not only the human body but also the soils.

Our modeling of the radiocesium transfer from the soils to the human body is based on the previous research regarding the vertical infiltration, the transfer factor to plants, and the residence time of cesium in the soils. We compute the intensity of radioactivity on the soils per area and the effective dose derived from the intake of radioactivity on the rice using actual measured radioactive fallout after the Fukushima Daiichi NPP accident. The actual measured data show that the dose rate caused by the intake of radioactivity on agricultural products derived from fallout after the Fukushima Daiichi NPP accident is less than natural radiation.

4.2 Theory and Methods

4.2.1 Modeling of transfer from radioactive fallout to radioactivity on agricultural products

We consider the transfer of radiocesium from the soils to the intake of radioactivity on agricultural products by using the data of fallout. Fig.4.1 shows a schematic diagrams of the radiocesium transfer. Fig.4.1(a) describes radiocesium fallout into the soils. We can know the actual intensity of radioactivity per area from above mentioned data of Nuclear Regulation Authority [61].

Fig.4.1(b) describes the transfer of radiocesium from the soils to rice, which is a typical example of agricultural product in Japan. Here, we introduce the transfer factor η defined as:

$$\eta = \frac{\begin{array}{c}\text{The concentration of radioactivity}\\ \text{in agricultural products (Bq/kg)}\end{array}}{\begin{array}{c}\text{The concentration of radioactivity}\\ \text{in the soils (Bq/kg)}\end{array}}. \tag{4.1}$$

There are many reports regarding the transfer factor [63–65]. The unit conversion from intensity of radioactivity per area (Bq/m^2) to the concentration of radioactivity in the soils (Bq/kg) is also required. Their treatment in this paper is described in detail later.

Fig.4.1(c) describes the intake of radiocesium on the rice. We can calculate the effective dose by using the method the authors had proposed. We recognize the transfer of radiocesium formally up to here.

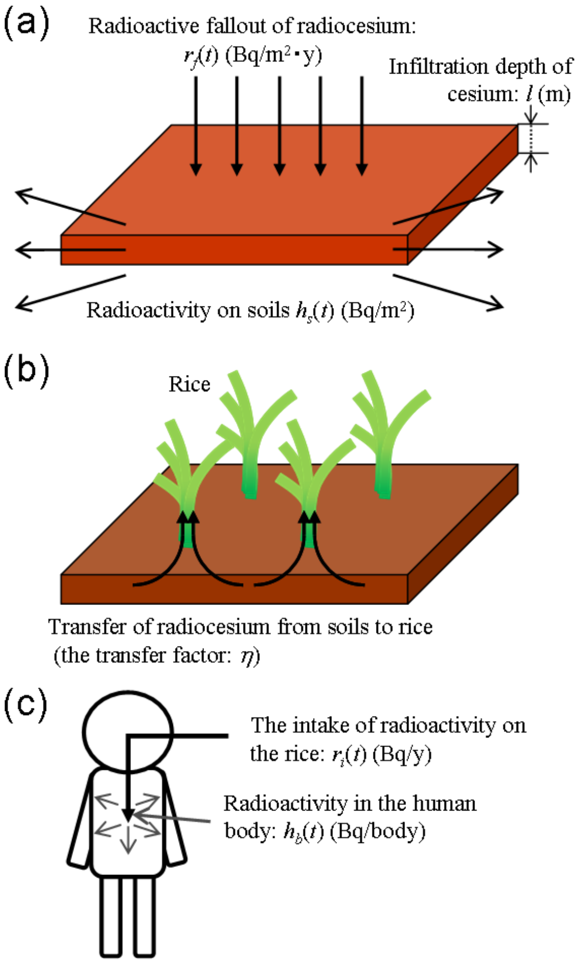

Fig 4.1: Schematic diagrams of the radiocesium transfer, (a) from the radioactive fallout to the soils, (b) from the soils to the rice and (c) the intake of radioactivity on the rice.

4.2.2 The attenuation of radioactivity on the soils

In the followings, we consider the procedure of the calculation. Fig.4.2 shows the flowchart to calculate radioactivity in the human body from the radiocesium fallout through the soils and the rice. At first, we can know the fallout of radiocesium from the above mentioned data [61]. The intensity of radioactivity per area on the soils $h_s(t)$ is calculated from the residence half-time T_r, which is defined as the time the intensity of radioactivity on the soils decreases by half, and so: $h_s(T_r) = h_s(0)/2$. The specific value of T_r is referenced previous studies [66–69].

The existence of residence half-time suggests that intensity of radioactivity per area on the soils decreases exponentially. We can also say that runoff is linear with respect to the storage. The authors have studied the method to calculate intensity of radioactivity in the human body by solving first order linear ordinary differential equation. Here, we suggests that intensity of radioactivity on the soils also can be described by first order linear ordinary differential equation. This is the aim of this paper, we attempt to expand the subject of our method from the human body to the soils.

Recall that again, intensity of radioactivity per area on the soils decreases exponentially and runoff is linear with respect to the storage. Fig.4.2(a) gives the relationship of total

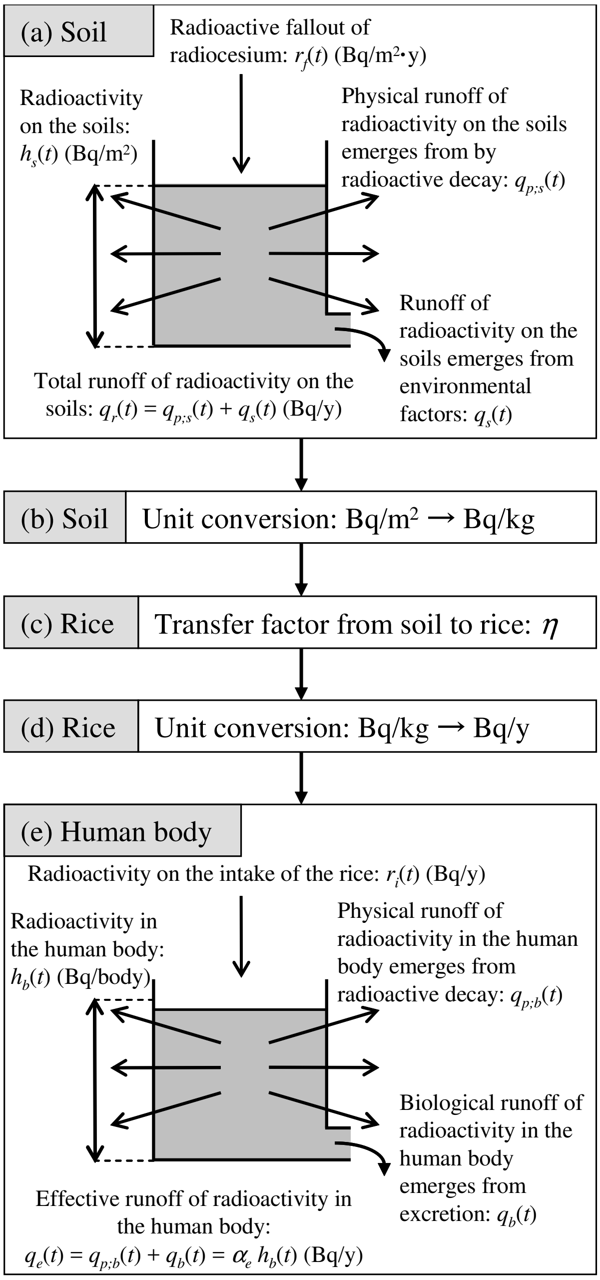

Fig 4.2: The flowchart to calculate radioactivity in the human body from the radiocesium fallout through the soils and rice.

runoff $q_r(t)$ with physical runoff $q_{ps}(t)$ and runoff by environmental factors on the soils $q_s(t)$ as:

$$q_r(t) = q_{ps}(t) + q_s(t) = (\alpha_p + \alpha_s)h_s(t) = \alpha_r h_s(t), \tag{4.2}$$

where α_s and α_r are attenuation constant on the soil and residence attenuation constant. That is, the relationship between residence half-time T_r, physical half-life T_p, and ”half-life on the soils” (named here for convenience) T_s is:

$$\frac{1}{T_r} = \frac{1}{T_p} + \frac{1}{T_s} \tag{4.3}$$

as with the human body shown in Eq.(2.7).

From Fig.4.2(a), we can denote the continuity equation of radioactivity in the soils as:

$$\frac{dh_s(t)}{dt} = r_f(t) - q_r(t), \tag{4.4}$$

From Eq.(4.2), Eq.(2.4) is rewritten as:

$$\frac{dh_s(t)}{dt} = r_f(t) - \alpha_r h_s(t), \tag{4.5}$$

that is a first order linear ordinary differential equation. We can get the solution of Eq.(4.5) analytically as:

$$h_s(t) = \int_0^t e^{-\alpha_r(t-\tau)} r_f(\tau) d\tau + h_{s0} e^{-\alpha_r t} \tag{4.6}$$

under the initial condition $h_s(0) = h_{s0}$.

4.2.3 The vertical infiltration of cesium in the soils

We discuss the vertical infiltration of cesium in the soils. It reasons the unit conversion from intensity of radioactivity per area (Bq/m^2) to the concentration of radioactivity in the soils (Bq/kg). It is reported that cesium remains the shallow topsoils [70, 71]. If we can know the infiltration depth of cesium on the topsoils l (m) and the unit weight of the soils ρ (kg/m^3), then we can obtain the concentration of radioactivity in the soils as:

$$\begin{array}{c}\text{The concentration of radio-}\\\text{activity in the soils}\end{array} (\text{Bq/kg}) = \frac{h_s(t)\ (\text{Bq/m}^2)}{\rho\ (\text{kg/m}^3)\ l\ (\text{m})}. \tag{4.7}$$

Eq.(4.7) describes the unit conversion from intensity of radioactivity per area (Bq/m^2) to the concentration of radioactivity in the soils (Bq/kg).

4.2.4 Transfer factor of radiocesium from the soils to polished rice

From the definition of the transfer factor Eq.(4.1),

$$\begin{array}{c}\text{The concentration of radio-}\\\text{activity in the polished rice}\end{array} (\text{Bq/kg}) = \eta \frac{h_s(t)}{\rho l}. \tag{4.8}$$

The specific value of η is referenced previous studies [63–65, 72].

4.2.5 The annual intake of radioactivity on the rice

If we can know the average annual rice consumption per capita of Japanese adults κ (kg/y), then we can obtain the annual intake of radioactivity on the rice $r_i(t)$ as:

$$r_i(t) = \kappa\eta \frac{h_s(t)}{\rho l}\ (\text{Bq/y}). \tag{4.9}$$

Eq.(4.9) is simplified as:

$$r_i(t) = \lambda h_s(t), \tag{4.10}$$

where $\lambda = \kappa\eta/\rho l$. Thus, the annual intake of radioactivity on the rice $r_i(t)$ is proportional to radioactivity on the soils $h_s(t)$ for the proportionality coefficient λ (m^2/y).

Table 4.1: The parameters regarding T_r, l, ρ, η and κ used for specific calculation

The parameter	Symbol	The value
Residence half-time	T_r	18.1 y
Infiltration depth of cesium on the topsoils	l	0.03 m
Unit weight of the soils	ρ	2000 kg/m^3
Transfer factor from the soils to rice	η	0.01, 0.1, 0.4
Average annual rice consumption per capita of Japanese adults	κ	60 kg/y

4.2.6 Mathematical description of transfer from radioactive fallout to radioactivity on polished rice

Our previous study describes the intensity of radioactivity in the human body $h_b(t)$ (Bq/-body) by the following continuity equation:

$$\frac{dh_b(t)}{dt} = r_i(t) - \alpha_e h_b(t), \tag{4.11}$$

Solving Eq.(4.11), we can get $h_b(t)$ analytically as a function of time t as:

$$h_b(t) = \int_0^t e^{-\alpha_e(t-\tau)} r_i(\tau) d\tau \tag{4.12}$$

under the initial condition $h_b(0) = 0$.

$$h_b(t) = \lambda \int_0^t \left(e^{-\alpha_e(t-\tau)} \int_0^\tau r_f(\tau') e^{-\alpha_r(\tau-\tau')} d\tau' + h_{s0} e^{-\alpha_r \tau} \right) d\tau \tag{4.13}$$

4.2.7 Specific parameters regarding the transfer of radiocesium from fallout to the rice for internal dose calculation

Table 4.1 shows the parameters regarding T_r, l, ρ, η and κ used for concrete calculation. The parameters are justified by the below references. Residence half-time T_r=18.1 (year) is average across Japan [69]. It is employed as a typical example. Tanihata [71] suggests that the depth which the radioactivity of radiocesium decreases to one-tenth is 31 mm, and so we employ the infiltration depth of cesium on the topsoils as l =0.03 m. The unit weight of the soils ρ=2000 kg/m^3 is represent value for convenience. For example, Braja [73] notes that The unit weight of the soils is 1970.3 kg/m^3. It is known that the transfer factor from the soils to rice η is fluctuated due to the environmental factors [63–65]. National Institute for Agro-Environmental Sciences reports that the range of transfer factor is $3 \times 10^{-4} \leq \eta \leq 0.18$ [72], and the average is 2.6×10^{-3}. Here, we assume η=0.01, 0.1, 0.4, as plausible values. In the case of η=0.4, it is larger than double of the maximum measurement value. We can say that we estimate sufficiently more unsafe dose when η=0.4. The average annual rice consumption of Japanese adults $\kappa$is referenced Abdullah [74], set to be 60 kg/y.

Then, the proportionality coefficient λ is branched as:

$$\lambda = \begin{cases} 0.01 \ \ \mathrm{m}^2/\mathrm{y} & (\eta = 0.01) \\ 0.1 \ \ \mathrm{m}^2/\mathrm{y} & (\eta = 0.1) \\ 0.4 \ \ \mathrm{m}^2/\mathrm{y} & (\eta = 0.4) \end{cases} . \tag{4.14}$$

4.3 Results and Discussion

4.3.1 The calculation of radioactivity on the soil using the measured fallout

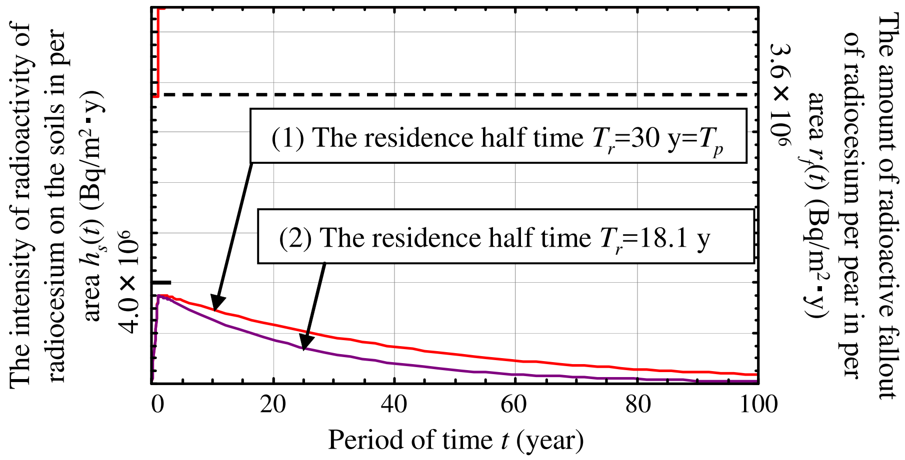

Fig 4.3: The time series of the annual amount of radiocesium fallout and the intensity of radioactivity per area on the soil $h_s(t)$.

Fig.4.3 shows the time series of the annual amount of radiocesium fallout and the intensity of radioactivity per area on the soil. The annual amount of radioactive fallout is estimated by the summation of measured monthly amount of fallout [61]. It should be noticed that the annual amount of fallout after the April 2014 (i.e. $t > 3$) is extrapolated. Fig.4.3 suggests that almost fallout had fallen with in 1 year after the accident.

Fig.4.3 (1) shows the intensity of radioactivity per area on the soil $h_s(t)$ in the case of the residence half-time $T_r = T_p = 30$ y, that is, the case runoff emerges from environmental factors are neglected. It is the improbable case realistically. We can also say Fig.4.3 (1) is maximum intensity of radioactivity.

Fig.4.3 (2) shows the intensity of radioactivity per area on the soil $h_s(t)$ in the case of the residence half-time $T_r = 18.1$ y. It is the average of Japan [69] as mentioned above. Here, we employ the value as a typical example. In the followings, we discuss using the solution of the intensity of radioactivity per area on the soil $h_s(t)$ shown in Fig.4.3 (2).

4.3.2 Radioactivity in the human body caused by the intake of radiocesium on the polished rice

The relationship between $h_s(t)$ with $r_i(t)$ is described in Eq.(4.10). Substituting Eq.(4.10) into Eq.(4.11), we can get the intensity of radioactivity in the human body $h_b(t)$. Fig.4.4 shows the time series of $h_b(t)$ according to the transfer factor η. It should be noted that we calculate $h_b(t)$ in the case of there is no artificial decontamination.

In the next section, we calculate the effective dose based on the result shown in Fig.4.4.

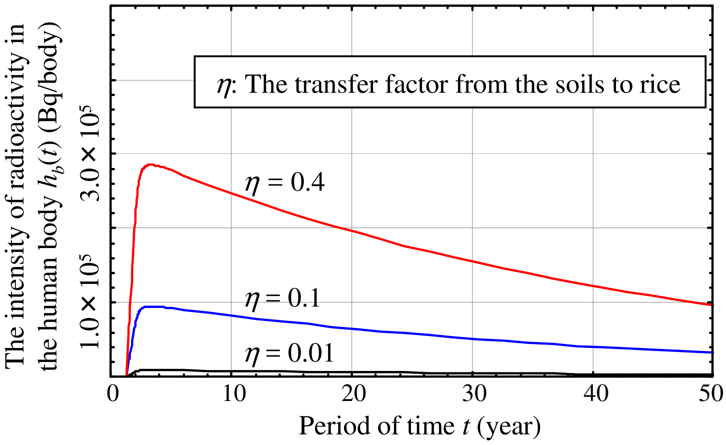

Fig 4.4: The time series of the intensity of radioactivity in the human body $h_b(t)$ according to the transfer factor η.

4.3.3 The effective dose calculation due to the intake of radioactivity on the rice derived from fallout

Substituting the solution of Eq.(4.11) into Eq.(2.21), cumulative number of radioactive decay $I(t)$ is obtained. Subsequently, substituting $I(t)$ into Eq.(3.9), we can get the effective dose $H_{\mathrm{eff}}(t)$. Fig.4.5 shows the time series of the effective dose according to η. If η=0.4, the dose rate is 5.8 mSv/y. We should notice that this calculation excluding artificial decontamination activity completely. It should be noted that the rice exceeds the standard limit for radiocesium in Japan (100 Bq/kg) is not on the market.

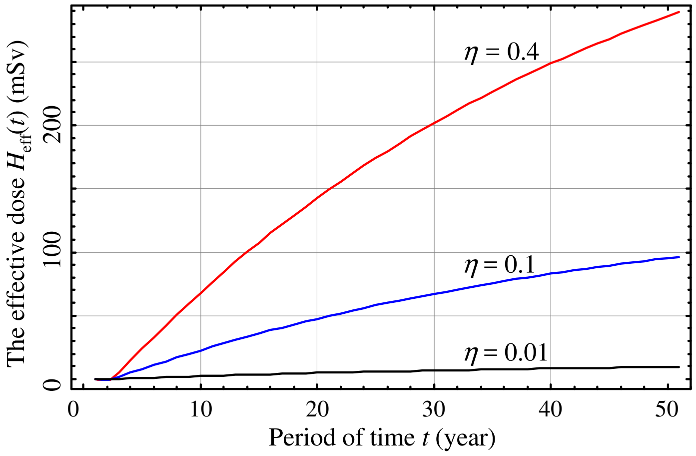

Fig 4.5: The time series of the effective dose according to the transfer factor η.

For reference to compare, the United Nations Scientific Committee on the Effects of Atomic Radiation (UNSCEAR) states that the total exposure dose due to natural radiation (world average) is 2.4 mSv/y [59]. UNSCEAR also noted that a dose rate less than 100 mSv/y indicates a linear no-threshold model [60]. It is said that the dose rate does not provide a significant indication of cancer risk [47]. We can say that the order of dose rate caused by the intake of radioactivity on the rice is same with those of natural radiation in the case of η=0.4.

4.4 Conclusion

In this chapter, we calculate the effective dose due to the intake of radioactivity on agricultural products derived from fallout. The scope of our research is expanded from the human body to the soils. Radioactivity on the soils is described as a single vessel as with the human body. Selecting rice as a typical example of agricultural products, we discuss the transfer of radiocesium from the soils to the rice based on previous studies. Thus, we calculate the effective dose due to the intake of radioactivity on agricultural products derived from fallout according to our previously proposed method. The results are summarized below.

1. Radioactivity on soils is computed hydrologic-based single vessel as with the human body. Radioactivity on soils is described by the continuity equation which includes radioactive fallout and residence half-time. We use actual measurement of radioactive fallout after the Fukushima Daiichi NPP accident. As with the effective half-life in the human body, residence half-time is determined by physical half-life and half-life on the soils.

2. Transfer of radiocesium from the soils to rice is modeled taking account for the vertical filtration of cesium in the soils, transfer factor of cesium from the soils to rice, and average annual rice consumption per capita. The annual intake of radioactivity on rice is proportionally corresponded with radioactivity in the soils.

3. By using the annual intake of radioactivity on rice obtained in that way, we compute the effective dose due to the intake of radioactivity on agricultural products derived from fallout. In most cases, the dose rate is less than natural radiation. Even if the transfer factor η is larger than double of the maximum measure value, the order of the dose rate is same with those of natural radiation. It should be noted that we calculate the effective dose due to the intake of the rice in the case of there is no artificial decontamination.

Thus, we studied radioactivity in the soils as an example of the application of the present hydrologic-based internal dose calculation in this chapter. We suggest that the methodology proposed in this article has a potential to discuss the behavior of radioisotopes and internal dose in various applications because of the simplicity of the fundamental equation.

5 Stochastic calculus of the internal dose calculation

5.1 Introduction

Up to previous chapters, we consider one-to-one correspondence between ingestion of radioactivity with intensity of radioactivity and the effective dose. That is, for any input, output is determined uniquely. The average behavior of the ensemble can be grasped enough by above method. However, we are dissatisfied with deterministic discussion. Radioactivity in the human body should be indeed distributed by individuals because of difference in physical or biological factors, for example different intake of radioactivity and metabolism. In fact, Hayano et al. studied the radiocesium body burden distribution by using whole-body counter in Fukushima prefecture after the Fukushima Daiichi NPP accident [77]. If we would like to discuss the probability density function of radioactivity in the human body, we shall introduce stochastic viewpoint including random fluctuation. We suggest that radioactivity in the human body not only deterministically but also stochastically considering the uncertainty and randomness.

In this chapter, we discuss the probability distribution of radioactivity in the human body and the effective dose based on stochastic calculus. We discuss the applications of stochastic calculus of the internal dose calculation based on lumped hydrologic method in the body of this thesis. We will study the fundamental and detailed theory of stochastic calculus in Appendix 5, major subjects are only discussed in the body of the thesis.

We introduce the stochastic process for internal dose calculation including random fluctuation. The Wiener process is typical and standard stochastic process to know the Gaussian-white random fluctuation such as Brownian motion [75,76]. We treat the random fluctuation of radioactivity in the human body as white noise as presupposition for simplicity. It is known that the stochastic fluctuations can be described by three kinds of formalisms, that is, the Langevin equations, Stochastic differential equations (SDEs), and the Fokker-Planck equation. These three formalisms are equivalent, they can be converted optionally each other (the equivalence is discussed in Appendix 5). We review the fundamental forms of these three formalisms. Based on each fundamental forms, we consider the application to radioactivity in the human body. The Langevin equation is adopted as the starting point of discussion adding the random fluctuation term to continuity equation. The probability density function of radioactivity in the human body is computed based on the Fokker-Planck formalism.

The effective dose can be computed by time integral of radioactivity in the human body. While the effective dose can be calculated uniquely from radioactivity in the human body, the inverse operation is non-unique cause of integral operation. In that reason, it seems to be difficult that we denote the effective dose by the stochastic process from the complete mathematical operation. As an alternative strategy, we propose the stochastic process of the effective dose which satisfies the convergence in the average. In the same way with radioactivity in the human body, we compute the probability density function of the effective dose by solving the Fokker-Planck equation.

While the previously deterministic methodology studied in chapter 2-3 is enough to plan radiological protection for the public, the probability distribution such as shown by Hayano et. al. [77] is also practically important to know actual internal dose. The method proposed in this article will provide stochastic viewpoint for radiological protection based on lumped hydrologic method. Although the main subject of this article is radiation exposure, the hydrologic-based methodology proposed in this article has a potential to discuss general hydrological phenomena including uncertainty and randomness.

5.2 Theory and method

5.2.1 The Langevin equation which describes fluctuation of radioactivity in the human body

First, we review the fundamental form of the Langevin equation, after that, we attempt to apply the Langevin equation to radioactivity in the human body.

The fundamental form of the Langevin equation is described as in the following:

$$\frac{d\dot{x}(t)}{dt} = -\gamma\dot{x}(t) + \xi(t), \tag{5.1}$$

where the second term of rhs of Eq.(5.1); $\xi(t)$ stands for the random fluctuating term.

The Langevin equation was originally introduced to discuss random motion of minute particles in a fluid [78]. The Langevin equation, however, can describe various phenomena which includes random fluctuation and noise. For example, the random fluctuation of electric charge of capacitor in an electric circuit can be discussed by the Langevin equation [79] which describes the conservation law of electric charge and Johnson-Nyquist noise (random fluctuation of electric charge caused by the agitation of electrons). In our hydrologic-based internal dose calculation, we compute the continuity equation of radioisotopes by analogy with lumped hydrological method. Because of the similarity of the fundamental equations, it is natural idea that we attempt to apply the Langevin equation to radioactivity in the human body when we take account for the random fluctuation of the intake of radioisotopes.

Hence, we will study the application of the Langevin equation to radioactivity in the human body. The continuity equation of radioactivity in the human body is again described as:

$$\frac{dh(t)}{dt} = r(t) - \alpha h(t) \tag{5.2}$$

as with chapter 2.

Adding the random fluctuation term $\xi(t)$ to Eq.(5.2), then we obtain

$$\frac{dh(t)}{dt} = r(t) + \xi(t) - \alpha h(t). \tag{5.3}$$

Eq.(5.3) is the Langevin equation of radioactivity in the human body. As shown in Fig.5.1, the Langevin equation which denotes conservation relationship of radioactivity in the human body (Eq.(5.3)) is schematically described by a single vessel as with lumped hydrological method.

We scrutinize Eq.(5.3). Let be intake of radioactivity $r(t)$ and $\xi(t)$

$$r(t) = \begin{cases} \langle r \rangle & (0 \le t \le t_m) \\ 0 & (t > t_m) \end{cases} \tag{5.4}$$

$$\xi(t) = \begin{cases} \eta(t) & (0 \le t \le t_m) \\ 0 & (t > t_m) \end{cases}, \tag{5.5}$$

where $\langle \cdot \rangle$ stands for the average. Here, the average intake of radioactivity $\langle r \rangle$ is treated an any constant. Eqs.(5.4) and (5.5) state that while there is intake of radioactivity including random fluctuation until the time t_m, there is not intake of radioactivity after the time t_m.

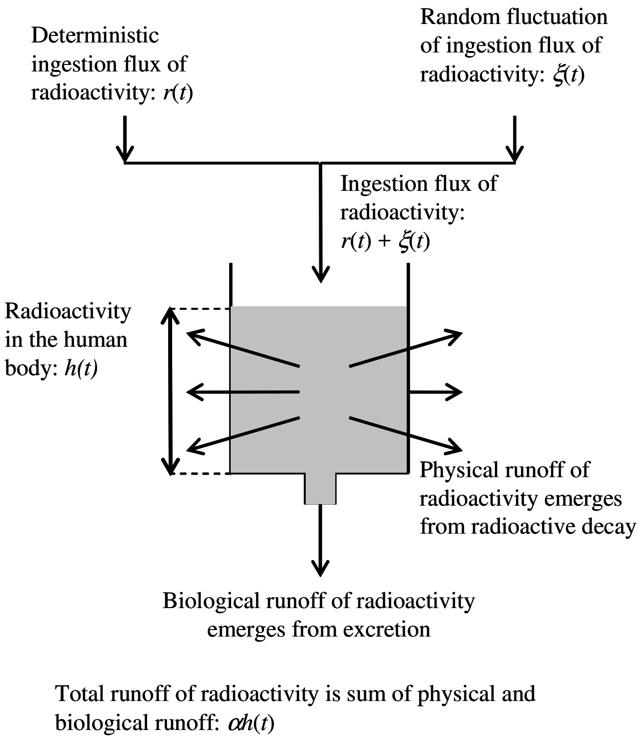

Fig 5.1: A single vessel similar to lumped hydrological method which describes the conservation relationship of radioactivity in the human body including random fluctuation of intake of radioactivity. It is denoted by the Langevin equation of radioactivity in the human body as Eq.(5.3).

The fluctuation term $\eta(t)$ satisfies

$$\langle \eta(t) \rangle = 0, \tag{5.6}$$

and

$$\langle \eta(t)\eta(t') \rangle = \frac{2D\gamma}{\mu}\delta(t-t'), \tag{5.7}$$

where D and μ are a diffusion constant and the mobility respectively[4]. Eq.(5.6) states that the fluctuations cancel each other. Eq.(5.7) states that the fluctuations are time independent. Eqs.(5.6) and (5.7)are important properties of random fluctuation term, that is discussed again from the different viewpoints later.

Intake of radioactivity described by Eqs.(5.4-5.7) is shown in Fig.5.2. Intake of radioactivity is distributed obeying to normal distribution with the average $\langle r \rangle$ and the standard deviation $2D\int_0^t d\tau(1-\frac{1}{\gamma}e^{-\gamma t})^2 = \frac{D}{2\gamma}(1-e^{-2\gamma t})$ (Section 5.3.4 discussed the variance in detail.). Fig.5.2 states that we cannot know pulse-like intake of radioactivity of each time deterministically, however, we can know the probability distribution.

When Eqs(5.4) and (5.5) are satisfied, then the Langevin equation of radioactivity in the human body is denoted by:

$$\frac{dh(t)}{dt} = \begin{cases} \langle r \rangle - \alpha h(t) + \eta(t) & (0 \leq t \leq t_m) \\ -\alpha h(t) & (t > t_m) \end{cases}. \tag{5.8}$$

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[4] $v = \dot{x} = \mu f$, where f is external force.

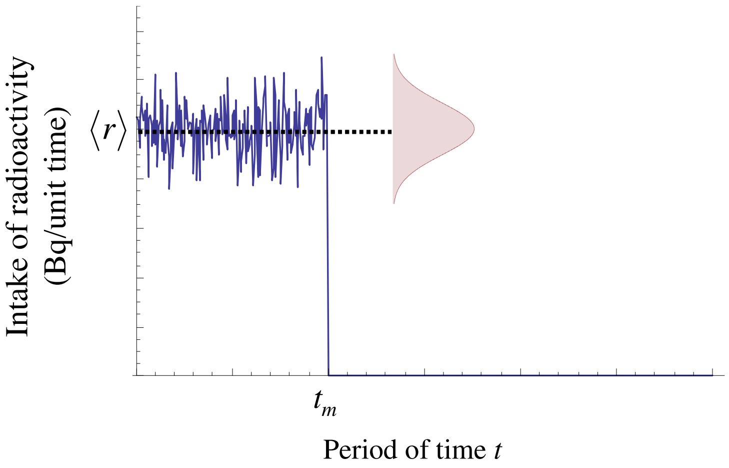

Fig 5.2: Intake of radioactivity including random fluctuation.

Recall that in case of there is no random term, then the differential equation can be solved elementally. The non-random solution can be regarded as the average of a lot of random paths because the random term $\xi(t)$ is white-Gaussian noise with the average zero. We can also regard the non-random solution as the deterministic solution shown in Eq.(2.26).

That is, we can put on par with non-random solution, the average, and the deterministic solution. These relationship is denoted by:

$$\frac{d\,\langle h(t)\rangle}{dt} = r(t) - \alpha\,\langle h(t)\rangle\,, \tag{5.9}$$

random fluctuation term is vanished by averaging. It is difficult to know behavior of the individual samples including random fluctuation, however, it is easy to know the average behavior of a lot of samples. This fundamental idea is critically important to discuss the phenomena which have randomness and uncertainty.

Solving Eq.(5.9), we obtain

$$\langle h(t)\rangle = \begin{cases} \dfrac{\langle r\rangle}{\alpha}(1-e^{-\alpha t}) & (0 \leq t \leq t_m) \\ \dfrac{\langle r\rangle}{\alpha}(-1+e^{\alpha t_m})e^{-\alpha t} & (t > t_m) \end{cases}. \tag{5.10}$$

Eq.(5.10), indeed, agrees with deterministic solution shown in Eq.(2.26) which calculates radioactivity in the human body deterministically.

5.2.2 The stochastic differential equation of radioactivity in the human body

As with the Langevin equation, first we review the fundamental form of the SDE.

The theory of Itô calculus [80–83] allows us to denote the Langevin equation, Eq.(5.1), by the stochastic integral equation as:

$$x(t) = \int_0^t \mu(\tau, x(\tau)) + \sqrt{2D}\int_0^t dW(t), \tag{5.11}$$

where μ corresponds with the average $\langle x(t)\rangle$ and $W(t)$ stands for the standard Wiener process respectively. Wiener process is the typical mathematical method describing random fluctuation, we introduce the Wiener process to discuss random fluctuation of radioactivity in the human body.

In Eq.(5.11), the random term treated as the functional integration of the Wiener process $W(t)$. It should be noted that while the first term of rhs of Eq.(5.11) is the ordinary Riemann integral, the second term of rhs of Eq.(5.11) is the Itô integral.

Expressing Eq.(5.11) in the differential form, we obtain the SDE as:

$$dx(t) = \mu(t, x(t)) + \sqrt{2D}dW(t). \tag{5.12}$$

The Wiener process $W(t)$ satisfies $W(0) = 0$ and $W(t) \sim N(0, t)$, where $N(\mu, \sigma^2)$ stands for the normal distribution with average μ and variance σ^2. We can interpret that the Wiener process $W(t)$ describes the probability density function is the normal distribution for a particle which exists at the origin point at the time $t = 0$. It is noteworthy that the Wiener process represents Fick's law of distribution and Fourier's heat conduction law. That is, the standard deviation of the probability density function is proportional to $\sqrt{t}$.

We cannot differentiate $W(t)$ everywhere, but the infinitesimal change $dW(t)$ can be calculated obeying Ito's rules;

$$\{dW(t)\}^2 = dt \text{ and } dW(t)dt = (dt)^2 = 0. \tag{5.13}$$

It is natural idea that we neglect second or more order terms of infinitesimal changes. Therefore, the Ito's rule, $\{dW(t)\}^2 = dt$, seems to be irregular. The rule, however, enables to describe consistently diffusion process of which standard deviation is proportional $\sqrt{t}$.

Up to here, we reviewed foundation of SDE formalism. Next, we attempt to apply the SDE to radioactivity in the human body. Expressing the Langevin equation, Eq.(5.8), with SDE, we obtain:

$$dh(t) = \begin{cases} [r - \alpha h(t)]dt + \sqrt{2D}dW(t) & (0 \leq t \leq t_m) \\ -\alpha h(t)dt & (t > t_m) \end{cases}. \tag{5.14}$$

Eq.(5.14) is the SDE of radioactivity in the human body.

5.2.3 The Fokker-Planck equation and the probability density function of radioactivity in the human body

The Langevin equation and SDE provide equivalent Fokker-Planck equation for the probability density function. The standard form of the Fokker-Planck equation is in the following:

$$\frac{\partial p(t, X)}{\partial t} = -\frac{\partial}{\partial X}[\mu(t, X)p(t, x)] + D\frac{\partial^2 p(t, X)}{\partial X^2}, \tag{5.15}$$

which corresponds to Eq.(5.1) (Langevin formalism) and Eq. (5.12) (SDE formalism). $p(t, X)$ stands for the probability density function of the random variable X. It is repeatedly emphasized that the Langevin equation, SDE and the Fokker-Planck equation are equivalent each other. They describe the same one from different viewpoints. We should choose appropriate formalism as necessary.

The Fokker-Planck formalism is useful when we would like to get the probability density functions. The aim of this article is to know probability distribution of radioactivity in the human body, therefore, it is reasonable to employ the Fokker-Planck formalism. In this article, we compute the Fokker-Planck equation choosing the Langevin equation as a starting point of discussion.

The Fokker-Planck equation of radioactivity in the human body is described as:

$$\begin{cases} \dfrac{\partial p_h(t,H)}{\partial t} = -\dfrac{\partial}{\partial H}\{[\langle r\rangle - \alpha H]p_h(t,H)\} + D\dfrac{\partial^2 p_h(t,H)}{\partial H^2} & (0 \le t \le t_m) \\ \dfrac{\partial p_h(t,H)}{\partial t} = \alpha\dfrac{\partial}{\partial H}[Hp_h(t,H)] & (t > t_m) \end{cases} \tag{5.16}$$

where $p_h(t, H)$ is the probability density function of radioactivity in the human body, H stands for the random variable of radioactivity in the human body. Capital letters are used to distinguish clearly random variables in this article.

Generally, the Fokker-Planck equation cannot be solved analytically. We can solve the Fokker-Planck equation analytically only in the special cases. Eq.(5.16) cannot be solved at the interval $0 \le t \le t_m$. However, Eq.(5.16) can be solved analytically at the interval $t > t_m$.

Our strategy to compute Eq.(5.16) is numerical calculation at the interval $0 \le t \le t_m$, and we compute Eq.(5.16) analytically at the interval $t > t_m$ using the boundary condition at $t = t_m$ given by above numerical calculation.

The initial condition and the boundary conditions should be $p_h(0, H) = \delta(H)$ and $\lim_{H\to\pm\infty} p_h(t, H) = 0$. However, being numerical calculation, they are substituted to be $p_h(0, H) = N(0, \varepsilon)$ and $p_h(t, \pm\Lambda) = 0$, where $\varepsilon \ll 1$ and $\Lambda \gg r/\alpha$.

5.2.4 The stochastic differential equations and the Fokker-Planck equations of the effective dose

Up to previous section, we discuss the stochastic process and probability density function of radioactivity in the human body, however, we wish we could calculate those of the effective dose from the viewpoint of radiological protection. In this section, we attempt to compute the stochastic process and probability density function of the effective dose from two inferences. Although we cannot derive them by strict mathematical operation, we state the reasons why these two inferences are reasonable and likely.

Radioactivity in the human body $h(t)$ is the number of radioactive decay per unit time, hence, cumulative number of radioactive decay $\varphi(t)$ is defined by time integral of radioactivity in the human body as:

$$\varphi(t) = \int_0^t h(\tau)d\tau. \tag{5.17}$$

The effective dose $s(t)$ is proportional to cumulative number of radioactive decay as:

$$s(t) = \beta\varphi(t), \tag{5.18}$$

where β is a constant which divides absorbed energy per one radioactive decay $\langle E\rangle$ by body weight ($\beta = \langle E\rangle$ (MeV)/weight (kg)=[J/kg] = [Sv]). The proportional constant β is treated to depend on only the kind of nuclides.

When $h(t)$ is not including random term, in other words deterministic, we can compute $\varphi(t)$ and $s(t)$ elementary. It is difficult, however, to calculate the stochastic process and the probability density function of $\varphi(t)$ and $s(t)$ in case of $h(t)$ includes the random term. This is solely derived from $\varphi(t)$ cannot map $h(t)$ in one-to-one correspondence because of the integral operation. For any value of $\varphi(t)$, countless paths of $h(t)$ are allowed to exist, therefore, we cannot uniquely determine correspondence between $h(t)$ with $\varphi(t)$. It will be difficult to denote $\varphi(t)$ in the SDE from by:

$$d\varphi(t) = \mu_\varphi(t) + \sqrt{2D_\varphi(t)}dW(t). \tag{5.19}$$

However, convergence in average is clue to consider the stochastic process of $\varphi(t)$. Repeatedly saying, the average agrees with non-random solution and deterministic solution. We can denote the relationship by:

$$\langle\varphi(t)\rangle = \int_0^t \langle h(\tau)\rangle \, d\tau, \tag{5.20}$$

in the sense of convergence in the average. That is, we can compute $\langle\varphi(t)\rangle$ as:

$$\langle\varphi(t)\rangle = \begin{cases} \dfrac{r}{\alpha^2}(\alpha t - 1 + e^{-\alpha t}) & (0 \le t \le t_m) \\ \dfrac{r}{\alpha^2}[\alpha t_m + e^{-\alpha t}(1 - e^{\alpha t_m})] & (t > t_m) \end{cases}, \tag{5.21}$$

by substituting Eq.(5.10) into Eq.(5.20). Eq.(5.21) indeed agrees with deterministic path of $\varphi(t)$ shown in Eq.(2.29), which calculates the cumulative number of radioactive decay deterministically.

The non-random solution should agree with the average, hence, μ_φ in Eq.(5.19) should be

$$\mu_\varphi = \langle h(t)\rangle \, . \tag{5.22}$$

Next, we attempt to redefine the random term in Eq.(5.19). There is a suggestive example. The Wiener process is non-differentiable, but integrable as:

$$\int_0^t W(\tau)d\tau = \int_0^t (t-\tau)dW(\tau) \sim N(0, t^3/3). \tag{5.23}$$

Corresponding stochastic process to Eq.(5.23) is $tdW(t)$. So, we may denote random term in Eq.(5.19) by:

$$D_\varphi = DtdW(t). \tag{5.24}$$

In Eq.(5.19), the deterministic term $\mu_\varphi dt$ and the random term $\sigma_\varphi dW(t)$ are independent. Therefore, we propose as the candidate SDE of $\varphi(t)$ in the following:

$$d\varphi(t) = \langle h(t)\rangle \, dt + \sqrt{2D}tdW(t). \tag{5.25}$$

As discussed above, Eq.(5.25) satisfies the convergence in the average and the integral of Wiener process. These two property are enough to be trustworthy that Eq.(5.25) is not a bad guess as the SDE of $\varphi(t)$.

The effective dose is proportional to $\varphi(t)$, the SDE is denoted by:

$$ds(t) = \beta d\varphi(t) = \beta[\langle h(t)\rangle]dt + \sqrt{2D}tdW(t)]. \tag{5.26}$$

Substituting Eqs.(5.4) and (5.5) into Eq.(5.26), we obtain

$$ds(t) = \begin{cases} \beta[\langle h(t)\rangle \, dt + \sqrt{2D}tdW(t)] & (0 \le t \le t_m) \\ \beta \, \langle h(t)\rangle \, dt & (t > t_m) \end{cases}, \tag{5.27}$$

Eq.(5.27) is the SDE of the effective dose.

Corresponding Fokker-Planck equation of the effective dose is denoted by

$$\begin{cases} \dfrac{\partial p_s(t,S)}{\partial t} = -\dfrac{\partial}{\partial S}\left[\beta \left\langle h(t)\right\rangle p_s(t,S)\right] + Dt^2\beta \dfrac{\partial^2 p_s(t,S)}{\partial S^2} & (0 \le t \le t_m) \\ \dfrac{\partial p_s(t,S)}{\partial t} = -\dfrac{\partial}{\partial S}\left[\beta \left\langle h(t)\right\rangle p_s(t,S)\right] & (t > t_m) \end{cases}, \tag{5.28}$$

where S and $p_s(t,S)$ are the random variable and the probability density function of the effective dose.

As with Eq.(5.16), we compute Eq.(5.28) numerically at the interval $0 \le t \le t_m$. They can be solved analytically at the interval $t > t_m$, only the boundary condition at $t = t_m$ is given the numerical calculation.

5.2.5 Two variables Fokker-Planck equation

We consider the simultaneous stochastic differential equations in the followings [5]:

$$\begin{cases} dx = ydt \\ dy = (-\gamma y + f(t))dt + \sqrt{2D}dW(t) \end{cases}. \tag{5.29}$$

Eq.(5.29) is denoted in the vector notation by:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dW(t) \tag{5.30}$$

Eq.(5.30) is denoted in tensor notation by:

$$dx^i(t) = A^i_j x^j(t)dt + B^i dt + C^i dW(t) \quad (i,j = 1,2), \tag{5.31}$$

where $x^i(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $A^i_j = \begin{pmatrix} 0 & 1 \\ 0 & -\gamma \end{pmatrix}$, $B^i = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$, and $C^i = \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix}$, respectively. Indexes in Eq.(5.31) is summed according to the Einstein convention.

Corresponding Fokker-Planck equation with Eq.(5.31) is denoted by:

$$\frac{\partial p(t,x^i)}{\partial t} = -\partial_j A^j_k x^k p(t,x^i) - \partial_j B^j p(t,x^i) + \frac{1}{2}\delta^{jk}\partial_j\partial_k C^l C_l p(t,x^i) \tag{5.32}$$

---

[5] The property of random fluctuation $\eta(t)$ (Eqs.(5.6) and (5.7)) and the Wiener process give the average properties of x_1 and x_2

$$\begin{aligned} \langle \Delta x\rangle &= y\Delta t, \\ \langle \Delta y\rangle &= (-\gamma y + f(t))\Delta t, \\ \left\langle (\Delta x)^2\right\rangle &= 0, \\ \langle \Delta x \Delta y\rangle &= 0, \\ \left\langle (\Delta y)^2\right\rangle &= 2D\Delta t. \end{aligned}$$

The Einstein's relation for diffusion coefficient $D = \mu k_b T = \dfrac{\Gamma}{\gamma^2}$ is used. When x diffuses with diffusion constant D and fluctuation $\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$, then y diffuses with diffusion constant DS.

That is, the Fokker-Planck equation corresponding with Eq.(5.31) is denoted by

$$\begin{aligned}\frac{\partial p(t,X,Y)}{\partial t} =& -Y\frac{\partial p(t,X,Y)}{\partial X} - \frac{\partial}{\partial Y}\{(-\gamma Y + f(t))p(t,X,Y)\} \\ &+ D\frac{\partial^2 p(t,X,Y)}{\partial Y^2}.\end{aligned} \tag{5.33}$$

Eq.(5.33) is the two variables Fokker-Planck equation. Integrating Eq.(5.33) by X, we have the Fokker-Planck equation which yields probability distribution function of Y. Conversely, we have the Fokker-Planck equation which yields the probability distribution function of X by integrating Eq.(5.33) by Y.

We discuss to apply two variables Fokker-Planck equation Eq.(5.33) to radioactivity in the human body and effective dose. Eq.(5.18) gives

$$ds(t) = \beta h(t)dt. \tag{5.34}$$

Taking simultaneous equations with Eq.(5.34) and Eq.(5.14), we have the simultaneous SDEs of radioactivity in the human body and effective dose as:

$$\begin{cases} ds(t) = \beta h(t)dt \\ dh(t) = (r(t) - \alpha h(t))dt + \sqrt{2D}dW(t) \end{cases}. \tag{5.35}$$

The Fokker-Planck equation of radioactivity in the human body and effective dose corresponding with simultaneous SDEs Eq.(5.35) is denoted by:

$$\begin{aligned}\frac{\partial p(t,S,H)}{\partial t} =& -\beta H\frac{\partial p(t,S,H)}{\partial S} - \frac{\partial}{\partial H}\{(r(t) - \alpha H)p(t,S,H)\} \\ &+ D\frac{\partial^2 p(t,S,H)}{\partial H^2}.\end{aligned} \tag{5.36}$$

In principle, Eq.(5.36) yields the probability density function of radioactivity in the human body and effective dose setting aside the difficulty of computation. Being a many variables partial differential equation, it can be calculated analytically only special cases.

5.2.6 Estimation of average intake amount of radioactivity and diffusion constant of radioactivity in the human body

Hayano et al.(2014) showed that the centroid and width of ^{134}Cs body burden distribution are 825 Bq and 480 Bq, four months after the Fukushima Daiichi NPP accident in Fukushima [77]. By using the actual measurement radioactivity in the human body radiocesium, we estimate average intake of radiocesium and diffusion constant of radioactivity in the human body.

The measurement with whole-body counter was carried out between July 11, 2011 and July 29, 2011 [77], that is, between 122-140 days after the Fukushima Daiichi NPP accident. Here, time t is set to be $t = 131$ days, taking intermediate for simplicity of computation. Effective half-life of ^{134}Cs is 95 days, that is $\alpha = \ln 2/T = \ln 2/95$ (day). Substituting $t = 131$ days and $\langle h(t)\rangle = 825$ Bq into Eq.(5.10), we obtain

$$825\ (\mathrm{Bq}) = \frac{\langle r\rangle\ (\mathrm{Bq/day})}{\ln 2/95\ (\mathrm{day})}(1 - e^{-\ln 2/95\ (\mathrm{day})\times 131\ (\mathrm{day})}). \tag{5.37}$$

Computing Eq.(5.37), we obtain average intake of radioactivity as:

$$\langle r \rangle = 9.8 \text{ (Bq/day)}. \tag{5.38}$$

The diffusion constant D and standard deviation σ have a relationship

$$\sqrt{2Dt} = \sigma \tag{5.39}$$

according to the property of a Wiener process as detailed in appendix E.5 (It can be also interpreted as fundamental property of diffusion equation.). The diffusion constant of radioactivity in the human body D is estimated to substitute $t = 131$ days and standard deviation $\sigma = 480$ Bq into Eq.(5.39) as:

$$\sqrt{2D \times 131 \text{ (day)}} = 480 \text{ (Bq)}. \tag{5.40}$$

Computing Eq.(5.40), we have

$$D = 879 \text{ (Bq}^2\text{/day)}. \tag{5.41}$$

5.2.7 Transformation of random variables of the probability density function from effective half-life to radioactivity in the human body and effective dose

Although we have studied the uncertainty and probability distribution of intake of radioactivity up to previous section, we study the probability distribution of effective half-life in this section. The intake of radioactivity r is fixed as a constant for convenience.

As shown in Eqs.(2.26), radioactivity in the human body is saturated as:

$$\lim_{(t,t_m \to \infty,\infty)} h(t) = \lim_{(t,t_m \to \infty,\infty)} \left[\frac{r}{\alpha}(1 - e^{-\alpha t})\right] = \frac{r}{\alpha}. \tag{5.42}$$

Let define saturated radioactivity in the human body h_{sat} as below:

$$h_{\text{sat}} = \frac{r}{\alpha} = \frac{rT}{\ln 2}., \tag{5.43}$$

where T stands for effective half-life.

When we know the probability density function of effective half-life, then we can know the probability density function of h_{sat} according to the transform rule of probability density function. We will study the transformation of the probability density function in the followings.

Let be a monotonic function $y = f(x)$, that is, the variables x and y have an order and one-to-one correspondence. When we have p_x; the probability density function of x, the probability that x exists between an any infinitesimal interval dx is denoted by:

$$p_x dx. \tag{5.44}$$

In the same way, the probability that y exists between an any infinitesimal interval dy is denoted by:

$$p_y dy. \tag{5.45}$$

Being $y = f(x)$ is monotonic, both probabilities must be agreed:

$$p_x dx = p_y dy \tag{5.46}$$

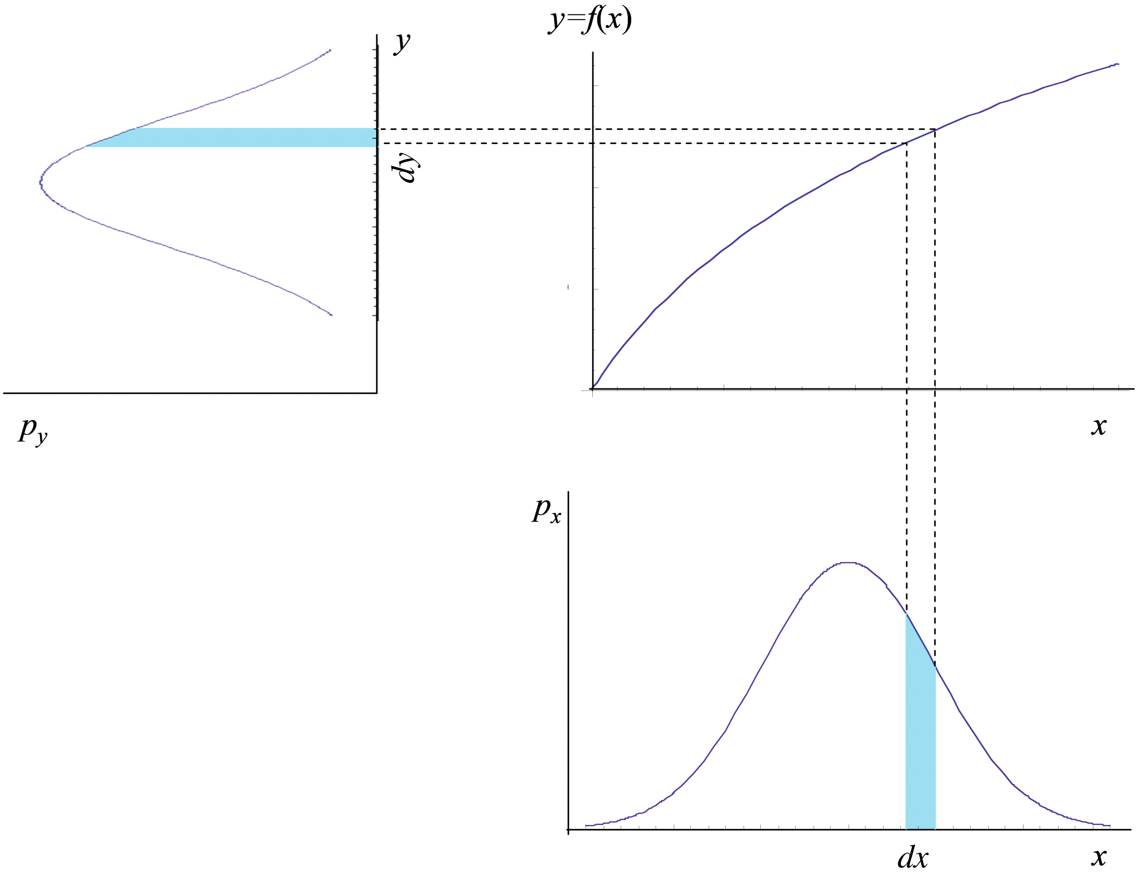

Fig 5.3: Transformation of probability density function from p_x to p_y for a monotonic function $y = f(x)$.

as shown in Fig.5.3. Eq.(5.46) yields

$$p_y(y) = p_x(f^{-1}(y)) \frac{df^{-1}(y)}{dy}, \tag{5.47}$$

where $x = f^{-1}(y)$ stands for an inverse function. Eq.(5.47) is the transformation of random variables of probability density function.

As shown in Eq.(5.43), saturated radioactivity in the human body h_{sat} is monotonically increasing for effective half-life T , hence, the transform rule of probability density function Eq.(5.47) can be applied. When we have the probability density function of the effective half-life $p_T(T)$, then we can obtain the probability density function of saturated radioactivity in the human body $p_{hs}(H_s)$ as:

$$\begin{aligned} p_{hs}(H_s) &= \frac{dh_{\mathrm{sat}}}{dT} p_T(h_{\mathrm{sat}}^{-1}(T)) \\ &= \frac{\ln 2}{r} p_T\left(\frac{\ln 2}{r} H_s\right), \end{aligned} \tag{5.48}$$

where H_s stands for a random variable of saturated radioactivity in the human body. Eq.(5.48) is the transformation of the probability density function from effective half-life to saturated radioactivity in the human body.

Next, we study the probability distribution of effective dose caused by the difference of biological half-life by individuals. Eq.(5.21) states that the effective dose $s(t)$ can be approximated as

$$s(t) = \beta\varphi(t) \simeq \beta\frac{rt_m}{\alpha}, \tag{5.49}$$

in the case of t is sufficiently larger than t_m as $t \gg t_m$.

Let be

$$s_\star = \beta\frac{rt_m}{\alpha} = \beta\frac{rt_m}{\ln 2}T. \tag{5.50}$$

Then the effective dose $s_\star$ is monotonically increasing with respect to effective half-life T, hence, we can take transformation of probability density function according to Eq.(5.47). The probability density function of the effective dose $p_{s_\star}$ is denoted by:

$$p_{s_\star}(S_\star) = \frac{dT}{ds_\star}p_T(T)\Big|_{T=\frac{\ln 2}{\beta rt_m}s_\star} = \frac{\ln 2}{\beta rt_m}p_T\left(\frac{\ln 2}{\beta rt_m}S_\star\right). \tag{5.51}$$

Eq.(5.51) is the transformation of the random variables from effective half-life to effective dose.

In the section 5.3.6, we will study the probability density function of radioactivity in the human body and effective dose by using actual measured distribution of effective half-lives.

5.3 Results and Discussion

5.3.1 The sample paths and the probability density functions of radioactivity in the human body

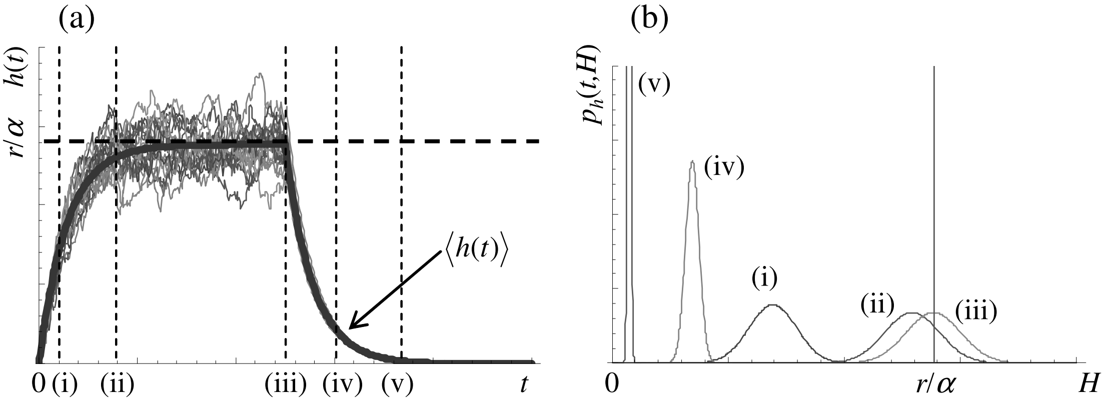

Fig 5.4: (a) The sample paths of radioactivity in the human body $h(t)$ and its average $\langle h(t)\rangle$. (b) The probability density function of radioactivity in the human body $p_h(t, H)$. The Roman numerals stands for the $p_h(t, H)$ for each time t described in (a).

We reproduce the sample paths of radioactivity in the human body by using pseudo-random number generated by the computer. A sample path is one of the possible path of the stochastic process. The stochastic process of radioactivity in the human body is denoted by the SDE formalism in Eq.(5.14).

Fig.5.4 (a) shows the sample paths of radioactivity in the human body. The bold line in Fig.5.4 (a) is the average $\langle h(t)\rangle$ described in Eq.(5.10). The average $\langle h(t)\rangle$ of a large number of sample paths agrees with the deterministic path not including random fluctuation. It should be noted that the agreement is by no means trivial, derived from the properties of Gaussian white noise and the Wiener process focused in this article.

Computing the Fokker-Planck equation of radioactivity in the human body Eq.(5.16), we can get the probability density function of radioactivity in the human body as shown in Fig.5.4 (b). The centroid of $p_h(t, H)$ also obeys the average $\langle h(t)\rangle$. At the interval $0 \leq t \leq t_m$, the centroid of $p_h(t, H)$ approaches r/α, which is the asymptote of $\langle h(t)\rangle$. The centroid of $p_h(t, H)$ is almost stationary at the interval $T_{\text{half}} \ll t < t_m$, the width of $p_h(t, H)$ increases because of random fluctuation. After the end of ingestion of radioactivity $t > t_m$, the centroid of $p_h(t, H)$ decreases to 0 exponentially. The width of $p_h(t, H)$ also decreases monotonically, it is expected that $\lim_{t\to\infty} p_h(t, H) = \delta(H)$. This result agrees with our intuitive understanding that radioactivity in the human body is removed almost surely after sufficient long time has passed from the end of ingestion of radioactivity.

5.3.2 The sample paths and the probability density functions of the effective dose

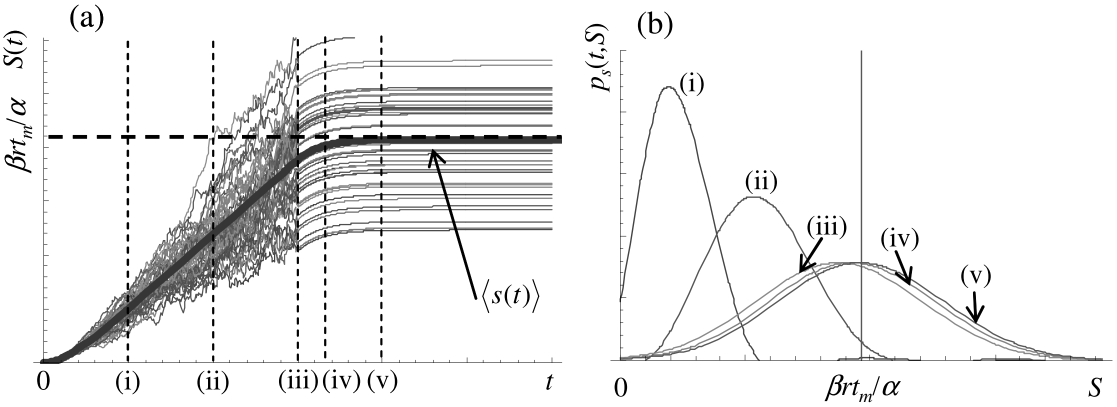

Fig 5.5: (a) The sample paths of the effective dose $s(t)$ and its average $\langle s(t)\rangle$. (b) The probability density function of the effective dose $p_s(t, S)$. The Roman numerals stands for the $p_s(t, S)$ for each time t described in (a).

Fig.5.5 (a) shows the sample paths of the effective dose, corresponding SDE is Eq.(5.26). During the ingestion period $0 \leq t \leq t_m$, paths are distributed randomly around the average (bold line) $\langle s(t)\rangle$. After the end of ingestion of radioactivity $t > t_m$, the paths become stationary thorough smooth increase.

As shown in Fig.5.5 (b), the centroid of the probability density function of the effective dose $p_s(t, S)$ obeys the average $\langle s(t)\rangle$. During ingestion period $0 \leq t \leq t_m$, the width of $p_s(t, S)$ increases faster than that of $p_h(t, H)$ (recall Eq.(5.23)). After the end of ingestion $t > t_m$, the centroid and width of $p_s(t, S)$ become stationary through some increase.

5.3.3 Solution of the linear Fokker-Planck equation (Ornstein-Uhlenbeck process)

We attempt to solve

$$\frac{\partial p(t,X)}{\partial t} = \alpha\frac{\partial X p(t,X)}{\partial X} + D\frac{\partial^2 p(t,X)}{\partial X^2}. \tag{5.52}$$

Eq.(5.52) is the linear Fokker-Planck equation, corresponding stochastic process is called the Ornstein-Uhlenbeck process. Eq.(5.16) at the interval $t > t_m$ agrees with Eq.(5.52) when the diffusion constant $D = 0$. Taking the Fourier transform both sides of Eq.(5.52), we have

$$\frac{\partial \tilde{p}(t,K)}{\partial t} = \alpha \int_{-\infty}^{\infty} dX e^{-iKX} \frac{\partial X p(t,X)}{\partial X} - DK^2 \tilde{p}(t,K). \tag{5.53}$$

where Fourier transform and inverse Fourier transform are defined as:

$$\tilde{p}(t,K) = \int_{-\infty}^{\infty} dX e^{-iKX} p, \tag{5.54}$$

$$p(t,X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dK e^{iKX} \tilde{p}, \tag{5.55}$$

respectively.

The first term of RHS of Eq.(5.53) is computed by integral by parts in the following:

$$\begin{aligned} \alpha \int_{-\infty}^{\infty} e^{-iKX} \frac{\partial X p(t,X)}{\partial X} &= \left[e^{-iKX} \alpha X p(t,x) \right]_{-\infty}^{\infty} + i\alpha K \tilde{p}(t,K) \\ &= - \int_{-\infty}^{\infty} dX e^{-iKX} \frac{\partial}{\partial K} (\alpha K p(t,X)) \\ &= \alpha K \frac{\partial \tilde{p}(t,K)}{\partial K}. \end{aligned} \tag{5.56}$$

Hence, Eq.(5.53) is denoted by:

$$\frac{\partial \tilde{p}(t,K)}{\partial t} = \alpha K \frac{\partial \tilde{p}(t,K)}{\partial K} - DK^2 \tilde{p}(t,K). \tag{5.57}$$

Solving Eq.(5.57) under the initial condition $\tilde{p}(0,K) = e^{iKX_0}$ (i.e. $p(0,X) = \delta(X - X_0)$), we obtain [6]

$$\tilde{p}(t,K) = \exp\left(- \frac{DK^2(1 - e^{-2\alpha t}) - 2i\alpha K X_0}{2\alpha} \right). \tag{5.58}$$

Taking inverse Fourier transform of both sides of Eq.(5.58), we have

$$\begin{aligned} p(t,X) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dK e^{iKX} \tilde{p}(t,K) \\ &= \frac{1}{\sqrt{\pi D (1 - e^{-2\alpha t})/\alpha}} \exp\left(- \frac{(e^{\alpha t} X - X_0)^2 \alpha}{D(-1 + e^{2\alpha t})} \right). \end{aligned} \tag{5.59}$$

Eq.(5.59) is the solution of the linear Fokker-Planck equation.

---

[6] $\frac{d\tilde{p}}{ds} = -\alpha K \frac{\partial \tilde{p}}{\partial K} + \frac{\partial \tilde{p}}{\partial t} = -DK^2 \tilde{p}$, where $\frac{dK}{ds} = -\alpha k$ and $\frac{dt}{ds} = 1$. These are yielded from the formula $d\tilde{p} = \frac{\partial \tilde{p}}{\partial K} dK + \frac{\partial \tilde{p}}{\partial t} dt$

5.3.4 Analytical solution of the Langevin equation of radioactivity in the human body in the special case

In this section, we discuss the probability density function of radioactivity in the human body in case the Langevin equation can be solved analytically [84]. Solving the Langevin equation of radioactivity in the human body

$$\frac{dh(t)}{dt} = r(t) + \xi(t) - \alpha h(t) \tag{5.60}$$

under the initial condition $h(0) = 0$, we have radioactivity in the human body $h(t)$ as:

$$\begin{aligned} h(t) &= \int_0^t d\tau e^{-\alpha(t-\tau)} r(\tau) + \int_0^t d\tau e^{-\alpha(t-\tau)} \eta(\tau) \\ &= \langle h(t) \rangle + \theta(t) \end{aligned} \tag{5.61}$$

informally, where $\theta(t)$ and $\psi(\tau)$ are defined as:

$$\theta(t) = \int_0^t d\tau e^{-\alpha(t-\tau)} \eta(\tau) = \int_0^t d\tau \psi(t) \eta(\tau), \tag{5.62}$$

$$\psi(\tau) = e^{-\alpha(t-\tau)}, \tag{5.63}$$

respectively.

Hence, we can know the probability distribution function of h when we know probability distribution function of θ by subtracting $\langle h \rangle$. The relation of random variables H and Θ are

$$H = \langle h(t) \rangle + \Theta. \tag{5.64}$$

The Wiener process $W(t)$ is defined by the time integral of fluctuation as:

$$W(t) = \int_0^t d\tau \eta(\tau). \tag{5.65}$$

The probability density function of the Wiener process is normal distribution as:

$$p(t, X) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{X^2}{4Dt}\right). \tag{5.66}$$

Eq.(5.66) is important property of Wiener process.

$$\begin{aligned} \theta &= \sum_j \psi_j \eta_j \Delta t_j = \sum_j \psi_j W(\Delta t_j) = \sum_j a_j \\ &= \int_0^t \psi(\tau) dW(\tau) = \int_0^t da(\tau), \end{aligned} \tag{5.67}$$

where $a_j = \psi_j W(\Delta t_j)$ and $a(t) = \psi(t) W(t)$. Eq.(5.66) yields the probability density function of a_j as:

$$p_a(A_j) = \frac{1}{\sqrt{4\pi l_j^2}} \exp\left(-\frac{A_j^2}{4l_j^2}\right), \tag{5.68}$$

where l_j is defined as:

$$l_j = D\psi_j^2 \Delta t_j. \tag{5.69}$$

The probability density function of $\theta = \sum_j a_j$ is denoted by:

$$p_\theta(\Theta_j) = \frac{1}{\sqrt{4\pi \sum_j l_j^2}} \exp\left(-\frac{\Theta^2}{4\sum_j l_j^2} \right) \tag{5.70}$$

from the central limit theorem, where

$$\sum_j l_j^2 = D \sum_j \psi_j^2 \Delta t_j = D \int_0^t d\tau (\psi(\tau))^2 \tag{5.71}$$

Hence, the probability distribution function of θ is denoted by:

$$p_\theta(t, \Theta) = \frac{1}{\sqrt{4\pi D_\theta \displaystyle\int_0^t d\tau (\psi(\tau))^2}} \exp\left(-\frac{\Theta^2}{4D_\theta \displaystyle\int_0^t d\tau (\psi(\tau))^2} \right). \tag{5.72}$$

In case of our discussion of radioactivity in the human body shown in Eq.(5.62),

$$\begin{aligned} \int_0^t d\tau (\psi(\tau))^2 &= \int_0^t d\tau e^{-2\alpha(t-\tau)} = \left[\frac{1}{2\alpha} e^{-2\alpha(t-\tau)} \right]_0^t \\ &= \frac{1}{2\alpha}(1 - e^{-2\alpha t}). \end{aligned} \tag{5.73}$$

Substituting Eqs.(5.64) and (5.73) into Eq.(5.72), we obtain the probability density function of radioactivity in the human body as:

$$p_h(t, H) = \frac{1}{\sqrt{\dfrac{2\pi D_\theta}{\alpha}(1 - e^{-2\alpha t})}} \exp\left(-\frac{\alpha (H - \langle h(t) \rangle)^2}{2D_\theta (1 - e^{-2\alpha t})} \right). \tag{5.74}$$

When $r(t)$ and $\xi(t)$ satisfy Eqs.(5.4) and(5.5), then Eq.(5.74) is described as:

$$p_h(t, H) = \begin{cases} \dfrac{1}{\sqrt{\dfrac{2\pi D_\theta}{\alpha}(1 - e^{-2\alpha t})}} \exp\left(-\dfrac{\alpha \left(H - \dfrac{\langle r \rangle}{\alpha}(1 - e^{-\alpha t}) \right)^2}{2D_\theta (1 - e^{-2\alpha t})} \right) & (0 \leq t \leq t_m) \\ \dfrac{1}{\sqrt{\dfrac{2\pi D_\theta}{\alpha}(1 - e^{-2\alpha t_m})}} \exp\left(-\alpha(t_m - t) - \dfrac{\alpha \left(H e^{-\alpha(t_m - t)} - \frac{\langle r \rangle}{\alpha}(1 - e^{-\alpha t_m}) \right)^2}{2D_\theta (1 - e^{-\alpha(t_m - t)})} \right) & (t > t_m) \end{cases}. \tag{5.75}$$

Eq.(5.75) is calculated by substituting Eq.(5.10) at the interval $0 \leq t \leq t_m$. Eq.(5.75) is calculated at the interval $t > t_m$ by giving the boundary condition at $t = t_m$.

Taking limit $(t, t_m) \to (\infty, \infty)$, we have

$$\lim_{(t,t_m)\to(\infty,\infty)} p_h(t, H) = \frac{1}{\sqrt{2\pi D_\theta/\alpha}} \exp\left(-\frac{\left(H - \dfrac{\langle r\rangle}{\alpha}\right)^2}{2D_\theta/\alpha} \right) \sim N\left(\frac{\langle r\rangle}{\alpha}, \frac{D_\theta}{\alpha}\right) \tag{5.76}$$

Eq.(5.76) states that the steady state of the probability distribution after enough time from the beginning of intake of radioactivity is dependent on only biological half-life ($T_b = \ln 2/\alpha_b$) and average intake. The width of probability density function of radioactivity does not increase infinitely, likely maximum value and exceedance probability can be estimated.

5.3.5 Sample paths and probability density function employing estimated average intake of radioactivity and diffusion constant from actual measurement

Fig.5.6 (a) and (b) show respectively the sample paths and probability density function of radioactivity in the human body according to the SDE and the Fokker-Planck equation, by using diffusion constant D=879 (Bq^2/day) and average intake of radioactivity $\langle r\rangle$= 9.8 (Bq/day) estimated in section 5.2.6 from actual measurement. Comparing (ii) and (iv) of Fig.5.6 (b), the Fokker-Planck equation seems to provide the probability distribution which is more large kurtosis than Gaussian distribution. Although we assume Gaussian distribution for simplicity here, comparison with actual frequency distribution may be useful to discuss more accurate distribution of radioactivity in the human body for each individual. However, we keep the discussion only on the comparison with Gaussian distribution in this article.

5.3.6 Probability distribution of saturated radioactivity in the human body and effective dose caused by the difference of effective half-life

Up to previous sections, we studied the probability distribution of radioactivity in the human body caused by the difference of intake amount of radiocesium. In this section, we will study the probability distribution of radioactivity in the human body caused by the difference of effective half-life based on the transformation of probability density function discussed in the section 5.2.7. Actually, it is indicated by measurement that biological half-life of cesium is different for each individuals [39, 85]. Table 2 of Rühm et al. (1999) shows the difference of biological half-lives of each German individuals for 41 subjects [39]. Fig.5.7 (a) shows the histogram of effective half-lives of radiocesium according to Table 2 of Rühm et al. (1999) for 33 subjects, where 8 subjects under the age of 10 are removed for avoiding confusion of the age-dependence of biological half-life. Assuming Gaussian distribution, the average and standard deviation shown in the histogram Fig.5.7 (a) are 81.6 days and 19.1 days, respectively. Although the distribution of biological half-life will be different by racial factors, body weight or eating habits etc., this distribution is adopted in this article as an example.

Taking transformation of random variables of probability density function as shown in Eq.(5.48), we can obtain the probability density function of saturated radioactivity in the human body from the distribution of effective half-life. Fig.5.7 (b) shows the probability density of saturated radioactivity in the human body computed by Eq.(5.48). The intake amount of radioactivity r is fixed to 1 Bq/day for convenience. The average and standard

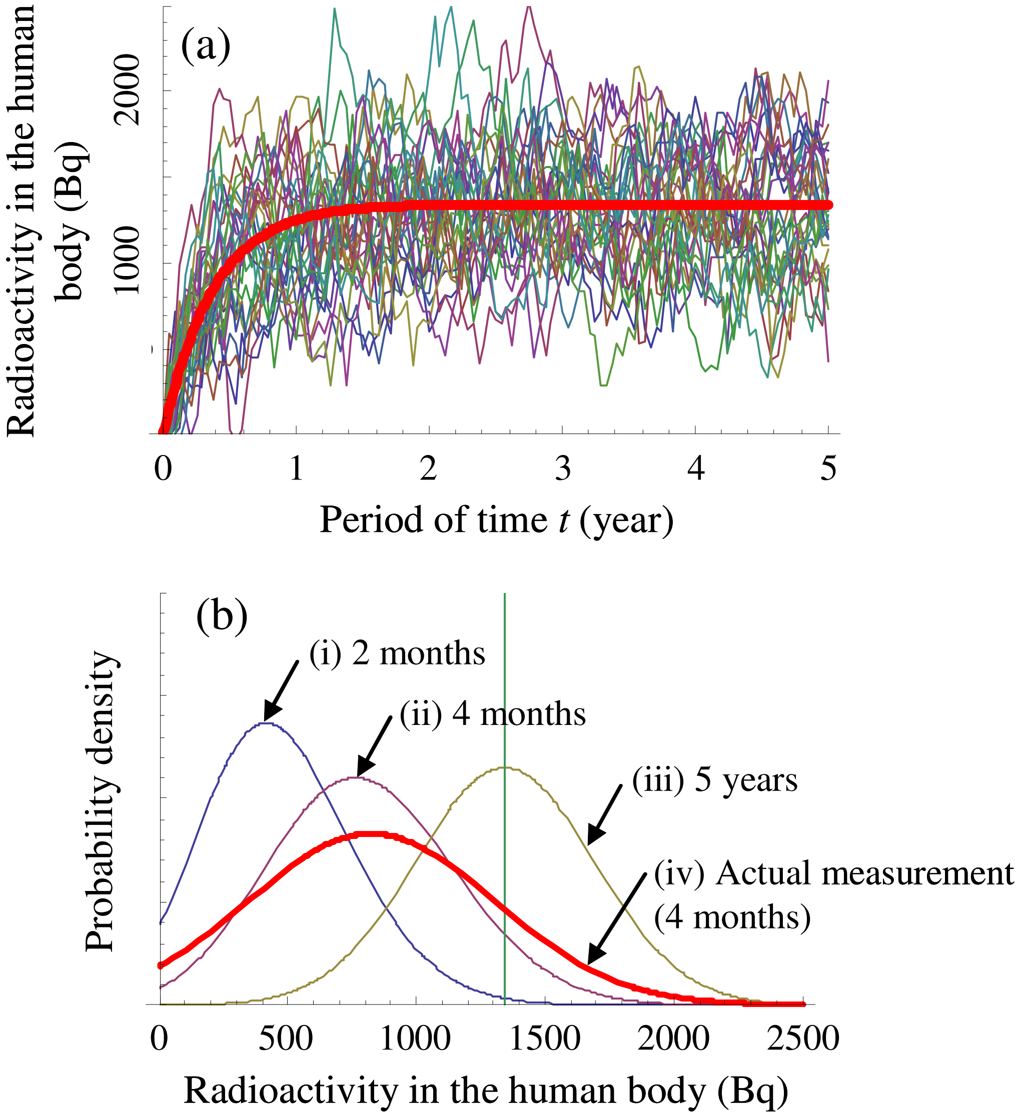

Fig 5.6: (a) The sample paths and (b) probability density functions of radioactivity in the human body, by using diffusion constant and average intake of radioactivity estimated in section 5.2.6 from actual measurement.

deviation of saturated radioactivity in the human body are 117.7 Bq and 27.6 Bq respectively. It should be noted that the average and standard deviation are linear respect with the intake amount of radioactivity r.

Similarly, the probability density function of effective dose can be calculated according to transformation of random variables as shown in Eq.(5.51), where $\beta = \langle E \rangle /\text{weight} = 0.465$ (Mev) $\times\ 1.6 \times 10^{-13}$ (J/MeV)/70 (kg) $= 1.1 \times 10^{-15}$ (mSv) regarding ^{137}Cs. Fig.5.8 shows the probability density function of effective dose caused by the distribution of effective half-life, where intake amount of radioactivity and ingestion period conveniently are $r = 1$ (Bq/year) $= 1/365$ (Bq/day) and t_m =365 days=1 year respectively for convenience. The average, standard deviation, and coefficient of variation of effective dose are 0.011 μSv, 0.0025 μSv, and 0.23 respectively. As shown here, the probability distribution of biological half-life yields the probability density function of radioactivity in the human body and effective dose

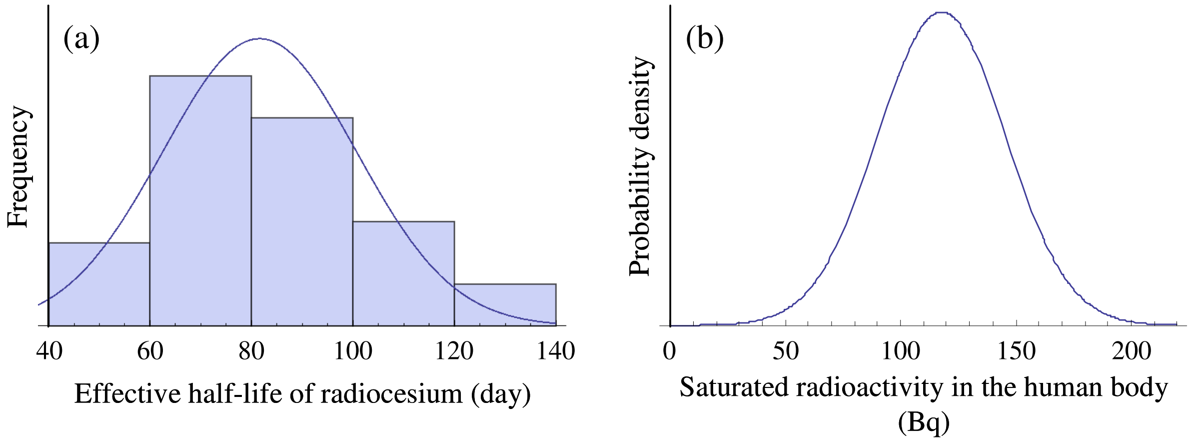

Fig 5.7: (a) Histogram of effective half-lives of radiocesium according to W. Rühm et al. (1999). (b) Probability density of saturated radioactivity in the human body computed by transformation of probability density function described in Eq.(5.48).

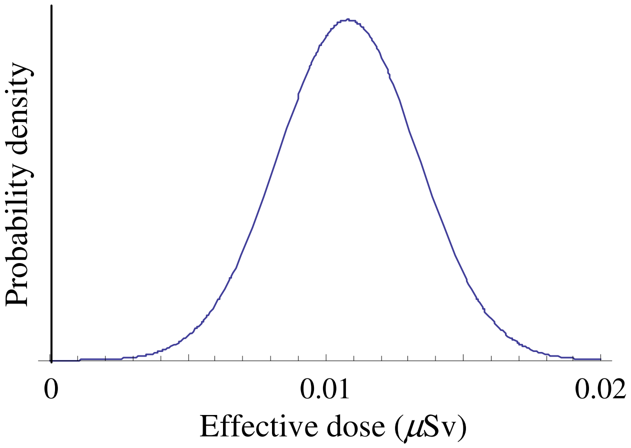

Fig 5.8: Probability density function of effective dose caused by the distribution of effective half-life taking transformation of random variables as described in Eq.(5.51).

in case of fixed intake amount of radioactivity r.

Fig.5.9 compares the committed effective dose of ^{137}Cs calculated by the present method with those determined according to ICRP [32] as with Fig.3.5, which includes the error bar according to the distribution shown in Fig.5.8. The length of the error bar is 1σ, Fig.5.10 shows the correspondence of the error bar and probability density function.

Fig.5.8 and Fig.5.9 enable to compute the exceedance probability of committed effective dose. For example, 0.3% of the people (3σ) receive the 0.018 μSv of effective dose for the intake of radioactivity of ^{137}Cs per 1 Bq. In the same way, we can compute one person of 10000 people receive 0.02 μSv of effective dose for the intake of radioactivity of ^{137}Cs per 1 Bq.

Thus, computation of the probability density function is useful to calculate the exceedance probability of effective dose and estimation of the number of the people who receive larger than a certain value which is much larger than average.

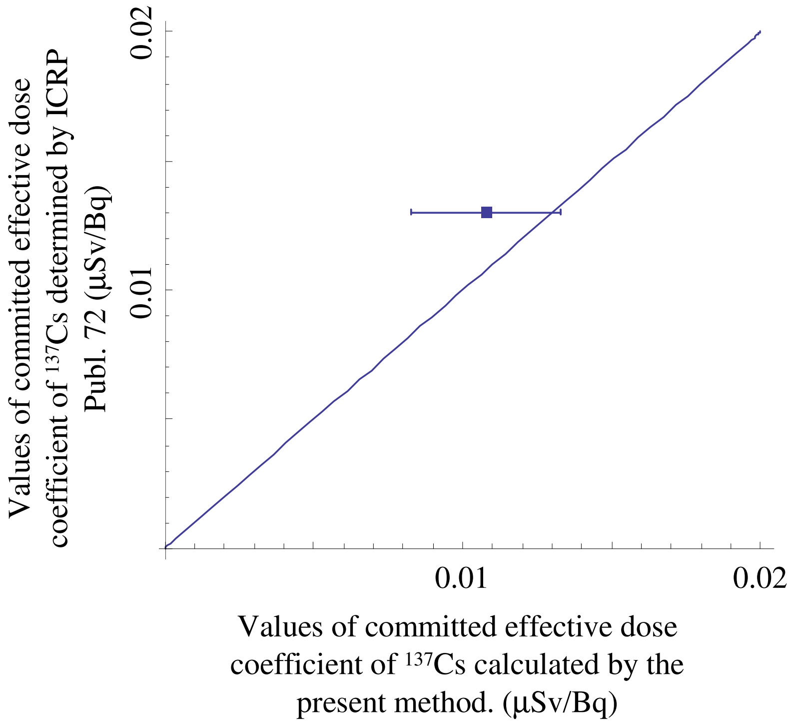

Fig 5.9: Comparison of the committed effective dose of ^{137}Cs calculated by the present method with those determined according to ICRP [32], which includes the error bar according to the distribution shown in Fig.5.8. The length of the error bar is 1σ.

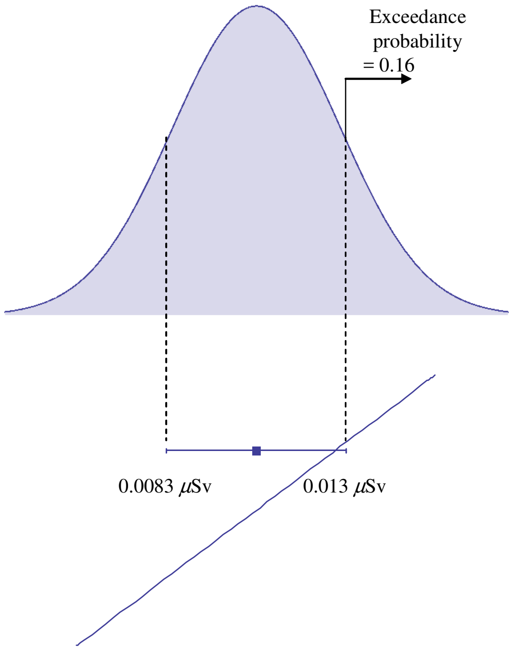

Fig 5.10: Correspondence of the error bar and probability density function.

5.4 Conclusion

In this chapter, we introduce the stochastic process to discuss the probability density function of radioactivity in the human body and the effective dose. We provide the Langevin equation of radioactivity in the human body adding random fluctuation term to the continuity equation previously studied method in this thesis. The results are summarized below.

1. The Langevin equation can be deformed to equivalent SDE and Fokker-Planck equation. Based on the SDE formalism, the sample paths of radioactivity in the human body are generated. Solving the Fokker-Planck equation, we obtain the probability density function of radioactivity in the human body. The probability density function of radioactivity in the human body is computed numerically during ingestion period. After the end of ingestion, the probability density function of radioactivity in the human body is solved analytically.

2. The candidate of stochastic process of the effective dose is proposed considering the convergence in the average and the integral of the Wiener process. In the same way with radioactivity in the human body, we produce sample paths of the effective dose based on SDE formalism. The probability density function of the effective dose is computed solving the Fokker-Planck equation.

3. By using the actual measurement of radioactivity in the human body, we estimate the average intake $\langle r \rangle$ of radioactivity and the diffusion constant D of radioactivity in the human body. It is shown that the probability density function obtained by computing the Fokker-Planck equation employing above $\langle r \rangle$ and D differs from the Gaussian distribution.

4. Considering the probability distribution of biological half-life, we compute the probability distribution functions of saturated radioactivity in the human body and the effective dose by using transformation of random variables. The method enables to compute the exceedance probability of effective dose and estimation of the number of the people who receive larger than a certain value which is much larger than average.

The method studied in this section has a potential to provide radiological protection based on the stochastic viewpoint including the uncertainty and randomness by using lumped hydrologic method. We suggest that it can be useful idea to consider the probability distribution rather than a certain value.

6 General conclusion

The objective of this thesis was to provide readily-understandable internal dose calculation based on lumped hydrologic method different from existing methods in radiology which uses sophisticated human body imaging. We compute the number of radioactive atoms in the human body solving the continuity equation, which is describe by the first order ordinary differential equation. A methodology for calculating intensity of radioactivity in the human body, cumulative number of radioactive decay, and effective dose is proposed based on the number of radioactive atoms in the human body.

We also compute radioactivity in the human body and the effective dose due to the intake of radioactivity on agricultural products derived from fallout according to above discussed hydrologic internal dose calculation.

The distribution of radioactivity in the human body is studied to treat the continuity equation as a stochastic process. The probability density function of radioactivity in the human body is computed by the Fokker-Planck formalism.

This thesis thus studies the internal dose calculation based on lumped hydrologic method, which may provide a readily-understandable knowledge of radiological protection.

The results are summarized as follows:

1. The number of radioactive atoms in the human body and radioactivity in the human body is described by readily-understandable method which corresponds to lumped hydrologic method. The method gives a continuity equation of the number of radioactive atoms in the human body as a first order ordinary differential equation. Radioactivity in the human body is also computed by solving the continuity equation in the same way. The cumulative number of radioactive decay is obtained by time integral of radioactivity in the human body.

2. Radioactivity in the human body depends on the effective half-life, which is determined by physical and biological half-lives as shown in Eq.(2.7). If one of the half-lives is markedly longer than the other, it nearly equals the effective half-life. For example, the biological half-life of ^{137}Cs is much shorter than the physical half-life; thus, the biological half-life is dominant in the derivation of effective half-life.

3. In the case of ^{137}Cs of which effective half-life is 70 days, radioactivity in the human body is saturated with in 1 year after the beginning of ingestion of radiocesium. Likewise, it is removed from the human body within 1 year after the end of ingestion . The cumulative number of radioactive decay increases during the ingestion period. After the end of ingestion, it becomes almost constant.

4. Considering the energy spectrum of β ray, the average energy of β ray is estimated to be one third of its maximum energy. The absorptivity of γ ray is estimated to be a half, because the order of energy of γ ray is about 0.01-0.1 MeV. In these reasons, the absorbed energy of β decay into the human body can be estimated rationally from the decay scheme. Effective dose can be computed to multiply the absorbed energy of β decay into the human body by cumulative number of radioactive decay.

5. The accuracy of the present method was demonstrated by comparison with previous research. Measurement value and analytic solution obtained by the present method are very similar regarding intensity of radioactivity in the human body as shown in

Fig.2.4(b). The energy absorbed per radioactive decay event calculated by the present method is similar to that in previous research as shown in Fig.3.3. The values of the coefficient of committed effective dose are comparable. For most nuclides, the results are similar to previous research. For some nuclides, however, there were discrepancies with previous research. Committed effective dose values for ^{59}Fe and ^{65}Zn, minerals required by the human body, are modified by taking into account adsorption into the human body from the digestive system. ^{90}Sr is homologous to calcium and is known to be a bone-seeker. Multiplying by the ratio of strontium concentrated in bone gave a better coefficient. The value for iodine was modified by taking into account its concentration in the thyroid gland, using a tissue weighting factor for the thyroid gland. The comparisons of committed effective dose coefficient calculated by the present method with previous research are summarized in Fig.3.5.

6. Concrete calculation of committed effective doses are shown as exercises; each involves ingestion of food and water containing radioactive isotopes. In Eqs. (3.15), (3.18) and (3.20), the values of committed effective dose are computed considering the standard limit for radioactive isotopes in general food and water in Japan in 2015. Even if food or water that contains radioactive isotopes at the standard limit is ingested for 50 years, the committed effective dose is less than 100 mSv. which does not cause significant difference in cancer risk epidemiologically. It is the value that no significant difference in cancer risk is likely to be found epidemiologically.

7. Transfer of radiocesium from the soil to agricultural products is modeled considering half-life on soil, the unit weight of the soils, transfer factor, and annual consumption per capita. We choose rice as a typical example. By using actual data of fallout, we compute the effective dose due to intake of radioactivity on rice derived from radiocesium fallout . It is estimated that the order of internal dose rate caused by intake of radioactivity on rice derived from radiocesium fallout is same with that of natural radiation.

8. We introduce a stochastic process to describe the probability distribution of radioactivity in the human body. Adding random term to the continuity equation of radioactivity in the human body, we obtain the Langevin equation to be a starting point of discussion. The Langevin equation is denoted by other two equivalent formalism, that is, stochastic differential equation and Fokker-Planck equation. The Fokker-Planck formalism is useful to compute the probability density function. Hence, the Langevin equation of radioactivity in the human body provides the Fokker-Planck equation of radioactivity in the human body. Solving the Fokker-Planck equation of radioactivity in the human body numerically, we obtain the probability density function of radioactivity in the human body.

9. Probability density functions of radioactivity in the human body and effective dose caused by the difference of biological half-life are computed by using transform of the probability density function. The probability density function of the effective dose is useful to calculate the exceedance probability and the number of the people who receive the effective dose larger than a certain dose.

The present methodology, which corresponds to a lumped hydrological model, indicates equal accuracy with previously proposed methods. These divide the human body into its constituent organs, which may correspond to a distributed hydrological model. The results of this study demonstrate that regarding the human body as a single large tissue is essentially equivalent to dividing the human body into its organs. The universality of knowledge developed in hydrology; that is, a hydrological distributed model and a hydrological lumped model, was shown to be essentially equivalent in the field of radiology. The present hydrologic-based methodology has a potential to expand the scope of discussion of internal dose calculation straightforwardly because of the simplicity of the fundamental equation. In this article, we studied the radioactivity on the soil and stochastic calculus of internal dose calculation as practical examples of application. In summary, the present study provides an understanding of the influence of radioactive substances on the human body that not only radiologists but also general technologists can comprehend.

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List of symbols

Chapter 2

- T_p: Physical half-life
- T_b: Biological half-life
- T_e: Effective half-life
- α_p: Physical attenuation constant; $T_p = \dfrac{\ln 2}{\alpha_p}$
- α_b: Biological attenuation constant; $T_b = \dfrac{\ln 2}{\alpha_b}$
- α_e: Effective attenuation constant; $T_e = \dfrac{\ln 2}{\alpha_e}$
- t: Time
- $N(t)$: The number of radioactive atoms in the human body
- $h(t)$: Intensity of radioactivity in the human body
- N_0: The number of radioactive atoms at time $t = 0$; i.e. initial condition $N(0) = N_0$
- $q_{pa}(t)$: Physical runoff of the number of radioactive atoms; $q_p(t) = \alpha_p N(t)$
- $q_{ba}(t)$: Biological runoff of the number of radioactive atoms; $q_b(t) = \alpha_b N(t)$
- $q_{ea}(t)$: Effective runoff of the number of radioactive atoms; $q_e(t) = q_p(t) + q_b(t) = \alpha_e N(t)$
- $r_a(t)$: Ingestion flux of radioactive atoms
- $r_r(t)$: Ingestion flux of intensity of radioactivity; $r_r(t) = \alpha_p r_a(t)$
- $q_{er}(t)$: Effective runoff of intensity of radioactivity; $q_{er}(t) = \alpha_e h(t)$
- r_{a0}: Mean ingestion flux of radioactive atoms during ingestion period
- r_{r0}: Mean ingestion flux of intensity of radioactivity during ingestion period; $r_{r0} = \alpha_p r_{a0}$
- t_m: Ingestion period
- $I(t)$: Internal total exposure dose
- τ: A convenient variable describes time
- $r_{\mathrm{ref}}(t)$: Mean intake of intensity of radioactivity based on a mean food consumption
- $r_{\mathrm{ind}}(t)$: Individual intake of intensity of radioactivity
- cr_{age}: Age-dependent and gender dependent food consumption rate

- b_{ind}: Factor to account for potential differences between individuals and mean food consumption
- a_1: Coefficient of short biological half-life
- a_2: Coefficient of long biological half-life; $(a_1 + a_2 = 1)$
- T_1: Short biological half-life
- T_2: Long biological half-life
- T_{e1}: Effective half-life emerged from short term excretion; $\dfrac{1}{T_{e1}} = \dfrac{1}{T_p} + \dfrac{1}{T_1}$
- T_{e2}: Effective half-life emerged from long term excretion; $\dfrac{1}{T_{e2}} = \dfrac{1}{T_p} + \dfrac{1}{T_2}$
- α_{e1}: Effective attenuation constant emerged from short term excretion; $\alpha_{e1} = \dfrac{\ln 2}{T_{e1}}$
- α_{e2}: Effective attenuation constant emerged from long term excretion; $\alpha_{e2} = \dfrac{\ln 2}{T_{e2}}$
- $h_1(t)$: Intensity of radioactivity in the human body emerged from short term excretion
- $h_2(t)$: Intensity of radioactivity in the human body emerged from long term excretion
- q_{br1}: Runoff emerged from short term excretion; $q_{br1} = \alpha_1 h_1(t)$
- q_{br2}: Runoff emerged from long term excretion; $q_{br2} = \alpha_1 h_2(t)$

Chapter 3

- n: Neutron
- p: Proton
- e^-: Electron / β ray
- $\bar{\nu}_{\mathrm{e}}$: Anti electron neutrino
- γ: γ ray / Photon / Electromagnetic wave
- $K_e^{\max}$: Maximum kinematic energy of β ray
- A: Mass number
- Z: Atomic number, i.e. the number of protons
- $M(A, Z)$: Mass of a nuclide
- c: Velocity of light in a vacuum
- $\langle K_e \rangle$: Mean kinematic energy of β ray

- ξ: $\langle K_e \rangle$-$K_e^{\max}$ ratio
- ζ: Absorptivity of γ ray into the human body
- i: Index of summation regarding energy of β ray, i.e. the number of β decay path
- j: Index of summation regarding energy of γ ray, i.e. the number of γ decay path
- μ: Probability of β decay
- ν: Emission ratio of γ ray
- e_γ: Energy of γ ray
- R: Index of summation regarding radiation types
- T: Index of summation regarding the tissues or organs
- $D_{\mathrm{T,R}}(t)$: Absorbed dose
- $H_{\mathrm{T}}(t)$: Equivalent dose
- $H_{\mathrm{eff}}(t)$: Effective dose
- C_{CED}: Committed effective dose coefficient
- r_{tot}: Total ingested intensity of radioactivity

Chapter 4

- η: Transfer factor from the soils to agricultural products
- $h_s(t)$: Intensity of radioactivity per area on the soils
- T_r: Residence half-time in the soils
- $q_r(t)$: Total runoff of radioactivity on the soils
- $q_{p;s}$: Physical runoff of radioactivity on the soils
- $q_s(t)$: Runoff of radioactivity by environmental factors on the soils
- α_s: Attenuation constant; $T_s = \dfrac{\ln 2}{\alpha_s}$
- α_r: Residence attenuation constant of in the soils; $T_r = \dfrac{\ln 2}{\alpha_r}$
- T_r: Residence half-time in the soils
- T_s: Half-life on the soils
- $r_f(t)$: Inflow flux of radioactive fallout
- h_{s0} Initial value of $h_s(t)$ as :$h_{s0} = h_s(0)$

- l: Infiltration depth of cesium on the topsoils
- ρ: Unit weight of the soils
- κ: Average annual rice consumption per capita of Japanese
- $r_i(t)$: Annual intake of radioactivity on the rice
- λ: Constant as: $\lambda = \kappa\eta/\rho l$
- $h_b(t)$: Radioactivity in the human body

Chapter 5

- $\dot{x}(t)$: Any function (in original Langevin equation, velocity of minute particles)
- γ: Any constant (in original Langevin equation, drag coefficient)
- $\xi(t)$: Random fluctuation term
- $\eta(t)$: Random fluctuation term at the interval $(0 \leq t \leq t_m)$
- $\langle r \rangle$: Average ingestion flux of radioactivity which is treated as an any constant
- Γ: Any constant
- $\langle h \rangle (t)$: Average of radioactivity in the human body
- $x(t)$: Any function including stochastic integral
- $\mu(t, x(t))$: Drift term
- $W(t)$: Standard Wiener process
- X :Any random variable
- H: Random variable of radioactivity in the human body
- $p_h(t, H)$: Probability density function of radioactivity in the human body
- ε: Sufficiently small constant as: $\varepsilon \ll 1$
- Λ: Sufficiently large constant as: $\Lambda \gg r/\alpha$
- $\varphi(t)$: Cumulative number of radioactive decay
- $s(t)$: Effective dose
- β: Energy absorbed into the human body per one radioactive decay which depends on each nuclide
- $\mu_\varphi(t)$: Drift term of effective dose
- $\sigma_\varphi(t)$: Diffusion term of effective dose
- $\langle \varphi(t) \rangle$: Average of cumulative number of radioactive decay

- $p_\varphi(t, \Phi)$: Probability density function of cumulative number of radioactive decay
- Φ: Random variable of cumulative number of radioactive decay
- $p_h(t, S)$: Probability density function of effective dose
- S: Random variable of effective dose

List of achievements

1 Original papers (Refereed papers)

[1] **佐々木翔太**, 山田正, 山田朋人, 水文学的モデルを用いた放射性同位体の経口摂取による人体における放射性同位体の原子数，放射能強度および内部被曝総量の推定手法の提案 (Proposal of a hydrological model to estimate a number of radioactive atoms in the human body, radioactivity and internal total exposure caused by ingestion of radioactive isotopes), 水文・水資源学会誌. Vol. 27, No. 4, pp. 170-181 (2014, in Japanese).

[2] **S. Sasaki**, H. Yoneda, T. Yamada, T.J. Yamada, An application of hydrologic methodology to the internal dose calculation: the intake of radioactivity on agricultural products derived from fallout, Journal of Japan Society of Civil Engineering, Ser. B1 (Hydraulic Engineering), Vol. 71, No. 4, I_49-I_55 (2015).

[3] **S. Sasaki**, M. Choi, T. Yamada, T.J. Yamada, COMPUTATION OF THE PROBABILITY DENSITY FUNCTION OF RADIOACTIVITY IN THE HUMAN BODY AND THE EFFECTIVE DOSE BY USING STOCHASTIC PROCESSES BASED ON LUMPED HYDROLOGICAL METHOD, Journal of Japan Society of Civil Engineering, Ser. B1 (Hydraulic Engineering), Vol. 72 (accepted).

2 Papers (Non-refereed papers)

[1] **S. Sasaki**, H. Yoneda, T. Yamada, T.J. Yamada, A Hydrologic Approach Relating to the Influences of Radioactive Substance in a Human Body II, Environmental Science and Technology Vol. 2, pp 185-190 (2013).

[2] T. Yamada, T.J. Yamada, **S. Sasaki**, Introduction to Radiological Hydrology: A Kinematic Model to Estimate the Effective Dose of Radioactive Substances in the Human Body for Radiological Protection, Proceedings of 2013 IAHR World Congress, Z1003 (2013).

[3] **S. Sasaki**, T. Yamada, T.J. Yamada, THE INTERNAL DOSE CALCULATION BY USING HYDROLOGICAL BASED LUMPING METHOD, International Atomic Energy Agency International Symposium on Isotope Hydrology, Revisiting Foundations and Exploring Frontiers Book of extended Synopses, Poster session 2, ID 120, pp. 61-64 (2015).

3 International conference presentation

[1] **S. Sasaki**, H. Yoneda, T. Yamada, T.J. Yamada, A HYDROLOGIC APPROACH RELATING TO THE INFLUENCES OF RADIOACTIVE SUBSTANCE IN A HUMAN BODY, The Sixth International Conference on Environmental Science and Technology, 05-1 (Oral), Texas, United States of America, June 27, 2012.

[2] **S. Sasaki**, A hydrological model for the behavior and its influences of radioactive substances in a human body, Chuo University Program for Capacity Building in Global Water-Environmental Engineering Vietnam Symposium, Mini-Research-Symposium

between Viet students and Japanese Students (Oral), Ho Chi Minh City, Vietnam, November 27, 2012.

[3] **S. Sasaki**, T. Yamada, T.J. Yamada, A Kinematic Model To Estimate Effective Dose Of Radioactive Substances In A Human Body, Chuo University 1st International Symposium on Water Environment, Poster session, Tokyo, Japan, March 22, 2013.

[4] **S. Sasaki**, T. Yamada, T.J. Yamada, A Kinematic Model to Estimate Effective Dose of Radioactive Substances in the Human Body for Radiological Protection, 2013 AGU meeting of Americas, H33B-14 (Poster), Cancun, Mexico, May 15, 2013.

[5] **S. Sasaki**, H. Yoneda, T. Yamada, T.J. Yamada, A kinematic model to estimate the effective dose of radioactive substances in the human body for radiological protection, 6th Asia Pacific Association of Hydrology and Water Resources Conference, B115 (Oral), Seoul, Korea, August 20, 2013.

[6] **S. Sasaki**, T. Yamada, T.J. Yamada, A kinematic model to estimate the effective dose of radioactive isotopes in the human body for radiological protection, 2013 AGU fall meeting, GC059 (Poster), California, United States of America, December 15, 2013.

[7] **S. Sasaki**, T. Yamada, T.J. Yamada, A kinematic-based methodology for radiological protection: Runoff analysis to calculate the effective dose for internal exposure caused by ingestion of radioactive isotopes, The EGU General Assembly 2014, oral-44 # 11064 (Oral), Vienna, Austria, April 29, 2014.

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[9] **S. Sasaki**, T. Yamada, T.J. Yamada, A hydrologic-based lumped method for stochastic internal dose calculation and the time evolution of the probability density distribution of radioactivity in the human body, 2015 Joint Assembly, 33026 (Oral), Montreal, Canada, May 7, 2015.

[10] **S. Sasaki**, T. Yamada, T.J. Yamada, International Atomic Energy Agency International Symposium on Isotope Hydrology: Revisiting Foundations and Exploring Frontiers (Conference ID: 46527 (CN-225)), ID 120 (Poster), Vienna, Austria, May 13, 2015.

[11] **S. Sasaki**, T. Yamada, T.J. Yamada, The 26th the International Union of Geodesy and Geophysics (IUGG) general assembly 2015, IUGG-2618 (Oral), Prague, Czech, June 25, 2015.

4 Domestic conference presentation

[1] 佐々木翔太, 米田駿星, 山田正, 山田朋人, 印旛沼土地改良区設立60周年記念祝賀会, 放射線防護のための運動学的方法論による放射性同位体の経口摂取時に受ける実効線量推定手法の提案 (Poster), Chiba, Japan, June 15, 2013.

[2] **佐々木翔太**, 山田正, 山田朋人, 放射線防護のための人体における放射性同位体に起因する内部被曝による実効線量を計算する運動学的方法論, 第53回日本核医学会学術総会, M2IIID3 (Oral), Fukuoka, Japan, November 9, 2013.

[3] **S. Sasaki**, H. Yoneda, T. Yamada, T.J. Yamada, An application of hydrologic methodology to the internal dose calculation: the intake of radioactivity on agricultural products derived from fallout, 第59回水工学講演会, R102 (oral, international session), Tokyo, Japan, March 11, 2015.

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Appendix

A Mathematical theory of half-life

A.1 The reason why radioactivity decreases exponentially

In the present section, we show mathematically why radioactivity decreases exponentially due to radioactive decay. Let's consider the probability which a nuclide decays between time t with $t+\varepsilon$. Here we set the decay probability $D(\varepsilon)$ [7] , and take ε is time interval sufficiently small. The decay probability is linear respect with the time interval ε as:

$$D(\varepsilon) = \alpha\varepsilon, \tag{a.1}$$

where α is attenuation constant which depends only the kind of nuclide [8].

For readily understanding, we imagine a roulette that determines whether or not occur radioactive decay. We assume if a ball falls into the pocket 00, then radioactive decay occurs. The attenuation constant α can be regarded as the spinning frequency of the roulette. If attenuation constant α is large, then the number of times spinning the roulette increases. And so, the probability which a ball falls into the pocket 00 also increases of course. Again, if the time interval ε increases, then the probability which a ball falls into the pocket 00 increases in the same reason. From these analogies, it seems that the imaginary roulette reproduces the radioactive decay.

We consider the probability which radioactive decay does not occur within the time interval ε. The probability $p(\varepsilon)$ is complementary event of $D(\varepsilon)$,

$$p(\varepsilon) = 1 - D(\varepsilon) = 1 - \alpha\varepsilon. \tag{a.2}$$

It should be that $p(2\varepsilon)$ is the probability which radioactive decay does not occur between time t with $t+\varepsilon$ and between time $t+\varepsilon$ with $t+2\varepsilon$, that is,

$$p(2\varepsilon) = \{1 - D(\varepsilon)\}^2 = (1 - \alpha\varepsilon)^2. \tag{a.3}$$

Eq. (a.3) is generalized substituting $t = n\varepsilon$ as:

$$p(t) = \{1 - D(\varepsilon)\}^n = (1 - \alpha\varepsilon)^n = \left(1 - \frac{\alpha t}{n}\right)^n. \tag{a.4}$$

Here, t is fixed. When the time interval $\varepsilon \to 0$, then $n \to \infty$. Taking limit, we can get[9]

$$p(t) = \lim_{n\to\infty}\left(1 - \frac{\alpha t}{n}\right)^n = e^{-\alpha t}. \tag{a.5}$$

Eq. (a.5) represents exponential decrease of radioactivity itself. Eq. (a.5) shows that the probability of radioactive decay does not depend on the past state, depends on only the current state. Such process is said to be memoryless process.

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[7] It is implied that D does not depend on time t.

[8] In terms of mathematical accuracy, Eq. (a.1) is inaccurate. When $\varepsilon > 1/\alpha$, $D(\varepsilon)$ becomes greater than 1. It is nonsense as the probability of course. The probability $D(\varepsilon)$ should be strictly described as $D(\varepsilon) = \alpha\varepsilon + O(\varepsilon^2)$. This means that when ε is sufficiently close to 0, then $O(\varepsilon^2)$ is much smaller than constant times ε enough to neglect. For simplicity, we omit $O(\varepsilon^2)$ in the main text intentionally.

[9] $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$.

We confirm that radioactive decay is memoryless. Recall the attenuation of radioactivity and the definition of half-life, we can get

$$h(t+T_p) = h(0)e^{-\alpha_p(t+T_p)} = h(0)e^{-\alpha_p t}e^{-\alpha_p T_p} = \frac{h(t)}{2}. \tag{a.6}$$

Eq. (a.6) shows that radioactive decay is memoryless, that is, depends on only the current state.

Above discussion is all based on the experimental fact that radioactivity decreases exponentially. The exponential decrease proves the memorylessness and stochasticity of a physical phenomenon, such as radioactive decay.

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B First-order linear ordinary differential equation

B.1 Variation of constants

General non-homogeneous first-order linear ordinary differential equation is describes as

$$\frac{dy(t)}{dt} + a(t)y(t) = b(t). \tag{b.1}$$

At first, homogeneous differential equation (i.e. identically $b(t) = 0$)

$$\frac{dy(t)}{dt} + a(t)y(t) = 0 \tag{b.2}$$

is solved to solve Eq. (b.1). By separation of variables, Eq. (b.2) is deformed as

$$\frac{1}{y(t)}dy(t) = -a(t)dt. \tag{b.3}$$

To integrate both sides of Eq. (b.3),

$$\ln|y(t)| = -\int a(t)dt + C \tag{b.4}$$

is obtained. C is a constant of integration, redefined accordingly. Eq. (b.4) is deformed as

$$y(t) = Ce^{-\int a(t)dt}. \tag{b.5}$$

Eq. (b.5) is the general solution of the homogeneous differential equation Eq. (b.2). Under the initial condition $y(t_0) = y_0$,

$$y(t) = y_0 e^{-\int_{t_0}^{t} a(\tau)d\tau} \tag{b.6}$$

Again, solution of the non-homogeneous differential equation Eq. (b.1) is discussed. Variation of constants is one of a method to solve non-homogeneous differential equation. Constant of integration C is replaced with $C(t)$, function of the variable t. Eq. (b.5) is rewritten as

$$y(t) = C(t)e^{-\int a(t)dt}. \tag{b.7}$$

To differentiate both sides of Eq. (b.7),

$$\frac{dy(t)}{dt} = \frac{dC(t)}{dt}e^{-\int a(t)dt} - C(t)a(t)e^{-\int a(t)dt} \tag{b.8}$$

is obtained. Product rule $\frac{d}{dx}fg = dfracdfdxg + fdfracdgdx$ is used. To substituted Eqs. (b.7) and (b.8) to Eq. (b.1),

$$\frac{dC(t)}{dt} = b(t)e^{-\int a(t)dt} \tag{b.9}$$

is obtained. To integrate both sides of Eq. (b.9),

$$C(t) = \int b(t)e^{\int a(t)dt}dt + C. \tag{b.10}$$

To substitute Eq. (b.10) to Eq. (b.7),

$$y(t) = e^{-\int a(t)dt}\Big(\int b(t)e^{\int a(t)dt}dt + C\Big) \tag{b.11}$$

is obtained. Eq. (b.11) is general solution of the non-homogeneous first-order linear ordinary differential equation Eq. (b.1).

If interval of integral and initial condition are set as $[t_0, t]$ and $y(t_0) = y_0$ respectively, Eq. (b.11) is rewritten as

$$y(t) = e^{-\int_{t_0}^{t} a(\tau)d\tau}\Big(\int_{t_0}^{t} b(\tau)e^{\int_{t_0}^{t} a(\tau)d\tau}d\tau + y_0\Big). \tag{b.12}$$

Eq. (b.12) is one of a particular solution of Eq. (b.1)

As addressed in the main of this thesis, if $a(t)$ is time constant (i.e. $a(t) = \alpha = \text{const}$), Eq. (b.1) is rewritten as

$$\frac{dy(t)}{dt} + \alpha y(t) = b(t). \tag{b.13}$$

If smaller endpoint of interval of integral is set to be $t_0 = 0$ and initial condition $y_0 = 0$,

$$y(t) = \int_0^t b(\tau)e^{-\alpha(t-\tau)}d\tau. \tag{b.14}$$

Eq. (b.14) is the particular solution of equation of continuity discussed in the main of this thesis.

B.2 Laplace transform

Laplace transform is widely used solving linear ordinary differential equations, it is much dissimilar with variation of constants. In many cases, Laplace transform is easier because the differential equation is transformed to the algebraic equation.

Laplace transform of a function $f(t)$ ($t \in \mathbb{R}$) is denoted as

$$\mathcal{L}[f(t)], \tag{b.15}$$

which is linear operator that transforms $f(t)$ to a function $F(s)$ ($s \in \mathbb{C}$). Laplace transform is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt. \tag{b.16}$$

Inverse Laplace transform is denoted as

$$\mathcal{L}^{-1}[F(S)] = f(t). \tag{b.17}$$

Eq. (b.1) is solved by using Laplace transform as an example. In this problem, property of differentiation

$$\mathcal{L}\Big[\frac{df(t)}{dt}\Big] = sF(s) - f(0), \tag{b.18}$$

property of convolution

$$\mathcal{L}\Big[\int_0^t f(\tau)g(t-\tau)\Big] = F(s)G(s), \tag{b.19}$$

and the formula

$$\mathcal{L}[e^{-\alpha t}] = \frac{1}{s+\alpha} \quad (x \geq 0) \tag{b.20}$$

are used.

The Laplace transform is taken to both sides of Eq. (b.1),

$$sY(s) - y_0 + \alpha Y(s) = B(s), \tag{b.21}$$

where $y_0 = y(0)$. Solving Y(s), Eq. (b.21) is arranged as

$$Y(s) = \frac{B(s)}{s+\alpha} + \frac{y_0}{s+\alpha}. \tag{b.22}$$

The inverse Laplace transform is taken to find the solution,

$$y(t) = \int_0^t b(\tau) e^{-\alpha(t-\tau)} d\tau + y_0 e^{-\alpha t}. \tag{b.23}$$

If the initial condition is set to be $y_0 = 0$, the same result with Eq. (b.14) can be obtained.

Thus, Laplace transform is one of useful method to solve differential equations.

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C Diffusion equation

C.1 Fundamental form of the diffusion equation

The diffusion equation is the partial differential equation which describes the diffusion of material density. The equation is usually written as:

$$\frac{\partial \rho(t,x)}{\partial t} = D\frac{\partial^2 \rho(t,x)}{\partial x^2}, \tag{c.1}$$

where D is said to be the diffusion constant. Let solve the diffusion equation by using the Fourier transform.

C.2 Fourier transform

In this article, we define the Fourier transform $\mathcal{F}$ as:

$$\mathcal{F}[f(x)] = \tilde{f}(k) = \int_{-\infty}^{\infty} dx f(t) e^{-ikx}, \tag{c.2}$$

and the inverse Fourier transform $\mathcal{F}^{-1}$ as:

$$\mathcal{F}^{-1}[\tilde{f}(k)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx}, \tag{c.3}$$

respectively [10].

C.3 Fundamental solution of diffusion equation

We solve the diffusion equation Eq.(c.1) by using the Fourier transform. Taking the Fourier transform of both sides of Eq.(c.1), we obtain

$$\frac{\partial \tilde{\rho}(t,k)}{\partial t} = D\frac{\partial^2 \tilde{\rho}(t,k)}{\partial x^2}. \tag{c.4}$$

At first, we solve Eq.(c.4). After that, the solution is taken the inverse Fourier transform. Thus, we obtain the solution of the original differential equation Eq.(c.1).

It can be computed that

$$\begin{aligned} \frac{\partial \tilde{\rho}(t,k)}{\partial x} &= \int_{-\infty}^{\infty} dx e^{-ikx}\frac{\partial \rho(t,x)}{\partial x} \\ &= [e^{-ikx}p(t,x)]_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} dx e^{-ikx}\rho(t,x) = ik\tilde{p}(t,k), \end{aligned} \tag{c.5}$$

where $\lim\limits_{|x|\to\infty} \rho(t,x) = 0$. That is, Eq.(c.5) states that differential in x-space corresponds to ik times in k-space. Likewise,

$$\frac{\partial^2 \tilde{\rho}(t,k)}{\partial x^2} = -k^2\tilde{\rho}(t,k) \tag{c.6}$$

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[10] It should be noted that they are defined by the pair. It is needed to return the original function for inverse transform.

is satisfied when $\lim_{|x|\to\infty} \frac{\partial \rho(t,x)}{\partial x} = 0$.

Thus, Eq.(c.4) is denoted by:

$$\frac{\partial \tilde{\rho}(t,k)}{\partial t} = -k^2 \tilde{\rho}(t,k). \tag{c.7}$$

Solving Eq.(c.7), we obtain

$$\frac{\tilde{\rho}(t,k)}{\partial t} = Ce^{-Dk^2 t}. \tag{c.8}$$

Taking the inverse Fourier transform of Eq.(c.8), we can compute

$$\begin{aligned} \rho(t,x) &= \frac{C}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} e^{-Dk^2 t} \\ &= \frac{C}{2\pi} \int_{-\infty}^{\infty} \exp\left(-Dt \left(k - \frac{i}{2D} \right)^2 - \frac{x^2}{4Dt} \right) \\ &= \frac{C}{2\pi} \exp\left(-\frac{x^2}{4Dt} \right) \int_{-\infty + \frac{i}{2D}}^{\infty + \frac{i}{2D}} d\xi \exp(-Dt\xi^2) \\ &= \frac{C}{2\pi} \exp\left(-\frac{x^2}{4Dt} \right) \sqrt{\frac{\pi}{Dt}} \\ &= \frac{C}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt} \right), \end{aligned} \tag{c.9}$$

where we use the variable transformation $\xi = k - \frac{i}{2D}$ and Gaussian integral [11] $\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$.

If $\int_{-\infty}^{\infty} dx \rho(t,x) = 1$,

$$\rho(t,x) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt} \right). \tag{c.10}$$

Eq.(c.10) is the fundamental solution of the diffusion equation. The initial condition is given as $\rho(0,x) = \delta(x)$, which is also called a Green's function or a propagator.

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[11] The gamma function $\Gamma(s)$ is useful to calculate the Gaussian integral. It defined as: $\Gamma(s) = \int_0^{\infty} dx\ x^{s-1} e^{-x}$. The Gaussian integral is given by $\Gamma(1/2) = \sqrt{\pi}$.

D Advection equation

D.1 Fundamental form and the solution of advection equation

The Fundamental form of advection equation is described in the following:

$$\frac{\partial\psi(t,x)}{\partial t} = -v\frac{\partial\psi(t,x)}{\partial x}. \tag{d.1}$$

The formula of total derivative

$$d\psi = \frac{\partial\psi}{\partial t}dt + \frac{\partial\psi}{\partial x}dx \tag{d.2}$$

provides

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial t}\frac{dt}{ds} + \frac{\partial\psi}{\partial x}\frac{dx}{ds}. \tag{d.3}$$

When the new variable s satisfies

$$\frac{dt}{ds} = 1 \quad \text{and} \quad \frac{dx}{ds} = v, \tag{d.4}$$

then Eq.(d.3)=0 agrees with Eq.(d.1). When the initial conditions at $s = 0$ are

$$t(s)|_{s=0} = 0 \quad \text{and} \quad x(s)|_{s=0} = x_0, \tag{d.5}$$

then Eq.(d.4) provides

$$t = s \quad \text{and} \quad x = x_0 + vt. \tag{d.6}$$

Eq.(d.3) is rewritten as:

$$\frac{d\psi}{ds} = 0, \tag{d.7}$$

that is, $\psi(s)$ is constant with s. When we have the initial condition $\psi(s)_{s=0} = \psi(0, x_0) = f(x)$, then the general solution of advection equation $psi(t, x)$ is denoted by:

$$\psi(t,x) = f(t, x - vt). \tag{d.8}$$

D.2 Advection equation velocity is not constant

We solve the partial differential equation

$$\frac{\partial\psi(t,x)}{\partial t} = \alpha x\frac{\partial\psi(t,x)}{\partial x}, \tag{d.9}$$

In the same way with previous section, we have

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial t}\frac{dt}{ds} + \frac{\partial\psi}{\partial x}\frac{dx}{ds} = 0, \tag{d.10}$$

where

$$\frac{dt}{ds} = 1 \quad \text{and} \quad \frac{dx}{ds} = -\alpha x. \tag{d.11}$$

When the initial conditions at $s = 0$ are

$$t(s)|_{s=0} = 0 \quad \text{and} \quad x(s)|_{s=0} = x_0, \tag{d.12}$$

then

$$t = s \quad \text{and} \quad x = x_0 e^{-\alpha t} \tag{d.13}$$

When we have the initial condition $\psi(s)_{s=0} = \psi(0, x_0) = f(x)$, then the general solution of advection equation $psi(t, x)$ is denoted by:

$$\psi(t, x) = f(xe^{\alpha t}). \tag{d.14}$$

Eq.(d.14) agrees with the Fokker-Planck equation of radioactivity in the human body shown in Eq.(5.16) at the interval $t > t_m$.

D.3 Advection-diffusion equation

We solve the partial differential equation

$$\frac{\partial \psi(t, x)}{\partial t} = \alpha x \frac{\partial \psi(t, x)}{\partial x} - Dx^2 f(t, x), \tag{d.15}$$

In the same way with previous section, we have

$$\frac{d\psi}{ds} = \frac{\partial \psi}{\partial t}\frac{dt}{ds} + \frac{\partial \psi}{\partial x}\frac{dx}{ds} = -Dx^2\psi(s), \tag{d.16}$$

where

$$\frac{dt}{ds} = 1 \quad \text{and} \quad \frac{dx}{ds} = -\alpha x, \tag{d.17}$$

When the initial conditions at $s = 0$ are

$$t(s)|_{s=0} = 0 \quad \text{and} \quad x(s)|_{s=0} = x_0, \tag{d.18}$$

then

$$t = s \quad \text{and} \quad x = x_0 e^{-\alpha t}. \tag{d.19}$$

Solving

$$\frac{d\psi(s)}{ds} = -Dx^2\psi(s) = -Dx_0^2 e^{-2\alpha s}\psi(s), \tag{d.20}$$

under the initial condition $\psi(0, x) = f(x_0)$, we have

$$\psi(s) = f(x_0) \exp\left(\frac{Dx_0^2 e^{-2\alpha s}}{2\alpha}\right) \tag{d.21}$$

$$\psi(t, x) = f(xe^{\alpha t}) \exp\left(\frac{Dx^2}{2\alpha}\right) \tag{d.22}$$

Eq.(d.22) is computed to solve the linear Fokker-Planck equation as shown in Eqs.(5.16) and (5.57)

[1] S.J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover Publications, New York (1993). (translated in Japanese by M. Iri and Y. Iri)

E Stochastic calculus

E.1 Requirements of randomness

At first, we review fundamental knowledge of randomness and its requirements. The term "randomness" should satisfy the following two requirements. Let consider a six-sided dice as a familiar example of randomness.

1. Equability
   Randomness requires equability. In the example of a six-sided dice, the six pips appear equally as [12]:
   $$5, 3, 6, 2, 3, 2, 3, 6, 6, 5, 2, 5, 5, 5, 6 \cdots . \tag{e.1}$$
   It is not allowed that a specific pip appears frequently [13].

2. Irregularity
   Another requirement of randomness is irregularity. In the example of a six-sided dice, the sequence
   $$1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 1, 2, 3 \cdots . \tag{e.2}$$
   has regularity. Such trial cannot be treated random.

These two requirements state that randomness is unpredictable essentially.

E.2 Brownian motion

Brownian motion [14] is the random and jagged motion of small particles in a fluid as shown in Fig. E.1 (a). It should be noted that Brownian motion is completely random and non-differentiable everywhere. The jaggedness leaves as it stands even if in the infinitesimal displacement, we cannot define the derivative dx/dt.

We introduce a thought experiment for understanding of the existing probability of a Brownian particle. Let be the 1-dimensional space coordinate $x(t)$, where t is time. The initial condition is $x(0) = 0$. The random trial such as dice throwing is repeated n times. The time interval of the trial is τ. That is, the period of time $t = n\tau$. Let be $\hat{x}_i$ a random variable. The index i stands for the number of time steps. The displacement of a Brownian particle $x(t)$ is described as:

$$x(t) = \hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_n. \tag{e.3}$$

Let be

$$\hat{x}_i = \begin{cases} a & \text{(probability: } p) \\ -a & \text{(probability: } p) \\ 0 & \text{(probability: } 1-2p) \end{cases} . \tag{e.4}$$

That is, the displacement $x(t)$ is the sum of random variables. Here, we consider the density of a number of Brownian particles $\rho(t, x)$. Focus the space interval a around the space coordinate x. In the time interval τ, the inflow number of particles and the outflow number

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[12] The author generated the sequence by actual dice throwing!

[13] Such dice will be said to be cheating.

[14] Robert Brown studied random motion of pollen grains at first. He found that the same random motion was also observed for coal dust and fossils and metal powder as with pollen grains. He concluded that the random motion is universal phenomenon which does not depend on a material.

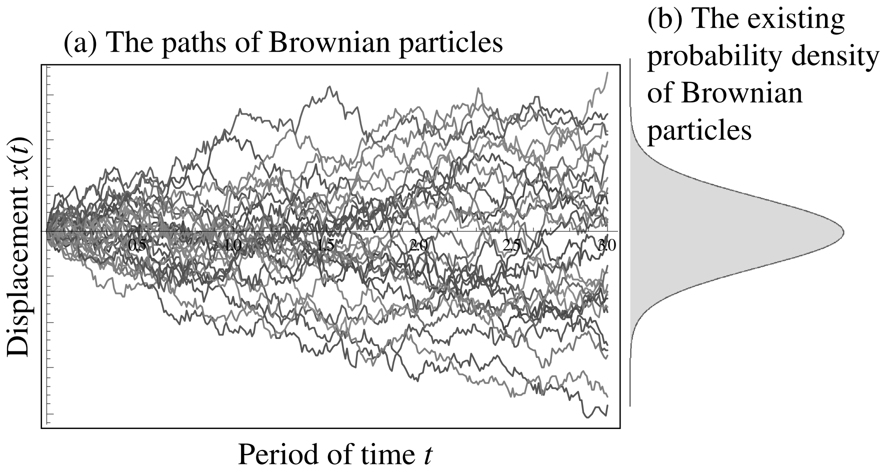

Fig E.1: (a) The paths of Brownian particles which move according to the Wiener Process. (b) The existing probability density of Brownian particles. It distribute according to a normal distribution.

of particles are described as $pa\rho(t,x-a)+pa\rho(t,x+a)$ and $2pa\rho(t,x)$, respectively. We can get

$$a\rho(t+\tau,x) = a\rho(t,x) - 2pa\rho(t,x) + pa\rho(t,x-a) + pa\rho(t,x+a) \tag{e.5}$$

The Taylor expansion of $\rho(t,x)$ gives

$$\rho(t+\tau,x) \simeq \rho(t,x) + \tau\frac{\partial\rho(t,x)}{\partial t} \tag{e.6}$$

and

$$\rho(t,x\pm a) \simeq \rho(t,x) \pm a\frac{\partial\rho(t,x)}{\partial x} + a^2\frac{\partial^2\rho(t,x)}{\partial x^2}. \tag{e.7}$$

Substituting Eqs.(e.6) and (e.7) into the Eq.(e.5) and taking the limit $a \to 0$ and $\tau \to 0$ while remaining a^2/t constant, we can get the diffusion equation,

$$\frac{\partial\rho(t,x)}{\partial t} = D\frac{\partial\rho(t,x)}{\partial x^2}, \tag{e.8}$$

where D is the diffusion constant which satisfies $D = pa^2/\tau$. The fundamental solution of Eq.(e.8) is [15]

$$\rho(t,x) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right) \tag{e.9}$$

under the initial condition $\rho(0,x) = \delta(x)$. Eq.(e.9) states that random Brownian particles obey the orderly motion in the ensemble average. Although each particles move randomly, the set of great number particles are extremely predictable according to a normal distribution as shown in Fig.E.1 (b). We can also understand the operation and result by using Pascal ' s triangle elementary.

In Eqs.(e.6) and (e.7), it seems to be much intentionally that we compute first order Taylor series of time t in spite of we compute second order Taylor series of x. However, the reason is justified by the Ito's rules described later.

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[15] See Appendix.C for detailed solution method.

E.3 Einstein-Smoluchowski relation

The Einstein-Smoluchowski relation connects macroscopic diffusion process with microscopic molecular motion through discussion of Brownian motion. The equation relation is generally denoted by:

$$D = \mu k_B T, \tag{e.10}$$

where k_B stands for the Boltzmann's constant, T is temperature. The mobility μ describes the thermal drift velocity to an applied force f as $v = \dot{x} = \mu f$.

Let prove Einstein-Smoluchowski relation Eq.(e.10). The advection-diffusion equation is generally denoted by:

$$\frac{\partial \rho}{\partial t} = v\frac{\partial \rho}{\partial x} + D\frac{\partial^2 \rho}{\partial x^2}. \tag{e.11}$$

as shown in D.3. In equilibrium state, the time change of density ρ is vanished as:

$$v\frac{d\rho}{dx} = -D\frac{d^2\rho}{dx^2}. \tag{e.12}$$

Solving Eq.(e.12), we have

$$\rho = \rho_0 \exp\left(-\frac{v}{D}x\right) \tag{e.13}$$

under the boundary condition[16] $\rho|_{x=0} = \rho_0$. The potential energy of a particle which is applied to external force f is fx. Hence, the average potential energy of particles is computed as:

$$\frac{f\displaystyle\int_0^\infty dx\ x\rho_0 \exp\left(-\frac{v}{D}x\right)}{\displaystyle\int_0^\infty dx\ \rho_0 \exp\left(-\frac{v}{D}x\right)} = \frac{f\dfrac{1}{(v/D)^2}}{v/D} = f\frac{D}{v}. \tag{e.14}$$

On the other hand, the average energy of particles is given as: $k_B T$ from the law of equipartition of energy regarding one-dimension. That is, we can obtain

$$f\frac{D}{v} = k_B T \Rightarrow D = \mu k_B T. \tag{e.15}$$

Thus, the Einstein-Smoluchowski relation is derived from the advection-diffusion equation and the law of equipartition of energy. It is known that the Boltzmann's constant is denoted by $k_B = R/N_A$, where R and N_A stand for gas constant and Avogadro constant. The Einstein-Smoluchowski relation interestingly suggests that we can know microscopic Avogadro constant from macroscopic diffusion phenomena.

E.4 Langevin equation

We attempts to describe the random motion of Brownian particles by adding random force term to the Newton's equation of motion as:

$$\frac{d\dot{x}(t)}{dt} = -\beta\dot{x}(t) + \eta(t), \tag{e.16}$$

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[16] The boundary condition corresponds with the "bottom" in the potential.

where β is constant and $\eta(t)$ stands for random force. Eq.(e.16) is said to be the Langevin equation. The random force $\eta(t)$ satisfies

$$\langle \eta(t) \rangle = 0, \tag{e.17}$$

and

$$\langle \eta(t)\eta(t') \rangle = \beta k_B T \delta(t - t') = \frac{\beta D}{\mu}\delta(t - t'). \tag{e.18}$$

Eq.(e.17) states that random forces are canceled each other. Eq.(e.18) states that random forces are time-independent and are not affected each other. The random force $\eta(t)$ is pulse-like and non-differentiable everywhere. It seems to be difficult to treat such pathological function $\eta(t)$ in consistent way. Thus, we will study the Wiener process and Ito's rules how to describe the random fluctuation $\eta(t)$ in the next section.

E.5 Wiener process and Ito's rules

A stochastic process is a **set** of time-evolving random function. The Wiener process $W(t)$ is typical one, which obeys the following properties:

(i) $W(t)$ is continuous everywhere. It does not jump to other points discontinuously.

(ii) $W(0) = 0$. That is, the initial point is fixed as the origin point.

(iii) $W(t+\Delta t)$ and $W(t'+\Delta t')$ are is independent respectively, where the interval $[t, t+\Delta t]$ and $[t' + \Delta t']$ do not overlap.

(iv) $W(t+\Delta t)$ is distributed Gaussian as: $N(0, \Delta t) = \frac{1}{\sqrt{2\pi\Delta t}}\exp(-\frac{(\Delta W)^2}{2\Delta t})$. The average $\langle \Delta W(t) \rangle = 0$ and the variance $V(\Delta W(t)) = V(W(\Delta t)) = \Delta t$.

The properties (iii) and (iv) state the scale-invariance of a Wiener process.

Here, we compute the following integral

$$\int_0^t W(\tau) dW(\tau) \tag{e.19}$$

to characterize a Wiener process. Eq.(e.19) is said to be stochastic integral. Let be

$$\Delta W(t_i) = W(t_{i+1}) - W(t_i), \tag{e.20}$$

where $t_{i+1} = t_i + \Delta t_i$. Eq.(e.20) gives

$$\begin{aligned} \{\Delta W(t_i)\}^2 &= \{W(t_{i+1})\}^2 + \{W(t_i)\}^2 - 2W(t_{i+1})W(t_i) \\ &= \{W(t_{i+1})\}^2 - \{W(t_i)\}^2 - 2\{W(t_{i+1}) - W(t_i)\}W(t_i) \\ &= \{W(t_{i+1})\}^2 - \{W(t_i)\}^2 - 2\Delta W(t_i)W(t_i), \end{aligned} \tag{e.21}$$

hence,

$$\{W(t_{i+1})\}^2 - \{W(t_i)\}^2 = \{\Delta W(t_i)\}^2 + 2\Delta W(t_i)W(t_i) \tag{e.22}$$

is satisfied.

Then, stochastic integral Eq.(e.19) is defined as:

$$\sum_i [\{W(t_{i+1})\}^2 - \{W(t_i)\}^2] = \sum_i \{\Delta W(t_i)\}^2 + 2\sum_i \Delta W(t_i)W(t_i). \tag{e.23}$$

Taking $\Delta t_i \to 0$, Eq.(e.23) is denoted by:

$$\{W(t)\}^2 = \int_0^t \{dW(\tau)\}^2 + 2\int_0^t W(\tau)dW(\tau). \tag{e.24}$$

The definition of stochastic integral is similar with Riemann integral. However, everywhere non-differentiable Wiener process cannot be determined rectangular area in the naive sense. Computation rules different from ordinary integral will be required.

We turn to the properties of the Wiener process to compute stochastic integral. The variance of Wiener process is computed as:

$$V(\Delta W) = \left\langle (\Delta W - \langle \Delta W \rangle)^2 \right\rangle = \left\langle (\Delta W)^2 \right\rangle - \langle \Delta W \rangle^2 . \tag{e.25}$$

The property of the Wiener process (iv) states that $\langle \Delta W \rangle = 0$ and $V(\Delta W) = \Delta t$. Considering infinitesimal change, Eq.(e.25) requires

$$\{dW(t)\}^2 = dt. \tag{e.26}$$

t seems to be strange that the square of the infinitesimal change cannot be neglected. The property of a Wiener process which is distributed Gaussian, however, requires that the square of the infinitesimal change of a Wiener process $\{dW(t)\}^2$ has a same order with dt. Eq.(e.26) is the important computation rule to define the stochastic integral. On the other hand,

$$dtdW(t) = 0 \text{ and } (dt)^2 = 0. \tag{e.27}$$

Eqs.(e.26) and (e.27) are said to be the Ito's rules.

Following Ito's rules, Eq.(e.24) yields

$$\begin{aligned} \int_0^t W(\tau)dW(\tau) &= \frac{1}{2}\{W(t)\}^2 - \frac{1}{2}\int_0^t \{W(\tau)\}^2 d\tau \\ &= \frac{1}{2}\{W(t)\}^2 - \frac{1}{2}\int_0^t d\tau \\ &= \frac{1}{2}\{W(t)\}^2 - \frac{1}{2}t. \end{aligned} \tag{e.28}$$

Different from ordinary integral[17], new term $-t/2$ appears.

In this section, we have studied the property of the Wiener process and its calculus. We will again study the Langevin equation by using the Wiener process and the Ito's rules in the next section.

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[17] In this thesis, it means Riemann integral.

E.6 Stochastic differential equation

The Wiener process enables to denote the Langevin equation by:

$$dx(t) = \mu(t,x)dt + \sigma(t,x)dW(t). \tag{e.29}$$

Eq.(e.29) is said to be the stochastic differential equation (SDE) or Ito process. It should be noted that SDE denotes the change dx (displacement) itself rather than the derivative function dx/dt (velocity). The first term of RHS of Eq.(e.29) describes the deterministic external force which does not include randomness, μ is said to be a drift. The second term of RHS Eq.(e.29) describes the random term, σ is said to be a volatility. In physical term, a volatility corresponds to a diffusion constant D as: $D = 1/2\sigma^2$. Their correspondence selectively used as appropriate.

While Eq.(e.29) is denoted by differential form, it can be also denoted by integral form as:

$$x(t) = x(0) + \int_0^t \mu(\tau, x(\tau))d\tau + \int_0^t \sigma(\tau, x(\tau))dW(\tau). \tag{e.30}$$

The first term of RHS of Eq.(e.30) is ordinary integral. The second term of RHS of Eq.(e.30), on the other hand, is Ito integral. It is important idea that the random term is integrated as functional of the Wiener process.

We consider the function $f(t,x)$, where x is the stochastic process denoted by Eqs.(e.29) and (e.30). The Taylor expansion of $f(t,x)$ yields

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + O((dx)^3). \tag{e.31}$$

Substituting Eq.(e.29) into Eq.(e.31), we can get

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu dt + \sigma dW) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu dt + \sigma dW)^2 + O((dx)^3). \tag{e.32}$$

Following the Ito's rules Eqs.(e.26) and (e.27), Eq.(e.32) is described as:

$$df = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dW. \tag{e.33}$$

Eq.(e.33) is said to be the Ito's formula, which states that we obtain another SDE by substituting a SDE into any function.

E.7 Fokker-Planck equation

The Fokker-Planck equation is the partial differential equation which describes the probability density function including random fluctuation. When a random variable follows the SDE, then its probability density function is described by the Fokker-Planck equation. The Langevin equation, SDE and the Fokker-Planck equation are mathematically equivalent each other, they state same one from different viewpoints.

The equivalence is demonstrated by computing the average of an any smooth function $\varphi(x)$ in two ways. Let be $p(t,x)$ is the probability density function of $\varphi(x)$, then the average of $\varphi(x)$ at $t + \delta t$ is generally denoted by

$$\begin{aligned} \langle\varphi\rangle &= \int_{-\infty}^{\infty} \varphi(x)p(t+\delta t, x)dx \\ &= \int_{-\infty}^{\infty} \left(\varphi(x)\frac{\partial p(t,x)}{\partial t}\delta t\right)dx. \end{aligned} \tag{e.34}$$

where infinitesimal change is substituted to δ to distinguish the integral variable. Eq.(e.29) also provides the average of $\varphi(x)$ as:

$$\langle\varphi\rangle = \left\langle \int_{-\infty}^{\infty} \varphi(x+\delta x)p(t,x)dx \right\rangle . \tag{e.35}$$

In Eq.(e.35), the Wiener process term is vanished by averaging, That is, Eq.(e.29) is the constraint itself to establish Eq.(e.34) equals to Eq.(e.35).

The Ito's rules allow us to compute

$$\begin{aligned} \varphi(x+\delta x) &= \frac{\partial\varphi(x)}{\partial x}\delta x + \frac{1}{2}\frac{\partial^2\varphi(x)}{\partial x^2}(\delta x)^2 \cdots \\ &= \frac{d\varphi(x)}{dx}(\mu(t,x)\delta t + \sigma(t,x)\delta W(t)) + \frac{1}{2}\{\sigma(t,x)\}^2\frac{\partial^2\varphi(x)}{\partial x^2}, \end{aligned} \tag{e.36}$$

Eq.(e.35) can be denoted by:

$$\langle\varphi\rangle = \int_{-\infty}^{\infty} \left(\mu(t,x)\frac{d\varphi(x)}{dx} + \frac{1}{2}\sigma^2(t,x)\frac{d^2\varphi(x)}{dx^2}\right)\delta t p(t,x)dx. \tag{e.37}$$

Integral by parts [18] deforms Eq.(e.37) to

$$\langle\varphi\rangle = \int_{-\infty}^{\infty} \left(-\frac{\partial(\mu(t,x)p(t,x))}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma^2(t,x)p(t,x)}{\partial x^2}\right)\varphi(x)dx. \tag{e.38}$$

Eq.(e.34) equals to Eq.(e.38) identically, then

$$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial\{\mu(t,x)p(t,x)\}}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma^2(t,x)p(t,x))}{\partial x^2} \tag{e.39}$$

should be satisfied. Eq.(e.39) is said to be the Fokker-Planck equation, which describes the probability density function of paths that obey the SDE Eq.(e.29).

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[18] When $\lim_{x\to\pm\infty} f(x) = 0$ and $\lim_{x\to\pm\infty} \frac{df(x)}{dx} = 0$, then $\int_{-\infty}^{\infty} dx \frac{dg(x)}{dx} f(x) = [g(x)f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx \frac{df(x)}{dx} g(x) = -\int_{-\infty}^{\infty} dx \frac{df(x)}{dx} g(x)$ and $\int_{-\infty}^{\infty} dx \frac{d^2g(x)}{dx^2} f(x) = \left[\frac{dg(x)}{dx}\frac{df(x)}{dx}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx \frac{d^2f(x)}{dx^2} g(x) = \int_{-\infty}^{\infty} dx \frac{d^2f(x)}{dx^2} g(x)$.

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F Vector calculus

F.1 Linear combination and vector space

Let us consider n scalers c_1, c_2, $\cdots$, c_n and n vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\cdots$, $\boldsymbol{x}_n$. Then,

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n \tag{f.1}$$

is said to be linear combination.

If

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n = \boldsymbol{0}, \tag{f.2}$$

and then,

$$c_1 = c_2 = \cdots = c_n = 0 \tag{f.3}$$

is satisfied, set of vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\cdots$, $\boldsymbol{x}_n$ is said to be linearly dependent. Otherwise, the set of vectors is linearly independent [19].

Here we define n-dimensional vector space as linear combination of linearly independent n vectors. For example, spacetime treated in special relativity is 4-dimensional space consisted of real components [20], while quantum state treated in quantum mechanics is infinity dimensional space consisted of complex components.

F.2 Vector field

The scalar field (e.g. temperature) is a scaler-valued function in space. Each point gives each scalar. The vector field (e.g. air flow) is a vector-valued function in space. Each point in space gives each vector. Here, we shall introduce 3 important operations regarding scalar and vector field [21].

1. The gradient
   The gradient is defined as:

$$\text{grad } f = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{bmatrix} \tag{f.4}$$

   for scalar field $f(x, y, z)$. The gradient indicates the difference in the scalar quantity between two nearby points.

2. The divergence
   The divergence is defined as:

$$\text{div}\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{f.5}$$

   for a vector field $\boldsymbol{A}(x, y, z) = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix}$. The divergence indicates outflow flux of a vector (inflow flux is expressed as negative value).

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[19] In terms of elementary geometry, all vectors are not parallel each other.

[20] In this thesis, we do not use imaginary time.

[21] Although we restrict ourselves to 3-dimensional Cartesian coordinate system in this section for simplicity, these operations can be applied to more dimensional space.

3. The rotation
   The rotation is defined as:

$$\begin{aligned}\mathrm{rot}\boldsymbol{A} &= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \begin{bmatrix} \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} & \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} & \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \end{bmatrix}\end{aligned} \tag{f.6}$$

   for vector $\boldsymbol{A}$ ($\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ are unit bases parallel to x, y, z axes). The rotation is the circulation density [22] of a vector field.

We can treat the differential operator as a vector. The operator ∇ is defined as:

$$\nabla = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{bmatrix}. \tag{f.7}$$

Then, above mentioned 3 operations are rewritten as:

$$\mathrm{grad}\ f = \nabla f, \tag{f.8}$$

$$\mathrm{div}\boldsymbol{A} = \nabla \cdot \boldsymbol{A}, \tag{f.9}$$

and

$$\mathrm{rot}\boldsymbol{A} = \nabla \times \boldsymbol{A}, \tag{f.10}$$

respectively.

The useful formulas of vector calculus are enumerated in the followings [23]:

$$\begin{aligned}
\mathrm{div\ grad}\ f &= \nabla^2 f, && \text{(f.11)}\\
\mathrm{rot\ grad}\ f &= 0, && \text{(f.12)}\\
\mathrm{div\ rot}\boldsymbol{A} &= 0, && \text{(f.13)}\\
\mathrm{rot\ rot}\boldsymbol{A} &= \mathrm{grad\ div}\boldsymbol{A} - \nabla^2\boldsymbol{A}, && \text{(f.14)}\\
\mathrm{div}(f\boldsymbol{A}) &= f\mathrm{div}\boldsymbol{A} + \boldsymbol{A}\cdot\mathrm{grad}\ f, && \text{(f.15)}\\
\mathrm{rot}(f\boldsymbol{A}) &= f\mathrm{rot}\boldsymbol{A} - \boldsymbol{A}\times\mathrm{grad}\ f, && \text{(f.16)}\\
\mathrm{div}(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{B}\cdot\mathrm{rot}\boldsymbol{A} - \boldsymbol{A}\cdot\mathrm{rot}\boldsymbol{B}, && \text{(f.17)}\\
\mathrm{grad}(\boldsymbol{A}\cdot\boldsymbol{B}) &= (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times\mathrm{rot}\boldsymbol{B} + \boldsymbol{B}\times\mathrm{rot}\boldsymbol{A}, && \text{(f.18)}\\
\mathrm{rot}(\boldsymbol{A}\times\boldsymbol{B}) &= (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}\mathrm{div}\boldsymbol{B} + \boldsymbol{B}\mathrm{div}\boldsymbol{A}. && \text{(f.19)}
\end{aligned}$$

As an example, Eq.(f.14) is proved.

$$\mathrm{rot\ rot}\boldsymbol{A} = \nabla\times(\nabla\times\boldsymbol{A}) = \nabla\times\begin{bmatrix} \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} & \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} & \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}. \end{bmatrix} \tag{f.20}$$

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[22] The circulation is defined as: $\oint_\Gamma \boldsymbol{A}\cdot d\boldsymbol{s}$, where Γ is any closed loop and $\boldsymbol{s}$ is a unit vector comes in contact with Γ.

[23] Although the calculation is complicated, the derivation itself is easy according to the definition of gradient, divergence and rotation.

We calculate x component of Eq.(f.20) at first.

$$\begin{aligned}
x \text{ component of Eq.(f.20)} &= \frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \partial z} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A_x \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A_x \\
&= \frac{\partial}{\partial x}(\nabla \cdot \boldsymbol{A}) - \nabla^2 A_x = \frac{\partial}{\partial x}\mathrm{div}\boldsymbol{A} - \nabla^2 A_x.
\end{aligned} \tag{f.21}$$

We can calculate y and z components in the same way as:

$$y \text{ component of Eq.(f.20)} = \frac{\partial}{\partial y}\mathrm{div}\boldsymbol{A} - \nabla^2 A_y, \tag{f.22}$$

$$z \text{ component of Eq.(f.20)} = \frac{\partial}{\partial z}\mathrm{div}\boldsymbol{A} - \nabla^2 A_z, \tag{f.23}$$

respectively. Summarizing Eqs.(f.21-f.23), we can prove Eq.(f.14).

F.3 Covariance and contravariance of vectors

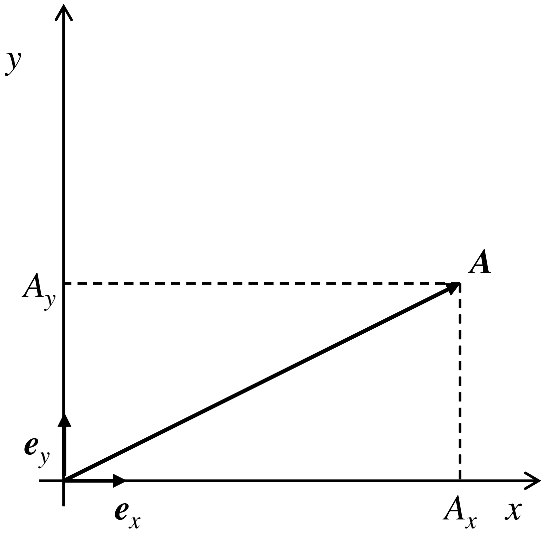

Fig F.1: A vector described on a Cartesian coordinate system.

In 2-dimensional Cartesian coordinate system (Fig.F.1), a vector $\boldsymbol{A}$ is described by the linear combination of unit bases as:

$$\boldsymbol{A} = A_x \boldsymbol{e}_x + A_y \boldsymbol{e}_y. \tag{f.24}$$

Next, we discuss 2-dimensional oblique coordinate system (Fig.F.2). A vector $\boldsymbol{A}$ can be described by different 2 ways.

One way is to use contravariant components and covariant vectors. The meaning of covariance and contravariance is mentioned later in detail. As shown in Fig.F.2(a), A^1 and A^2 are the projections of the vector $\boldsymbol{A}$ parallel with each axis. For example, A^1 is the projection of the vector $\boldsymbol{A}$ parallel with x^2 axis. $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are bases parallel with each axis. A^i is said to be contravariant component. And, $\boldsymbol{e}_i$ is said to be covariant basis. We shall grasp the superscript indicates contravariance, and the subscript indicates covariance. The vector $\boldsymbol{A}$ is described as:

$$\boldsymbol{A} = A^1\boldsymbol{e}_1 + A^2\boldsymbol{e}_2 \tag{f.25}$$

by contravariant components and covariant vectors.

Another way is to use covariant components and contravariant vectors. As shown in Fig.F.2(b) A_1 and A_2 are the projections of the vector $\boldsymbol{A}$ orthogonal with each axis. For example, A_1 is the projection of the vector $\boldsymbol{A}$ orthogonal with x^1 axis. $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$ are bases orthogonal with each axis. It should be noted that the norm of vectors $|\boldsymbol{e}_i| \neq 1$. A_i is said to be covariant component. And, $\boldsymbol{e}^i$ is said to be contravariant basis. The relationship the covariant basis $\boldsymbol{e}_i$ between the contravariant basis $\boldsymbol{e}^i$ is said to be the dual basis. The dual basis are defined to satisfy:

$$\boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}, \tag{f.26}$$

where δ^i_j is said to be the Kronecker delta. The norm of contravariant vectors are defined by Eq. (f.26). The vector $\boldsymbol{A}$ is described as:

$$\boldsymbol{A} = A_1\boldsymbol{e}^1 + A_2\boldsymbol{e}^2 \tag{f.27}$$

by covariant components and contravariant vectors. Thus, a vector is described by 2 different way.

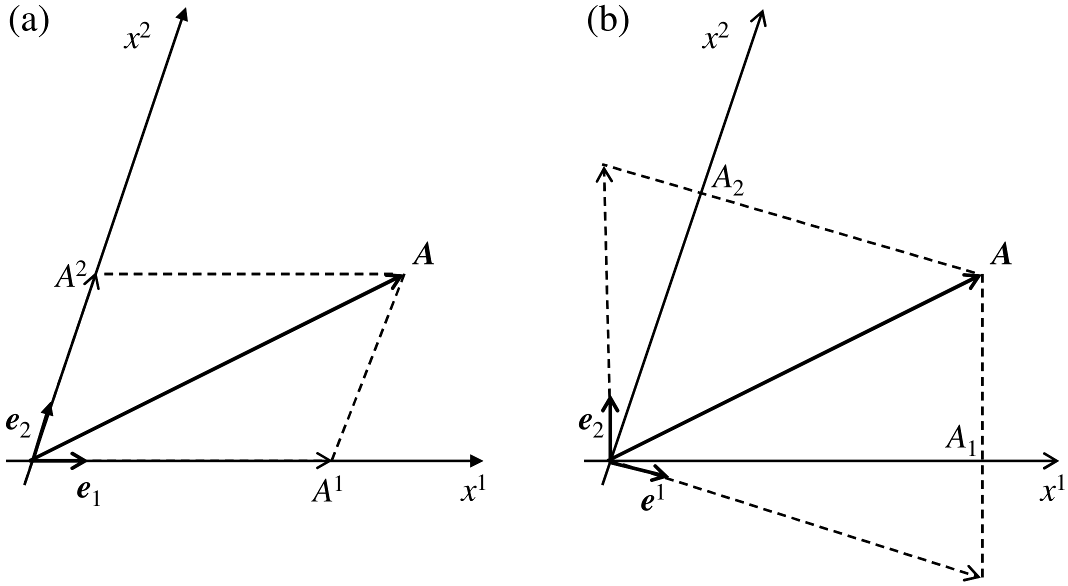

Fig F.2: A vector on a Cartesian coordinate system.

We introduce the Einstein notation defined as below [24]:

$$a^i b_i = \sum_{i=0}^{j} a^i b_i = a^0 b_0 + a^1 b_1 + \cdots + a^j b_j. \tag{f.28}$$

Although we set the range of the summation as $\{0, 1, \cdots, j\}$ at Eq. (f.28), it should be noticed that the range of the summation is voluntary.

The vector $\boldsymbol{A}$ is described as:

$$\boldsymbol{A} = A^i \boldsymbol{e}_i = A_i \boldsymbol{e}^i \tag{f.29}$$

by Einstein notation.

F.4 Transformation law of vectors

Here, we discuss the transformation law of the coordinate system. For simplicity, we treat 3-dimensional space such as $(x, y, z) \to (x', y', z')$ as an example [25]. The base of new coordinate system (x', y', z') are not linear combinations of the base of original coordinate system (x, y, z) [26]. However, the transformation law of the base can be described by linear combination in differential form. We use the following formula of a partial derivative

$$dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy + \frac{\partial x'}{\partial z} dz. \tag{f.30}$$

The right side of Eq. (f.30) is rewritten as:

$$(\text{The right side of Eq. (f.30)}) = \frac{\partial x'}{\partial x^1} dx^1 + \frac{\partial x'}{\partial x^2} dx^2 + \frac{\partial x'}{\partial x^3} dx^3 = \frac{\partial x'}{\partial x^i} dx^i. \tag{f.31}$$

We can calculate y' and z' in the same way as x'.

Thus, we can get

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \tag{f.32}$$

by using Einstein notation [27]. Eq. (f.32) is the transformation law of infinitesimal displacement. We define contravariant components of a vector A^i as the components follow the same transformation law with that of infinitesimal displacement (Eq.(f.32)) as:

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j = a^i_j A^j, \tag{f.33}$$

where a^i_j is described as:

$$a^i_j = \begin{bmatrix} \dfrac{\partial x'^1}{\partial x^1} & \dfrac{\partial x'^1}{\partial x^2} & \dfrac{\partial x'^1}{\partial x^3} \\ \dfrac{\partial x'^2}{\partial x^1} & \dfrac{\partial x'^2}{\partial x^2} & \dfrac{\partial x'^2}{\partial x^3} \\ \dfrac{\partial x'^3}{\partial x^1} & \dfrac{\partial x'^3}{\partial x^2} & \dfrac{\partial x'^3}{\partial x^3} \end{bmatrix} \tag{f.34}$$

---

[24] An index which is summed as i, which is said to be dummy index. An index which is not summed is said to be free index.

[25] We can apply the discussion to more dimensional space of course.

[26] For example, we consider the transformation of spherical coordinate system (r, θ, ϕ) to Cartesian coordinate system (x, y, z). $x = r \sin\theta \cos\phi$, $y = r \sin\theta \cos\phi$, and $z = r \sin\phi$. These transformation laws are not linear combination obviously.

[27] Here, free index i is not summed up, dummy index j is summed up.

in detail, a^i_j is defined as the two rank mixed tensor [28]. Eq. (f.33) is the transformation law of contravariant components of vectors. We can say that the tensor a_{ij} is the linear operator which the contravariant components of a vector A^i map A'^i [29].

Next, we discuss the transformation law of covariant components of a vector. Let's consider a scalar field $f(x, y, z)$,

$$\begin{aligned}\begin{bmatrix}\dfrac{\partial}{\partial x'} & \dfrac{\partial}{\partial y'} & \dfrac{\partial}{\partial z'}\end{bmatrix} f = \nabla' f = \begin{bmatrix}\dfrac{\partial f}{\partial x'} & \dfrac{\partial f}{\partial y'} & \dfrac{\partial f}{\partial z'}\end{bmatrix} \\ = \frac{\partial f}{\partial x'^i} = \begin{bmatrix}\dfrac{\partial f}{\partial x'^1} & \dfrac{\partial f}{\partial x'^2} & \dfrac{\partial f}{\partial x'^3}\end{bmatrix} \\ = \begin{bmatrix}\dfrac{\partial f}{\partial x^j}\dfrac{\partial x^j}{\partial x'^1} & \dfrac{\partial f}{\partial x^j}\dfrac{\partial x^j}{\partial x'^2} & \dfrac{\partial f}{\partial x^j}\dfrac{\partial x^j}{\partial x'^3}\end{bmatrix}.\end{aligned} \tag{f.35}$$

is satisfied. As an example, we denote the first component of Eq.(f.35) in detail as:

$$\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^1} = \frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial x'^1} + \frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial x'^1} + \frac{\partial f}{\partial x^3}\frac{\partial x^3}{\partial x'^1}. \tag{f.36}$$

In the same way, we can obtain second component and third component of Eq.(f.35). Thus, Eq.(f.35) can be rewritten as:

$$\frac{\partial f}{\partial x'^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^i}. \tag{f.37}$$

We define covariant components of a vector A_i as the components follow the same transformation law with that of the gradient of a scalar field (Eq.(f.37)) as:

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j = b^j_i A_j. \tag{f.38}$$

The mixed tensor b^j_i is describe as:

$$b^j_i = \begin{bmatrix}\dfrac{\partial x^1}{\partial x'^1} & \dfrac{\partial x^2}{\partial x'^1} & \dfrac{\partial x^3}{\partial x'^1} \\ \dfrac{\partial x^1}{\partial x'^2} & \dfrac{\partial x^2}{\partial x'^2} & \dfrac{\partial x^3}{\partial x'^2} \\ \dfrac{\partial x^1}{\partial x'^3} & \dfrac{\partial x^2}{\partial x'^3} & \dfrac{\partial x^3}{\partial x'^3}\end{bmatrix} \tag{f.39}$$

in detail.

Although we restrict ourselves to three-dimension in this section for simplicity, we can expand the discussion to more-dimension straightforwardly. In theory of relativity, the transformation law of four-dimensional spacetime corresponding to coordinate transformation is essential for discussion of invariance of physical laws. We should sufficiently notice that a vector itself is not changed by coordinate transformation. Only the description of a vector is changed by coordinate transformation.

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[28] A tensor a_{ij} is said to be a covariant tensor. A tensor a^{ij} is said to be a contravariant tensor.

[29] A vector does not depend on the coordinate, and so $d\boldsymbol{r} = d\boldsymbol{r}'$. And so, $\boldsymbol{e}'_i dx'^i = \boldsymbol{e}_j dx^j$. A vector $\boldsymbol{A}$ is described as $\boldsymbol{A} = \boldsymbol{e}_i A^i$. We can show the transformation law Eq.(f.33) from these relationship.

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G Special relativity

G.1 Maxwell's equations

Maxwell's equations are set of four governing equations which describe electromagnetism, they are summarized as follows:

$$\mathrm{div}\boldsymbol{B} = 0, \tag{g.1}$$

$$\mathrm{div}\boldsymbol{D} = \rho, \tag{g.2}$$

$$\mathrm{rot}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{g.3}$$

$$\mathrm{rot}\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{j}, \tag{g.4}$$

where $\boldsymbol{E}$ is electric field intensity, $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ is electric flux density, $\boldsymbol{H}$ is magnetic field intensity, and $\boldsymbol{B} = \mu \boldsymbol{H}$ is magnetic flux density, respectively. ρ is charge density, $\boldsymbol{j}$ is current density, ε is permittivity, and μ is magnetic permeability Eqs. (g.1)-(g.4) give wave equations of electric field and magnetic field, i.e. electromagnetic wave or light.

The identity of vector calculus

$$\mathrm{rot}\ \mathrm{rot}\boldsymbol{A} = \mathrm{grad}\ \mathrm{div}\boldsymbol{A} - \nabla^2 \boldsymbol{A} \tag{g.5}$$

is used to derive wave equations (again, see Eq.(f.14)).

We consider the space there are no charge $\rho = 0$ and no current $\boldsymbol{i} = 0$, i.e. there are no wave sources such as vacuum. Then, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ (permittivity and magnetic permeability of vacuum). Taking rotation ($\nabla \times$) of Eq. (g.3),

$$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = \nabla \times \left(-\frac{\partial \boldsymbol{B}}{\partial t} \right) \tag{g.6}$$

is obtained from Eq. (g.5).

From Eq. (g.2), left side of Eq. (g.6) is descried as

$$(\text{left side of Eq. (e.6)}) = -\nabla^2 \boldsymbol{E}. \tag{g.7}$$

From Eq. (g.4), right side of Eq. (g.6) is descried as

$$\begin{aligned}(\text{right side of Eq. (e.6)}) &= -\mu_0 \frac{\partial}{\partial t}\left(\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \\ &= \varepsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \end{aligned} \tag{g.8}$$

Eqs. (g.7), (g.8), and velocity of light in vacuum $c = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$ provide the wave equation of electric field;

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0. \tag{g.9}$$

In the same way as electric field $\boldsymbol{E}$, the wave equation of magnetic field $\boldsymbol{H}$ can be obtained by taking rotation of Eq. (g.4) as

$$\nabla^2 \boldsymbol{H} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0. \tag{g.10}$$

Eq. (g.1) is used for deriving Eq.(g.10).

Eq.(g.9) gives

$$\frac{\partial^2 E_x(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_x(x,t)}{\partial t^2} = 0, \tag{g.11}$$

where $E_x(x,t)$ is the x component of the electric field $\boldsymbol{E}$, and a function of two variables; displacement x and time t.

Let's solve the wave equation Eq.(g.11) according to d'Alembert's formula [30]. We introduce new two variables as: $\xi(x,t) = x - ct$ and $\eta(x,t) = x + ct$. The electric field is replaced as $E_x(x,t) = E_x(\xi, \eta)$. Then, we can get

$$\frac{\partial E_x}{\partial t} = \frac{\partial E_x}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial E_x}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial E_x}{\partial \xi} + c\frac{\partial E_x}{\partial \eta} \tag{g.12}$$

and

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial E_x}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial E_x}{\partial \xi} + \frac{\partial E_x}{\partial \eta} \tag{g.13}$$

by applying the chain rule[31]. Eqs.(g.12-g.13) give

$$\frac{\partial^2 E_x}{\partial t^2} = c^2\frac{\partial^2 E_x}{\partial \xi^2} - 2c^2\frac{\partial^2 E_x}{\partial \xi \partial \eta} + c^2\frac{\partial^2 E_x}{\partial \eta^2} \tag{g.14}$$

and

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{\partial^2 E_x}{\partial \xi^2} + \frac{\partial^2 E_x}{\partial \xi \partial \eta} + \frac{\partial^2 E_x}{\partial \eta^2}, \tag{g.15}$$

respectively. Substituting Eqs.(g.14-g.15) into Eq.(g.11),

$$\frac{\partial^2 E_x}{\partial \xi^2} + \frac{\partial^2 E_x}{\partial \xi \partial \eta} + \frac{\partial^2 E_x}{\partial \eta^2} - \frac{1}{c^2}\left(c^2\frac{\partial^2 E_x}{\partial \xi^2} - 2c^2\frac{\partial^2 E_x}{\partial \xi \partial \eta} + c^2\frac{\partial^2 E_x}{\partial \eta^2}\right) = 0, \tag{g.16}$$

which is simplified into

$$\frac{\partial^2 E_x}{\partial \xi \partial \eta} = 0. \tag{g.17}$$

The general solution of Eq.(g.17) is:

$$E_x(x,t) = f(\xi) + g(\eta) = f(x - ct) + g(x + ct). \tag{g.18}$$

Indeed, Eq.(g.18) describes the waveform $E_x(x, t_0)$ goes in the velocity c without deformation. The first term of Eq.(g.18), $f(x - ct)$, stands for the forward wave, the second term of Eq.(g.18), $g(x + ct)$, stands for the backward wave.

We can obtain y component E_y and z component E_z of the electric field $\boldsymbol{E}$ in the same way. Furthermore, the solution of Eq.(g.10), the magnetic field $\boldsymbol{H}$ can be also obtained in the same way.

As shown in Eqs. (g.9) and (g.10), the velocity of electromagnetic wave (light) dose not depend on the observer, depends on only permittivity ε_0 and magnetic permeability μ_0. This result contradicts with Newtonian mechanics which sets the absolute time and space. The inconsistency is explained by the principle of invariant light speed and the special principle of relativity, that is special relativity.

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[30] We can also derive d'Alembert's formula by Fourier transform. Originally, Fourier transform was invented to solve the partial differential equation of heat diffusion.

[31] If $z = f(x,y)$, $x(t) = g(t)$ and $y(t) = h(t)$, then $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$.

G.2 Lorentz transformation

In Newtonian mechanics, regarding inertial systems S and S', time is independent of inertial frames of reference, $t = t'$. Galilean transformation associates inertial systems S with S' as:

$$x' = x - vt, \tag{g.19}$$

$$y' = y, \tag{g.20}$$

$$z' = z, \tag{g.21}$$

$$t' = t, \tag{g.22}$$

where the inertial frame of reference S' is in motion toward S at velocity v along the x-direction. Although Galilean transformation is sufficiently correct in the case of $v/c \ll 1$, contradicts with electromagnetism as shown in G.1.

Einstein proposed the principle of invariant light speed, which claims that velocity of light is independent of the observer. And, the principle of relativity claims that every physical laws are equivalent for every frames of reference. Lorentz transformation shows how to associate with the coordinates of two frames of reference. Lorentz transformation is described as

$$ct' = \gamma(ct - \beta x) \tag{g.23}$$

$$x' = \gamma(x - \beta ct) \tag{g.24}$$

$$y' = y \tag{g.25}$$

$$z' = z, \tag{g.26}$$

where $\beta = \dfrac{v}{c}$, $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$. γ is said to be Lorentz factor. It is emphasized that Lorentz's work preceded Einstein's work, and was done independently.

Lorentz transformation Eq. (g.26) can be written in matrix form:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}. \tag{g.27}$$

In special relativity, time and space is comparable, a coordinate system is redefined as a four-dimensional vector. The four-dimensional vector of spacetime is said to be four-vector. Four-vector is written as

$$x^\mu = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}. \tag{g.28}$$

Lorentz transformation Eq. (g.26) can be written concisely in the Einstein notation:

$$x'^\mu = L^\mu{}_\nu x^\nu \tag{g.29}$$

where μ and ν are 0,1,2,3, and

$$L^\mu{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{g.30}$$

Although the interval of two points is $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ in three-dimension space, the interval is also redefined by including time to the coordinates in four-dimensional spacetime. The spacetime interval Δs is defined as

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2. \tag{g.31}$$

Eq. (g.31) is rewritten infinitesimally as

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2. \tag{g.32}$$

The spacetime interval is invariant to Lorentz transformation,

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2 \tag{g.33}$$

is satisfied identically regarding two inertial frames of reference S and S'. $(ds)^2$ is invariant to any inertial systems.

Eq. (g.33) is written in the Einstein summation convention as

$$\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu, \tag{g.34}$$

where

$$\eta_{\mu\nu} = \begin{cases} 1 & (\mu = \nu = 0) \\ 0 & (\mu \neq \nu) \\ -1 & (\mu = \nu = 1, 2, 3) \end{cases}, \tag{g.35}$$

$\eta_{\mu\nu}$ is said to be Minkowski metric [32].

G.3 Relativistic mechanics

In Eq. (g.33), if $dx' = dy' = dz' = 0$,

$$(cdt')^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{g.36}$$

is satisfied. $\tau = t'$ is said to be proper time, which describes difference of elapsed time between two inertial systems S and S'. Then, S' can be regarded as an inertial frame of reference of a fixed observer.

Dividing Eq. (g.36) into dt,

$$\frac{d\tau}{dt} = \sqrt{1 - \beta^2} = \gamma \tag{g.37}$$

is obtained, where $\beta = \frac{1}{c}\frac{d\mathbf{v}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$.

Eq. (g.36) in the Einstein summation convention as

$$(cd\tau)^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{g.38}$$

Dividing Eq. (g.38) into $(d\tau)^2$,

$$c^2 = \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{g.39}$$

---

[32] A metric tensor is defined by the inner product of two base as: $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$. In a curved space, the metric tensor for each point on the coordinate $g_{ij}(x)$ characterizes the curved state of the space. A curved spacetime is essential idea of general relativity; we shall not discuss general relativity here. In addition, Minkowski metric is the metric tensor of plane spacetime.

is obtained, where $\dot{x}$ is $\dfrac{dx}{d\tau}$.

In Newtonian mechanics, momentum is defined as

$$p^i = m\frac{dx^i}{dt} \ (i = 1, 2, 3). \tag{g.40}$$

In special relativity, momentum is also redefined as the four-vector. Straightforward expansion of momentum to spacetime is

$$p^\mu = m\frac{dx^\mu}{d\tau} \ (\mu = 0, 1, 2, 3), \tag{g.41}$$

p^μ is said to be four-momentum. Then,

$$p^0 = m\frac{cdt}{d\tau} = \frac{mc}{\sqrt{1-\beta^2}} = mc\gamma. \tag{g.42}$$

When Newtonian limit, $v \ll c$ i.e. $\beta \to 0$, is taken,

$$p^0 = mc(1-\beta^2)^{-\frac{1}{2}} \approx mc(1 + \frac{1}{2}\beta^2) = mc + \frac{1}{c}\cdot\frac{1}{2}mv^2. \tag{g.43}$$

From Eq. (g.43),

$$cp^0 \approx mc^2 + \frac{1}{2}mv^2 \tag{g.44}$$

is obtained. cp^0 agree with energy of the mass point. The first term mc^2 is the rest mass energy, the second term $\dfrac{1}{2}mv^2$ is the kinetic energy. The first term shows mass-energy equivalence.

G.4 Mass-energy equivalence: $E = mc^2$

In the followings, mass-energy equivalence is discussed in detail. Multiplying m^2 by both sides of Eq. (g.39)

$$m^2c^2 = \eta_{\mu\nu}(m\dot{x}^\mu)(m\dot{x}^\nu). \tag{g.45}$$

From definition Eq.(g.41), Eq. (g.45) is arranged as

$$m^2c^2 = \eta_{\mu\nu}p^\mu p^\nu. \tag{g.46}$$

Substituting $\boldsymbol{p} = p^i$ $(i = 1, 2, 3)$, i.e. space components of p^μ, to Eq.(g.46),

$$m^2c^2 = (p^0)^2 - (\boldsymbol{p})^2 \tag{g.47}$$

is obtained. Arranging Eq.(g.47),

$$(cp^0)^2 = (mc^2)^2 + (c\boldsymbol{p})^2 = E^2, \tag{g.48}$$

i.e.

$$E = \pm\sqrt{(mc^2)^2 + (c\boldsymbol{p})^2}. \tag{g.49}$$

When $\boldsymbol{p} = 0$ and $E > 0$,

$$E = mc^2 \tag{g.50}$$

is satisfied. Eq. (g.50) is the famous equation which describes mass-energy equivalence achieved by Einstein.

G.5 Velocity of β ray

When the kinetic energy of an electron is similar or larger than rest mass energy of an electron 0.5 MeV, the velocity of an electron v is asymptotic to velocity of light c. The energy of an electron E_e is described to

$$E_e = \sqrt{(m_e c^2)^2 + (cp)^2} \tag{g.51}$$

The momentum p is

$$p = \frac{m_e v}{\sqrt{1-(v/c)^2}}, \tag{g.52}$$

relativistically. Eq. (g.52) is arranged by v,

$$v = c\sqrt{1 - \frac{m_e^2 c^4}{E_e^2}} \tag{g.53}$$

In the case of β decay of ^{137}Cs, kinetic energy of an electron is maximally 1.174 MeV. When kinetic energy of an electron is 1.174 MeV, $v \simeq 0.95c$.

G.6 Four-current

Current is also redefined relativistically as:

$$j^\mu = \begin{bmatrix} j^0 \\ j^1 \\ j^2 \\ j^3 \end{bmatrix} = \begin{bmatrix} c\rho \\ j_x \\ j_y \\ j_z \end{bmatrix}. \tag{g.54}$$

j^μ is said to be four-current.

Taking divergence of both sides of Eq. (g.4), we can get charge conservation as:

$$\mathrm{div}\boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0, \tag{g.55}$$

which is the continuity equation of the charge density.

Here, let's define [33]

$$\partial_\mu = \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix}. \tag{g.56}$$

Then, Eq. (g.55) can be rewritten as:

$$\partial_\mu j^\mu = 0. \tag{g.57}$$

Eq. (g.57) must be satisfied in any inertial systems, i.e.

$$\partial_\mu j^\mu = \partial'_\mu j'^\mu = 0. \tag{g.58}$$

Eq. (g.58) shows that charge conservation is invariant to the transformation of inertial systems [34]. The Lorentz transformation of the four-current j^μ is shown to be

$$c\rho' = \gamma(c\rho - \beta j_x), \tag{g.59}$$

$$j'_x = \gamma(j_x - \beta c\rho), \tag{g.60}$$

$$j'_y = j_y, \tag{g.61}$$

$$j'_z = j_z. \tag{g.62}$$

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[33] A partial differential operator is covariant.

[34] Furthermore, Maxwell's equations themselves are covariant to the Lorentz transformation.

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H Quantum mechanics

H.1 Schrödinger equation

Quanta, for example photons and electrons, have wave-particle duality. Einstein claimed that light has wave-particle duality, he proposed an idea of light particle, it is called photon now. Energy of a photon E depends on its frequency ν or angular frequency ω as

$$E = h\nu = \hbar\omega \tag{h.1}$$

and momentum of photon p depends on its wave length λ or wave number k as

$$p = \frac{h}{\lambda} = \hbar k, \tag{h.2}$$

where h is Planck's constant, $\hbar = h/2\pi$ is reduced Planck's constant.

De-Broglie claims that electrons also have wave-particle duality, the same relationship with light can be expanded to any particles. Eqs. (h.1) and (h.2) are Einstein-De Broglie relations.

The wave function of de-Broglie wave $\psi(x, t)$ is described by typical plane wave as

$$\psi(x, t) = Ae^{i(kx-\omega t)}. \tag{h.3}$$

Eq. (h.3) rewritten as

$$\psi(x, t) = Ae^{i(px-Et)/\hbar} \tag{h.4}$$

from Einstein-De Broglie relations; Eqs. (h.1) and (h.2).

The energy of an particle E is written as

$$E = \frac{p^2}{2m} + V, \tag{h.5}$$

where m is mass and V is potential energy respectively.

Correspondence of Eqs. (h.4) with (h.5) is discussed.

Differentiating wave function h.4 twice with respect to x,

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\psi. \tag{h.6}$$

Differentiating wave function (h.4) with respect to t,

$$\frac{\partial\psi}{\partial t} = -i\frac{E}{\hbar}\psi. \tag{h.7}$$

Substituting Eqs. (h.6) and (h.7) to Eq. (h.5),

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi, \tag{h.8}$$

Schrödinger equation Eq. (h.8) can be obtained.

Schrödinger equation, Eq. (h.8), can be interpreted the law of the conservation of energy in quantum mechanics. The energy and the momentum become differential operators as

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \tag{h.9}$$

and

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}, \tag{h.10}$$

respectively. In quantum mechanics, "physical quantities" is operator not real number. Such operators are said to be observables.

The Hamiltonian is also defined by an operator as

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \tag{h.11}$$

according to canonical quantization. Eq.(h.8) is rewritten as

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{h.12}$$

When ψ does not depend on time t (i.e. steady state),

$$\hat{H}\psi = E\psi, \tag{h.13}$$

where E is energy eigenvalue of an operator $\hat{H}$. That is, the Hamiltonian is the operator which generates energy eigenvalue by taking the inner product with the wave function.

Thus far, one-dimensional Schrödinger equation is discussed. Expansion to three-dimensional is straightforward, only position x is redefined

$$x = \boldsymbol{x} = (x, y, z) \tag{h.14}$$

as a three-dimensional vector. The momentum and the Hamiltonian is redefined as

$$\hat{p} = -i\hbar\nabla, \tag{h.15}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V, \tag{h.16}$$

by using $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, respectively. Eq. (h.8) can be rewritten as:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{h.17}$$

Although the interpretation of the wave function $\psi(x, t)$ is important problem, it is generally accepted that square of the absolute value represents probability density of the particle. Indeed, Schrödinger equation, Eq. (h.17), can be also treated as the continuity equation of the probability density. Here, taking Eq. (h.17) $\times\psi^*$ [35],

$$i\hbar\psi^*\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi\psi^*. \tag{h.18}$$

Additionally, taking the complex conjugate of Eq. (h.17) $\times\psi$,

$$-i\hbar\psi\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\psi\nabla^2\psi^* + V\psi\psi^*. \tag{h.19}$$

---

[35] Regarding the complex number $\psi = a + ib$, $\psi^* = (a + ib)^* = a - ib$, thus $*$ means the complex conjugate.

Subtracting Eq. (h.19) from Eq. (h.18),

$$\psi\frac{\partial\psi^*}{\partial t}+\psi^*\frac{\partial\psi}{\partial t}\frac{i\hbar}{2m}(\psi\nabla^2\psi^*-\psi^*\nabla^2\psi)=0. \tag{h.20}$$

Let define the probability density ρ and the probability current $\boldsymbol{j}$ as:

$$\rho=\psi^*\psi=|\psi|^2, \tag{h.21}$$

and

$$\boldsymbol{j}=\frac{i\hbar}{2m}(\psi\nabla\psi^*-\psi^*\nabla\psi), \tag{h.22}$$

respectively. By Eqs.(h.21) and (h.22), Schrödinger equation (Eq. (h.17)), can be rewritten as [36] :

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{j}=0. \tag{h.23}$$

Eq. (h.23) is the continuity equation of the probability density [37].

H.2 Linear algebra for quantum mechanics

Basic linear algebra including eigenvalue problem is necessary for understanding of quantum mechanics.

If square matrix A satisfies

$$A\boldsymbol{x}=\lambda\boldsymbol{x}, \tag{h.24}$$

λ and $\boldsymbol{x}$ are said to be eigenvalue and eigenvector respectively only $\det A\neq 0$ and $\boldsymbol{x}=\boldsymbol{0}$. Eq. (h.24) is rewritten as

$$(A-\lambda I)\boldsymbol{x}=\boldsymbol{0}, \tag{h.25}$$

where I is an identity matrix. By solving the equation

$$\det(A-\lambda I)=0, \tag{h.26}$$

eigenvalue of matrix A, λ is obtained. From eigenvalue λ, eigenvector $\boldsymbol{x}$ can be obtained.

Eigenvalue λ and eigenvector $\boldsymbol{x}$ are determined by square matrix A, then A acts as an operator which multiplies constant by eigenvector $\boldsymbol{x}$.

Here, we introduce bra-ket notation. In quantum mechanics, quantum states are described by infinite dimension complex vector space, whose basic vectors are set to be orthonormal basis as

$$|\psi\rangle=\begin{bmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\\ \vdots\end{bmatrix}. \tag{h.27}$$

$|\psi\rangle$ is said to be ket vector. Complex conjugate transpose of ket is

$$|\psi\rangle\dagger=\begin{bmatrix}\psi_1^* & \psi_2^* & \cdots\psi_n^*\end{bmatrix}=\langle\psi|, \tag{h.28}$$

---

[36] $\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial t}\psi^*\psi=\psi^*\frac{\partial}{\partial t}\psi+\psi\frac{\partial}{\partial t}\psi^*$

[37] Eq. (h.23) is the same form with the continuity equation in the narrow sense of fluid dynamics: $\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=0$.

where $\langle\psi|$ is said to be bra vector.

Bra-ket notation is a standard notation for describing quantum state,

$$\begin{aligned}\langle\psi|\phi\rangle &= \text{ inner product of ket}|\psi\rangle \text{ with ket}|\phi\rangle \\ &= \begin{bmatrix}\psi_1^* & \psi_2^* & \cdots & \psi_n^* & \cdots\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\\ \vdots\end{bmatrix} = \sum_i \psi_i^*\phi_i.\end{aligned} \tag{h.29}$$

Then,

$$\langle\psi|\psi\rangle = ||\psi\rangle|^2 \tag{h.30}$$

is satisfied.

A quantum state $|\psi\rangle$ can be described by linear combination of ket vectors as

$$\begin{aligned}|\psi\rangle &= c_1|e_1\rangle + c_2|e_2\rangle + \cdots + c_n|e_n\rangle + \cdots = \sum_i c_i|e_i\rangle \\ &= |\psi_1\rangle + |\psi_2\rangle + \cdots + |\psi_n\rangle + \cdots = \sum_i |\psi_i\rangle,\end{aligned} \tag{h.31}$$

where $c_1, c_2, \cdots, c_n, \cdots$ are complex number, and $|e_1\rangle, |e_2\rangle \cdots, |e_n\rangle, \cdots$ are unit vector linearly independent as

$$|e_1\rangle = \begin{bmatrix}1\\0\\ \vdots\\0\\ \vdots\end{bmatrix}, \quad |e_2\rangle = \begin{bmatrix}0\\1\\ \vdots\\0\\ \vdots\end{bmatrix}, \cdots, \quad |e_n\rangle = \begin{bmatrix}0\\0\\ \vdots\\1\\ \vdots\end{bmatrix}, \cdots. \tag{h.32}$$

respectively.

The linear combination of kets $|\psi\rangle$ is said to be superposition. The square of the absolute value of the wave function is interpreted as the probability,

$$\sum_{i=1}^{\infty} |c_i|^2 = \sum_{i=1}^{\infty} |\psi_i|^2 = 1 \tag{h.33}$$

is required.

Any bra vectors and ket vectors satisfies the requirements of orthonormal basis. Eq. (h.31) satisfies the following 3 requirements.

[i] The orthogonal system satisfies that the inner products of each different basis are 0. It is described as:

$$\langle\psi_i|\psi_j\rangle = \delta_{i,j}, \tag{h.34}$$

where $\delta_{i,j}$ is said to be Kronecker delta. The orthogonality can be shown by Eq. (h.31).

[ii] The normality is satisfied as Eq. (h.33), i.e. the norm is 1.

[iii] The complete system is defined that the vector is described by linear combination of vectors as Eq. (h.31). Complete system satisfies the following completeness relation as

$$\sum_i |\psi_i\rangle\langle\psi_i| = 1. \tag{h.35}$$

These properties of orthonormal basis shown in the above are important in quantum mechanics.

We will study quantum mechanics from the viewpoints of linear algebra in the following sections.

H.3 Quantum state vector

Using bra-ket notation, Schrödinger equation is denoted by [38]:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \tag{h.36}$$

where the Hamiltonian $\hat{H}$ is an operator which yields the energy eigenvalues by taking the inner product of the state vector $|\psi(t)\rangle$. Wave functions are describe as bra vectors, which is called quantum state vectors (or simply state vectors).

In the same way, steady Schrödinger equation is described as

$$\hat{H}|\psi\rangle = E|\psi\rangle, \tag{h.37}$$

where E is the energy eigenvalue. As shown in Eq. (h.16), steady Schrödinger equation results in eigenvalue problem itself.

H.4 Expectation of observables

According to bra-ket notation, the expectation of an observable $\hat{A}$ is denoted by:

$$\langle\psi|\hat{A}|\psi\rangle = \sum_{i,j} a_i c_i c_j^* \langle\psi_i|\psi_j\rangle = \sum_i a_i|c_i|^2 = \text{expectation of A} = \langle A\rangle\,. \tag{h.38}$$

Eq.(h.38) is useful, often used hereinafter. Newtonian mechanics is derived as the expectation in quantum mechanics.

H.5 Quantum superposition

As shown in Eq.(h.31), state vector is describe as the linear combination of each state. It is the quantum superposition of state vectors. We consider the superposition of two state

$$|\psi\rangle = c_1|\psi_1\rangle + c_1|\psi_2\rangle \tag{h.39}$$

for simplicity. The expectation of observable $\hat{A}$ in the state $|\psi\rangle$ is described as:

$$\begin{aligned}\langle\psi|\hat{A}|\psi\rangle &= (c_1|\psi_1\rangle + c_1|\psi_2\rangle)(c_1|\psi_1\rangle + c_1|\psi_2\rangle)^\dagger \\ &= |c_1|^2\langle\psi_1|\hat{A}|\psi_1\rangle + |c_2|^2\langle\psi_2|\hat{A}|\psi_2\rangle + c_1^*c_2\langle\psi_1|\hat{A}|\psi_2\rangle + c_1c_2^*\langle\psi_2|\hat{A}|\psi_1\rangle\end{aligned} \tag{h.40}$$

---

[38] Eq.(h.36) is represented by the Schrödinger picture, which considers the state vectors evolve in time but the observables are constant with time. Inverse formulation is the Heisenberg picture.

The first and second terms of Eq.(h.40) are linear combination of expectation of $\hat{A}$ in each states $|\psi_1\rangle$ and $|\psi_2\rangle$. In classical mechanics, the expectation $\langle A\rangle$ agrees with the sum of the first and second terms of Eq.(h.40). However, the third and fourth terms of Eq.(h.40) yields inconsistency of quantum mechanics with classical mechanics. These terms are called interference terms, which account for quantum phenomena such as electron interference.

Incidentally, Schrödinger's cat is a well-known thought experiment regarding quantum superposition. Although a cat that may be both alive and dead seems counter-intuitive, superposition principle explains the conflict. The quantum state of a cat $|\psi_{\text{cat}}\rangle$ is described as

$$|\psi_{\text{cat}}\rangle = |\psi_{\text{alive}}\rangle + |\psi_{\text{dead}}\rangle, \tag{h.41}$$

i.e., $|\psi_{\text{cat}}\rangle$ is the superposition of alive state $|\psi_{\text{alive}}\rangle$ and dead state $|\psi_{\text{dead}}\rangle$. $|\psi_{\text{alive}}\rangle$ and $|\psi_{\text{dead}}$ are linearly independent, there is no irrational point. Then,

$$\langle\psi_{\text{dead}}|\psi_{\text{alive}}\rangle = 0, \tag{h.42}$$

$$\langle\psi_{\text{cat}}|\psi_{\text{cat}}\rangle = 1 \tag{h.43}$$

are satisfied respectively. Thus, the state of a cat is understood as quantum superposition.

H.6 Commutation relation

When operators $\hat{A}$ and $\hat{B}$ satisfy $\hat{A}\hat{B} - \hat{B}\hat{A} = 0$, then $\hat{A}$ and $\hat{B}$ are said to be commutative. We define the commutator as:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{h.44}$$

When $[\hat{A}, \hat{B}] = 0$ is satisfied, then operators $\hat{A}$ and $\hat{B}$ are called commutative. Also, an operator which satisfies $\hat{A}^\dagger = \hat{A}$ is said to be a Hermitian operator. We will study the commutation relation of Hermitian operator in this section.

Let be

$$\Delta\hat{A} = \hat{A} - \langle A\rangle \text{ and } \Delta\hat{B} = \hat{B} - \langle B\rangle\,. \tag{h.45}$$

Being $\langle A\rangle$ and $\langle B\rangle$ are constant, we obtain the commutation relation

$$[\Delta\hat{A}, \Delta\hat{B}] = [\hat{A} - \langle A\rangle\,, \hat{B} - \langle B\rangle] = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A}, \hat{B}]. \tag{h.46}$$

The expectation of the variance of $\left\langle \hat{A} \right\rangle$ is denoted by:

$$\begin{aligned} \left\langle (\Delta\hat{A})^2 \right\rangle &= \left\langle (\hat{A})^2 - 2\hat{A}\,\langle A\rangle + \langle A\rangle^2 \right\rangle = \left\langle (\hat{A})^2 \right\rangle - \langle A\rangle^2 \\ &= \langle\psi|(\Delta\hat{A})^2|\psi\rangle, \end{aligned} \tag{h.47}$$

This also applies to $\Delta\hat{B}$.

When operators $\hat{A}$ and $\hat{B}$ are non-commutative as

$$[\hat{A}, \hat{B}] = ik \tag{h.48}$$

and Hermitian operators, then $\left\langle (\Delta\hat{A})^2 \right\rangle \left\langle (\Delta\hat{B})^2 \right\rangle$ has a positive-real minimum value. It is proved in the next section.

H.7 Uncertainty principle

Let consider $\Delta\hat{A} + i\lambda\Delta\hat{B}$. The probability is positive for any λ, therefore, it is computed as:

$$\begin{aligned}||(\Delta\hat{A} + i\lambda\Delta\hat{B})|\psi\rangle|^2 &= ((\Delta\hat{A} + i\lambda\Delta\hat{B})|\psi\rangle)^\dagger(\Delta\hat{A} + i\lambda\Delta\hat{B})|\psi\rangle \\ &= \langle\psi|(\Delta\hat{A} - i\lambda\Delta\hat{B})(\Delta\hat{A} + i\lambda\Delta\hat{B})\ \psi\rangle \\ &= \langle\psi|(\Delta\hat{A})^2 + i\lambda(\Delta\hat{A}\Delta\hat{B} - \Delta\hat{B}\Delta\hat{A}) + \lambda^2(\Delta\hat{B})^2|\psi\rangle \\ &= \left\langle(\Delta\hat{A})^2\right\rangle + i\lambda\left\langle[\Delta\hat{A}, \Delta\hat{B}]\right\rangle + \lambda^2\left\langle(\Delta\hat{B})^2\right\rangle \\ &\geq 0.\end{aligned} \tag{h.49}$$

Substituting Eqs.(h.46) and (h.48) into Eq.(h.49),

$$\left\langle(\Delta\hat{A})^2\right\rangle - k\lambda + \lambda^2\left\langle(\Delta\hat{B})^2\right\rangle \geq 0. \tag{h.50}$$

is obtained. Discriminant of quadratic equation gives

$$k^2 - 4\left\langle(\Delta\hat{A})^2\right\rangle\left\langle(\Delta\hat{B})^2\right\rangle \leq 0, \tag{h.51}$$

finally, we obtain

$$\left\langle(\Delta\hat{A})^2\right\rangle\left\langle(\Delta\hat{B})^2\right\rangle \geq \frac{|k|}{2}. \tag{h.52}$$

In quantum mechanics, position operator $\hat{x}$ and momentum operator $\hat{p}$ has a commutation relation

$$[\hat{x}, \hat{p}] = i\hbar, \tag{h.53}$$

that is,

$$\left\langle(\Delta\hat{x})^2\right\rangle\left\langle(\Delta\hat{p})^2\right\rangle = \sigma_x\sigma_p \geq \frac{\hbar}{2} \tag{h.54}$$

is satisfied. Eq.(h.54) is the uncertainty principle. Uncertainty principle claims that both $\sigma_x = 0$ and $\sigma_p = 0$ are not satisfied st the same time. When position x is completely determined as $\sigma_x \to 0$, then momentum is not determined at all as $\sigma_p \to \infty$. We cannot know both position and momentum in deterministic principally. If we would like to know both A and B deterministically, they should be commutative as $[A, B] = 0$. Nature defines that the position and momentum are not commutative, their variances are defined by Planck constant.

H.8 Perturbation theory

Let see again Schrödinger equation by using bra-ket notation Eq. (h.36).

It is postulated that $\hat{H}$ can be divided as

$$\hat{H} = \hat{H}_0 + \hat{V}. \tag{h.55}$$

The energy eigenvalues E_n and the eigenvector $|\varphi_n\rangle$ of unperturbed Hamiltonian $\hat{H}_0$ are assumed to be known exactly. When $\hat{V}$ is sufficiently small, the energy eigenvalues and the eigenvector of $\hat{H}$ will be almost the same with those of $\hat{H}_0$. Then, $\hat{V}$ is said to be the perturbation. It is assumed that $\hat{H}_0$ does not degenerate, i.e. energy eigenvalues E_n are all different from each other.

Eq. (h.36) is rewritten in Schrödinger picture as:

$$i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle = \hat{H}|\psi_S(t)\rangle. \tag{h.56}$$

Eq. (h.56) emphasizes that the equation is described in Schrödinger picture by subscript S. Schrödinger picture is one of pictures which describes time evolution of the system as time evolution of wave function, the operator is time-independent. In fact, Hamiltonian $\hat{H}$ is time-independent in Eq, (h.56).

Interaction picture describes time evolution of the system as time evolution of both wave function and operator. Relationship between Schrödinger picture with interaction picture is described in the followings. Wave function in interaction picture $|\psi_I(t)\rangle$ and the perturbation in interaction picture $\hat{V}_I(t)$ are defined as

$$|\psi_I(t)\rangle = \exp\left(-\frac{\hat{H}_0}{i\hbar}t\right)|\psi_S(t)\rangle \tag{h.57}$$

$$\hat{V}_I(t) = \exp\left(-\frac{\hat{H}_0}{i\hbar}t\right)\hat{V}_S\exp\left(\frac{\hat{H}_0}{i\hbar}t\right), \tag{h.58}$$

respectively. It should be noticed that $\hat{V}_S = \hat{V}$, time-independent. Substituting Eq. (h.57) to Eq. (h.56),

$$i\hbar\frac{d}{dt}\left(\exp\left(\frac{\hat{H}_0}{i\hbar}t\right)|\psi_I(t)\rangle\right) = (\hat{H}_0 + \hat{V}_S)\exp\left(\frac{\hat{H}_0}{i\hbar}t\right)|\psi_I(t)\rangle \tag{h.59}$$

is obtained. As a consequence, Schrödinger equation in interaction picture

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = \hat{V}_I(t)|\psi_I(t)\rangle. \tag{h.60}$$

is obtained [39]. Wave function in interaction picture $|\psi_I(t)\rangle$ is expanded as

$$|\psi_I(t)\rangle = \sum_n c_n(t)|\varphi_n\rangle. \tag{h.61}$$

To solve Eq. (h.60), both sides of Eq. (h.60) are integrated from 0 to t as:

$$|\psi_I(t)\rangle = |\psi_I(0)\rangle + \frac{1}{i\hbar}\int_0^t \hat{V}_I(t')|\psi_I(t')\rangle dt' \tag{h.62}$$

Substituting $|\psi(t')\rangle_I$ to $|\psi(t)\rangle_I$ described as Eq. (h.62) itself,

$$|\psi_I(t)\rangle = |\psi_I(0)\rangle + \frac{1}{i\hbar}\int_0^t \hat{V}_I(t')dt'|\psi_I(0)\rangle + \left(\frac{1}{i\hbar}\right)^2\int_0^t \hat{V}_I(t')dt'\int_0^{t'} \hat{V}_I(t'')dt'' \tag{h.63}$$

---

[39] Eq. (h.59) is deformed as:
$\hat{H}_0\exp\left(\frac{\hat{H}_0}{i\hbar}t\right)|\psi_I(t)\rangle + \exp\left(\frac{\hat{H}_0}{i\hbar}t\right)\frac{d}{dt}|\psi_I(t)\rangle = \hat{H}_0\exp\left(\frac{\hat{H}_0}{i\hbar}t\right)|\psi_I(t)\rangle + \hat{V}_S\exp\left(\frac{\hat{H}_0}{i\hbar}t\right)|\psi_I(t)\rangle$
from chain rule of differential calculus. Consequently,
$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = \exp\left(-\frac{\hat{H}_0}{i\hbar}t\right)\hat{V}_S\exp\left(\frac{\hat{H}_0}{i\hbar}t\right)|\psi_I(t)\rangle.$
As a result, Eq. (h.60) is given.

is obtained. By successive approximation by the repetition of the same operation, $|\psi_I(t)\rangle$ is described as the infinite series:

$$|\psi_I(t)\rangle = \left(1 + \sum_{k=1}^{\infty} \frac{1}{(i\hbar)^k} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \hat{V}_I(t_1) \cdots \hat{V}_I(t_k) dt_1 \cdots dt_k \right) |\psi_I(0)\rangle. \tag{h.64}$$

Here, Eq. (h.64) is approximated by up to first-order term of the perturbation as

$$|\psi_I(t)\rangle \simeq |\psi_I(0)\rangle + \frac{1}{i\hbar} \int_0^t \hat{V}_I(t') |\psi_I(0)\rangle dt'. \tag{h.65}$$

Taking inner product $\langle \varphi_n|$ with both sides of Eq. (h.65),

$$c_n(t) \simeq c_n(0) + \sum_m \frac{1}{i\hbar} \int_0^t V_{nm}(t') c_m(0) dt' \tag{h.66}$$

is obtained [40] from completeness $\sum_m |\varphi_m\rangle\langle\varphi_m| = 1$, where

$$V_{nm}(t) = \langle \varphi_n | \hat{V}_I(t) | \varphi_m \rangle. \tag{h.67}$$

Eq. (h.67) is described in interaction picture, rewritten in Schrödinger picture as:

$$V_{nm}(t) = \exp\left(\frac{E_m - E_n}{i\hbar} t\right) \langle \varphi_n | \hat{V}_S | \varphi_m \rangle \tag{h.68}$$

from definition Eq. (h.58).

H.9 Fermi's golden rule

Description of time evolution of quantum state by perturbation shows transition rate of quantum state. Here, the initial quantum state

$$|\psi_I(0)\rangle = |\varphi_i\rangle \tag{h.69}$$

is postulated. Then,

$$|\psi_I(0)\rangle = |\psi_I(0)\rangle + \frac{1}{i\hbar} \int_0^t \hat{V}_I(t') |\varphi_i\rangle dt' \tag{h.70}$$

from approximation Eq. (h.66).

Taking inner product $\langle \varphi_i|$ with both sides of Eq. (h.70),

$$c_i(t) = 1 + \frac{1}{i\hbar} \int_0^t V_{ii}(t') dt' \tag{h.71}$$

---

[40] In detail, the intermediate calculation is
$\langle \varphi_n | \psi_I(t) \rangle \simeq \langle \varphi_n | \psi_I(0) \rangle + \frac{1}{i\hbar} \int_0^t \langle \varphi_n | \hat{V}_I(t') | \psi_I(0) \rangle dt'$,
$c_n(t) \simeq c_n(0) + \sum_m \frac{1}{i\hbar} \int_0^t \sum_m |\varphi_m\rangle\langle\varphi_m| \langle \varphi_n | \hat{V}_I(t') | \psi_I(0) \rangle dt'$.
The second term of the right side is deformed as:
$\frac{1}{i\hbar} \int_0^t \sum_m \langle \varphi_n | \hat{V}_I(t') | \varphi_m \rangle \langle \varphi_m | \psi_I(0) \rangle dt' = \frac{1}{i\hbar} \int_0^t \langle \varphi_n | \exp\left(-\frac{E_n}{i\hbar} t\right) \hat{V}_S \exp\left(\frac{E_m}{i\hbar} t\right) | \varphi_m \rangle c_m(0) dt'$.

and

$$c_i(0) = 1. \tag{h.72}$$

Taking inner product $\langle\varphi_f|$ $(i \neq f)$ with both sides of Eq. (h.70),

$$c_f(t) = \frac{1}{i\hbar}\int_0^t V_{fi}(t')dt' \quad (i \neq f) \tag{h.73}$$

and

$$c_f(0) = 0. \tag{h.74}$$

Eqs. (h.71-h.74) say that initial quantum state $|\varphi_i\rangle$ changes to superposition of $|\varphi_i\rangle$ and final quantum state $|\varphi_i\rangle$ by time evolution.

Eq. (h.73) is deformed as:

$$\begin{aligned} c_f(t) &= \frac{1}{i\hbar}\int_0^t V_{fi}(t')dt' = \frac{1}{i\hbar}\int_0^t \langle\varphi_f|\hat{V}_I(t)|\varphi_i\rangle dt' \\ &= \frac{1}{i\hbar}\int_0^t \exp\left(\frac{E_i - E_f}{i\hbar}t'\right)\langle\varphi_f|\hat{V}|\varphi_i\rangle dt' \\ &= \frac{\langle\varphi_f|\hat{V}|\varphi_i\rangle}{i\hbar}\left[\frac{i\hbar}{E_i - E_f}\exp\left(\frac{E_i - E_f}{i\hbar}t'\right)\right]_0^t \\ &= \frac{\langle\varphi_f|\hat{V}|\varphi_i\rangle}{E_i - E_f}\left\{\exp\left(\frac{E_i - E_f}{i\hbar}t\right) - 1\right\}. \end{aligned} \tag{h.75}$$

Squaring both sides of Eq. (h.75),

$$|c_f(t)|^2 = \left\{\frac{2\sin\dfrac{E_f - E_i}{2\hbar}t}{E_f - E_i}\right\}^2 |\langle\varphi_f|\hat{V}|\varphi_i\rangle|^2 \tag{h.76}$$

is obtained [41].

Fig. H.1 shows the graph of $f(x,t) = \left(\dfrac{\sin xt}{x}\right)^2$ as function of x. When $t \to \infty$, $f(x,t)$ will be proportional to Dirac delta function $\delta(x)$ which is defined as below:

$$\delta(x) = \begin{cases} \infty & (x = 0) \\ 0 & (x \neq 0) \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(x)dx = 1. \tag{h.77}$$

$\int_{-\infty}^{\infty} f(x,t)dx$ is calculated to determine the proportional constant.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x,t)dx &= \int_{-\infty}^{\infty}\left(\frac{\sin xt}{x}\right)^2 dx = t^2\int_{-\infty}^{\infty}\left(\frac{\sin xt}{xt}\right)^2 dx \\ &= t\int_{-\infty}^{\infty}\left(\frac{\sin x'}{x'}\right)^2 dx' = t\pi, \end{aligned} \tag{h.78}$$

---

[41] $|\exp(i\theta) - 1|^2 = 2 - 2\cos\theta = \left\{2\sin\dfrac{\theta}{2}\right\}^2$, $(x-y)^2 = (y-x)^2$, and $\sin^2\theta = \sin^2(-\theta)$ were used for deformation.

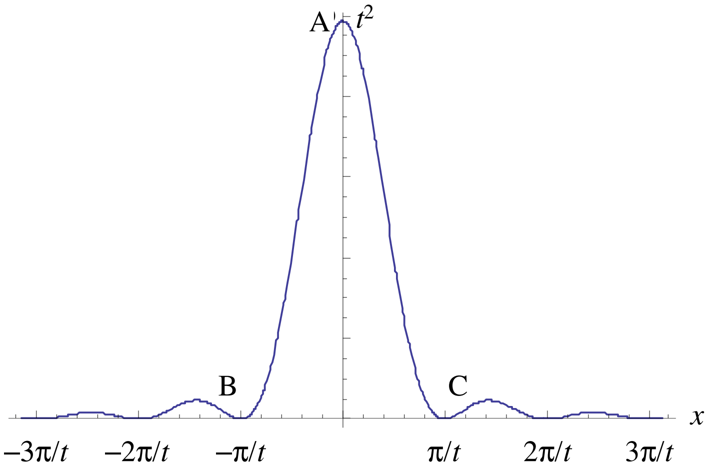

Fig H.1: The graph of $f(x,t) = \left(\dfrac{\sin xt}{x}\right)^2$.

Eq. (h.78) [42] provides

$$\lim_{t\to\infty} t\left(\frac{\sin xt}{x}\right)^2 = \pi\delta(x). \tag{h.79}$$

Substituting $x = \dfrac{E_f - E_i}{2\hbar}$ in Eq. (h.79), Eq. (h.76) is rewritten as [43]:

$$|c_f(t)|^2 \simeq \frac{t\pi}{\hbar^2}\delta\left(\frac{E_f - E_i}{2\hbar}\right)|\langle\varphi_f|\hat{V}|\varphi_i\rangle|^2 = \frac{2t\pi}{\hbar}\delta(E_f - E_i)|\langle\varphi_f|\hat{V}|\varphi_i\rangle|^2. \tag{h.80}$$

Eq. (h.80) is accurately satisfied when t is sufficiently large as the time scale we normally observed.

Eq. (h.80) is the probability density of the quantum state $|\varphi_f\rangle$ at time t. Dividing both sides of Eq. (h.80) by t, the transition rate of the quantum state $|\varphi_i\rangle \to |\varphi_f\rangle$ per unit time $dw_{i\to f}$ is provided as:

$$w_{i\to f} = \frac{2\pi}{\hbar}|\langle\varphi_f|\hat{V}|\varphi_i\rangle|^2\delta(E_f^{(0)} - E_i^{(0)}). \tag{h.81}$$

Eq. (h.81) can be also described as:

$$w_{i\to f} = \frac{2\pi}{\hbar}|\langle\varphi_f|\hat{V}|\varphi_i\rangle|^2\rho(E_f), \tag{h.82}$$

where $\rho(E)$ is said to be the level density. The number of possible transition states within an interval E between $E + dE$; dn can be written:

$$dn = \rho(E)dE. \tag{h.83}$$

---

[42] $f(x,t)$ drastically increases nearby $x = 0$. When $t \to \infty$, the area $\int_{-\infty}^{\infty} f(x,t)dx$ will accord with the area of the triangle $\triangle ABC$. Thus, $\int_{-\infty}^{\infty} f(x,t)dx = \frac{2\pi}{t}t^2\frac{1}{2} = t\pi$.

[43] $\delta(ax) = \delta(x)/a$. In addition, dimension of $\delta(x)$ is $[x]^{-1}$ because non-dimensional $\int_{-\infty}^{\infty}\delta(x)dx = 1$ is required (Dirac delta function satisfies the requirement of probability distribution function).

Eqs. (h.81 - h.82) is said to be Fermi's golden rule, useful to discuss the transition of quantum state of β decay.

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I Nuclear physics

I.1 General properties of atoms and atomic nuclei

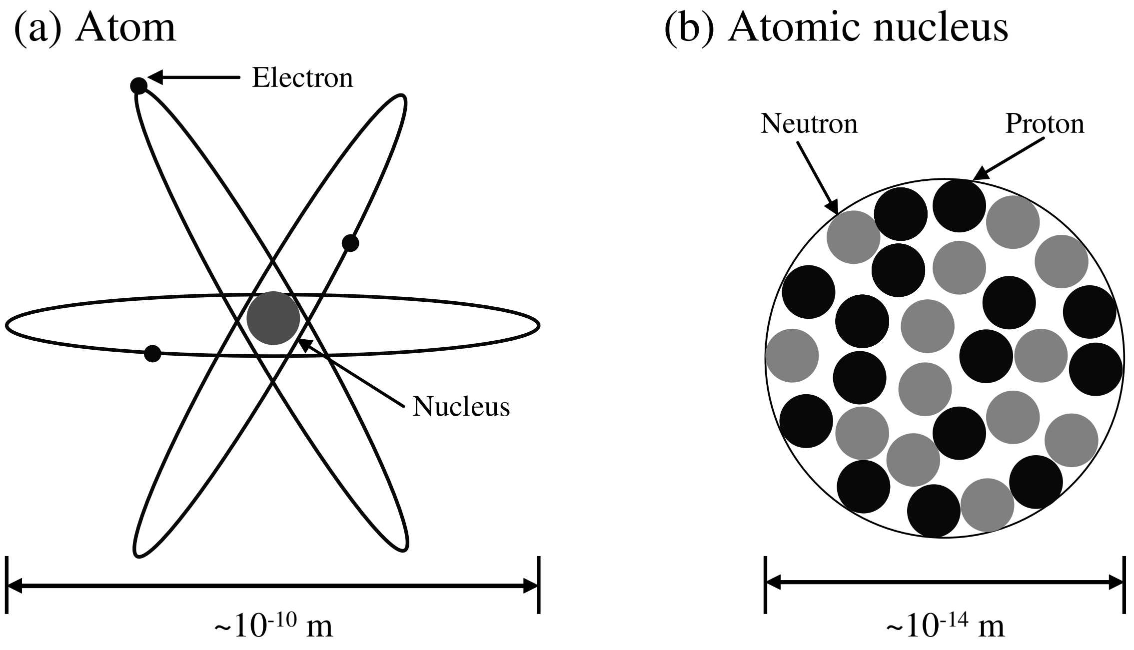

Fig I.1: The schematic structure of (a) an atom, (b) an atomic nucleus.

Fig. I.1 (a) shows the structure of an atom; a generic atomic planetary model (the Rutherford model). There is an atomic nucleus at the central of the atom. Although electrons are described as orbiting the nucleus, in fact, electrons are distributed like cloud stochastically. The diameter of an atom is similar to 10^{-10} m $= 1$ Å. The diameter of an nucleus is similar to 10^{-14} m. Thus, the nucleus is localized at within the atom.

Fig. I.1 (b) shows the structure of an atomic nucleus. A nucleus is consisted of protons and neutrons. Although protons are electrically positive and protons receive repulsive force each other, nuclei are bound together by the strong force which emerged from interchange of mesons. Mass of a neutron m_n is slightly larger than mass of a proton m_p as $m_n \simeq 1.0014 m_p$. m_p is much larger than mass of an electron m_e as $m_p \simeq 1836 m_e$ Thus, almost all of the mass of an atom is located in the nucleus. Neutron and proton are said to be nucleon.

Mass of a neutron, a proton and an electron is summarized as follows:

$$\begin{aligned} m_n &= 939.57 \quad (\mathrm{MeV}/c^2), \\ m_p &= 938.27 \quad (\mathrm{MeV}/c^2), \\ m_e &= 0.5110 \quad (\mathrm{MeV}/c^2). \end{aligned} \tag{i.1}$$

The mass of a hydrogen atom m_H, which is consisted of a proton and an electron, is also used conveniently,

$$m_H = m_p + m_e = 938.78 \quad (\mathrm{MeV}/c^2). \tag{i.2}$$

I.2 Binding energy of nuclei

As mentioned in section 3.2.1, the kinetic energy of β ray is emerged from the difference of rest mass energy of parent nucleus with daughter nucleus. Eq. (i.1) says that the difference of rest mass energy of a neutron with a proton is:

$$m_n - m_p \simeq 1.3 \quad (\mathrm{MeV}/c^2). \tag{i.3}$$

Although Eq. (i.3) appears to suggest that the maximum kinetic energy of β ray does not depend on nuclides, it indeed depends on nuclides as shown in Table 3.1.

The fact that the maximum kinetic energy of β ray depends on nuclides is caused by the mass defect due to the binding energy of nuclides. The mass of atom $M(A, Z)$, which the proton number is Z and the neutron number is N, is slightly smaller than $Zm_H + Nm_n$. The binding energy of nucleus is defined by the slight difference of mass as:

$$B(A, Z) = (Zm_H + Nm_n - M(A, Z))c^2, \tag{i.4}$$

the binding energy produces the mass defect and the difference of the kinetic energy of β ray each nuclides. Eq. i.4 shows that the binding energy is the energy required to divide a nucleus into Z protons and N neutrons.

The properties of mass of nuclei and binding energy which are observed by experiments are summarized in the followings.

1. The binding energy of a nucleus is proportional to the mass number $A = N + Z$. The binding energy of per a nucleon is almost constant with mass number as $B(A, Z)/A \sim$ 7.4 MeV to 8.8 MeV except nuclei whose mass number is small.
2. The binding energy of per a nucleon is maximum at $A \approx 60$. If mass number is larger than 60, $B(A, Z)/A$ monotonically decreases.
3. If proton number Z changes at the same mass number A, $B(A, Z)$ can be approximated as a parabola.
4. The binding energy of a nucleus which has even proton number is slightly larger than that has odd proton number Z at the same mass number A.
5. A nucleus tends to have the same proton number and neutron number ($Z = N$). It is remarkable when mass number of the nucleus is small.

The semi-empirical mass formula (also referred to as the Weizs"acker-Bethe formula) approximately explains mass properties of nuclei. A nucleus is treated as a liquid drop, the binding energy has the following five terms.

[i] **Volume energy**
Nucleons within a nucleus attracts each other (strong force), and a nucleus is condensed. The energy of nucleon is smaller than if it is isolated. The binding energy increases in proportion to the volume, i.e. the mass number A as:

$$E_{\mathrm{vol}} = a_{\mathrm{vol}} A, \tag{i.5}$$

where E_{vol} is volume energy and a_{vol} is a coefficient. Eq. (i.5) corresponds with the 1st property mentioned above.

[ii] **Surface energy**
Nucleons which exists in surface of a nucleus is not entirely surrounded by other nucleons. The binding energy decreases in proportion to the surface area of nucleus, i.e. $A^{\frac{2}{3}}$ [44] as:

$$E_{\text{surf}} = -a_{\text{surf}} A^{\frac{2}{3}}, \tag{i.6}$$

where E_{surf} is the surface energy.

[iii] **Coulomb energy**
Although nucleons within a nucleus attract each other by strong force, protons repulse each other by Coulomb force at the same time. The Coulomb potential energy decreases the binding energy. The Coulomb potential energy between two electrically charged particle at a distance r is $\frac{q_1 q_2}{4\pi\varepsilon_0 r}$. Therefore the total Coulomb potential energy E_{Coul} can be described as [45] :

$$E_{\text{Coul}} = -a_{\text{Coul}} \frac{Z(Z-1)}{A^{\frac{1}{3}}}. \tag{i.7}$$

[iv] **Asymmetry energy**
A nucleus tends to have the same proton number and neutron number as 5th property mentioned above. Asymmetry energy is emerged from Pauli exclusion principle (Both neutrons and protons are Fermion). Asymmetry energy E_{sym} can be described as:

$$E_{\text{asym}} = -a_{\text{asym}} \frac{(N-Z)^2}{A} = -a_{\text{asym}} \frac{(A-2Z)^2}{A}. \tag{i.8}$$

Eq. (i.8) indeed equals to minimum value 0 at $N = Z$.

[v] **Pairing energy**
Nuclei tend to have even neutron number and even proton number as 4th property mentioned above. The binding energy increases when both neutron number and proton number are even, and decreases when both neutron number and proton number are odd. The pairing energy $\delta(A, Z)$ is described as:

$$\delta(A, Z) = \begin{cases} -\frac{a_{\text{pair}}}{\sqrt{A}} & Z,\, N \text{ even} \\ 0 & A = N + Z \text{ odd} \\ \frac{a_{\text{pair}}}{\sqrt{A}} & Z,\, N \text{ odd} \end{cases}, \tag{i.9}$$

As a conclusion, the Weizs"acker-Bethe formula is described as:

$$\begin{aligned} B(A, Z) &= E_{\text{vol}} + E_{\text{surf}} + E_{\text{Coul}} + E_{\text{asym}} - \delta(A, Z) \\ &= a_{\text{vol}} A - a_{\text{surf}} A^{\frac{2}{3}} - a_{\text{Coul}} \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_{\text{asym}} \frac{(A-2Z)^2}{A} - \delta(A, Z). \end{aligned} \tag{i.10}$$

---

[44] Let think a sphere. Regarding the volume V and the radius r, $V \propto r^3$. Regarding the surface area S and radius r, $S \propto r^2$. So $S \propto V^{2/3}$.

[45] a_{Coul} can be calculated analytically based on fundamental electromagnetism.

The coefficients are calculated by fitting to many experimentally measured masses of nuclei. As an example, the coefficients

$$\begin{aligned} a_{\mathrm{vol}} &= 15.75 \quad (\mathrm{MeV}), \\ a_{\mathrm{surf}} &= 17.8 \quad (\mathrm{MeV}), \\ a_{\mathrm{Coul}} &= 0.711 \quad (\mathrm{MeV}), \\ a_{\mathrm{asym}} &= 23.7 \quad (\mathrm{MeV}), \\ a_{\mathrm{pair}} &= 11.18 \quad (\mathrm{MeV}), \end{aligned} \tag{i.11}$$

are used.

I.3 Fermi theory of beta decay

Fermi proposed an analogy between proton-γ reaction $p \to p + \gamma$ and β decay $n \to p + e^- + \bar{\nu}_e$. It can be said that the association of electromagnetic interaction and interaction describing β decay (today, known as weak interaction).

The Hamiltonian density of proton-γ reaction H_{em} is described as:

$$H_{\mathrm{em}} = e j_\mu^{\mathrm{em}} \boldsymbol{A}_\mu, \tag{i.12}$$

where e is elementary charge, j_μ^{em} is electromagnetic four-current of proton, $\boldsymbol{A}_\mu$ is Dirac spinor of photon (electromagnetic four-potential).

Fermi replaced current j_μ^{em} with transition of the neutron into a proton as:

$$j_\mu^{\mathrm{em}} \longrightarrow j_\mu^{n \to p} = i(\bar{\psi}_p \gamma_\mu \psi_n), \tag{i.13}$$

and replaced $\boldsymbol{A}_\mu$ with emission of the electron and the neutrino as:

$$\boldsymbol{A}_\mu \longrightarrow j_{e \to \nu}^\mu = i(\bar{\psi}_e \gamma^\mu \psi_\nu). \tag{i.14}$$

Further, the charge e is also replaced with the coupling constant G_F as:

$$e \longrightarrow -G_F, \tag{i.15}$$

which is now known as the Fermi's coupling constant [46].

As a result, Eqs. (i.13 - i.15) replace Eq. (i.12) as

$$H_\beta = G_F (\bar{\psi}_p \gamma_\mu \psi_n)(\bar{\psi}_e \gamma^\mu \psi_\nu). \tag{i.16}$$

Fermi claimed that H_β is the interaction which causes β decay.

I.4 The shape of the beta-spectrum

As shown in Eq. (h.82), the probability of the β ray which has energy E_e emitted per unit time $P(E_e)dE_e$ can be described as:

$$P(E_e)dE_e = \frac{2\pi}{\hbar} |H_\beta|^2 \frac{dn}{dE_0}, \tag{i.17}$$

---

[46] Today, G_F is measured experimentally as: $\dfrac{G_F}{(\hbar c)^3} = 1.1637 \times 10^{-5}$. ($\mathrm{GeV}^{-2}$)

where E_0 is total energy of a β ray and a neutrino, dn/dE_0 is its level density and $|H_\beta|$ is the matrix element of β decay.

The number of states dn_1 which a particle exists in volume Ω and the momentum is interval of p between $p + dp$ is described as:

$$dn_1 = \frac{4\pi p^2 dp\Omega}{h^3} = \frac{p^2 dp\Omega}{2\pi^2\hbar^3}. \tag{i.18}$$

The number of states dn which the total energy of a β ray and a neutrino is interval of E_0 between $E_0 + dE_0$ is described as:

$$dn = \frac{p_e^2 dp_e\Omega}{2\pi^2\hbar^3}\frac{q^2 dq\Omega}{2\pi^2\hbar^3} = \Omega^2\frac{p_e^2 dp_e q^2 dq}{4\pi^4\hbar^6}, \tag{i.19}$$

where p_e is the momentum of a β ray, q is the momentum of a neutrino. From Eq. (i.19), the level density of E_0 is described as:

$$\frac{dn}{dE_0} = \Omega^2\frac{p_e^2 dp_e q^2 dq}{4\pi^4\hbar^6 dE_0}. \tag{i.20}$$

Let transform momentum to energy. The emission energy E_0 is scrambled with E_e and E_ν,

$$E_0 = E_e^{\max} = E_e + E_\nu, \tag{i.21}$$

$$E_e = \sqrt{m_e^2c^4 + c^2p_e^2}, \tag{i.22}$$

and

$$E_\nu \simeq cq \tag{i.23}$$

are satisfied respectively. m_e is the mass of an electron, the mass of a neutrino is approximated to 0.

Differentiating both sides of Eq. (i.22) by p_e,

$$\frac{dE_e}{dp_e} = \frac{c^2p_e}{E_e}. \tag{i.24}$$

Differentiating both sides of Eq. (i.23) by E_0,

$$\frac{dq}{dE_0} = \frac{1}{c} \tag{i.25}$$

from Eq. (i.21).

Substituting Eqs. (i.24) and (i.25) to Eq. (i.20),

$$\begin{aligned}\frac{dn}{dE_0} &= \frac{\Omega^2}{4\pi^4\hbar^6}\frac{\frac{1}{c^2}E_e dE_e\frac{1}{c^2}(E_0 - E_e)^2\frac{1}{c}dE_0}{dE_0} = \frac{\Omega^2}{4\pi^4\hbar^6c^5}p_eE_e(E_0 - E_e)^2 dE_e \\ &= \frac{\Omega^2}{4\pi^4\hbar^6c^6}\underbrace{\sqrt{E_e^2 - m_e^2c^4}E_e(E_0 - E_e)^2}_{\text{statistical factor}}dE_e\end{aligned} \tag{i.26}$$

is obtained. The brace part in Eq. (i.26) is said to be a statistical factor. The shape of the beta-spectrum mostly depends on the statistical factor.

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J Conservation law

J.1 Euler-Lagrange equation of motion

Let consider the continuous and smooth function $f(x)$. We can approximate $f(x)$ around a as:

$$f(a + \Delta x) \simeq f(a) + \Delta x f'(a), \tag{j.1}$$

where $f'(x) = \dfrac{df(x)}{dx}$. Eq. (j.1) states that if $f'(a) = 0$ then $f(x)$ is stationary for a change of x around a. The point a which gives $f'(a) = 0$ is said to be a stationary point.

With this in mind, we consider $f(y(x), y'(x))$, which is the function of two functions of x, $y(x)$ and $y'(x)$. $I(y(x), y'(x))$ called the functional is defined as:

$$I(y(x), y'(x)) = \int_a^b dx f(y(x), y'(x)). \tag{j.2}$$

Here, we discuss the case y and y' occurs small change as:

$$\bar{y} = y + \delta y \tag{j.3}$$

and

$$\bar{y'} = y' + \delta y', \tag{j.4}$$

where δy and $\delta y'$ is said to be the variations of y and y'. The symbol δ stands for the variation, which differs from the infinitesimal change as dx.

The variation of the functional I is described as:

$$\delta I = \int_a^b dx f(y + \delta y, y' + \delta y') - \int_a^b dx f(y, y'). \tag{j.5}$$

Rewriting Eq. (j.5) [47] [48],

$$\begin{aligned} \text{r.h.s of Eq.(j.5)} &= \int_a^b dx \frac{\partial f}{\partial y}\delta y + \int_a^b dx \frac{\partial f}{\partial y'}\delta y' \\ &= \int_a^b dx \frac{\partial f}{\partial y}\delta y + \left[\frac{\partial f}{\partial y'}\delta y\right]_a^b - \int_a^b dx \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\delta y\right). \end{aligned} \tag{j.6}$$

At the end points a and b, the variations are 0, therefore the second term of Eq. (j.6) is vanished. That is,

$$\delta I = \int_a^b dx \frac{\partial f}{\partial y}\delta y - \int_a^b dx \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\delta y\right). \tag{j.7}$$

If the variation $\delta I = 0$, we can get

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}. \tag{j.8}$$

Eq.(j.8) is said to be the Euler-Lagrange equation of motion. We can say that Eq.(j.8) is the function that gives the stationary point to the functional $I(y(x), y'(x))$.

---

[47] $df = \dfrac{\partial f}{\partial x_1}dx_1 + \dfrac{\partial f}{\partial x_2}dx_2 + \cdots$.

[48] We use the integration by parts $\int_a^b dx f g' = [fg]_a^b - \int_a^b dx f' g$.

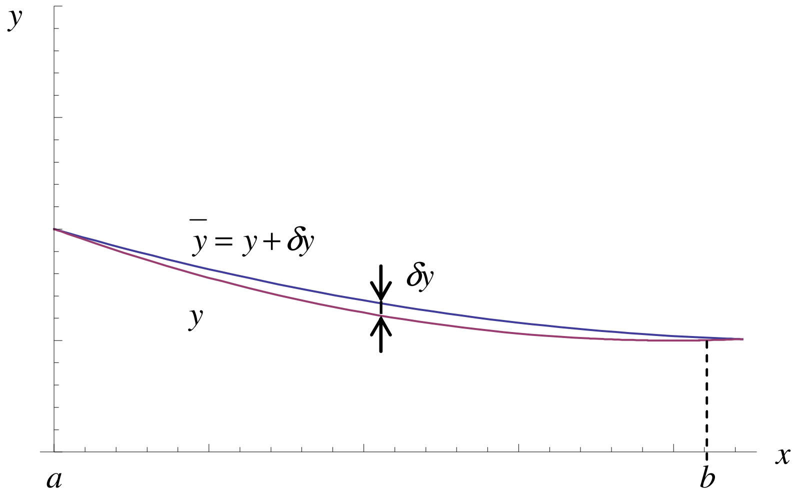

Fig J.1: The variation of the function $y(x)$.

J.2 Principle of least action

Replacing Eq.(j.8) as the functional $f(y, y')$ to the function $L(q, \dot{q})$ and the variable x to time t, we can get

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}, \tag{j.9}$$

where L is the function said to be Lagrangian, and $\dot{q} = \dfrac{dq}{dt}$. We can demonstrate that Eq.(j.9) reproduces the Newton's second law $m\dfrac{d^2x}{dt^2} = f$ in the range of Newtonian mechanics. The Lagrangian is scaler, Eq. (j.9) is covariant with a coordinate transformation. The merit of Euler-Lagrange equation of motion is to be completed in any coordinate systems.

We define the action S as the time integral of the Lagrangian:

$$S = \int dtL. \tag{j.10}$$

The variation of the action satisfies

$$\delta S = 0 \tag{j.11}$$

from the Euler-Lagrange equation of motion, Eq. (j.8). Eq.(j.11) is the principle of least action.

The Lagrangian L is the function which gives $\delta S = 0$. We can also say that the real particle moves on the path which satisfies $\delta S = 0$. The principle of least action is the most important guiding principle of physics. It can apply to not only classical mechanics but also electromagnetism, quantum mechanics and relativity etc.

J.3 Analytical mechanics for a field theory and the Lagrangian

The principle of least action derives the field equation as with a particle. For example, the Maxwell's equations

$$\Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = 0 \tag{j.12}$$

are the second order partial differential equations for the electromagnetic field A_μ. The Lagrangian density $\mathcal{L}$ [49] and the action S which derive Eq.(j.12) are:

$$S = \int dV \mathcal{L}(A_a, \partial_\mu A_a), \tag{j.13}$$

where dV is infinitesimal volume element. The field equation is derived from $\delta S = 0$, that is,

$$\begin{aligned} \delta S &= \int dV \mathcal{L} = \int dV \left\{ \frac{\partial \mathcal{L}}{\partial A_a} \delta A_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_a)} \right\} \\ &= \int dV \left\{ \frac{\partial \mathcal{L}}{\partial A_a} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_a)} \right\} \delta A_a + \int dV \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_a)} \delta A_a \right) = 0. \end{aligned} \tag{j.14}$$

The second term of Eq.(j.14) can be rewritten from volume integral dV to surface integral dS_μ according to Gauss's theorem [50] as:

$$(\text{The second term of Eq.(j.14)}) = \int dS_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_a)} \delta A_a \tag{j.15}$$

It should be satisfied that $\delta A_a = 0$ on the surface of the integral area, Eq.(j.14) can be rewritten as:

$$\frac{\partial \mathcal{L}}{\partial A_a} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_a)} = 0. \tag{j.16}$$

Eq.(j.16) is said to be the Lagrange's equation for field theory.

J.4 Noether's theorem

At first, let mention the statement of the Noether's theorem.

Noether's theorem claims that if the Lagrangian, or action, is invariant to a certain transform, then the corresponding conservation law exists on the path of the equation of motion.

We denote the change in the action as:

$$\begin{aligned} \delta_L S = S' - S &= \int d\tau \left\{ L'(x^\mu, \dot{x}^\mu) - L(x^\mu, \dot{x}^\mu) \right\} \\ &= \int d\tau \left(\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} \right) \delta x^\mu + \int d\tau \frac{d}{d\tau} \left(\frac{\partial L}{\partial x'^\mu} \delta x^\mu \right) \end{aligned} \tag{j.17}$$

It should be satisfied that $\delta_L S = 0$ from the principle of least action . On the path of the equation of motion, the first term of Eq.(j.17) must be 0, inevitably, the second term of Eq.(j.17) is

$$\delta_L S = \int d\tau \left(\frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu \right) = 0. \tag{j.18}$$

That is,

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu \right) = 0. \tag{j.19}$$

Eq.(j.19) is the conservation law itself. In the followings, the examples of Noether's theorem are enumerated.

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[49] $L = \int dV \mathcal{L}$. In field theory, $\mathcal{L}$ is simply said to be Lagrangian without notice.

[50] Gauss's theorem is $\int_V \nabla \cdot \boldsymbol{A} = \oint_S dS \boldsymbol{A} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is unit vector on the boundary.

1. Conservation of Energy and momentum
   Let be invariant
   $$\delta x^\mu = \begin{cases} a & \text{(one component)} \\ 0 & \text{(otherwise components)} \end{cases}, \tag{j.20}$$
   where a is an infinitesimal constant. Then,
   $$\frac{d}{d\tau}(p_\mu a) = 0, \quad \therefore \frac{dp_\mu}{d\tau} = 0. \tag{j.21}$$
   where $p_m u$ is generalized momentum [51]. Eq.(j.21) is the conservation of energy for time component $\mu = 0$, and the conservation of momentum for space components $\mu = 1, 2, 3$.

2. Conservation of angular momentum
   Let be
   $$\delta x^\mu = \omega^{\mu\nu} x_\nu, \tag{j.22}$$
   where $\omega^{\mu\nu}$ is an antisymmetric infinitesimal tensor ($\omega^{\mu\nu} = -\omega^{\nu\mu}$), let be
   $$\omega^{\mu\nu} = \begin{cases} \omega & (\mu = 2, \nu = 1) \\ -\omega & (\mu = 1, \nu = 2) \\ 0 & \text{(otherwise)} \end{cases}. \tag{j.23}$$
   Then,
   $$\frac{d}{d\tau}(p_\mu a) = 0, \quad \therefore \frac{d}{d\tau}(x_1 p_2 \omega - x_2 p_1 \omega) = 0. \tag{j.24}$$
   Eq.(j.24) is the conservation of angular momentum.

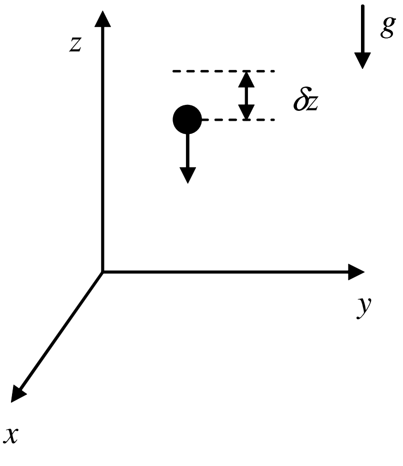

Fig J.2: The symmetry with x, y-axes and the asymmetry with z-axis in the equable gravity field.

Noether's theorem states the symmetry of a system. In the case of shown in Fig.XX, the x and y directions have symmetry, the z does not have symmetry as:

$$\delta z = z' - z \Rightarrow L' - L = -mg(z' - z) = -mg\delta z, \tag{j.25}$$

$$\delta x = x' - x \Rightarrow L' - L = 0. \tag{j.26}$$

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[51] $p_\mu = \dfrac{\partial L}{\partial \dot{x}^\mu}$.

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