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REAL HYPERSURFACES OF A PSEUDO RICCI SYMMETRIC COMPLEX PROJECTIVE SPACE

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ABSTRACT. Real hypersurfaces of a pseudo Ricci symmetric complex projective space are classified and we obtained that there are no real hypersurfaces of such type complex projective space.

1. INTRODUCTION

Let CP^n, $n \geq 3$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4 and let M be a real hypersurface of CP^n. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure of CP^n. Many differential geometers have been studied real hypersurfaces of a complex projective space such as Bejancu and Deshmukh [2], Cecil and Ryan [3], Cho and Ki [6], Deshmukh [7], Hamada ([9],[10],[11]), Ikuta [12], Kimura ([14], [15], [16], [17]), Kimura and Maeda [18], Maeda ([19], [20], [21]), Matsuyama ([22], [23], [24]), Okumura [25], Perez et. al ([28], [29], [30], [31]), Takagi ([33],[34], [35]), Wang [36] and others.

It is well known that there does not exist a real hypersurface M of CP^n satisfying the condition that the second fundamental tensor A of M is parallel. Again in [10] Hamada use the condition that the second fundamental tensor A is recurrent, i.e. there exists an 1-form α such that $\nabla A = A \otimes \alpha$. And Hamada [10] proved that there are no real hypersurfaces of a complex projective space with recurrent second fundamental tensor. Again many differential geometers studied real hypersurfaces of complex projective space satisfying some condition of Ricci tensor. In [13] Ki proved that there are no real hypersurfaces of a complex projective space with parallel Ricci tensor. Again in [11], Hamada studied the real hypersurfaces of a complex projective space with recurrent Ricci tensor and proved that there are no real hypersurfaces with recurrent Ricci tensor of CP^n under the condition that ξ is a principal curvature vector.

A Riemannian space is said to be Ricci symmetric if its Ricci tensor S of type (0,2) satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. During the last five decades, the notion of Ricci symmetry has been weakened by many authors in several ways to a different extent such as Ricci recurrent space [27], Ricci semisymmetric space [32], pseudo Ricci symmetric space by Deszcz [8], pseudo Ricci symmetric space by

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Chaki [4].

A non-flat Riemannian space (M^n, g) is said to be pseudo Ricci symmetric [4] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(Y, X), \tag{1.1}$$

for any vector field X, Y, Z, where α is a nowhere vanishing 1-form and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional space is denoted by $(PRS)_n$. The pseudo Ricci symmetric spaces have been also studied by Arslan et. al [1], Chaki and Saha [5], Özen [26] and many others.

The relation (1.1) can be written as

$$(\nabla_X Q)Y = 2\alpha(X)QY + \alpha(Y)QX + S(Y, X)\rho, \tag{1.2}$$

where ρ is the vector field associated to the 1-form α such that $\alpha(X) = g(X, \rho)$ and Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y.

Motivated by the above studies the present paper deals with the study of real hypersurfaces of a *pseudo Ricci symmetric complex projective space*. A complex projective space CP^n, $n \geq 3$ is called *pseudo Ricci symmetric complex projective space* if its Ricci operator Q satisfies the relation (1.2). The paper is organized as follows. Section 2 is concerned with some preliminaries. Section 3 is devoted to the study of real hypersurfaces of a pseudo Ricci symmetric complex projective space and it is proved that there are no real hypersurfaces of a pseudo Ricci symmetric complex projective space CP^n.

2. Preliminaries

Let M be a real hypersurface of CP^n. In a neighbourhood of each point, we take a unit normal vector field N in CP^n. The Riemannian connections $\widetilde{\nabla}$ in CP^n and ∇ in M are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{2.1}$$

$$\widetilde{\nabla}_X N = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the Fubini-Study metric G of CP^n and A is the second fundamental tensor of M in CP^n. Let TM be the tangent bundle of M. An eigenvector X of the second fundamental tensor A is called a principal curvature vector. Also an eigenvalue λ of A is called a principal curvature. It is known that M has an almost contact metric structure induced from the Kähler structure J on CP^n, that is we define a tensor field ϕ of type (1,1), a vector field ξ and an 1-form η on M by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, N)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = g(\xi, \xi) = 1, \quad \phi\xi = 0. \tag{2.3}$$

Also it follows from (2.1) that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \tag{2.4}$$

$$\nabla_X\xi = \phi AX. \tag{2.5}$$

Let $\widetilde{R}$ and R be the curvature tensors of CP^n and M respectively. From the expression of the curvature tensor of CP^n, we see that the curvature tensor, Codazzi equation and the Ricci tensor of type (1,1) are given by

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &- 2g(\phi X,Y)\phi Z + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi, \tag{2.7}$$

$$QX = (2n+1)X - 3\eta(X)\xi + hAX - A^2X, \tag{2.8}$$

where $h = traceA$.
Again we have

$$\begin{aligned} (\nabla_X Q)Y &= -3g(\phi AX,Y)\xi - 3\eta(Y)\phi AX + (Xh)AY \\ &+ h(\nabla_X A)Y - A(\nabla_X A)Y - (\nabla_X A)AY. \end{aligned} \tag{2.9}$$

Also we recall the following:

Lemma 2.1. [19] *If ξ is a principal curvature vector, then the corresponding principal curvature a is locally constant.*

Lemma 2.2. [19] *Assume that ξ is a principal curvature vector and the corresponding principal curvature is a. If $AX = \lambda X$ for $X \perp \xi$, then we have $A\phi X = \overline{\lambda}$, where $\overline{\lambda} = \frac{(a\lambda+2)}{(2\lambda-a)}$.*

Theorem 2.1. [3] *Let M be a connected real hypersurface of CP^n, $n \geq 3$, whose Ricci tensor S satisfies $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$ for some smooth functions a and b on M. Then M is locally congruent to one of the following:*
(i) *a geodesic hypersurface,*
(ii) *a tube of radius r over a totally geodesic CP^k, $1 \leq k \leq n-2$, where $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \frac{k}{n-k-1}$,*
(iii) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$ and $\cot^2 2r = n-2$.*

Theorem 2.2. [33] *Let M be a homogeneous real hypersurface of CP^n. Then M is a tube of radius r over one of the following Kähler submanifolds:*
(A_1) hyperplane CP^{n-1}, where $0 < r < \frac{\pi}{2}$,
(A_2) totally geodesic CP^k, $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,
(B) complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,
(C) $CP^1 \times CP^{\frac{n-1}{2}}$, where $0 < r < \frac{\pi}{4}$ and $n(\geq 5)$ is odd,

(D) complex Grassman $cG_{2,5}$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 9$,
(E) Hermitian symmetric space $SO(10)/U(5)$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 15$.

Theorem 2.3. [15] *Let* M *be a real hypersurface of* CP^n. *Then* M *has constant principal curvatures and* ξ *is a principal curvature vector if and only if* M *is locally congruent to a homogeneous real hypersurface.*

Theorem 2.4. [13] *There are no real hypersurfaces with parallel Ricci tensor of a complex space form* cM^n, $c \neq 0$.

3. Real hypersurfaces of a pseudo Ricci symmetric complex projective space CP^n

In this section, we have studied real hypersurfaces of a pseudo Ricci symmetric complex projective space CP^n and prove the following:

Lemma 3.1. *Let* M *be a connected real hypersurface of a pseudo Ricci symmetric complex projective space* CP^n. *If all eigenvalues of the Ricci operator* Q *are constant then the Ricci tensor* S *of* M *is parallel.*

Proof: Let $QX = \lambda X$, $QY = \mu Y$ and $QZ = \nu Z$. Then we have

$$g((\nabla_X Q)Y, Z) = (\mu - \nu)g(\nabla_X Y, Z).$$

Again we have from (1.2) that

$$g((\nabla_X Q)Y, Z) = 2\alpha(X)\mu g(Y, Z) + \alpha(Y)\nu g(X, Z) + \alpha(Z)\mu g(X, Y).$$

If $\lambda \neq \mu$, $\lambda \neq \nu$ and $\mu \neq \nu$ then $g((\nabla_X Q)Y, Z) = 0$. Also in the case of $\mu = \nu$, we obtain $g((\nabla_X Q)Y, Z) = 0$.
Assume that $\mu \neq \nu$, $\lambda = \nu$. Then $g((\nabla_Z Q)X, Y) = 0$.
On the other hand, we have

$$g((\nabla_Z Q)X, Y) = 2\alpha(Z)\mu g(X, Y) + \left[\alpha(X)S(Z, Y) + \alpha(Y)S(X, Z)\right].$$

Thus we obtain $\alpha(Z)\mu g(X, Y) = 0$. Hence $g((\nabla_X Q)Y, Z) = 0$. Consequently, the Ricci tensor S of M is parallel.
From (2.8) and since ξ is principal, the principal curvature vector will also be eigenvectors of S. Thus Ricci tensor of a homogeneous real hypersurface has constant eigenvalues. Again the hypersurface listed in Theorem 2.2 do not have parallel Ricci tensor. Thus from Lemma 3.1 and Theorem 2.3, we obtain

Proposition 3.1. A homogeneous real hypersurface of CP^n can not be pseudo Ricci symmetric.

So by using Theorem 2.1, we have

Corollary 3.1. A real hypersurface of CP^n, $n \geq 3$ whose Ricci tensor S satisfies $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$ for some smooth functions a and b on M, can not be pseudo Ricci symmetric.

Now we prove the following:

Theorem 3.1. *There are no real hypersurfaces of pseudo Ricci symmetric complex projective space CP^n under the condition that ξ is a principal curvature vector.*

Proof: Let us take a real hypersurface of pseudo Ricci symmetric complex projective space CP^n. Then by virtue of (2.8) it follows from (1.2) that

$$\begin{aligned} g((\nabla_X Q)Y, Z) &= 2\alpha(X)\big[(2n+1)g(Y,Z) - 3\eta(Y)\eta(Z) \\ &+ hg(AY,Z) - g(A^2Y,Z)\big] \\ &+ \alpha(Y)\big[(2n+1)g(X,Z) - 3\eta(X)\eta(Z) \\ &+ hg(AX,Z) - g(A^2X,Z)\big] \\ &+ \alpha(Z)\big[(2n+1)g(X,Y) - 3\eta(X)\eta(Y) \\ &+ hg(AX,Y) - g(A^2X,Y)\big]. \end{aligned} \tag{3.1}$$

Using (2.9) in (3.1), we get

$$\begin{aligned} & 2\alpha(X)\big[(2n+1)g(Y,Z) - 3\eta(Y)\eta(Z) + hg(AY,Z) - g(A^2Y,Z)\big] \\ + \; & \alpha(Y)\big[(2n+1)g(X,Z) - 3\eta(X)\eta(Z) + hg(AX,Z) - g(A^2X,Z)\big] \\ + \; & \alpha(Z)\big[(2n+1)g(X,Y) - 3\eta(X)\eta(Y) + hg(AX,Y) - g(A^2X,Y)\big] \\ + \; & 3\eta(Z)g(\phi AX,Y) + 3\eta(Y)g(\phi AX,Z) - (Xh)g(AY,Z) \\ - \; & hg((\nabla_X A)Y,Z) + g(A(\nabla_X A)Y,Z) + g((\nabla_X A)AY,Z) = 0 \end{aligned} \tag{3.2}$$

for any tangent vectors X, Y, Z.
Putting $Y = \xi$ and $Z = \phi X$ in (3.2), we get

$$\begin{aligned} & 2\alpha(X)\big[hg(A\xi,\phi X) - g(A^2\xi,\phi X)\big] + \alpha(\xi)\big[hg(AX,\phi X) - g(A^2X,\phi X)\big] \\ + \; & \alpha(\phi X)\big[2(n-1)\eta(X) + hg(AX,\xi) - g(A^2X,\xi)\big] \\ + \; & 3g(AX,X) - 3\eta(AX)\eta(X) - (Xh)g(A\xi,\phi X) \\ - \; & hg((\nabla_X A)\xi,\phi X) + g(A(\nabla_X A)\xi,\phi X) + g((\nabla_X A)A\xi,\phi X) = 0. \end{aligned} \tag{3.3}$$

Let us assume $A\xi = a\xi$. Then by Lemma 2.1, we have a is constant and hence we get

$$(\nabla_X A)\xi = a\phi AX - A\phi AX. \tag{3.4}$$

Using (3.4) in (3.3), we obtain

$$\begin{aligned} & \alpha(\xi)\big[hg(AX,\phi X) - g(A^2X,\phi X)\big] + 2(n-1)\alpha(\phi X)\eta(X) \\ + \; & 3g(AX,X) - 3a\big(\eta(X)\big)^2 - hag(\phi AX,\phi X) \\ + \; & hg(A\phi AX,\phi X) - g(A\phi AX, A\phi X) + a^2 g(\phi AX,\phi X) = 0 \end{aligned} \tag{3.5}$$

for any tangent vector X on M. We choose X as a unit principal curvature vector orthogonal to ξ and by virtue of Lemma 2.2, we have

$$AX = \lambda X \quad \text{and } A\phi X = \overline{\lambda} X,$$

where $\overline{\lambda} = \frac{a\lambda+2}{2\lambda-a}$. Therefore we obtain

$$\lambda\big[\overline{\lambda}^2 - \{h + \alpha(\xi)\}\overline{\lambda} - (a^2 - ha + 3)\big] = 0. \tag{3.6}$$

Again from Lemma 2.2, we may write

$$\lambda\overline{\lambda} = \frac{\lambda + \overline{\lambda}}{2}a + 1. \tag{3.7}$$

If $\lambda = \overline{\lambda}$ then (3.7) yields

$$\lambda^2 = a\lambda + 1. \tag{3.8}$$

If 0 occurs as a principal curvature (for a principal vector orthogonal to ξ), then (3.7) shows that all principal curvature must be constant.
Next assuming that 0 is not a principal curvature (again we consider only directions orthogonal to ξ), the relation (3.6) shows that there are at most two distinct principal curvatures. If λ and $\overline{\lambda}$ are distinct then we have

$$\lambda + \overline{\lambda} = h + \alpha(\xi) \quad \text{and } \lambda\overline{\lambda} = -(a^2 - ha + 3),$$

which yields

$$-(a^2 - ha + 3) = \frac{\{h + \alpha(\xi)\}a}{2} + 1,$$

i.e.

$$a^2 - \frac{\{h - \alpha(\xi)\}a}{2} + 4 = 0.$$

Thus the coefficients in (3.6) are constants and hence so are λ and $\overline{\lambda}$. The final possibility is that all principal curvatures (with principal vectors orthogonal to ξ) satisfy (3.8) and are again constant.
So by Theorem 2.3 and Proposition 3.1, we get the desired result.
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