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Symplectic volumes of double weight varieties associated with SU(3), II

by
Taro Suzuki

DEPARTMENT OF MATHEMATICS
CHUO UNIVERSITY
BUNKYOKU TOKYO JAPAN

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Abstract

We consider double weight varieties associated with $SU(3)$, that is, symplectic torus quotients for a direct product of two coadjoint orbits of $SU(3)$. In the previous paper, we gave an explicit formula for the symplectic volumes of them in the case where both of two coadjoint orbits are flag manifolds of $SU(3)$. In this paper, we generalize the volume formula so that we can also apply it to the case where both or one of two orbits is the complex projective plane. Moreover, using the volume formula, we concretely express the symplectic volume in some typical cases.

1 Introduction.

This paper is a continuation of [16]. In [16], we introduced a new class of symplectic manifolds, called double weight varieties. We first review the definition of double weight varieties.

Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$, and T a maximal torus of G with Lie algebra $\mathfrak{t}$. Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ be the dual of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Under an invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, we identify $\mathfrak{g}$ and $\mathfrak{t}$ with $\mathfrak{g}^*$ and $\mathfrak{t}^*$, respectively. By this identification, we regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$.

Under the left coadjoint action of G on $\mathfrak{g}^*$, let $\mathcal{O}_\lambda$ be a coadjoint orbit of G through a point $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$, which has a natural symplectic structure. Then the T-action on $\mathcal{O}_\lambda$ is Hamiltonian.

In general, let M be a compact symplectic manifold with an action of a torus T. Suppose that the T-action on M is Hamiltonian with moment map $\Phi : M \to \mathfrak{t}^*$. For any regular value $\mu \in \mathfrak{t}^*$ of Φ, the symplectic quotient at μ is defined by $M//_\mu T := \Phi^{-1}(\mu)/T$.

In particular, $\mathcal{O}_\lambda//_\mu T$ is called a *weight variety* ([13]). For $G = SU(n)$, Guillemin-Lerman-Sternberg gave some formulas for the volume of weight varieties ([6]), and

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Goldin described the cohomology ring of weight varieties ([4]). For compact semisimple Lie groups except of type A, weight varieties are orbifolds in general and are used as model spaces to verify some invariants of orbifolds ([5]).

Let $\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}$ be two coadjoint orbits. Then the diagonal T-action on the direct product $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$ is Hamiltonian, and we have the symplectic quotient $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$. We call it a *double weight variety*.

In this paper, we consider the case $G = SU(3)$. Except for the orbit consisting only of the origin, each coadjoint orbit of $SU(3)$ is diffeomorphic to either the flag manifold $SU(3)/T$ or the complex projective plane $\mathbb{P}^2$.

There are two goals in this paper. The first is to generalize the volume formula given in [16]. In [16], we gave an explicit formula for the symplectic volumes of double weight varieties $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ in the case where both of two coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are flag manifolds $SU(3)/T$. In this paper, we also consider the case where both or one of two coadjoint orbits is the complex plane $\mathbb{P}^2$.

The second is to give more concrete expression of the symplectic volume in several cases by using the volume formula obtained above. To this end, we draw the image of the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$, and investigate the position of μ in it. The set of regular values of the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$ is decomposed into several connected components, which are called alcoves of Φ. The topological type of a double weight variety $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ depends on the alcove to which μ belongs. By the theorem of Duistermaat-Heckman ([3]), the symplectic volume on each alcove become a polynomial function of μ. Thus in order to study the volume of all double weight varieties, we must detect all of the alcove of $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$. However, it seems to be complicated even in the case of $G = SU(3)$. Therefore, we restrict ourselves to some typical cases.

In order to state the first result in this paper, we use the notation as follows. Let $\mathfrak{W}$ be the Weyl group of $SU(3)$. We denote by $\mathfrak{t}^*_+$ a positive Weyl chamber, and by $\mathfrak{t}^*_{++}$ the interior of the Weyl chamber $\mathfrak{t}^*_+$. Let $\alpha_1, \alpha_2 \in \mathfrak{t}^*$ be simple roots, and $\Lambda_1, \Lambda_2 \in \mathfrak{t}^*$ fundamental weights, which satisfy $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 2$, $\langle \alpha_1, \alpha_2 \rangle = -1$, and $\langle \alpha_i, \Lambda_j \rangle = \delta_{ij}$ $(i, j = 1, 2)$. We set $\rho = \alpha_1 + \alpha_2 = \Lambda_1 + \Lambda_2$, $\gamma_1 = \mathbb{R}_{>0}\alpha_1 + \mathbb{R}_{>0}(\alpha_1 + \alpha_2)$, and $\gamma_2 = \mathbb{R}_{>0}(\alpha_1 + \alpha_2) + \mathbb{R}_{>0}\alpha_2$. For further details of the notation, see Section 2.

Theorem. (See Corollary 3.7 below.) *Let $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+ - \{0\}$ and $\mu \in \mathfrak{t}^*$ satisfy the following two assumptions.*

(A1) *μ is in the convex full of $\{w_1\lambda_1 + w_2\lambda_2 | w_1, w_2 \in \mathfrak{W}\}$,*

(A2) *For any $w_1, w_2 \in \mathfrak{W}$,*

$$\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1 \rangle \neq 0, \ \ \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2 \rangle \neq 0,$$
$$\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1 \rangle \neq \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2 \rangle.$$

Then we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1}\times\mathcal{O}_{\lambda_2})//_{\mu}T)=\sum_{w_1,w_2\in\mathfrak{W}}\varepsilon(w_1)\varepsilon(w_2)F(\lambda_1,\lambda_2,\mu;w_1,w_2),$$

where $\varepsilon(w)=\pm1$ *is the signature of* $w\in\mathfrak{W}$, *and* $F=F(\lambda_1,\lambda_2,\mu;w_1,w_2)$ *is given as follows. We write*

$$A=\langle w_1\lambda_1+w_2\lambda_2-\mu,\Lambda_1\rangle\ ,\ B=\langle w_1\lambda_1+w_2\lambda_2-\mu,\Lambda_2\rangle,$$
$$C=\langle w_1\rho+w_2\rho,\Lambda_1\rangle\ ,\ D=\langle w_1\rho+w_2\rho,\Lambda_2\rangle$$

for brevity.

(1) *If* $\lambda_1,\lambda_2\in\mathfrak{t}^*_{++}$, *then*

$$F=\begin{cases}\dfrac{1}{12}B^3(2A-B) & (\textit{if }\ w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_1),\\ \dfrac{1}{12}A^3(-A+2B) & (\textit{if }\ w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_2),\\ 0 & (\textit{otherwise}).\end{cases}$$

(2) *If* $\lambda_1\in\mathfrak{t}^*_{++},\lambda_2\in\mathfrak{t}^*_{+}-\mathfrak{t}^*_{++}-\{0\}$, *then*

$$F=\begin{cases}\dfrac{1}{6}B^2(B(C-2D)+3AD) & (\textit{if }\ w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_1),\\ \dfrac{1}{6}A^2(A(-2C+D)+3BC) & (\textit{if }\ w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_2),\\ 0 & (\textit{otherwise}).\end{cases}$$

(3) *If* $\lambda_1,\lambda_2\in\mathfrak{t}^*_{+}-\mathfrak{t}^*_{++}-\{0\}$, *then*

$$F=\begin{cases}\dfrac{1}{12}B(B(-6D^2+6CD+1)+2A(3D^2-1)) & \\ & (\textit{if }\ w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_1),\\ \dfrac{1}{12}A(A(-6C^2+6CD+1)+2B(3C^2-1)) & \\ & (\textit{if }\ w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_2),\\ 0 & (\textit{otherwise}).\end{cases}$$

The second result in this paper is concerned with more concrete expression of the symplectic volume in the following three cases.

Case A: $\lambda_1,\lambda_2\in\mathfrak{t}^*_{+}-\mathfrak{t}^*_{++}-\{0\}$ and $\mu\in\mathfrak{t}^*$.

Case B: $\lambda_1,\lambda_2\in\mathfrak{t}^*_{+}-\{0\},\mu\in\mathfrak{t}^*$ and μ is sufficiently close to $\lambda_1+\lambda_2$.

Case C: $\lambda_1,\lambda_2\in\mathfrak{t}^*_{+}-\{0\},\mu\in\mathfrak{t}^*$ and μ is sufficiently close to 0.

In Case A, both of two coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are the complex projective planes $\mathbb{P}^2$, while the position of μ is general. On the other hand, in Cases B and C, the orbit type of coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are general, while μ is in a particular alcove. Here in this introduction, we only state the result for Case B. For Cases A and C, see Section 4.

Corollary. (See Corollary 4.1 below.) *Let* $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+ - \{0\}$ *and* $\mu \in \mathfrak{t}^*$ *satisfy the assumptions* (A1) *and* (A2). *Furthermore assume that* μ *is sufficiently close to* $\lambda_1 + \lambda_2$. *Then we have*

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = \sum_{w_1 \in \mathfrak{W}_{\lambda_1}, w_2 \in \mathfrak{W}_{\lambda_2}} \varepsilon(w_1)\varepsilon(w_2) F(\lambda_1, \lambda_2, \mu; w_1, w_2),$$

where for $\lambda \in \mathfrak{t}^*_+$, *we denote* $\mathfrak{W}_\lambda = \{w \in \mathfrak{W} | w\lambda = \lambda\}$.

Let us set $\mu = x\alpha_1 + y\alpha_2 \in \mathfrak{t}^*$. We can express the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T)$ as a polynomial of p, q, r, s, u, v, x, y as follows.

(1) Let us set $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+ - \mathfrak{t}^*_{++} - \{0\} = \mathbb{R}_{>0}\Lambda_1 \sqcup \mathbb{R}_{>0}\Lambda_2$.

(1a) If $\lambda_1 = u(2\alpha_1 + \alpha_2), \lambda_2 = v(2\alpha_1 + \alpha_2) \in \mathbb{R}_{>0}\Lambda_1$, then we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = (u + v - y)(u + v - x + y).$$

Note that $\lambda_1 + \lambda_2 - \mu$ is not in γ_2.

(1b) If $\lambda_1 = u(2\alpha_1 + \alpha_2) \in \mathbb{R}_{>0}\Lambda_1, \lambda_2 = v(\alpha_1 + 2\alpha_2) \in \mathbb{R}_{>0}\Lambda_2$, then we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = \begin{cases} \frac{1}{2}(u + 2v - y)^2 & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{2}(2u + v - x)^2 & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}$$

(1c) If $\lambda_1 = u(\alpha_1 + 2\alpha_2), \lambda_2 = v(\alpha_1 + 2\alpha_2) \in \mathbb{R}_{>0}\Lambda_2$, then we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = (u + v - x)(u + v + x - y).$$

Note that $\lambda_1 + \lambda_2 - \mu$ is not in γ_1.

(2) Let us set $\lambda_1 \in \mathfrak{t}^*_{++}, \lambda_2 \in \mathfrak{t}^*_+ - \mathfrak{t}^*_{++} - \{0\}$.

(2a) If $\lambda_1 = p\alpha_1 + q\alpha_2 \in \mathfrak{t}^*_{++}, \lambda_2 = u(2\alpha_1 + \alpha_2) \in \mathbb{R}_{>0}\Lambda_1$, then we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = \begin{cases} \frac{1}{6}(q + u - y)^2(3p - 2q + 4u - 3x + 2y) \\ \qquad\qquad\qquad\qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{6}(p + 2u - x)^3 \qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}$$

(2b) If $\lambda_1 = p\alpha_1 + q\alpha_2 \in \mathfrak{t}^*_{++}, \lambda_2 = u(\alpha_1 + 2\alpha_2) \in \mathbb{R}_{>0}\Lambda_2$, then we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = \begin{cases} \frac{1}{6}(q + 2u - y)^3 \qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{6}(p + u - x)^2(-2p + 3q + 4u + 2x - 3y) \\ \qquad\qquad\qquad\qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}$$

(3) If $\lambda_1 = p\alpha_1 + q\alpha_2, \lambda_2 = r\alpha_1 + s\alpha_2 \in \mathfrak{t}^*_{++}$, then we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = \begin{cases} \frac{1}{12}(q + s - y)^3(2p - q + 2r - s - 2x + y) \\ \qquad\qquad\qquad\qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{12}(p + r - x)^3(-p + 2q - r + 2s + x - 2y) \\ \qquad\qquad\qquad\qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}$$

This paper is organized as follows. In Sections 2 and 3, we review the definition of a double weight variety and the result in [16] for the weight multiplicity. In section 3, we prove the theorem of the symplectic volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$. Finally, in Section 4, we consider several examples and write down the symplectic volume for them more explicitly.

2 Preliminaries.

2.1 The representation theory of $SU(3)$ and notation.

In this subsection, we review some standard facts about the representation theory of $SU(3)$ in order to recall the notation used in [16]. We refer to [2] for the generalities on compact Lie groups and their representations.

Let $G = SU(3)$, $\mathfrak{g} = \mathfrak{su}(3)$, T the standard maximal torus of G consisting of diagonal matrices in G, and $\mathfrak{t}$ its Lie algebra. Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ be the duals of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. We define the $\mathrm{Ad}G$-invariant positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by

$$\langle X, Y\rangle := -\frac{1}{4\pi^2}\mathrm{Tr}(XY) \qquad (X, Y \in \mathfrak{g}).$$

We identify $\mathfrak{g}^*$ with $\mathfrak{g}$, or $\mathfrak{t}^*$ with $\mathfrak{t}$ by the inner product $\langle\ ,\ \rangle$. We define simple roots $\alpha_1, \alpha_2 \in \mathfrak{t}^* = \mathfrak{t}$ by

$$\alpha_1 := 2\pi\sqrt{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \alpha_2 := 2\pi\sqrt{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which satisfy $\langle\alpha_1, \alpha_1\rangle = \langle\alpha_2, \alpha_2\rangle = 2,\ \langle\alpha_1, \alpha_2\rangle = -1$. Let us set

$$Q := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2,$$
$$\gamma_1 := \mathbb{R}_{>0}\alpha_1 + \mathbb{R}_{>0}(\alpha_1 + \alpha_2),\ \gamma_2 := \mathbb{R}_{>0}(\alpha_1 + \alpha_2) + \mathbb{R}_{>0}\alpha_2,$$
$$\bar{\gamma_1} := \mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2),\ \bar{\gamma_2} := \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2) + \mathbb{R}_{\geq 0}\alpha_2.$$

We define fundamental weights $\Lambda_1, \Lambda_2 \in \mathfrak{t}^*$ by

$$\Lambda_1 := \frac{2\alpha_1 + \alpha_2}{3},\ \Lambda_2 := \frac{\alpha_1 + 2\alpha_2}{3},$$

which satisfy $\langle\alpha_i, \Lambda_j\rangle = \delta_{ij}$ $(i, j = 1, 2)$. Let us set

$$\mathfrak{t}^*_+ := \mathbb{R}_{\geq 0}\Lambda_1 + \mathbb{R}_{\geq 0}\Lambda_2,\ \mathfrak{t}^*_{++} := \mathbb{R}_{>0}\Lambda_1 + \mathbb{R}_{>0}\Lambda_2,$$
$$P := \mathbb{Z}\Lambda_1 + \mathbb{Z}\Lambda_2,\ P_+ := \mathbb{Z}_{\geq 0}\Lambda_1 + \mathbb{Z}_{\geq 0}\Lambda_2,\ P_{++} := \mathbb{Z}_{>0}\Lambda_1 + \mathbb{Z}_{>0}\Lambda_2.$$

Let $\mathfrak{W} \cong \mathfrak{S}_3$ be the Weyl group of $G = SU(3)$. The Weyl group $\mathfrak{W}$ is generated by $s_1, s_2 \in \mathfrak{W}$, where s_1, s_2 are reflections on $\mathfrak{t}^*$ defined by $s_i\lambda := \lambda - \langle\alpha_i, \lambda\rangle\alpha_i$ for $\lambda \in \mathfrak{t}^*$ and $i = 1, 2$. The set $\mathfrak{t}^*_+$ is called a positive Weyl chamber and forms a fundamental domain of the action of the Weyl group $\mathfrak{W}$ on $\mathfrak{t}^*$.

Elements in P_+ are called dominant weights. For $\lambda \in P_+$, we denote by V_λ the irreducible representation of G with the highest weight $\lambda \in P_+$. Let us set $\rho := \alpha_1 + \alpha_2 = \Lambda_1 + \Lambda_2$.

We denote by W_μ the weight space associated with $\mu \in P$. For a representation V of T, we define the weight multiplicity of W_μ in V by

$$[V : W_\mu] := \dim_{\mathbb{C}}(V \otimes W_\mu^*)^T,$$

the complex dimension of the T-invariant subspace of $V \otimes W_\mu^*$. Then V decomposes the weight spaces:

$$V = \bigoplus_{\mu \in P} [V : W_\mu] W_\mu.$$

2.2 Double weight varieties.

In this subsection, we review the definition of the double weight variety introduced in [16]. Although we consider the case $G = SU(3)$, most of the following still holds when G is a general compact Lie group. For further details on coadjoint orbits, see, e.g., [11] and [14]. For general properties of symplectic quotients, see, e.g., [9],[12] and [15].

The left coadjoint action of G on $\mathfrak{g}^*$ is defined by $g \cdot f := \mathrm{Ad}^*(g) f$ for $g \in G$ and $f \in \mathfrak{g}^*$, where

$$\langle \mathrm{Ad}^*(g) f, X \rangle = \langle f, \mathrm{Ad}(g^{-1}) X \rangle \qquad (X \in \mathfrak{g}).$$

We denote by $\mathcal{O}_\lambda = G \cdot \lambda$ the coadjoint orbit through $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$. Then the intersection $\mathcal{O}_\lambda \cap \mathfrak{t}^*$ is the $\mathfrak{W}$-orbit through λ, and $\mathcal{O}_\lambda \cap \mathfrak{t}^*_+$ consists of a single point. In other words, coadjoint orbits are parametrized by elements in $\mathfrak{t}^*_+$.

For $G = SU(3)$, the topological type of coadjoint orbits $\mathcal{O}_\lambda$ are classified as follows, where G_λ denotes the isotropy subgroup at $\lambda \in \mathfrak{t}^*_+$ for the coadjoint action of G on $\mathfrak{g}^*$.

(1) If $\lambda \in \mathfrak{t}^*_{++}$, then $G_\lambda \cong T$ and $\mathcal{O}_\lambda$ is diffeomorphic to the flag manifold $SU(3)/T$.

(2) If $\lambda \in \mathfrak{t}^*_+ - \mathfrak{t}^*_{++} - \{0\} = \mathbb{R}_{>0}\Lambda_1 \sqcup \mathbb{R}_{>0}\Lambda_2$, then $G_\lambda \cong \{A \in U(1) \times U(2) | \det A = 1\}$ and $\mathcal{O}_\lambda$ is diffeomorphic to the complex projective plane $\mathbb{P}^2$.

(3) If $\lambda = 0$, then $G_\lambda = G$ and $\mathcal{O}_\lambda = \{0\}$.

In this paper, we assume $\lambda \neq 0$, that is, $\mathcal{O}_\lambda \neq \{0\}$.

On each coadjoint orbit $\mathcal{O}_\lambda$, there exists a natural G-invariant symplectic structure ω_λ, called the Kirillov-Kostant-Souriau symplectic form, defined by

$$(\omega_\lambda)_x(\tilde{X}, \tilde{Y}) = \langle x, [X, Y] \rangle \qquad (x \in \mathcal{O}_\lambda, X, Y \in \mathfrak{g}),$$

where $\tilde{X}$ is the vector field on $\mathcal{O}_\lambda$ given by

$$\tilde{X}_x := \left.\frac{d}{dt}\right|_{t=0} (\exp tX) \cdot x.$$

The action of G on $\mathcal{O}_\lambda$ is Hamiltonian and the associated moment map is given by the inclusion $\iota_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$, that is, we have $d\langle \iota, X\rangle = \omega_\lambda(\tilde{X}, \cdot)$.

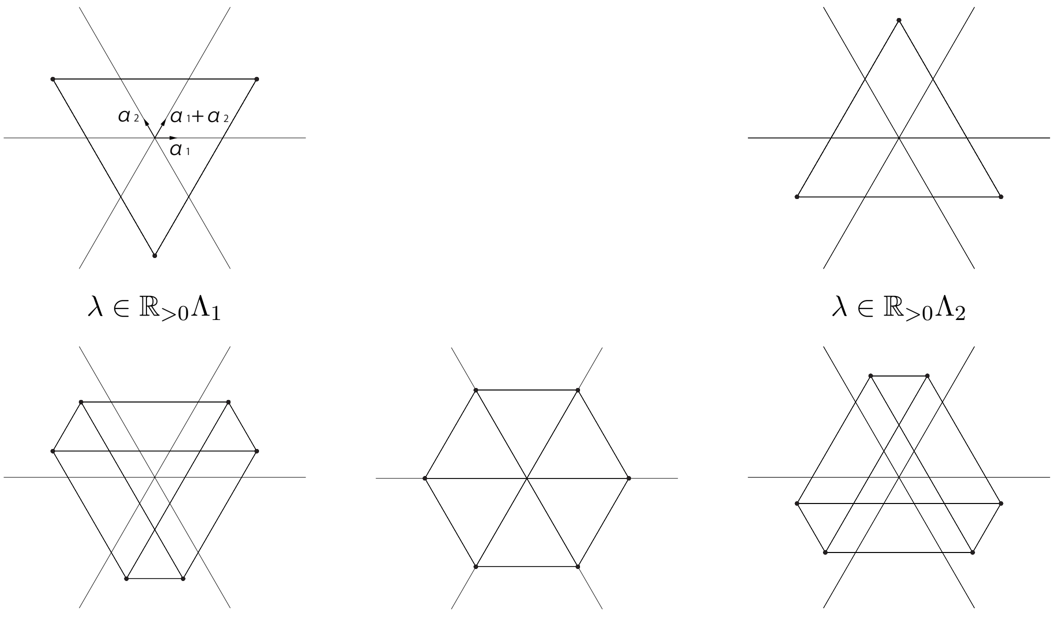
In addition, there exists a G-invariant complex structure J_λ on $\mathcal{O}_\lambda$, which is compatible with the symplectic structure ω_λ. Thus $\mathcal{O}_\lambda$ becomes a Kähler manifold.

Moreover, in the case $\lambda \in P_+$, there exists a G-equivariant holomorphic line bundle L_λ over $\mathcal{O}_\lambda$ such that $c_1(L_\lambda) = [\omega_\lambda]$. The Borel-Weil theorem (see, e.g., [2] and [10]) shows that

$$H^0(\mathcal{O}_\lambda, \mathcal{O}(L_\lambda)) \cong V_\lambda \ , \ \ H^k(\mathcal{O}_\lambda, \mathcal{O}(L_\lambda)) = 0 \ \ (k > 0)$$

as representation of G, where $\mathcal{O}(L_\lambda)$ denotes the sheaf of germs of holomorphic sections of L_λ.

The action of the maximal torus T of G on $\mathcal{O}_\lambda$ is also Hamiltonian, and the associated moment map $\pi_\lambda : \mathcal{O}_\lambda \to \mathfrak{t}^*$ is given by the composition of the inclusion $\iota_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$ and the projection $\mathfrak{g}^* \to \mathfrak{t}^*$. The image of the moment map $\pi_\lambda : \mathcal{O}_\lambda \to \mathfrak{t}^*$ is expressed by a triangle or hexagon and the inside of it as in Figure 1. Here vertices of each triangle or hexagon correspond to the fixed points $(\mathcal{O}_\lambda)^T = \{w\lambda | w \in \mathfrak{W}\}$ for the action of T on $\mathcal{O}_\lambda$, and the union of edges which connect two vertices and are parallel to α_1, α_2 and $\alpha_1 + \alpha_2$ is equal to the set of singular values of the moment map π_λ. For more generalities, see [1], [6] and [7].



$\lambda \in \mathbb{R}_{>0}\Lambda_1$ $\lambda \in \mathbb{R}_{>0}\Lambda_2$

$\lambda \in \mathfrak{t}^*_{++} \cap \gamma_1$ $\lambda \in \mathbb{R}_{>0}(\alpha_1 + \alpha_2)$ $\lambda \in \mathfrak{t}^*_{++} \cap \gamma_2$

Figure 1. The image of the moment map $\pi_\lambda : \mathcal{O}_\lambda \to \mathfrak{t}^*$.

Let $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+$. The diagonal action of the maximal torus T of G on the direct product $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$ is also Hamiltonian and its moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$ is given by $\Phi(x_1, x_2) = \pi_{\lambda_1}(x_1) + \pi_{\lambda_2}(x_2)$.

For $\mu \in \mathfrak{t}^*$, we define the symplectic quotient at μ by

$$\begin{aligned}(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T &:= \Phi^{-1}(\mu)/T \\ &= \{(x_1, x_2) \in \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} | \pi_{\lambda_1}(x_1) + \pi_{\lambda_2}(x_2) = \mu\}/T.\end{aligned}$$

We write $\mathcal{M} = (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ for brevity. Here we assume that for $\mu \in \mathfrak{t}^*$,

(A1) μ is in the convex full of $\{w_1\lambda_1 + w_2\lambda_2 | w_1, w_2 \in \mathfrak{W}\}$,

(A2) For any $w_1, w_2 \in \mathfrak{W}$,

$$\begin{aligned}&\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1\rangle \neq 0, \\ &\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2\rangle \neq 0, \\ &\langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1\rangle \neq \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2\rangle.\end{aligned}$$

By the convexity theorem of Hamiltonian torus actions on symplectic manifolds (see [1] and [7]), (A1) implies that μ is in the image of the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$, that is, $\Phi^{-1}(\mu) \neq \emptyset$. On the other hand, (A2) implies that μ is a regular value of the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$ (see e.g., [6]). Namely, $\Phi^{-1}(\mu)$ is a submanifold of $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$ and the maximal torus T acts locally free on $\Phi^{-1}(\mu)$. In the case $G = SU(3)$, this T-action becomes free. Hence it follows from (A2) that the quotient space $\Phi^{-1}(\mu)/T$ becomes a smooth manifold.

In this case, there exists a natural symplectic structure $\omega = \omega(\lambda_1, \lambda_2, \mu)$ and a compatible complex structure J on $\mathcal{M}$ induced from those on $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$, which make $\mathcal{M}$ a Kähler manifold. The complex dimension d of $\mathcal{M}$ is

$$d = \dim_{\mathbb{R}} G - \frac{1}{2}(\dim_{\mathbb{R}} G_{\lambda_1} + \dim_{\mathbb{R}} G_{\lambda_2}) - \dim_{\mathbb{R}} T.$$

We call $\mathcal{M} = (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ a *double weight variety.*

Now suppose $\lambda_1, \lambda_2 \in P_+$. Let L_{λ_i} be the T-equivariant holomorphic line bundle over $\mathcal{O}_{\lambda_i}$ as above, and let us set

$$\mathcal{L} = (L_{\lambda_1} \boxtimes L_{\lambda_2})//_\mu T := ((\mathrm{pr}_1^* L_{\lambda_1} \otimes \mathrm{pr}_2^* L_{\lambda_2} \otimes W_{-\mu})|_{\Phi^{-1}(\mu)})/T,$$

where $\mathrm{pr}_i : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathcal{O}_{\lambda_i}$ $(i = 1, 2)$ is the i-th projection.

We assume that

(A3) $\lambda_1, \lambda_2, \mu \in Q$.

In general, the action on $\mathrm{pr}_1^* L_{\lambda_1} \otimes \mathrm{pr}_2^* L_{\lambda_2} \otimes W_{-\mu}$ of the center $Z(G) \cong \mathbb{Z}/3\mathbb{Z} \subset T$ of $G = SU(3)$ is not trivial although this action on $\Phi^{-1}(\mu)$ is trivial. However, it follows from (A3) that this action on $\mathrm{pr}_1^* L_{\lambda_1} \otimes \mathrm{pr}_2^* L_{\lambda_2} \otimes W_{-\mu}$ becomes trivial, that is, $\mathcal{L}$ is a genuine holomorphic line bundle over $\mathcal{M}$. Then we have $c_1(\mathcal{L}) = [\omega(\lambda_1, \lambda_2, \mu)]$.

In [16], we showed that the Riemann-Roch number $\mathrm{RR}(\mathcal{M}, \mathcal{L})$ of $(\mathcal{M}, \mathcal{L})$ and the symplectic volume $\mathrm{vol}(\mathcal{M})$ of $(\mathcal{M}, \omega)$ can be expressed in terms of representation of T.

Proposition 2.1 ([16], Proposition 2.3)**.** *Suppose that* $\lambda_1, \lambda_2 \in P_+$ *and* $\mu \in P$ *satisfy the assumptions* (A1), (A2) *and* (A3)*. Then we have*

$$(1) \int_{\mathcal{M}} \mathrm{Ch}(\mathcal{L})\mathrm{Td}(\mathcal{M}) = [V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu],$$

$$(2) \int_{\mathcal{M}} \frac{\omega^d}{d!} = \lim_{k\to\infty} \frac{1}{k^d} \cdot [V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}],$$

where $\mathrm{Ch}(\mathcal{L})$ *denotes the Chern character of* $\mathcal{L}$ *and* $\mathrm{Td}(\mathcal{M})$ *denotes the Todd class of* $\mathcal{M}$*, and* k *runs over positive integers while going to infinity.*

In other words, $\mathrm{RR}(\mathcal{M}, \mathcal{L})$ is equal to the weight multiplicity $[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu]$, and $\mathrm{vol}(\mathcal{M})$ is equal to the leading term in $[V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}]$ as a function of positive integer k.

3 Formulas.

3.1 Formulas for the weight multiplicity and the Riemann-Roch number.

In this subsection, we recall the concrete description of the weight multiplicity $[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu]$ and the explicit formula for the Riemann-Roch number $\mathrm{RR}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T,\ (L_{\lambda_1} \boxtimes L_{\lambda_2})//_\mu T)$ obtained in [16].

By Proposition 2.1, the explicit formula for the weight multiplicity plays the key role in order to compute the symplectic volume of double weight varieties.

Proposition 3.1 ([16])**.** *Let* $\lambda_1, \lambda_2 \in P_+$ *and* $\mu \in P$ *satisfy the assumptions* (A1), (A2) *and* (A3)*. Then we have*

$$[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu] = \sum_{w_1, w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2)E(\lambda_1, \lambda_2, \mu; w_1, w_2), \tag{3.1}$$

where $\varepsilon(w) = \pm 1$ *is the signature of* $w \in \mathfrak{W}$ *and* $E(\lambda_1, \lambda_2, \mu; w_1, w_2)$ *is given as follows. We write*

$$A = \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1\rangle\ ,\ B = \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2\rangle,$$
$$C = \langle w_1\rho + w_2\rho, \Lambda_1\rangle\ ,\ D = \langle w_1\rho + w_2\rho, \Lambda_2\rangle$$

for brevity.

(1) *If* $w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho \in \bar{\gamma_1}$*, then*

$$E(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12}(B + D - 1)(B + D)(B + D + 1)(2(A + C) - (B + D)).$$

(2) *If* $w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho \in \bar{\gamma_2}$*, then*

$$E(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12}(A + C - 1)(A + C)(A + C + 1)(-(A + C) + 2(B + D)).$$

(3) *Otherwise, then*

$$E(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0.$$

Combining Propositions 2.1 and 3.1, we immediately obtain the following formula for the Riemann-Roch number $\mathrm{RR}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T,\ (L_{\lambda_1} \boxtimes L_{\lambda_2})/\!/_\mu T)$.

Corollary 3.2 ([16], Proposition 3.3)**.** *Let $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$ satisfy the assumptions* (A1), (A2) *and* (A3)*. Then the Riemann-Roch number* $\mathrm{RR}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T,\ (L_{\lambda_1} \boxtimes L_{\lambda_2})/\!/_\mu T)$ *is given by the right hand side of the formula* (3.1) *in Proposition* 3.1*.*

3.2 Volume formula.

In this subsection, we give an explicit formula for the symplectic volume of double weight varieties. We have already obtained the volume formula for the case $\lambda_1, \lambda_2 \in \mathfrak{t}^*_{++}$, that is, both of two coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are flag manifolds $SU(3)/T$ ([16, Corollary 3.5]). Here, we also consider the case where either λ_1 or λ_2 is in $\mathfrak{t}^*_+ - \mathfrak{t}^*_{++} - \{0\}$, that is, both or one of $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ is the complex projective plane $\mathbb{P}^2$.

We first give the volume formula in the case $\lambda_1, \lambda_2 \in P_+ - \{0\}$ and $\mu \in P$.

Theorem 3.3. *Let $\lambda_1, \lambda_2 \in P_+ - \{0\}$ and $\mu \in P$ satisfy the assumptions* (A1), (A2) *and* (A3)*. Then we have*

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = \sum_{w_1, w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2) F(\lambda_1, \lambda_2, \mu; w_1, w_2), \tag{3.2}$$

where $F(\lambda_1, \lambda_2, \mu; w_1, w_2)$ is given as follows. We write

$$A = \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1\rangle\ ,\ B = \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2\rangle,$$
$$C = \langle w_1\rho + w_2\rho, \Lambda_1\rangle\ ,\ D = \langle w_1\rho + w_2\rho, \Lambda_2\rangle$$

for brevity.

(1) *If $\lambda_1, \lambda_2 \in P_{++}$, then*

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \begin{cases} \frac{1}{12}B^3(2A - B) & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1), \\ \frac{1}{12}A^3(-A + 2B) & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2), \\ 0 & (\textit{otherwise}). \end{cases}$$

(2) *If $\lambda_1 \in P_{++}, \lambda_2 \in P_+ - P_{++} - \{0\}$, then*

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \begin{cases} \frac{1}{6}B^2(B(C - 2D) + 3AD) & \\ & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1), \\ \frac{1}{6}A^2(A(-2C + D) + 3BC) & \\ & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2), \\ 0 & (\textit{otherwise}). \end{cases}$$

(3) *If* $\lambda_1, \lambda_2 \in P_+ - P_{++} - \{0\}$, *then*

$$F(\lambda_1,\lambda_2,\mu;w_1,w_2) = \begin{cases} \dfrac{1}{12}B(B(-6D^2+6CD+1)+2A(3D^2-1)) & \\ & (\mathit{if}\ \ w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_1), \\ \dfrac{1}{12}A(A(-6C^2+6CD+1)+2B(3C^2-1)) & \\ & (\mathit{if}\ \ w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_2), \\ 0 & (\mathit{otherwise}). \end{cases}$$

Proof. According to Propositions 2.1 and 3.1, we first compute $E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2)$.

The condition $w_1(\lambda_1+\rho)+w_2(\lambda_2+\rho)-\mu-2\rho\in\bar{\gamma_1}$ in (1) of Proposition 3.1 means that $kB+D-2\geq 0$ and

$$\begin{aligned} 0 &\leq \langle w_1(k\lambda_1+\rho)+w_2(k\lambda_2+\rho)-2\rho-k\mu, \Lambda_1-\Lambda_2\rangle \\ &= k(A-B)+(C-D). \end{aligned}$$

Let us take k large enough. By the assumption (A2), these inequalities above mean that $A>B>0$, that is, $w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_1$. Then we have

$$\begin{aligned} &E(k\lambda_1,k\lambda_2,k\mu;w_1,w_2) \\ &= \frac{1}{12}(kB+D-1)(kB+D)(kB+D+1)(2(kA+C)-(kB+D)) \\ &= \frac{k^4}{12}B^3(2A-B)+\frac{k^3}{6}B^2(B(C-2D)+3AD) \\ &\quad+\frac{k^2}{12}B(B(-6D^2+6CD+1)+2A(3D^2-1)) \\ &\quad+\frac{k}{6}(-AD(D^2-1)+BC(3D^2-1)-BD(2D^2-1)) \\ &\quad+\frac{1}{12}D(D^2-1)(-2C+D). \end{aligned}$$

Similarly, when $k\gg 0$, the condition $w_1(\lambda_1+\rho)+w_2(\lambda_2+\rho)-\mu-2\rho\in\bar{\gamma_2}$ in (2) of Proposition 3.1 means that $B>A>0$, that is, $w_1\lambda_1+w_2\lambda_2-\mu\in\gamma_2$. Then we have

$$\begin{aligned} &E(k\lambda_1,k\lambda_2,k\mu;w_1,w_2) \\ &= \frac{1}{12}(kA+C-1)(kA+C)(kA+C+1)(-(kA+C)+2(kB+D)) \\ &= \frac{k^4}{12}A^3(-A+2B)+\frac{k^3}{6}A^2(A(-2C+D)+3BC) \\ &\quad+\frac{k^2}{12}A(A(-6C^2+6CD+1)+2B(3C^2-1)) \\ &\quad+\frac{k}{6}(-BC(C^2-1)-AC(2C^2-1)+AD(3C^2-1)) \\ &\quad+\frac{1}{12}C(C^2-1)(C-2D). \end{aligned}$$

Finally, when $k\gg 0$, the condition that $w_1(\lambda_1+\rho)+w_2(\lambda_2+\rho)-\mu-2\rho$ is not in $\bar{\gamma_1}$ and $\bar{\gamma_2}$ in (3) of Proposition 3.1 means that $A<0$ or $B<0$. Then we have

$$E(k\lambda_1,k\lambda_2,k\mu;w_1,w_2)=0.$$

Now, for $\lambda_1, \lambda_2 \in P_+ - \{0\}, \mu \in P$ and $w_1, w_2 \in \mathfrak{W}$, we define functions F_1, F_2 and F_3 as follows.

$$F_1(\lambda_1, \lambda_2, \mu; w_1, w_2) := \begin{cases} \frac{1}{12} B^3(2A - B) & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1), \\ \frac{1}{12} A^3(-A + 2B) & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2), \\ 0 & (\textit{otherwise}). \end{cases}$$

$$F_2(\lambda_1, \lambda_2, \mu; w_1, w_2) := \begin{cases} \frac{1}{6} B^2(B(C - 2D) + 3AD) & \\ & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1), \\ \frac{1}{6} A^2(A(-2C + D) + 3BC) & \\ & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2), \\ 0 & (\textit{otherwise}). \end{cases}$$

$$F_3(\lambda_1, \lambda_2, \mu; w_1, w_2) := \begin{cases} \frac{1}{12} B(B(-6D^2 + 6CD + 1) + 2A(3D^2 - 1)) & \\ & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1), \\ \frac{1}{12} A(A(-6C^2 + 6CD + 1) + 2B(3C^2 - 1)) & \\ & (\textit{if } w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2), \\ 0 & (\textit{otherwise}). \end{cases}$$

Note that the functions F_1, F_2 and F_3 are the coefficient of k^4, k^3 and k^2 in $E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2)$, respectively.

(1) For $\lambda_1, \lambda_2 \in P_{++}$ and $\mu \in P$ satisfying (A1), (A2) and (A3), we have

$$d = \dim_{\mathbb{C}}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu} T) = 4$$

and

$$\begin{aligned} &[V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}] \\ &= k^4 \sum_{w_1, w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2) F_1(\lambda_1, \lambda_2, \mu; w_1, w_2) + (\text{lower terms of } k). \end{aligned}$$

By Proposition 2.1, taking $F = F_1$, we obtain the assertion (1). See also [16, Theorem 3.4]

(2) For $\lambda_1 \in P_{++}, \lambda_2 \in P_+ - P_{++} - \{0\}$ and $\mu \in P$ satisfying (A1), (A2) and (A3), we have

$$d = \dim_{\mathbb{C}}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu} T) = 3.$$

On the other hand, for each $w_1, w_2 \in \mathfrak{W}$, $E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2)$ is given by the same formula as (1). In this case, we claim the following although each $F_1(\lambda_1, \lambda_2, \mu; w_1, w_2) \neq 0$ in general.

Claim 3.4. If $\lambda_1 \in P_+ - \{0\}, \lambda_2 \in P_+ - P_{++} - \{0\}$ and $\mu \in P$ satisfy (A1), (A2) and (A3), then we have

$$\sum_{w_1, w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2) F_1(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0.$$

By Claim 3.4, we have

$$\begin{aligned}
&[V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}] \\
&= k^3 \sum_{w_1, w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2) F_2(\lambda_1, \lambda_2, \mu; w_1, w_2) + (\text{lower terms of } k).
\end{aligned}$$

Taking $F = F_2$, we obtain the assertion (2).

PROOF OF CLAIM 3.4. Regarding A, B, C and D as functions with respect to $(w_1, w_2) \in \mathfrak{W}^2$, we denote

$$A = A(w_1, w_2),\ B = B(w_1, w_2),\ C = C(w_1, w_2),\ D = D(w_1, w_2).$$

Now we assume $\lambda_2 \in 3\mathbb{Z}_{>0}\Lambda_1 \subset (P_+ - P_{++} - \{0\}) \cap Q$. Because $s_2\lambda_2 = \lambda_2$ for $s_2 \in \mathfrak{W}$, we see that $w_1\lambda_1 + w_2\lambda_2 - \mu = w_1\lambda_1 + w_2 s_2 \lambda_2 - \mu$ and

$$A(w_1, w_2) = A(w_1, w_2 s_2),\ B(w_1, w_2) = B(w_1, w_2 s_2) \qquad (w_1, w_2 \in \mathfrak{W}).$$

Here, for $\lambda_1 \in P_+ - \{0\}, \lambda_2 \in 3\mathbb{Z}_{>0}\Lambda_1$ and $\mu \in P$, it is enough to prove

$$\varepsilon(w_1)\varepsilon(w_2) F_1(\lambda_1, \lambda_2, \mu; w_1, w_2) + \varepsilon(w_1)\varepsilon(w_2 s_2) F_1(\lambda_1, \lambda_2, \mu; w_1, w_2 s_2) = 0 \tag{3.3}$$

for any $w_1, w_2 \in \mathfrak{W}$.

If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$, then we have

$$\begin{aligned}
&\varepsilon(w_1)\varepsilon(w_2) F_1(\lambda_1, \lambda_2, \mu; w_1, w_2) + \varepsilon(w_1)\varepsilon(w_2 s_2) F_1(\lambda_1, \lambda_2, \mu; w_1, w_2 s_2) \\
&= \varepsilon(w_1)\varepsilon(w_2) \frac{1}{12} B(w_1, w_2)^3 (2A(w_1, w_2) - B(w_1, w_2)) \\
&\quad + \varepsilon(w_1)\varepsilon(w_2 s_2) \frac{1}{12} B(w_1, w_2 s_2)^3 (2A(w_1, w_2 s_2) - B(w_1, w_2 s_2)) \\
&= \varepsilon(w_1)\varepsilon(w_2) \frac{1}{12} B(w_1, w_2)^3 (2A(w_1, w_2) - B(w_1, w_2)) \\
&\quad - \varepsilon(w_1)\varepsilon(w_2) \frac{1}{12} B(w_1, w_2)^3 (2A(w_1, w_2) - B(w_1, w_2)) \\
&= 0
\end{aligned}$$

as claimed.

If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$, then we have

$$\begin{aligned}
&\varepsilon(w_1)\varepsilon(w_2)F_1(\lambda_1,\lambda_2,\mu;w_1,w_2) + \varepsilon(w_1)\varepsilon(w_2s_2)F_1(\lambda_1,\lambda_2,\mu;w_1,w_2s_2)\\
&= \varepsilon(w_1)\varepsilon(w_2)\frac{1}{12}A(w_1,w_2)^3(-A(w_1,w_2) + 2B(w_1,w_2))\\
&\quad + \varepsilon(w_1)\varepsilon(w_2s_2)\frac{1}{12}A(w_1,w_2s_2)^3(-A(w_1,w_2s_2) + 2B(w_1,w_2s_2))\\
&= \varepsilon(w_1)\varepsilon(w_2)\frac{1}{12}A(w_1,w_2)^3(-A(w_1,w_2) + 2B(w_1,w_2))\\
&\quad - \varepsilon(w_1)\varepsilon(w_2)\frac{1}{12}A(w_1,w_2)^3(-A(w_1,w_2) + 2B(w_1,w_2))\\
&= 0
\end{aligned}$$

as claimed.

If $w_1\lambda_1 + w_2\lambda_2 - \mu$ is not in γ_1 and γ_2, then it is obvious that (3.3) holds.

On the other hand, in the case $\lambda_2 \in 3\mathbb{Z}_{>0}\Lambda_2$, by $s_1\lambda_2 = \lambda_2$ for $s_1 \in \mathfrak{W}$, we see that $w_1\lambda_1 + w_2\lambda_2 - \mu = w_1\lambda_1 + w_2s_1\lambda_2 - \mu$ and

$$A(w_1,w_2) = A(w_1,w_2s_1),\ B(w_1,w_2) = B(w_1,w_2s_1) \qquad (w_1,w_2 \in \mathfrak{W}).$$

Then it is enough to prove

$$\varepsilon(w_1)\varepsilon(w_2)F_1(\lambda_1,\lambda_2,\mu;w_1,w_2) + \varepsilon(w_1)\varepsilon(w_2s_1)F_1(\lambda_1,\lambda_2,\mu;w_1,w_2s_1) = 0$$

for any $w_1, w_2 \in \mathfrak{W}$. This is showed by a similar argument to that in (3.3). We have completed the proof of Claim 3.4. $\square$

(3) If $\lambda_1, \lambda_2 \in P_+ - P_{++} - \{0\}$ and $\mu \in P$ satisfy (A1), (A2) and (A3), we have

$$d = \dim_{\mathbb{C}}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T) = 2.$$

Also in this case, for each $w_1, w_2 \in \mathfrak{W}$, $E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2)$ is given by the same formula as (1). We claim the following although each $F_i(\lambda_1,\lambda_2,\mu;w_1,w_2) \neq 0$ $(i = 1,2)$ in general.

Claim 3.5. If $\lambda_1, \lambda_2 \in P_+ - P_{++} - \{0\}$ and $\mu \in P$ satisfy (A1), (A2) and (A3), then we have

$$\text{(i)} \sum_{w_1,w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2)F_1(\lambda_1,\lambda_2,\mu;w_1,w_2) = 0,$$

$$\text{(ii)} \sum_{w_1,w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2)F_2(\lambda_1,\lambda_2,\mu;w_1,w_2) = 0.$$

By Claim 3.5, we have

$$\begin{aligned}
&[V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}]\\
&= k^2 \sum_{w_1,w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2)F_3(\lambda_1,\lambda_2,\mu;w_1,w_2) + (\text{lower terms of } k).
\end{aligned}$$

Taking $F = F_3$, we obtain the assertion (3). $\square$

Proof of Claim 3.5. (i) It follows from Claim 3.4.

(ii) We denote $\mathfrak{W}_\lambda = \{w \in \mathfrak{W} | w\lambda = \lambda\}$ for $\lambda \in P_+ - P_{++} - \{0\}$. Namely, $\mathfrak{W}_\lambda = \{e, s_2\}$ for $\lambda \in 3\mathbb{Z}_{>0}\Lambda_1$, and $\mathfrak{W}_\lambda = \{e, s_1\}$ for $\lambda \in 3\mathbb{Z}_{>0}\Lambda_2$. Then for any $w_1, w_2 \in \mathfrak{W}$, $w \in \mathfrak{W}_{\lambda_1}$ and $w' \in \mathfrak{W}_{\lambda_2}$, we have

$$w_1 w\lambda_1 + w_2 w'\lambda_2 - \mu = w_1\lambda_1 + w_2\lambda_2 - \mu$$

and

$$A(w_1w, w_2w') = A(w_1, w_2),\ B(w_1w, w_2w') = B(w_1, w_2).$$

It is enough to prove

$$\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w_1w)\varepsilon(w_2w')F_2(\lambda_1, \lambda_2, \mu; w_1w, w_2w') = 0 \tag{3.4}$$

for any $w_1, w_2 \in \mathfrak{W}$.

If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$, then we have

$$\begin{aligned}
&\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w_1w)\varepsilon(w_2w')F_2(\lambda_1, \lambda_2, \mu; w_1w, w_2w')\\
&= \sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w_1w)\varepsilon(w_2w')\frac{1}{6}B(w_1w, w_2w')^2\\
&\cdot \big\{B(w_1w, w_2w')\big(C(w_1w, w_2w') - 2D(w_1w, w_2w')\big) + 3A(w_1w, w_2w')D(w_1w, w_2w')\big\}\\
&= \varepsilon(w_1)\varepsilon(w_2)\frac{1}{6}B(w_1, w_2)^2\\
&\cdot \Big\{B(w_1, w_2)\Big(\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w)\varepsilon(w')\big(C(w_1w, w_2w') - 2D(w_1w, w_2w')\big)\Big)\\
&\qquad + 3A(w_1, w_2)\Big(\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w)\varepsilon(w')D(w_1w, w_2w')\Big)\Big\}.
\end{aligned}$$

Since

$$\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w)\varepsilon(w')(w_1w\rho + w_2w'\rho) = 0,$$

we have

$$\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w)\varepsilon(w')C(w_1w, w_2w') = \sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w)\varepsilon(w')D(w_1w, w_2w') = 0,$$

which proves the equation (3.4).

If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$, then we have

$$
\begin{aligned}
&\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w_1w)\varepsilon(w_2w')F_2(\lambda_1,\lambda_2,\mu;w_1w,w_2w') \\
&= \sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w_1w)\varepsilon(w_2w')\frac{1}{6}A(w_1w,w_2w')^2 \\
&\cdot \big\{A(w_1w,w_2w')\big(-2C(w_1w,w_2w')+D(w_1w,w_2w')\big)+3B(w_1w,w_2w')C(w_1w,w_2w')\big\} \\
&= \varepsilon(w_1)\varepsilon(w_2)\frac{1}{6}A(w_1,w_2)^2 \\
&\cdot \Big\{A(w_1,w_2)\Big(\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w)\varepsilon(w')\big(-2C(w_1w,w_2w')+D(w_1w,w_2w')\big)\Big) \\
&\qquad\qquad + 3B(w_1,w_2)\Big(\sum_{w\in\mathfrak{W}_{\lambda_1}, w'\in\mathfrak{W}_{\lambda_2}} \varepsilon(w)\varepsilon(w')C(w_1w,w_2w')\Big)\Big\} \\
&= 0
\end{aligned}
$$

as claimed.

If $w_1\lambda_1 + w_2\lambda_2 - \mu$ is not in γ_1 and γ_2, then it is obvious that (3.4) holds. This completes the proof. $\square$

Remark 3.6. In general, for $\mu \in P$, there exist $w \in \mathfrak{W}$ and $\mu' \in P_+$ such that $\mu = w\mu'$. Since $[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu] = [V_{\lambda_1} \otimes V_{\lambda_2} : W_{\mu'}]$, we have $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T) = \mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_{\mu'} T$. But the computation of the right hand side of the formula (3.2) for $\mu' \in P_+$ becomes simpler than that for $\mu \in P$.

We can extend Theorem 3.3 to the general case $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+ - \{0\}$ and $\mu \in \mathfrak{t}^*$.

Corollary 3.7. *Suppose that $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+ - \{0\}$ and $\mu \in \mathfrak{t}^*$ satisfy the assumptions* (A1) *and* (A2). *Then the volume* $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T)$ *is given by the right hand side of the formula* (3.2), *where we replace P_+ and P_{++} by $\mathfrak{t}^*_+$ and $\mathfrak{t}^*_{++}$, respectively.*

Proof. By continuity of the symplectic form of symplectic quotients (see, e.g., [3] and [8]), it is enough to show in the case that $\lambda_1, \lambda_2 \in (P \otimes \mathbb{Q}) \cap \mathfrak{t}^*_+ - \{0\}$ and $\mu \in P \otimes \mathbb{Q}$ satisfy (A1) and (A2). Since $P \otimes \mathbb{Q} = Q \otimes \mathbb{Q}$, there exists $n \in \mathbb{Z}_{>0}$ such that $n\lambda_1, n\lambda_2 \in P_+ - \{0\}$ and $n\mu \in P$ satisfy (A3). Thus we have

$$\mathrm{vol}((\mathcal{O}_{n\lambda_1} \times \mathcal{O}_{n\lambda_2})/\!/_{n\mu} T) = \sum_{w_1, w_2 \in \mathfrak{W}} \varepsilon(w_1)\varepsilon(w_2)F(n\lambda_1, n\lambda_2, n\mu; w_1, w_2).$$

On the other hand, $(\mathcal{O}_{n\lambda_1} \times \mathcal{O}_{n\lambda_2})/\!/_{n\mu} T = (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T$ as complex manifolds, and $\omega(n\lambda_1, n\lambda_2, n\mu) = n^d \cdot \omega(\lambda_1, \lambda_2, \mu)$. Hence we have

$$\mathrm{vol}((\mathcal{O}_{n\lambda_1} \times \mathcal{O}_{n\lambda_2})/\!/_{n\mu} T) = n^d \cdot \mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T).$$

Furthermore, for $n \in \mathbb{Z}_{>0}$, the conditions $w_1(n\lambda_1) + w_2(n\lambda_2) - n\mu \in \gamma_1$ and $w_1(n\lambda_1) + w_2(n\lambda_2) - n\mu \in \gamma_2$ are equivalent to the conditions $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$ and $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$, respectively. Thus we have

$$F(n\lambda_1, n\lambda_2, n\mu; w_1, w_2) = n^d \cdot F(\lambda_1, \lambda_2, \mu; w_1, w_2).$$

Combining all the results above, we obtain the assertion. $\square$

Remark 3.8. As will be discussed in [17], using the volume formula (3.2), we can compute the generating function of the cohomological intersection product for double weight varieties $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$.

4 Examples.

In this section, as examples of Theorem 3.3 and Corollary 3.7, we concretely compute the volume of some special double weight varieties. As stated in introduction, the topological type of a double weight variety $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ depends on the alcove to which μ belongs. Thus in order to compute the volume of double weight varieties, we must detect all of the alcove of $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$. We investigate the following three cases.

Case A: λ_1 and λ_2 are in $\mathfrak{t}^*_+ - \mathfrak{t}^*_{++} - \{0\} = \mathbb{R}_{>0}\Lambda_1 \sqcup \mathbb{R}_{>0}\Lambda_2$.

Case B: $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+ - \{0\}$, $\mu \in \mathfrak{t}^*$ and μ is sufficiently close to $\lambda_1 + \lambda_2$.

Case C: $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+ - \{0\}$, $\mu \in \mathfrak{t}^*$ and μ is sufficiently close to 0.

In Case A, each of two coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ is the complex projective plane $\mathbb{P}^2$, while the position of μ is general. On the other hand, in Cases B and C, the orbit type of coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are general, while μ is in a particular alcove.

Assume that $\lambda_1, \lambda_2 \in \mathfrak{t}^*_+ - \{0\}$ and $\mu \in \mathfrak{t}^*$ satisfy the assumptions (A1) and (A2) in Section 3.

4.1 Case A: λ_1 and λ_2 are in $\mathfrak{t}^*_+ - \mathfrak{t}^*_{++} - \{0\}$.

We consider the following two cases.

Case I. $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}\Lambda_1 = \mathbb{R}_{>0}(2\alpha_1 + \alpha_2)$.

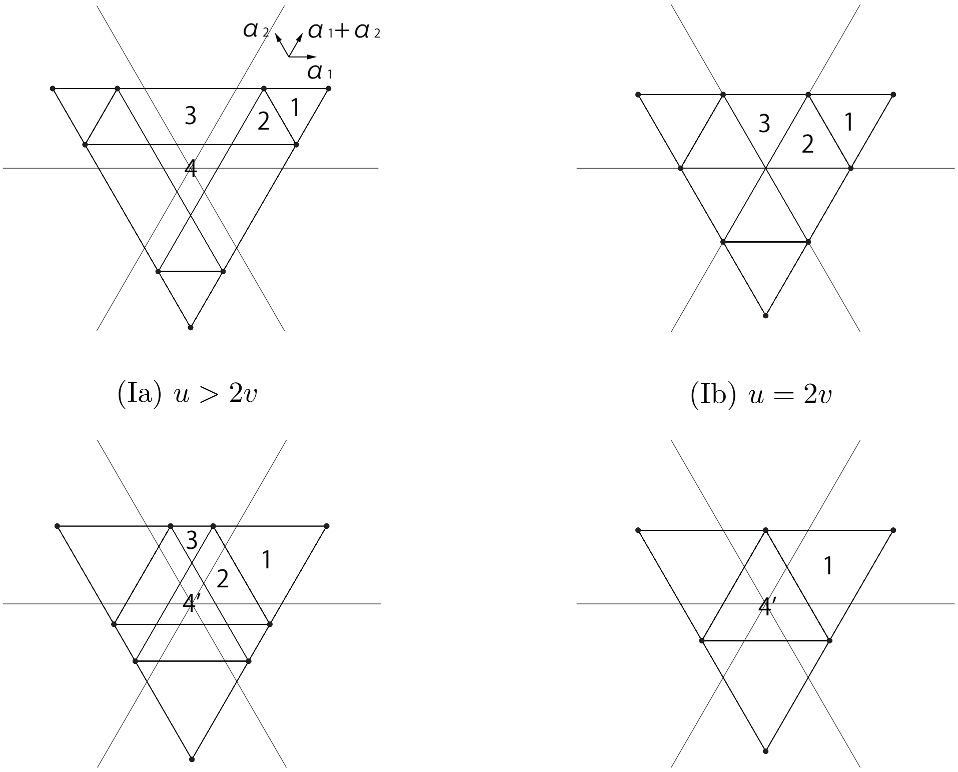
Case II. $\lambda_1 \in \mathbb{R}_{>0}\Lambda_1 = \mathbb{R}_{>0}(2\alpha_1 + \alpha_2)$, $\lambda_2 \in \mathbb{R}_{>0}\Lambda_2 = \mathbb{R}_{>0}(\alpha_1 + 2\alpha_2)$.

In Case I, we can set $\lambda_1 = u(2\alpha_1 + \alpha_2), \lambda_2 = v(2\alpha_1 + \alpha_2)$ $(u \geq v > 0)$ without loss of generality. Then Case I are divided into the following four cases.

(Ia) $u > 2v$, (Ib) $u = 2v$, (Ic) $v < u < 2v$, (Id) $u = v$.

In the four cases above, the image of the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$ is expressed by a triangle and the inside of it as in Figure 2.

By Remark 3.4, it is sufficient to consider only the case $\mu \in \mathfrak{t}^*_+$. If $\mu = x\alpha_1 + y\alpha_2 \in \mathfrak{t}^*_+$ is in the alcove 1 in Figure 2 (Ia), then $w_1, w_2 \in \mathfrak{W} \cong \mathfrak{S}_3$ which satisfy $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$ are $(w_1, w_2) = (e, e), (e, s_1), (s_1, e), (s_1, s_1)$, and $w_1, w_2 \in \mathfrak{W}$ which satisfy $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$ is nothing. Thus we can express the volume



(Ia) $u > 2v$ (Ib) $u = 2v$

(Ic) $v < u < 2v$ (Id) $u = v$

Figure 2.

$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ as a polynomial of u, v, x, y as follows.

$$\begin{aligned}
\mathrm{vol}_1 =& \frac{1}{12}(u+v-y)\{(u+v-y)\cdot 1 + 2(2u+2v-x)\cdot 11\} \\
&- \frac{1}{12}(u+v-y)\{(u+v-y)\cdot 7 + 2(2u+2v-x)\cdot 2\} \\
&- \frac{1}{12}(u+v-y)\{(u+v-y)\cdot 7 + 2(2u+2v-x)\cdot 2\} \\
&+ \frac{1}{12}(u+v-y)\{(u+v-y)\cdot 1 + 2(2u+2v-x)\cdot(-1)\} \\
=&(u+v-y)\{-(u+v-y)+(2u+2v-x)\} \\
=&u^2+2uv+v^2-ux-vx+xy-y^2,
\end{aligned}$$

where the symbol "vol_1" means the volume $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T$ for $\mu \in \mathfrak{t}^*$ which is in the alcove 1. In the following, we use similar symbols.

If $\mu = x\alpha_1 + y\alpha_2$ is in the alcove 2, 3, 4 in Figure 2 (Ia), then we have

$$\begin{aligned}
\mathrm{vol}_2 =& -u^2 + 4uv + \frac{1}{2}v^2 + ux - 2vx - \frac{1}{2}x^2 + xy - y^2, \\
\mathrm{vol}_3 =& -\frac{1}{2}u^2 + 2uv + \frac{5}{2}v^2 + uy - 2vy - \frac{1}{2}y^2, \\
\mathrm{vol}_4 =& \frac{9}{2}v^2,
\end{aligned}$$

respectively.

If μ is in the alcove 1, 2, 3 in Figure 2 (Ib) or (Ic), then the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ are equal to $\mathrm{vol}_1,\ \mathrm{vol}_2,\ \mathrm{vol}_3$, respectively.

If $\mu = x\alpha_1 + y\alpha_2$ is in the alcove $4'$ in Figure 2 (Ic), then we have

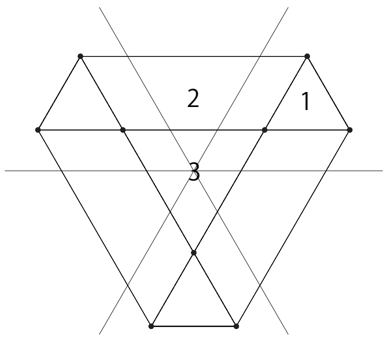
$$\mathrm{vol}_{4'} = -\frac{3}{2}u^2 + 6uv - \frac{3}{2}v^2 - x^2 + xy - y^2.$$

If μ is in the alcove 1, $4'$ in Figure 2 (Id), then the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ are equal to $\mathrm{vol}_1,\ \mathrm{vol}_4'$, respectively.

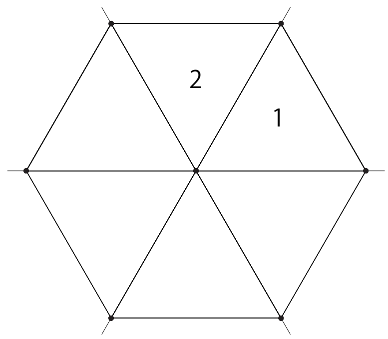
Similarly, in Case II, we set $\lambda_1 = u(2\alpha_1 + \alpha_2), \lambda_2 = v(\alpha_1 + 2\alpha_2)$ $(u, v > 0)$. Then Case II are divided into the following three cases.

(IIa) $u > v$, (IIb) $u = v$, (IIc) $u < v$.

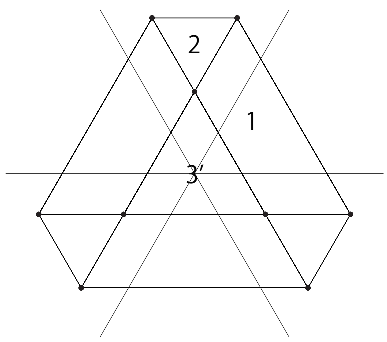
In the three cases above, we can draw the image of the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \to \mathfrak{t}^*$ as in Figure 3.



(IIa) $u > v$



(IIb) $u = v$



(IIc) $u < v$

Figure 3.

If $\mu = x\alpha_1 + y\alpha_2$ is in the alcove 1, 2, 3 in Figure 3 (IIa), then we have

$$\begin{aligned}
\mathrm{vol}_1 &= \frac{1}{2}(2u+v-x)^2 = 2u^2 + 2uv + \frac{1}{2}v^2 - 2ux - vx + \frac{1}{2}x^2, \\
\mathrm{vol}_2 &= \frac{1}{2}(u+2v-y)^2 = \frac{1}{2}u^2 + 2uv + 2v^2 - uy - 2vy + \frac{1}{2}y^2, \\
\mathrm{vol}_3 &= \frac{9}{2}v^2,
\end{aligned}$$

respectively.

If μ is in the alcove 1, 2 in Figure 3 (IIb) or (IIc), then the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ are equal to $\mathrm{vol}_1,\ \mathrm{vol}_2$, respectively.

If $\mu = x\alpha_1 + y\alpha_2$ is in the alcove $3'$ in Figure 3 (IIc), then we have

$$\mathrm{vol}_{3'} = \frac{9}{2}u^2.$$

4.2 Case B: μ is sufficiently close to $\lambda_1 + \lambda_2$.

If μ is in an alcove which is close to the vertex $\lambda_1+\lambda_2$, then we have $F(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0$ unless $w_1, w_2 \in \mathfrak{W}$ satisfy $w_1\lambda_1 = \lambda_1$ and $w_2\lambda_2 = \lambda_2$. Hence we obtain the following by Corollary 3.7.

Corollary 4.1. *Let $\lambda_1, \lambda_2 \in \mathfrak{t}_+^* - \{0\}$ and $\mu \in \mathfrak{t}^*$ satisfy the assumptions* (A1) *and* (A2)*. Furthermore assume that μ is sufficiently close to $\lambda_1 + \lambda_2$. Then we have*

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T) = \sum_{w_1 \in \mathfrak{W}_{\lambda_1}, w_2 \in \mathfrak{W}_{\lambda_2}} \varepsilon(w_1)\varepsilon(w_2)F(\lambda_1, \lambda_2, \mu; w_1, w_2),$$

where for $\lambda \in \mathfrak{t}_+^*$, we denote $\mathfrak{W}_\lambda = \{w \in \mathfrak{W} | w\lambda = \lambda\}$.

Let us set $\mu = x\alpha_1 + y\alpha_2 \in \mathfrak{t}^*$. We can express the volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ as a polynomial of p, q, r, s, u, v, x, y as follows.

(1) Let us set $\lambda_1, \lambda_2 \in \mathfrak{t}_+^* - \mathfrak{t}_{++}^* - \{0\} = \mathbb{R}_{>0}\Lambda_1 \sqcup \mathbb{R}_{>0}\Lambda_2 = \mathbb{R}_{>0}(2\alpha_1+\alpha_2) \sqcup \mathbb{R}_{>0}(\alpha_1 + 2\alpha_2)$.

(1a) If $\lambda_1 = u(2\alpha_1+\alpha_2), \lambda_2 = v(2\alpha_1+\alpha_2) \in \mathbb{R}_{>0}\Lambda_1$, then $\mathfrak{W}_{\lambda_1} = \mathfrak{W}_{\lambda_2} = \{e, s_2\}$. Thus we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T) = (u+v-y)(u+v-x+y).$$

Note that $\lambda_1 + \lambda_2 - \mu$ is not in γ_2. See Case I in Section 4.1.

(1b) If $\lambda_1 = u(2\alpha_1 + \alpha_2) \in \mathbb{R}_{>0}\Lambda_1, \lambda_2 = v(\alpha_1 + 2\alpha_2) \in \mathbb{R}_{>0}\Lambda_2$, then $\mathfrak{W}_{\lambda_1} = \{e, s_2\}, \mathfrak{W}_{\lambda_2} = \{e, s_1\}$. Thus we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T) = \begin{cases} \frac{1}{2}(u+2v-y)^2 & (\text{if } \lambda_1+\lambda_2-\mu \in \gamma_1), \\ \frac{1}{2}(2u+v-x)^2 & (\text{if } \lambda_1+\lambda_2-\mu \in \gamma_2). \end{cases}$$

See Case II in Section 4.1.

(1c) If $\lambda_1 = u(\alpha_1+2\alpha_2), \lambda_2 = v(\alpha_1+2\alpha_2) \in \mathbb{R}_{>0}\Lambda_2$, then $\mathfrak{W}_{\lambda_1} = \mathfrak{W}_{\lambda_2} = \{e, s_1\}$. Thus we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_{\mu}T) = (u + v - x)(u + v + x - y).$$

Note that $\lambda_1 + \lambda_2 - \mu$ is not in γ_1.

(2) Let us set $\lambda_1 \in \mathfrak{t}^*_{++}, \lambda_2 \in \mathfrak{t}^*_{+} - \mathfrak{t}^*_{++} - \{0\}$.

(2a) If $\lambda_1 = p\alpha_1 + q\alpha_2 \in \mathfrak{t}^*_{++}, \lambda_2 = u(2\alpha_1 + \alpha_2) \in \mathbb{R}_{>0}\Lambda_1$, then

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_{\mu}T) = \begin{cases} \frac{1}{6}(q + u - y)^2(3p - 2q + 4u - 3x + 2y) \\ \qquad\qquad\qquad\qquad\qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{6}(p + 2u - x)^3 \qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}$$

(2b) If $\lambda_1 = p\alpha_1 + q\alpha_2 \in \mathfrak{t}^*_{++}, \lambda_2 = u(\alpha_1 + 2\alpha_2) \in \mathbb{R}_{>0}\Lambda_2$, then

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_{\mu}T) = \begin{cases} \frac{1}{6}(q + 2u - y)^3 \qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{6}(p + u - x)^2(-2p + 3q + 4u + 2x - 3y) \\ \qquad\qquad\qquad\qquad\qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}$$

(3) If $\lambda_1 = p\alpha_1 + q\alpha_2, \lambda_2 = r\alpha_1 + s\alpha_2 \in \mathfrak{t}^*_{++}$, then we have

$$\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_{\mu}T) = \begin{cases} \frac{1}{12}(q + s - y)^3(2p - q + 2r - s - 2x + y) \\ \qquad\qquad\qquad\qquad\qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{12}(p + r - x)^3(-p + 2q - r + 2s + x - 2y) \\ \qquad\qquad\qquad\qquad\qquad (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases}$$

4.3 Case C: μ is sufficiently close to 0.

In this subsection, we consider the case where μ is in the alcove which contains the origin 0. Here, we assume that $\mu = 0$ is a regular value of the moment map Φ, that is, vertices $w_1\lambda_1 + w_2\lambda_2$ ($w_1, w_2 \in \mathfrak{W}$) do not lie in $\mathbb{R}\alpha_1, \mathbb{R}\alpha_2$ and $\mathbb{R}(\alpha_1 + \alpha_2)$.

Let us investigate the following six cases.

Case I. $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}\Lambda_1 = \mathbb{R}_{>0}(2\alpha_1 + \alpha_2)$.

Case II. $\lambda_1 \in \mathbb{R}_{>0}\Lambda_1 = \mathbb{R}_{>0}(2\alpha_1 + \alpha_2), \lambda_2 \in \mathbb{R}_{>0}\Lambda_2 = \mathbb{R}_{>0}(\alpha_1 + 2\alpha_2)$.

Case III. $\lambda_1 \in \mathfrak{t}^*_{++} \cap \gamma_1, \lambda_2 \in \mathbb{R}_{>0}\Lambda_1$.

Case IV. $\lambda_1 \in \mathfrak{t}^*_{++} \cap \gamma_1, \lambda_2 \in \mathbb{R}_{>0}\Lambda_2$.

Case V. $\lambda_1, \lambda_2 \in \mathfrak{t}^*_{++} \cap \gamma_1$.

Case VI. $\lambda_1 \in \mathfrak{t}^*_{++} \cap \gamma_1, \lambda_2 \in \mathfrak{t}^*_{++} \cap \gamma_2$.

For Cases I and II, we have already computed the volume of them in Section 4.1. For Cases III–VI, we first need to observe which vertices $w_1\lambda_1 + w_2\lambda_2$ ($w_1, w_2 \in \mathfrak{W}$) of $\Phi(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})$ are in the domains γ_1 and γ_2 so that we can compute $F(\lambda_1, \lambda_2, \mu; w_1, w_2)$ in Theorem 3.3 or Corollary 3.7. In order to describe them, we call each vertex

$w_1\lambda_1 + w_2\lambda_2$ as follows. First, as in Figure 4, we label each vertex $w\lambda$ ($w \in \mathfrak{W}$) of $\pi_\lambda(\mathcal{O}_\lambda) \subset \mathfrak{t}^*$ as “A”, “B”, $\cdots$, or “F”.

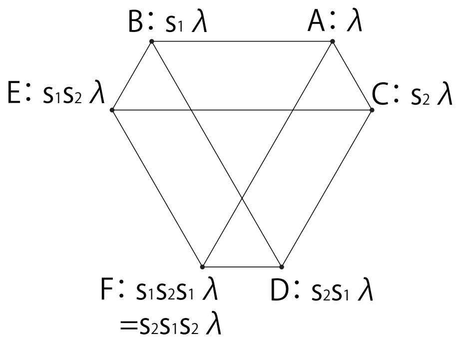


Figure 4.

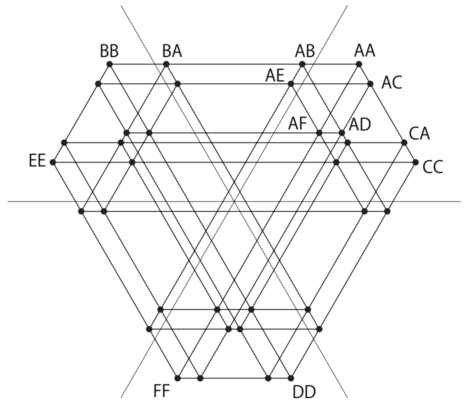


Figure 5.

Note that A=C, B=E and D=F for $\lambda \in \mathbb{R}_{>0}\Lambda_1$, and A=B, C=D and E=F for $\lambda \in \mathbb{R}_{>0}\Lambda_2$ as in Figure 1. For the vertices of $\Phi(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})$, for example, as in Figure 5, we label the vertex $\lambda_1 + \lambda_2$ as “AA”, $\lambda_1 + s_1\lambda_2$ as “AB”, $s_1\lambda_1 + \lambda_2$ as “BA” and so on.

Since we only observe which vertices are in the domains γ_1 and γ_2, for example, we do not distinguish two images in Figure 6.

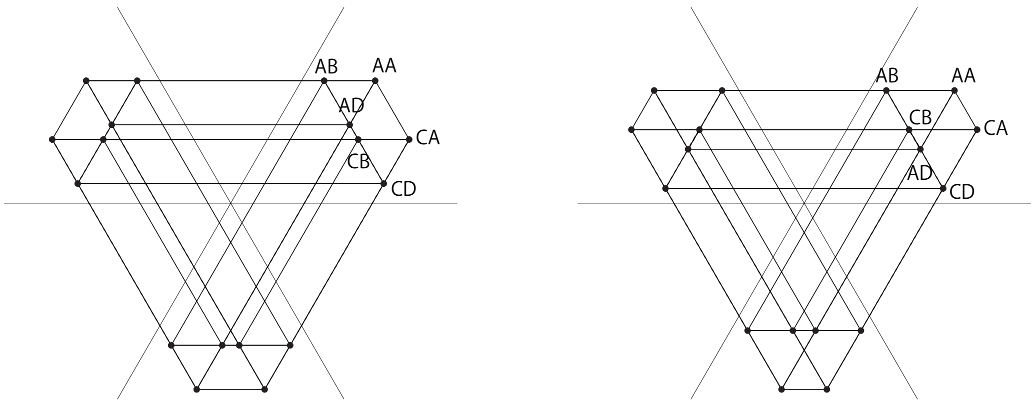
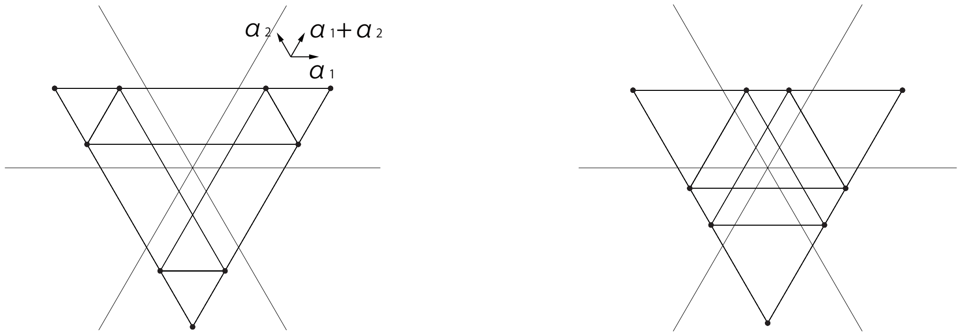


Figure 6.

We set $\mu = x\alpha_1 + y\alpha_2 \in \mathfrak{t}^*$.

Case I. $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}\Lambda_1 = \mathbb{R}_{>0}(2\alpha_1 + \alpha_2)$.

Let us set $\lambda_1 = u(2\alpha_1 + \alpha_2), \lambda_2 = v(2\alpha_1 + \alpha_2)$ $(u > v > 0)$. This case corresponds to the cases where μ belongs to the alcove 4 in Figure 2 (Ia), and the alcove $4'$ in Figure 2 (Ic). Thus the symplectic volumes are given as follows.



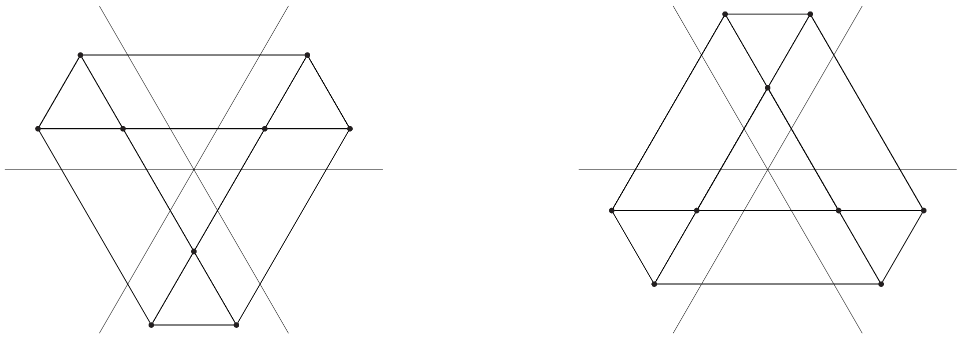
(Ia) $u > 2v$ (Ic) $v < u < 2v$

Figure 7.

$$\text{(Ia)}\quad \mathrm{vol} = \frac{9}{2}v^2. \qquad \text{(Ic)}\quad \mathrm{vol} = -\frac{3}{2}u^2 + 6uv - \frac{3}{2}v^2 - x^2 + xy - y^2.$$

Case II. $\lambda_1 \in \mathbb{R}_{>0}\Lambda_1 = \mathbb{R}_{>0}(2\alpha_1 + \alpha_2), \lambda_2 \in \mathbb{R}_{>0}\Lambda_2 = \mathbb{R}_{>0}(\alpha_1 + 2\alpha_2)$**.**

Let us set $\lambda_1 = u(2\alpha_1 + \alpha_2), \lambda_2 = v(\alpha_1 + 2\alpha_2)$ $(u, v > 0)$. This case corresponds to the cases where μ belongs to the alcove 3 in Figure 3 (IIa), and the alcove $3'$ in Figure 3 (IIc). Thus the symplectic volumes are given as follows.



(IIa) $u > v$ (IIc) $u < v$

Figure 8.

$$\text{(IIa)}\quad \mathrm{vol} = \frac{9}{2}v^2. \qquad \text{(IIc)}\quad \mathrm{vol} = \frac{9}{2}u^2.$$

Case III. $\lambda_1 \in \mathfrak{t}^*_{++} \cap \gamma_1, \lambda_2 \in \mathbb{R}_{>0}\Lambda_1$**.**

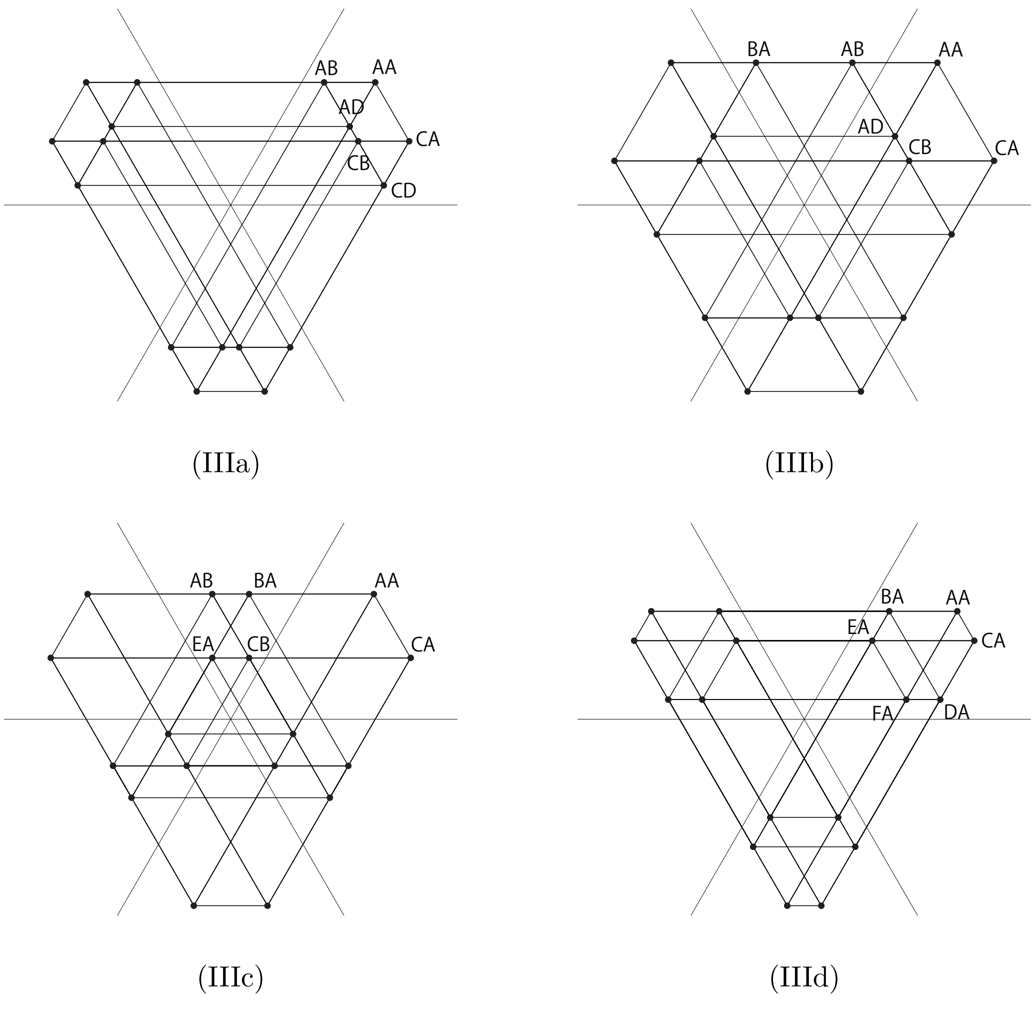
Varying $\lambda_1 \in \mathfrak{t}^*_{++} \cap \gamma_1, \lambda_2 \in \mathbb{R}_{>0}\Lambda_1$, we investigate which vertices are in the domains γ_1 and γ_2. In particular, we look at two vertices AB and AD.

(IIIa) If AB $\in \gamma_1$, then vertices AA, AB, AD, CA, CB, CD are in the domain γ_1, and no vertex is in γ_2 as in Figure 9 (IIIa).

(IIIb) If AB $\in \gamma_2$ and AD $\in \gamma_1$, then AA, AD, CA, CB are in γ_1 and AB, BA are in γ_2 as in Figure 9 (IIIb).

(IIIc) If AB $\in \gamma_2$ and AD $\notin \gamma_1 \cup \gamma_2$, then AA, CA are in γ_1 and AB, BA, CB, EA are in γ_2 as in Figure 9 (IIIc).

(IIId) If AB $\notin \gamma_1 \cup \gamma_2$, then AA, BA, CA, DA, EA, FA are in the domain γ_2, and no vertex is in γ_1 as in Figure 9 (IIId).



(IIIa) (IIIb)

(IIIc) (IIId)

Figure 9.

Thus we divide Case III into the four cases (IIIa)–(IIId) as in Figure 9.

Let us set $\lambda_1 = p\alpha_1 + q\alpha_2, \lambda_2 = u(2\alpha_1 + \alpha_2)$ $(2q > p > q > 0, u > 0)$. In each case, the symplectic volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T)$ is expressed as follows.

$$
\begin{aligned}
\text{(IIIa)}\quad 6 \cdot \mathrm{vol} =& (q+u-y)^2((q+u-y)\cdot(-2) + 3(p+2u-x)\cdot 2) \\
&- (q+u-y)^2(3(p+2u-x)) \\
&- (q+u-y)^2((q+u-y)\cdot(-3) + 3(p-u-x)\cdot 2) \\
&+ (q+u-y)^2((q+u-y)\cdot(-2) + 3(p-u-x)) \\
&+ (q-2u-y)^2(q-2u-y) \\
&- (p-q+u-y)^2(3(p+2u-x)) \\
&+ (p-q+u-y)^2((p-q+u-y)\cdot 2) \\
&+ (p-q+u-y)^2((p-q+u-y)\cdot(-1) + 3(p-u-x)) \\
&- (p-q-2u-y)^2((p-q-2u-y)\cdot 3 + 3(p-u-x)\cdot(-1)) \\
&+ (p-q-2u-y)^2((p-q-2u-y)\cdot 2 + 3(p-u-x)\cdot(-1)) \\
=& 27u^2(-p+2q). \\
\text{(IIIb)}\quad 6 \cdot \mathrm{vol} =& 3p^3 - 9p^2q + 9pq^2 - 3q^3 - 18p^2u + 36pqu - 18q^2u \\
&+ 9pu^2 + 18qu^2 - 24u^3 + 6px^2 - 6qx^2 - 12ux^2 \\
&- 6pxy + 6qxy + 12uxy + 3x^2y + 6py^2 - 6qy^2 \\
&- 12uy^2 - 3xy^2. \\
\text{(IIIc)}\quad 6 \cdot \mathrm{vol} =& 3p^3 - 9p^2q + 9pq^2 - 6q^3 - 18p^2u + 36pqt + 9pu^2 - 18qu^2 \\
&+ 6px^2 - 12qx^2 - 6pxy + 12qxy + 6py^2 - 12qy^2. \\
\text{(IIId)}\quad 6 \cdot \mathrm{vol} =& -6p^3 + 9p^2q + 9pq^2 - 6q^3 = 3(2q-p)(2p-q)(p+q).
\end{aligned}
$$

Case IV. $\lambda_1 \in \mathfrak{t}^*_{++} \cap \gamma_1, \lambda_2 \in \mathbb{R}_{>0}\Lambda_2$**.**

We look at three vertices AA, AC and AE.

(IVa) If AA $\in \gamma_1$, then vertices AA, AC, AE, CA, CC, CE are in the domain γ_1, and no vertex is in γ_2 as in Figure 10 (IVa).

(IVb) If AA, AE $\in \gamma_2$, then AC, CA are in γ_1 and, AA, AE, BA, BC are in γ_2 as in Figure 10 (IVb).

(IVc) If AA $\in \gamma_2$, AC $\in \gamma_1$ and AE $\notin \gamma_1 \cup \gamma_2$, then AC, CA, BC, DA are in γ_1 and AA, BA are in γ_2 as in Figure 10 (IVc).

(IVd) If AA $\in \gamma_2$ and AC, AE $\notin \gamma_1 \cup \gamma_2$, then AA, BA, CA, DA, EA, FA are in the domain γ_2, and no vertex is in γ_1 as in Figure 10 (IVd).

Thus we divide Case IV into the four cases (IVa)–(IVd) as in Figure 10.

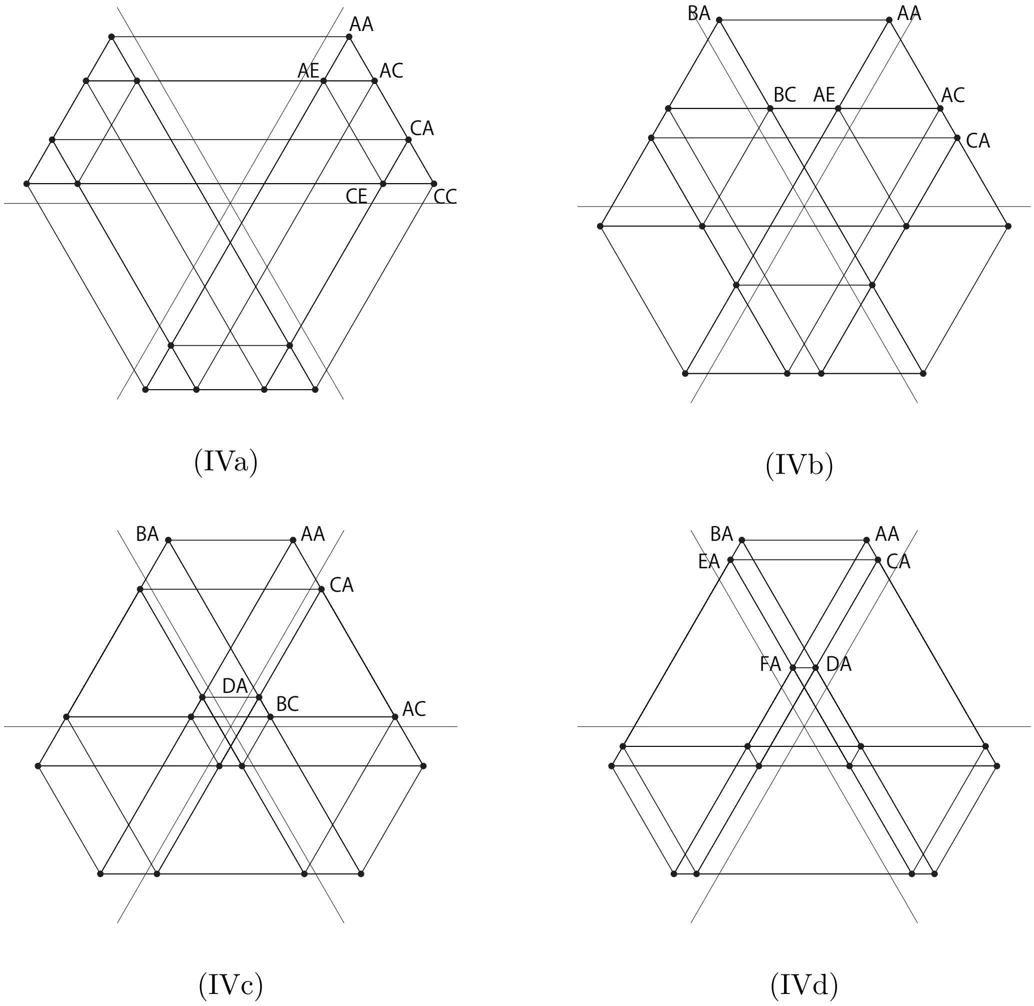


Figure 10.

Let us set $\lambda_1 = p\alpha_1 + q\alpha_2, \lambda_2 = u(\alpha_1 + 2\alpha_2)$ $(2q > p > q > 0, u > 0)$. In each case, the symplectic volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})/\!/_\mu T)$ is expressed as follows.

$$
\begin{aligned}
&\text{(IVa)} \quad 6 \cdot \mathrm{vol} = 27(-p+2q)u^2. \\
&\text{(IVb)} \quad 6 \cdot \mathrm{vol} = -3p^3 + 9p^2q - 9pq^2 + 3q^3 - 18p^2u + 36pqu - 18q^2u + 18pu^2 \\
&\qquad\qquad\qquad + 9qu^2 - 24u^3 - 6px^2 + 6qx^2 - 12ux^2 + 6pxy - 6qxy \\
&\qquad\qquad\qquad + 12uxy - 3x^2y - 6py^2 + 6qy^2 - 12uy^2 + 3xy^2. \\
&\text{(IVc)} \quad 6 \cdot \mathrm{vol} = -6p^3 + 9p^2q - 9pq^2 + 3q^3 + 36pqu - 18q^2u - 18pu^2 + 9qu^2 \\
&\qquad\qquad\qquad - 12px^2 + 6qx^2 + 12pxy - 6qxy - 12py^2 + 6qy^2. \\
&\text{(IVd)} \quad 6 \cdot \mathrm{vol} = -6p^3 + 9p^2q + 9pq^2 - 6q^3 = 3(2q-p)(2p-q)(p+q).
\end{aligned}
$$

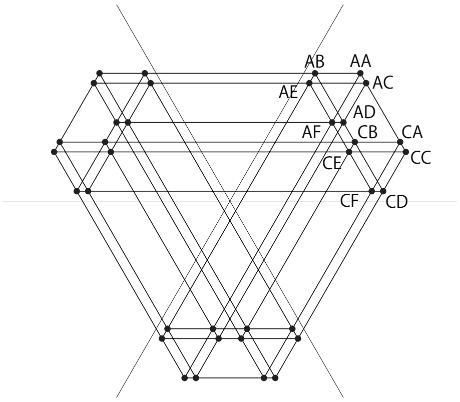
Case V. $\lambda_1, \lambda_2 \in \mathfrak{t}^*_{++} \cap \gamma_1$**.**

Let us set $\lambda_1 = p\alpha_1 + q\alpha_2, \lambda_2 = r\alpha_1 + s\alpha_2$ $(2q > p > q > 0, 2s > r > s > 0)$, and assume either $p > r$, or, $p = r$ and $q > s$. We look at two vertices AB and CB.

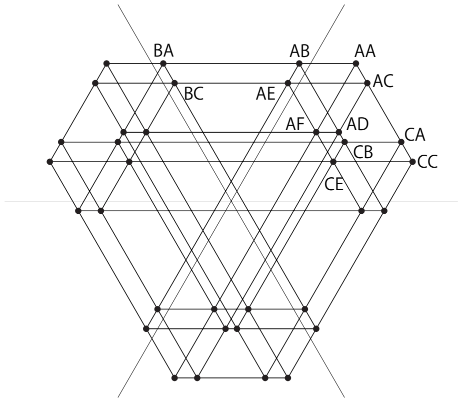
(Va) If AB $\in \gamma_1$, then vertices AA, AB, AC, AD, AE, AF, CA, CB, CC, CD, CE, CF are in the domain γ_1, and no vertex is in γ_2 as in Figure 11 (Va).

(Vb) If AB $\in \gamma_2$ and CB $\in \gamma_1$, then AA, AC, AD, AF, CA, CB, CC, CE are in γ_1, and AB, AE, BA, BC are in γ_2 as in Figure 11 (Vb).

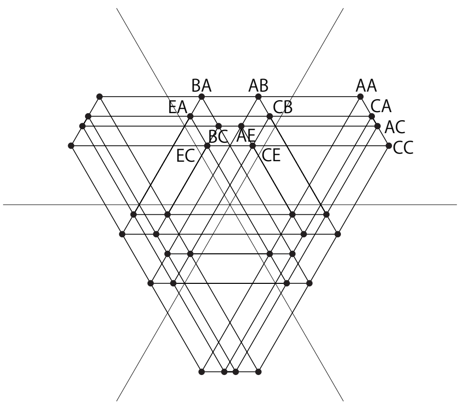
(Vc) If AB $\in \gamma_2$ and CB $\in \gamma_2$, then AA, AC, CA, CC are in γ_1, and AB, AE, BA, BC, CB, CE, EA, EC are in γ_2 as in Figure 11 (Vc).



(Va)



(Vb)



(Vc)

Figure 11.

Thus we divide Case V into the three cases (Va)–(Vc) as in Figure 11. In each case, the symplectic volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ is expressed as follows.

$$
\begin{aligned}
\text{(Va) } 12 \cdot \mathrm{vol} =& (q+s-y)^3(2(p+r-x)-(q+s-y)) \\
&- (q+s-y)^3(2(p-r+s-x)-(q+s-y)) \\
&- (q+r-s-y)^3(2(p+r-x)-(q+r-s-y)) \\
&+ (q-r-y)^3(2(p-r+s-x)-(q-r-y)) \\
&+ (q+r-s-y)^3(2(p-s-x)-(q+r-s-y)) \\
&- (q-r-y)^3(2(p-s-x)-(q-r-y)) \\
&- (p-q+s-y)^3(2(p+r-x)-(p-q+s-y)) \\
&+ (p-q+s-y)^3(2(p-r+s-x)-(p-q+s-y)) \\
&+ (p-q+r-s-y)^3(2(p+r-x)-(p-q+r-s-y)) \\
&- (p-q-r-y)^3(2(p-r+s-x)-(p-q-r-y)) \\
&- (p-q+r-s-y)^3(2(p-s-x)-(p-q+r-s-y)) \\
&+ (p-q-r-y)^3(2(p-s-x)-(p-q-r-y)) \\
=& 12pr^3 - 24qr^3 - 18pr^2s + 36qr^2s - 18prs^2 + 36qrs^2 + 12ps^3 - 24qs^3 \\
=& 6(2q-p)(-r+2s)(2r-s)(r+s).
\end{aligned}
$$

$$
\begin{aligned}
\text{(Vb) } 12 \cdot \mathrm{vol} =& -6p^3r + 18p^2qr - 18pq^2r + 6q^3r + 18p^2r^2 - 36pqr^2 + 18q^2r^2 - 6pr^3 \\
&- 6qr^3 + 6r^4 + 12p^3s - 36p^2qs + 36pq^2s - 12q^3s - 36p^2rs + 72pqrs \\
&- 36q^2rs + 18pr^2s - 12r^3s - 18prs^2 + 36qrs^2 + 12ps^3 - 24qs^3 \\
&- 12prx^2 + 12qrx^2 + 12r^2x^2 + 24psx^2 - 24qsx^2 - 24rsx^2 + 12prxy \\
&- 12qrxy - 12r^2xy - 24psxy + 24qsxy + 24rsxy - 6rx^2y + 12sx^2y \\
&- 12pry^2 + 12qry^2 + 12r^2y^2 + 24psy^2 - 24qsy^2 - 24rsy^2 \\
&+ 6rxy^2 - 12sxy^2.
\end{aligned}
$$

$$
\begin{aligned}
\text{(Vc) } 12 \cdot \mathrm{vol} =& -6p^3r + 18p^2qr - 18pq^2r + 12q^3r + 18p^2r^2 - 36pqr^2 - 6pr^3 \\
&+ 12qr^3 + 12p^3s - 36p^2qs + 36pq^2s - 24q^3s - 36p^2rs + 72pqrs \\
&+ 18pr^2s - 36qr^2s - 18prs^2 + 36qrs^2 + 12ps^3 - 24qs^3 \\
&- 12prx^2 + 24qrx^2 + 24psx^2 - 48qsx^2 + 12prxy - 24qrxy \\
&- 24psxy + 48qsxy - 12pry^2 + 24qry^2 + 24psy^2 - 48qsy^2.
\end{aligned}
$$

Case VI. $\lambda_1 \in \mathfrak{t}^*_{++} \cap \gamma_1, \lambda_2 \in \mathfrak{t}^*_{++} \cap \gamma_2$.

We look at five vertices AA, AB, CA, CB and BC.

(VIa) If AA $\in \gamma_1$ and AB $\in \gamma_1$, then vertices AA, AB, AC, AD, AE, AF, CA, CB, CC, CD, CE, CF are in the domain γ_1, and no vertex is in γ_2 as in Figure 12 (VIa).

(VIb) If AA $\in \gamma_1$, AB $\in \gamma_2$ and CB $\in \gamma_1$, then AA, AC, AD, AF, CA, CB, CC, CE are in γ_1, and AB, AE, BA, BC are in γ_2 as in Figure 12 (IVb).

(VIc) If AA $\in \gamma_1$, AB $\in \gamma_2$, CB $\in \gamma_2$ and BC $\in \gamma_2$, then AA, AC, CA, CC are in γ_1, and AB, AE, BA, BC, CB, CE, EA, EC are in γ_2 as in Figure 12 (VIc).

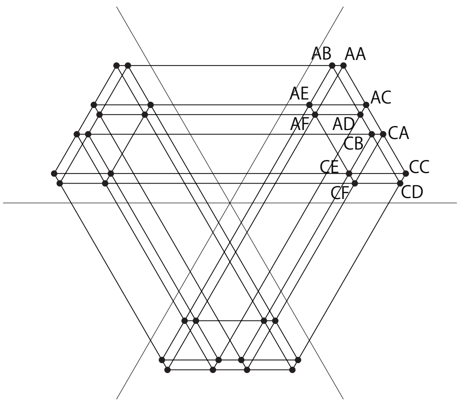
(VId) If AA $\in \gamma_1$, AB $\in \gamma_2$, CB $\in \gamma_2$ and BC $\in \gamma_1$, then AA, AC, BC, CA, CC, DA, EC, FA are in γ_1, and AB, BA, CB, EA are in γ_2 as in Figure 12 (VId).

(VIe) If AA $\in \gamma_2$, CA $\in \gamma_1$, CB $\in \gamma_1$ and BC $\in \gamma_1$, then AC, AD, BC, BD, CA, CB, DA, DB are in γ_1, and AA, AB, BA, BB are in γ_2 as in Figure 13 (VIe).

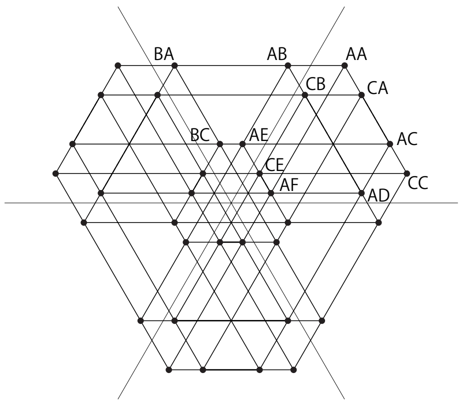
(VIf) If AA $\in \gamma_2$, CA $\in \gamma_1$, CB $\in \gamma_1$ and BC $\in \gamma_2$, then AC, AD, CA, CB are in γ_1, and AA, AB, BA, BB, BC, BD, DA, DB are in γ_2 as in Figure 13 (VIf).

(VIg) If AA $\in \gamma_2$, CA $\in \gamma_1$ and CB $\in \gamma_2$, then AC, AD, BC, DA are in γ_1, and AA, AB, BA, BB, CB, DB, EA, FA are in γ_2 as in Figure 13 (VIg).

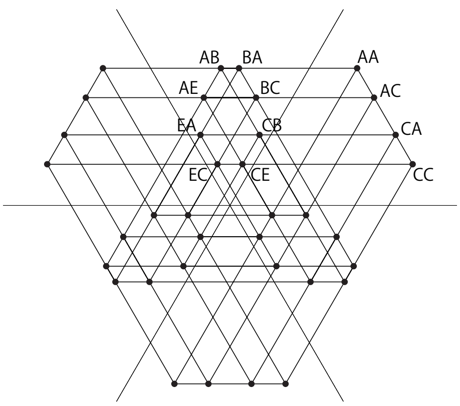
(VIh) If AA $\in \gamma_2$ and CA $\in \gamma_2$, then no vertex is in γ_1, and AA, AB, BA, BB, CA, CB, DA, DB, EA, EB, FA, FB are in γ_2 as in Figure 13 (VIh).



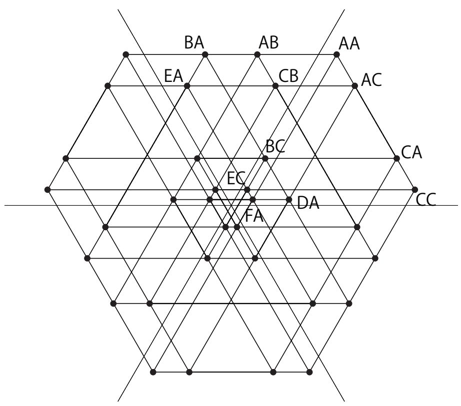
(VIa)



(VIb)



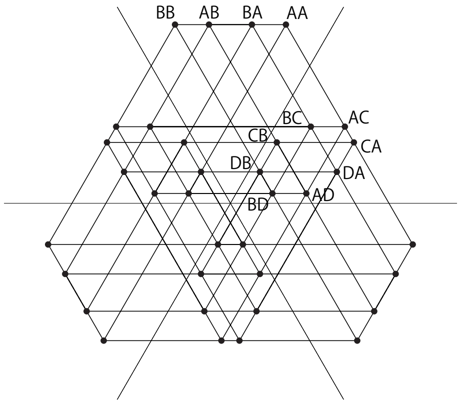
(VIc)



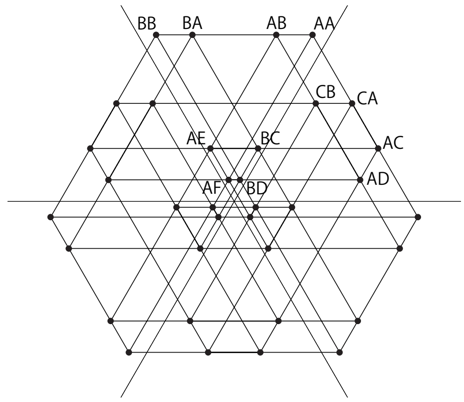
(VId)

Figure 12.

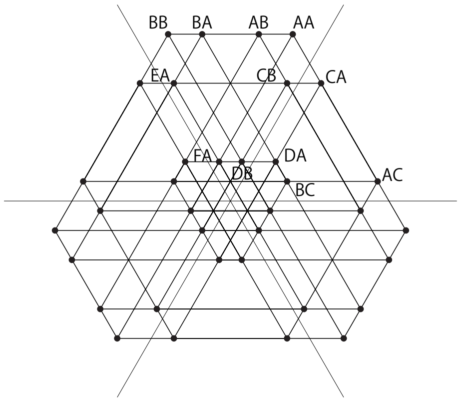
Thus we divide Case VI into the eight cases (VIa)–(VIh) as in Figure 12 and 13.



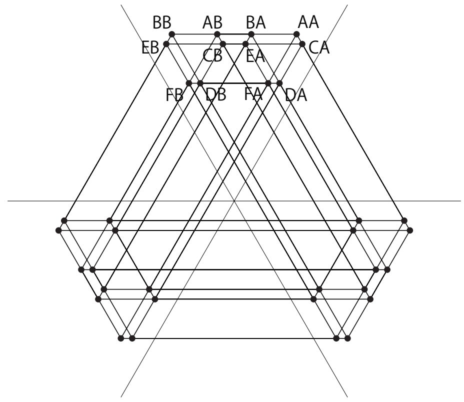
(VIe)



(VIf)



(VIg)



(VIh)

Figure 13.

Let us set $\lambda_1 = p\alpha_1 + q\alpha_2, \lambda_2 = r\alpha_1 + s\alpha_2$ $(2q > p > q > 0, 2r > s > r > 0)$. By Corollary 3.7, the symplectic volume $\mathrm{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ is expressed as follows.

$$
\begin{aligned}
\text{(VIa) } 12 \cdot \mathrm{vol} =& 12pr^3 - 24qr^3 - 18pr^2s + 36qr^2s - 18prs^2 + 36qrs^2 + 12ps^3 - 24qs^3 \\
=& 6(2q-p)(2s-r)(2r-s)(r+s).
\end{aligned}
$$

$$
\begin{aligned}
\text{(VIb) } 12 \cdot \mathrm{vol} =& -6p^3r + 18p^2qr - 18pq^2r + 6q^3r + 18p^2r^2 - 36pqr^2 + 18q^2r^2 \\
& - 6pr^3 - 6qr^3 + 6r^4 + 12p^3s - 36p^2qs + 36pq^2s - 12q^3s - 36p^2rs \\
& + 72pqrs - 36q^2rs + 18pr^2s - 12r^3s - 18prs^2 + 36qrs^2 + 12ps^3 \\
& - 24qs^3 - 12prx^2 + 12qrx^2 + 12r^2x^2 + 24psx^2 - 24qsx^2 - 24rsx^2 \\
& + 12prxy - 12qrxy - 12r^2xy - 24psxy + 24qsxy + 24rsxy \\
& - 6rx^2y + 12sx^2y - 12pry^2 + 12qry^2 + 12r^2y^2 + 24psy^2 \\
& - 24qsy^2 - 24rsy^2 + 6rxy^2 - 12sxy^2.
\end{aligned}
$$

$$
\begin{aligned}
\text{(VIc) } 12\cdot\mathrm{vol} =& -6p^3r+18p^2qr-18pq^2r+12q^3r+18p^2r^2-36pqr^2-6pr^3\\
&+12qr^3+12p^3s-36p^2qs+36pq^2s-24q^3s-36p^2rs+72pqrs\\
&+18pr^2s-36qr^2s-18prs^2+36qrs^2+12ps^3-24qs^3\\
&-12prx^2+24qrx^2+24psx^2-48qsx^2+12prxy-24qrxy\\
&-24psxy+48qsxy-12pry^2+24qry^2+24psy^2-48qsy^2.\\
\text{(VId) } 12\cdot\mathrm{vol} =& 6p^4-12p^3q-6p^3r+18p^2qr-18pq^2r+12q^3r+18p^2r^2-36pqr^2\\
&-6pr^3+12qr^3-6p^3s+36pq^2s-24q^3s-36p^2rs+72pqrs\\
&+18pr^2s-36qr^2s+18p^2s^2-36pqs^2-18prs^2+36qrs^2\\
&+6ps^3-12qs^3+12p^2x^2-24pqx^2-12prx^2+24qrx^2\\
&+12psx^2-24qsx^2-12p^2xy+24pqxy+12prxy-24qrxy\\
&-12psxy+24qsxy-6px^2y+12qx^2y+12p^2y^2-24pqy^2\\
&-12pry^2+24qry^2+12psy^2-24qsy^2+6pxy^2-12qxy^2.\\
\text{(VIe) } 12\cdot\mathrm{vol} =& -12p^3r+36p^2qr-36pq^2r+12q^3r-24pr^3+12qr^3+6p^3s\\
&-18p^2qs+18pq^2s-6q^3s-36p^2rs+72pqrs-36q^2rs+36pr^2s\\
&-18qr^2s+18p^2s^2-36pqs^2+18q^2s^2+18qrs^2-6ps^3-6qs^3\\
&-12rs^3+6s^4-24prx^2+24qrx^2+12psx^2-12qsx^2-24rsx^2\\
&+12s^2x^2+24prxy-24qrxy-12psxy+12qsxy+24rsxy\\
&-12s^2xy-12rx^2y+6sx^2y-24pry^2+24qry^2+12psy^2\\
&-12qsy^2-24rsy^2+12s^2y^2+12rxy^2-6sxy^2.\\
\text{(VIf) } 12\cdot\mathrm{vol} =& -24p^3r+36p^2qr-36pq^2r+12q^3r-24pr^3+12qr^3+12p^3s\\
&-18p^2qs+18pq^2s-6q^3s+72pqrs-36q^2rs+36pr^2s-18qr^2s\\
&-36pqs^2+18q^2s^2-36prs^2+18qrs^2+12ps^3-6qs^3\\
&-48prx^2+24qrx^2+24psx^2-12qsx^2+48prxy-24qrxy\\
&-24psxy+12qsxy-48pry^2+24qry^2+24psy^2-12qsy^2.\\
\text{(VIg) } 12\cdot\mathrm{vol} =& -12pq^3+6q^4-24p^3r+36p^2qr-6q^3r-36pqr^2+18q^2r^2-12pr^3\\
&+6qr^3+12p^3s-18p^2qs+18pq^2s-6q^3s+72pqrs-36q^2rs\\
&+36pr^2s-18qr^2s-36pqs^2+18q^2s^2-36prs^2+18qrs^2+12ps^3\\
&-6qs^3-24pqx^2+12q^2x^2-24prx^2+12qrx^2+24psx^2\\
&-12qsx^2+24pqxy-12q^2xy+24prxy-12qrxy-24psxy\\
&+12qsxy-12px^2y+6qx^2y-24pqy^2+12q^2y^2-24pry^2\\
&+12qry^2+24psy^2-12qsy^2+12pxy^2-6qxy^2.\\
\text{(VIh) } 12\cdot\mathrm{vol} =& -24p^3r+36p^2qr+36pq^2r-24q^3r+12p^3s-18p^2qs-18pq^2s+12q^3s\\
=& 6(-p+2q)(2p-q)(p+q)(2r-s).
\end{aligned}
$$

Remark 4.2. Using the concrete expression to symplectic volumes in Section 4, we can compute the Betti number and the Poincaré polynomial as well as the cohomological intersection product for each double weight variety $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T$ in Cases A, B and C. The details will be discussed in [17].

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Taro Suzuki
Graduate School of Science and Engineering
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112-8551, Japan
e-mail: suzuki@gug.math.chuo-u.ac.jp