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Helicity in differential topology and incompressible fluids on foliated 3-manifolds

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Abstract

We review the interpretation of the helicity of the velocity fields of incompressible fluids on closed 3-manifolds as the asymptotic linking pairing of vorticity fields and further develop this point of view. For codimension 1 foliated manifolds, this idea has a strong relation with the 1st foliated cohomology and the secondary invariants, such as the Godbillon-Vey invariant and the Reeb class. The main purpose of the present article is, based on these frameworks, to give a description of the space of velocity fields of incompressible fluids which are holonomically constrained to the leaves of a foliated 3-manifold. In particular for algebraic Anosov foliations we see how these ideas work effectively to understand the space of incompressible foliated flows.

Keywords: helicity ; asymptotic linking ; foliated cohomology ; algebraic Anosov foliations

1. Introduction

To understand a manifold or a geometric structure on it, quite often we set up a certain vector bundle and look at the space of sections and differential operators in order to reduce a geometric problem to a linear one, even though the space of sections is usually of infinite dimension. This is the point where the global analysis comes into the topology of manifolds. If we follow such an idea, it sounds quite natural to look at the fluid motions on a manifold to analyse geometric problems. However, the analytic foundations of fluid mechanics have not yet been well established, so that this idea might turn a difficult problem into far more difficult ones. Nevertheless, it is still tempting at least to think about the space of velocity fields, the equations of fluid motions, and stationary solutions for the Euler or the Navier-Stokes equations on manifolds with geometric structures.

In this article, we consider velocity fields of ideal fluids on a foliated manifold whose fluid particles are constrained to leaves of the foliation. We will mostly consider 2-dimensional foliations on closed 3-manifolds and introduce a framework to understand the space of such velocity fields, without detailed arguments. Definite results will be obtained for special classes of foliations.

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One of the key ideas is to treat the helicity as a symmetric bilinear form on the space of vorticity fields. This idea was originated in [2] and developed in [7]. Another source of ideas is found in an unpublished expository article [11]. For the foliation theory, the readers may refer to [5] or [8]. In this article all objects are assumed to be of class C^∞.

1.1. Problems and some notations

We will consider a closed manifold M, mainly of dimension 3, and a smooth foliation $\mathcal{F}$ of codimension 1 on M. $\mathcal{X}(M)$ denotes the space of smooth vector fields on M and $\mathcal{X}(M;\mathcal{F})$ that of smooth vector fields tangent to $\mathcal{F}$. When we fix a smooth volume form $dvol_M$ the space of divergence free vector fields is denoted by $\mathcal{X}_d(M)$.

The aim of this paper is to introduce a framework to understand the space of foliated divergence-free vector fields $\mathcal{X}_d(M;\mathcal{F}) = \mathcal{X}_d(M) \cap \mathcal{X}(M;\mathcal{F})$.

The key idea in the case of codimension 1 foliations on 3-manifolds is to look at slightly smaller spaces $\mathcal{X}_h(M)$ and $\mathcal{X}_h(M;\mathcal{F}) = \mathcal{X}_h(M) \cap \mathcal{X}_d(M;\mathcal{F})$, which are defined in later sections.

1.2. Two dimensional case

To start with, let us consider the space $\mathcal{X}_d(T^2;\mathcal{F})$ for one dimensional foliations on the 2-torus $T^2 = (\mathbb{R}/2\pi\mathbb{Z})^2 = \{(x,y)\}$.

First we assume that the area form is the standard one $dx \wedge dy$. If the foliation is linear, namely, it is given by the closed 1-form $\omega = dy - \lambda dx$, the rationality of the constant $\lambda \in \mathbb{R}$ determines the situation. If λ is a rational number, then it is easy to see that the foliation is area-preservingly diffeomorphic to the one with $\lambda = 0$. In this case, by a simple computation or just by a geometric argument, we can see easily that $\mathcal{X}_d(T^2;\mathcal{F}) = \{f(y)\frac{\partial}{\partial x}\} \cong C^\infty(S^1)$. However if λ is irrational, then every leaf is dense, and we can conclude

$$\mathcal{X}_d(T^2;\mathcal{F}) = \{c(\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y});\ c \in \mathbb{R}\} \cong \mathbb{R}\,.$$

Next let us consider a foliation with non-trivial holonomies. Take the 1-form $\omega = dy - \sin y\, dx$ and the foliation defined by ω. There are two compact leaves $\{y = 0\}$ and $\{y = \pi\}$ and the other leaves are accumulating to these two compact leaves. In this case we do not have any foliated divergence-free vector fields other than 0, namely we have

$$\mathcal{X}_d(T^2;\mathcal{F}) = \{0\}\,.$$

It is not too difficult to show this result by computation, but it is rather easier to argue in a geometric visual way if we look at the asymptotic nature of saturated regions. Therefore even if we change the area form, the same result follows immediately.

Now we come back to irrational linear foliations but with a non-standard area form $\varphi(x,y)dx\wedge dy$ for some positive smooth function φ. It is easy to see that the vector fields $c\varphi^{-1}(\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y})$ for constants $c \in \mathbb{R}$ are divergence-free and in fact there are no more. So we can conclude

$$\mathcal{X}_d(T^2;\mathcal{F}) \cong \mathbb{R}\,.$$

However the background situation differs depending on the character of the irrational number λ. Recall that an irrational number x is called Liouville if for any integer n there exists a pair of integers p and $q > 1$ such that $|x - p/q| < q^{-n}$. If λ is non-Liouville, namely "badly approximable by rationals", the foliation is area-preservingly diffeomorphic to the one with the standard area form. Therefore, for any non-Liouville irrational λ, regardless of area form, we have the following smooth conjugacy for some constnat $c > 0$.

$$c\varphi^{-1}(\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y}) \overset{C^\infty\text{conjugate}}{\sim} \frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y}\,. \tag{1}$$

On the other hand, for a Liouville irrational number λ, for generic area forms, we do not have smooth conjugacy but have only divergence-free vector fields.

The smooth conjugacy problem is reduced to the solvability of the following functional equation

$$g(y + \lambda) - g(y) = h(y) \tag{2}$$

on S^1 for a given $h(y)$ with $\int_0^{2\pi} h(y)dy = 0$. As is well-known, if λ is non-Liouville the equation has always a smooth solution $g(y)$ which is unique up to a constant.

1.3. Introduction to three dimensional case

Of course we can consider the case where the foliation has singularities. But it seems much more significant to work on three dimensional foliated manifolds. On 3-manifolds, two dimensional foliations might be far more interesting than one dimensional ones. In general, for a one dimensional foliation $\mathcal{F}$ once we find a good vetor field $X \in \mathcal{X}_d(M, \mathcal{F})$, any other is given as fX where f is a smooth function which is constant along the leaves.

Let us look at the local structure in the case of two dimensional foliations on closed 3-manifolds. For such a foliated manifold $(M, \mathcal{F})$ with any volume form, any point admits a neighbourhood with a local coordinate (x, y, z) for which the foliation is defined by dz and the volume form takes the standard presentation $dx \wedge dy \wedge dz$. Then we can easily see that there are a lot of vector fields in $\mathcal{X}_d(M, \mathcal{F})$ supported in the neighbourhood. Therefore $\mathcal{X}_d(M, \mathcal{F})$ contains a fairy large subspace $\mathcal{X}_{dloc}(M, \mathcal{F})$ which is spanned by such local vector fields.

The aim of this article is to determine and clarify how much $\mathcal{X}_d(M, \mathcal{F})$ exceeds $\mathcal{X}_{dloc}(M, \mathcal{F})$ as well as to describe $\mathcal{X}_{dloc}(M, \mathcal{F})$ not as the span of something but in a more direct way.

2. Helicity as asymptotic linking

Let us formulate an integral invariant 'helicity' as a symmetric bilinear form on the space of vorticity fields. This formulation has its origin in the work of Arnol'd [2].

2.1. Helicity of a velocity field as an invariant of the vorticity field

On an oriented Riemannian 3-manifold (M, g), where g denotes the Riemannian metric and $dvol_g$ the volume form defined by g, there are the following two standard bijective correspondences between vector fields and differential forms. Here $\Omega^k(M)$ denotes the space of k-forms on M.

$$\begin{aligned} m &: \mathcal{X}(M) \to \Omega^1(M)\,, \quad m(u) = g(u, \cdot)\,, \\ v &: \mathcal{X}(M) \to \Omega^2(M)\,, \quad v(u) = \iota_u dvol_g = dvol_g(u, \cdot, \cdot)\,. \end{aligned}$$

Remark here that by the second correspondence v the space $\mathcal{X}_d(M)$ of divergence-free vector fields precisely corresponds to the space $Z^2(M) \subset \Omega^2(M)$ of closed 2-forms. Now we define the space of homology-free vector fields as that of fields corresponding to exact 2-forms:

$$\mathcal{X}_h(M) = v^{-1}(B^2(M))\,.$$

One of the standard ways to define the helicity of a vector field $u \in \mathcal{X}(M)$ is as follows. First we define the vorticity field $\omega = \operatorname{curl} u = v^{-1}(d(m(u)))$. Then the integration

$$Hel(u) = \int_M g(u, \omega) dvol_g$$

gives the helicity of u. As a 3-form the integrand $hel(u)$ is written as

$$hel(u) = g(u, \omega) dvol_g = m(u) \wedge \iota_\omega dvol_g = m(u) \wedge d(m(u))$$

and is interpreted to be $d^{-1}(v(\omega)) \wedge v(\omega)$. From this we see that on a closed 3-manifold the integrals *Hel* does not depend on the choice of the representative of $d^{-1}(v(\omega))$ because different choices result in an exact 3-form as the difference of the integral. Therefore, the helicity $Hel(u)$ is determined only from the vorticity field ω, namely, if $\operatorname{curl} u_1 = \operatorname{curl} u_2$ then we have $Hel(u_1) = Hel(u_2)$. In this way, the helicity is considered to be a quadratic invariant for the vorticity field. The curl operator : $\mathcal{X}(M) \to \mathcal{X}_h(M)$ is surjective and remains so after being restricted to

$$\operatorname{curl}|_{\mathcal{X}_d(M)} : \mathcal{X}_d(M) \to \mathcal{X}_h(M)\,.$$

Therefore the space $\mathcal{X}_h(M)$ can be regarded as the space of vorticity fields from the (incompressible) fluid mechanical context.

2.2. Helicity (asymptotic linking) as a symmetric bilinear form

For a velocity field u the helicity is the integration of a pointwise determined quantity $hel(u)$. However, for the vorticity field the helicity is of more global nature and it is understood as the asymptotic self-linking ([2], [7]). In such a context, a remarkable fact is that the asymptotic linking is defined only with the volume form, we do not need the Riemannian metric. Now we can define the asymptotic linking as not only a quadratic form but also as a symmetric bilinear form on $\mathcal{X}_h(M)$. But the easiest way is to pass to the space of exact 2-forms by taking the volume dual v. For $d\alpha = v(X)$ and $d\beta = v(Y) \in B^2(M)$ where $X, Y \in \mathcal{X}_h(M)$, their asymptotic linking is defined by

$$lk(X,Y) = lk(d\alpha, d\beta) = \int_M \alpha \wedge d\beta \,. \tag{3}$$

As the symmetric bilinear form on $B^2(M)$, we do not need even the volume but only the orientation. If we start from a Riemannian metric, of course we have

$$Hel(u) = lk(\omega, \omega) = lk(v(\omega), v(\omega)) \,. \tag{4}$$

The asymptotic linking is a bilinear form on $\mathcal{X}_h(M)$ [*resp.* on $B^2(M)$] which is

- symmetric,
- non-degenerate, and
- invariant under volume-preserving [*resp.* orientation-preserving] diffeomorphisms.

Already almost 30 years have passed since these properties were recognized, while we have not yet fully succeeded in getting a benefit from the invariance.

One of natural desires concerning this pairing might be defining the signature, even though it should be a kind of '$\infty - \infty$', because if it could be possible, it would directly give us an invariant of the manifold. This idea is one of the origin for the arguments in the following sections.

3. Foliations and asymptotic linking pairing

3.1. Plane fields and asymptotic linking pairing

One basic idea to deduce topological informations of certain plane fields on 3-manifolds from the asymptotic linking is the following ([11]). To a given smooth non-singular plane field ξ on a closed 3-manifold M, assign the linear subspace

$$N(\xi) = \{d\alpha \,;\, \alpha|_\xi = 0,\, \alpha \in \Omega^1(M)\}$$

of the space of exact 2-forms $B^2(M)$. In [11] this was introduced to investigate the topology of contact plane fields, especially for a positive contact structure ξ which is (locally) defined by a 1-form α_ξ (ker $\alpha_\xi = \xi$) satisfying $\alpha_\xi \wedge d\alpha_\xi > 0$. While it has not yet been successful for contact topology, it turned out to be interesting for codimension 1 foliations on 3-manifolds. For a foliation $\mathcal{F}$ with $\xi = T\mathcal{F}$, the space $N(\xi)$ is also denoted by $N(\mathcal{F})$.

PROPOSITION 3.1. ([11])
1) For a foliation $\mathcal{F}$, $N(\mathcal{F})$ is a null subspace with respect to the asymptotic linking pairing.
2) For a positive contact structure ξ, $N(\xi)$ is a positive definite subspace.

These subspaces are fairy large and it could be expected that, for example, they are almost *maximal* among null subspaces or among positive definite subspaces. In fact, for some class of very special foliations it is a maximal null subspace. On the other hand, it is not quite true for any contact structures.

3.2. $N(\mathcal{F})$ and $N(\mathcal{F})^\perp$

To investigate the maximality and also for the search of possibility of defining the signature of the pairing, the orthonormal complement $N(\mathcal{F})^\perp$ of $N(\mathcal{F})$ with respect to the asymptotic linking pairing is a natural object to study. Remark that $N(\mathcal{F})^\perp$ contains $N(\mathcal{F})$ because $N(\mathcal{F})$ is a null subspace.

For a finite dimensional vector space V with a symmetric bilinear form λ, if we find a null subspace $N \subset V$ the computation of the signature can be reduced to that of another space whose dimension is smaller by twice the $\dim N$, because λ induces a natural symmetric bilinear form λ_N on $N^\perp/N$ and it is easy to see that $\operatorname{sgn}\lambda = \operatorname{sgn}\lambda_N$. In our case, while the computation of signature in general has not yet been justified, the space $N(\mathcal{F})^\perp/N(\mathcal{F})$ and the induced pairing $lk_{\mathcal{F}}$ have important meanings in the theory of characteristic classes of foliations. Before taking a glance at it, let us confirm their significance in our context of fluid mechanics. By careful computations we can verify the following facts.

PROPOSITION 3.2.

$$N(\mathcal{F})^\perp = v(\mathcal{X}_h(M,\mathcal{F}))\,, \quad \text{where } \mathcal{X}_h(M,\mathcal{F}) = \mathcal{X}(M,\mathcal{F}) \cap \mathcal{X}_h(M)\,. \tag{5}$$

$$N(\mathcal{F}) = v(\mathcal{X}_{dloc}(M,\mathcal{F}))\,. \tag{6}$$

(6) gives an understandable description. Our ultimate aim in this article is to understand $\mathcal{X}_d(M,\mathcal{F})$ in a certain case. Therefore our task is divided into the following two steps.

(a) To understand $\mathcal{X}_d(M,\mathcal{F})/\mathcal{X}_h(M,\mathcal{F})$.
(b) To understand $\mathcal{X}_h(M,\mathcal{F})/\mathcal{X}_{dloc}(M,\mathcal{F})$.

Concerning (a), as $\mathcal{X}_d(M,\mathcal{F})/\mathcal{X}_h(M,\mathcal{F})$ is at most $H_1(M;\mathbb{R}) \cong H^2(M;\mathbb{R})$, this part is of finite dimensional. On the other hand, (b) is highly non-trivial because the quotient space $\mathcal{X}_h(M,\mathcal{F})/\mathcal{X}_{dloc}(M,\mathcal{F}) \cong N(\mathcal{F})^\perp/N(\mathcal{F})$ is in most cases of infinite-dimensional and hard to compute. We introduce a special case where this part $\mathcal{X}_h(M,\mathcal{F})/\mathcal{X}_{dloc}(M,\mathcal{F})$ can be completely handled and in fact is equal to 0 or $\mathbb{R}$. It is related to the secondary invariant of the relevant foliation.

4. Leafwise cohomology and asymptotic linking

4.1. Leafwise cohomology

For a general foliated manifold $(M,\mathcal{F})$, the leafwise de Rham complex $(\Omega^*(M;\mathcal{F}),\, d_{\mathcal{F}})$ is defined as follows. Let $(\Omega^*(M;\mathcal{F}) = \Gamma^\infty(\Lambda^* T^*\mathcal{F})$ be the space of families of smooth differential forms on the leaves which vary smoothly in transverse directions, and $d_{\mathcal{F}}$ is the exterior differential along the leaves. This complex coincides with the quotient complex $(\Omega^*(M)/I^*(\mathcal{F}),\, d_{\mathcal{F}})$, where $I^*(\mathcal{F}) = \{\alpha \in I^*(\mathcal{F})\,;\, \alpha|_L = 0 \,\text{for}\,\forall\,\text{leaf}\, L\}$ is the differential ideal and $d_{\mathcal{F}}$ is naturally induced from d on $\Omega^*(M)$. This is valid for general smooth foliations. In the case of foliations of codimension 1, if the foliation $\mathcal{F}$ is defined by a single smooth non-singular 1-form ω, the ideal $I^*(\mathcal{F})$ is generated by ω, namely, $I^*(\mathcal{F}) = \langle\omega\rangle = \omega \wedge \Omega^*(M)$.

The cohomology of this complex is denoted by $H^*(M;\mathcal{F})$ and is called the *leafwise cohomology*. In this article we are particularly interested in $H^1(M;\mathcal{F})$.

Often it is also called the *foliated cohomology* while in some other contexts it can imply the cohomology of the ideal $I^*(\mathcal{F})$ or of other intermediate complexes. Therefore we call it leafwise cohomology in this paper. Remark that a priori we do not have the ellipticity of $d_{\mathcal{F}}$ in the transverse direction, so that $H^*(M;\mathcal{F})$ is in general hard to compute and quite often of infinite dimension.

4.2. Characteristic classes

For a general transversely oriented codimension 1 foliation $\mathcal{F}$, the famous Godbillon-Vey invariant $gv(\mathcal{F}) \in H^3(M;\mathbb{R})$ is defined as follows. First take a 1-form η so that $d\omega = \omega \wedge \eta$ for a defining 1-form ω. Then

$[\eta \wedge d\eta] = gv(\mathcal{F})$ gives the characteristic class. On a closed 3-manifold M the evaluation on $[M]$ (*i.e.*, the integration $\int_M \eta \wedge d\eta$) is denoted by $GV(\mathcal{F})$. In our context, $GV(\mathcal{F})$ is nothing but the asymptotic self-linking of $d\eta \in B^2(M)$. Taking a volume form, we see the corresponding flow lies in $\mathcal{X}_h(M; \mathcal{F})$ because $d\eta = \omega \wedge \zeta$ for some 1-form ζ.

Even though η is not necessarily a closed form on M, it is closed in the leafwise complex, so that it defines a leafwise cohomology class $[\eta] \in H^1(M; \mathcal{F})$ which is called the *Reeb class*. The Reeb class counts the transverse dilatation along the leaf loops.

PROPOSITION 4.1. ([4]) There is a symmetric bilinear pairing on $H^1(M; \mathcal{F})$

$$CJ : H^1(M; \mathcal{F}) \otimes H^1(M; \mathcal{F}) \to H^3(M; \mathbb{R}) \cong \mathbb{R} \tag{7}$$

which is defined by $CJ(\alpha \otimes \beta) = \alpha \wedge d\tilde{\beta}$ where $\tilde{\beta}$ is an extension of β as a 1-form on M.

The map *CJ* generalizes the relation between the Reeb class and the Godbillon-Vey class. On a closed oriented 3-manifold M by the integration $\int_M$, CJ is considered to be $\mathbb{R}$-valued.

4.3. $H^1(M; \mathcal{F})$ and $N(\mathcal{F})^\perp / N(\mathcal{F})$

Here we introduce one more important ingredient.

PROPOSITION 4.2. ([11]) There is a natural surjective mapping

$$\Phi : H^1(M; \mathcal{F}) \to N(\mathcal{F})^\perp / N(\mathcal{F}) \tag{8}$$

which intertwines the pairings CJ and $lk_{\mathcal{F}}$.

Thanks to this proposition, if we find a case where $H^1(M; \mathcal{F})$ is very small or the map Φ has a small image, we can settle down the step (b) in 3.2.

5. Algebraic Anosov foliations

Finally we introduce a class of foliated 3-manifolds with finite dimensional 1st leafwise cohomology. They are so called *algebraic Anosov* foliations.

5.1. Suspension Anosov flows

Take an integral unimodular 2×2 matrix $A \in SL(2; \mathbb{Z})$ with $\operatorname{trace} A > 2$, which naturally acts on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The solv 3-manifold M_A is obtained as the mapping torus $T^2 \times [0,1]/\sim$ where $(Ax, 0) \sim (x, 1)$. The suspension direction determines a vector field Y which generates the suspension flow. The matrix A has two independent eigen direction, one of which is expanding and the other contracting. We can choose vector fields U and S on each fibre T^2 along the eigen directions so as to have the following bracket relations.

$$[Y, U] = -U\,, \quad [Y, S] = S, \quad \text{and} \quad [S, U] = 0\,. \tag{9}$$

Here 'U' and 'S' stand for unstable and stable directions. The spans $E^{uu} = \langle U \rangle$ of U and $E^{ss} = \langle S \rangle$ of S are invariant under the flow generated by Y. A non-singular flow $\exp(tY)$ with an invariant continuous decomposition $TM = \langle Y \rangle \oplus E^{uu} \oplus E^{ss}$ by expanding and contracting sub-bundles E^{uu} and E^{ss} is called an *Anosov flow*. The span $\langle Y, U \rangle = E^u = T\mathcal{F}$ defines the unstable Anosov foliation $\mathcal{F} = \mathcal{F}^u$.

5.2. Geodesic flows of hyperbolic surfaces

Another class of typical Anosov flows is given as follows. Take a closed hyperbolic surface Σ_g of genus $g \geq 2$ and its unit tangent bundle $M = S^1(T\Sigma_g)$. The geodesic flow on M is also an Anosov flow. A family of geodesics which are getting infinitely closer in the past form a leaf of the unstable foliation.

To generalize his construction we take a compact quotient $M = \Gamma\backslash\widetilde{PSL(2;\mathbb{R})}$ by a co-compact discrete subgroup Γ of the universal covering group $\widetilde{PSL(2;\mathbb{R})}$ of $PSL(2;\mathbb{R})$. Then take left invariant vector fields

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{10}$$

from $psl(2;\mathbb{R})$ and we have the bracket relation

$$[Y,U] = -U\,, \quad [Y,S] = S, \quad \text{and} \quad [S,U] = Y\,. \tag{11}$$

5.3. *Leafwise cohomology*

In both cases, Y, U, and S form a global framing of TM. Let Y^*, U^*, and S^* be the dual framing for 1-forms. In both cases the unstable foliation $\mathcal{F}^u$ is defined by $\omega = S^*$ and from the bracket relations, we see that $dS^* = S^* \wedge Y^*$. So the Reeb class $[\eta]$ is exactly given by $[Y^*]$. In the suspension case, we have $dY^* = 0$ and especially the Godbillon-Vey class vanishes, while in the geodesic Anosov case, $Y^* \wedge dY^*$ is the standard volume form and thus the Godbillon-Vey is non-trivial. In fact this is the first examples in the history for non-trivial *GV* and is called Roussarie's example.

For these foliations the 1st leafwise cohomology was computed in [6] and [9].

THEOREM 5.1.
1) ([6]) For the suspension Anosov foliation, $H^1(M;\mathcal{F}) = H^1(M;\mathbb{R}) = \mathbb{R}[\eta]$.
2) ([9]) For the geodesic Anosov foliation, $H^1(M;\mathcal{F}) = H^1(M;\mathbb{R}) \oplus \mathbb{R}[\eta]$.

In the suspension Anosov case it is computed in a similar spirit to that of (1) and (2) in 1.2. From these computations and the propositions in the previous section, in both cases we can easily determine $N(\mathcal{F})^\perp/N(\mathcal{F})$ because Φ maps $H^1(M;\mathbb{R})$ trivially.

COROLLARY 5.2.
1) For the suspension Anosov foliation, $N(\mathcal{F})^\perp/N(\mathcal{F}) = 0$.
2) For the geodesic Anosov foliation, $N(\mathcal{F})^\perp/N(\mathcal{F}) = \mathbb{R}[d\eta]$.

As a conclusion, we obtain the following.

COROLLARY 5.3. For suspension algebraic Anosov foliations

$$\mathcal{X}_d(M;\mathcal{F}) = \mathcal{X}_{dloc}(M;\mathcal{F}) \oplus H_1(M;\mathbb{R})$$

and $H_1(M;\mathbb{R})$ is realized by the suspension Anosov flow Y.

CONJECTURE 5.4. In the geodesic Anosov case, we saw $\mathcal{X}_h(M;\mathcal{F}) = \mathcal{X}_{dloc}(M;\mathcal{F}) \oplus \langle Y \rangle$. It is also conjectured that

$$\mathcal{X}_d(M;\mathcal{F}) = \mathcal{X}_{dloc}(M;\mathcal{F}) \oplus \langle Y \rangle\,.$$

The reason why $H_1(M;\mathbb{R})$ is eliminated is still rigorously to be confirmed. In this class there are many rational homology 3-spheres, for which the conjecture holds because $H_1(M;\mathbb{R})$ does not matter.

It might look strange that these spaces are computed without fixing the volume form. However the leafwise cohomology explains that such a foliated manifold equipped with two different volume forms but with the equal total volume there is a foliated diffeomorphism which transforms one volume to the other.

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