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ABSTRACT. This paper is devoted to proving the global solvability of the Cauchy problem for the Kirchhoff equation in the Gevrey space $\gamma^\infty_{L^2}$. More precise results are derived for solutions in the s-Gevrey classes $\gamma^s_{L^2}$. Furthermore, similar results are obtained for the initial-boundary value problems in bounded domains and in exterior domains with compact boundary.

1. INTRODUCTION

G. Kirchhoff proposed the equation

$$\partial_t^2 u - \left(1 + \int_\Omega |\nabla u(t,y)|^2\, dy\right) \Delta u = 0 \quad (t \in \mathbb{R},\, x \in \Omega) \tag{1.1}$$

in his book on mathematical physics in 1883, as a model equation for transversal motion of the elastic string, where Ω is a domain in $\mathbb{R}^n$ (see Kirchhoff [26, Chap. 29, §7], and for finite dimensional approximation problem, see Nishida [37]). Since then, first it was in 1940 that Bernstein proved the existence of global in time analytic solutions on an interval of the real line in his celebrated paper [5]. After him, Pohozhaev extended Bernstein's result to several space dimensions (see [39]). Arosio and Spagnolo proved analytic well-posedness for the degenerate Kirchhoff type equation (see [4], and also D'Ancona and Spagnolo [11] and Kajitani and Yamaguti [25]).

As it is well known, this equation has a Hamiltonian structure, nevertheless it involves a challenging problem *whether or not, one can prove the existence of time global solutions corresponding to data in Gevrey classes, and standard Sobolev spaces without any smallness condition.* Up to now there is no solution to these problems, and even the existence of local solutions in Sobolev spaces H^σ for $1 \leq \sigma < 3/2$ is still unclear.

The global existence of quasi-analytic solutions is known from Nishihara, and Ghisi and Gobbino (see [38, 17]). Here quasi-analytic classes are a slight relaxation of the analytic class as opposed to the C^∞-class. Manfrin discussed the time global solutions in Sobolev spaces corresponding to non-analytic data having a spectral gap (see [29]), and a similar result is obtained by Hirosawa (see [21]).

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On the other hand, global well-posedness in the Sobolev space $H^{3/2}$, or H^2 with *small data* is well established in [6, 12, 13, 14, 19, 24, 30, 31, 41, 45, 46]. There, the classes of small data consist of compactly supported functions (see [19]), or more generally, they are characterised by some weight conditions (see [6, 12, 13, 14]) or oscillatory integrals (see [20, 24, 30, 31, 40, 41, 45, 46]). Recently, the authors studied the global well-posedness for Kirchhoff systems with small data (see [33]), and generalised all the previous results in the framework of small data, both for the Kirchhoff equation and Kirchhoff type systems. Here, the class of data in [33] consists of Sobolev space $(H^1)^m$, m being the order of system, and is characterised by some oscillatory integrals. The precise statements of the known results can be found in the survey paper [34]. Scattering results are also available, see [32].

The goal of this paper is to show the global existence of solutions to (1.2) with Gevrey data in $\mathbb{R}^n$ without smallness assumptions, see Theorem 1.1 and Theorem 1.2. Furthermore, we indicate how to modify the proof to also yield the global existence for the initial-boundary value problem in exterior domains, and in bounded domains (see Theorem 4.1 and Theorem 4.2, respectively).

The method of our proof is quite novel for this area. Namely, we assume that a solution blows up in finite time and arrive at a contradiction. The contradiction argument involves considering data in different regions (according to relations between their size and regularity), an energy estimate for local solutions with a controllable loss, the Schauder-Tychonoff fixed point theorem, lower bounds for life spans of local solutions, and the exploration of the Hamiltonian structure of the equation.

We note that our proof can be extended to treat more general equations of Kirchhoff type, for example of the form

$$\partial_t^2 u - \Phi\left(\int_{\mathbb{R}^n} |\nabla u(t,y)|^2\,dy\right)\Delta u = 0,$$

for a Lipschitz function $\Phi \ge \alpha > 0$. For the sake of simplicity here we treat the main example of such equations, the classical equation (1.1).

Thus, in this paper we consider the Cauchy problem for the Kirchhoff equation

$$\begin{cases} \partial_t^2 u - \left(1 + \int_{\mathbb{R}^n} |\nabla u(t,y)|^2\,dy\right)\Delta u = 0, & t>0, \quad x\in\mathbb{R}^n, \\ u(0,x)=u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Equation (1.2) has a Hamiltonian structure. More precisely, let us define the energy:

$$\mathscr{H}(u;t) := \frac{1}{2}\left\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\right\} + \frac{1}{4}\|\nabla u(t)\|_{L^2}^4.$$

Then we have

$$\mathscr{H}(u;t) = \mathscr{H}(u;0)$$

as long as a solution exists (see Lemma 3.1). We shall now recall the definition of Gevrey class of L^2 type. For $s \ge 1$, we denote by $\gamma^s_{L^2} = \gamma^s_{L^2}(\mathbb{R}^n)$ the Gevrey–Roumieu class of order s on $\mathbb{R}^n$:

$$\gamma^s_{L^2} = \bigcup_{\eta>0} \gamma^s_{\eta,L^2},$$

where f belongs to γ^s_{η,L^2} if and only if

$$\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} |\widehat{f}(\xi)|^2 \, d\xi < \infty.$$

Here $\widehat{f}(\xi)$ stands for the Fourier transform of $f(x)$. In the particular case $s = 1$, $\gamma^1_{L^2}$ corresponds to the analytic class. The space $\gamma^s_{L^2}$ is equipped with the inductive limit topology. As to the inclusion relation among the classes $\gamma^s_{L^2}$, we have

$$\gamma^{s_1}_{L^2} \subsetneq \gamma^{s_2}_{L^2} \qquad \text{if } 1 < s_1 < s_2$$

(see also Lemma A.1 in Appendix A). We will use the norms

$$\|f\|_{\gamma^s_{\eta,L^2}} = \left[\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} |\widehat{f}(\xi)|^2 \, d\xi \right]^{1/2}$$

and

$$\|(f,g)\|_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} = \left[\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} \left\{ |\widehat{f}(\xi)|^2 + |\widehat{g}(\xi)|^2 \right\} \, d\xi \right]^{1/2}$$

for $\eta > 0$. The Gevrey class $\gamma^\infty_{L^2}$ is defined as the union

$$\gamma^\infty_{L^2} = \bigcup_{s \geq 1} \gamma^s_{L^2}. \tag{1.3}$$

Locally the space $\gamma^\infty_{L^2}$ is 'almost' the space C^∞ of smooth functions but it appears to be more natural than C^∞ already for linear problems. For example, the Cauchy problem for the linear wave equation

$$\partial_t^2 v - a(t)\Delta v = 0, \quad v(0) = v_0, \ \partial_t v(0) = v_1, \tag{1.4}$$

with propagation speed $a = a(t)$ may not be well-posed in C^∞:

- if $v_0, v_1 \in C^\infty$ but $a > 0$ is Hölder C^α, $0 < \alpha < 1$, then (1.4) may have non-unique solutions, see Colombini, Jannelli and Spagnolo [8];
- if $v_0, v_1 \in C^\infty$ and $a \geq 0$ is in C^∞ then (1.4) may have no distributional solutions, see Colombini and Spagnolo [9].

However, the Cauchy problem (1.4) is still well-posed in Gevrey spaces, see Colombini, de Giorgi and Spagnolo [7]. While such situations do not happen in our case (the linearised nonlinearity is strictly positive and can be shown to have derivative in L^1_{loc}), it shows that classes of the type $\gamma^\infty_{L^2}$ may naturally enter such problems. Here, our first main result is:

Theorem 1.1. *Let $u_0, u_1 \in \gamma^\infty_{L^2}$. Then the Cauchy problem (1.2) admits a unique solution $u \in C^1\left([0,\infty); \gamma^\infty_{L^2}\right)$.*

Starting with this theorem, we can summarise the main results of this paper as follows:

- The global solvability of the Kirchhoff equation (1.2) in $\gamma^\infty_{L^2}$ is given in Theorem 1.1. There is no smallness assumption on data in this result.
- In turn, Theorem 1.1 is a consequence of a more refined solvability statement in classes $\gamma^s_{L^2}$ given in Theorem 1.2 below.

- While the statement of Theorem 1.2 is enough to yield the global solvability in $\gamma^{\infty}_{L^2}$, there is an arbitrarily small (but positive) loss of regularity for solutions in Theorem 1.2. This can be refined further and we can show that no such loss occurs for data in the 'more regular' part of $\gamma^s_{L^2}$ where the precise meaning of 'more regular' depends on the size of the Cauchy data: the 'larger' the data is, the higher regularity is required. However, all the regularity that comes under consideration here occurs within the same class $\gamma^s_{L^2}$, see Corollary 3.6 for a precise statement.
- In Section 4 we explain how these results can be extended to hold for initial-boundary value problem for the Kirchhoff equation in bounded domains and in exterior domains. Compared with the previous literature on exterior problems (for example for small data), the results in the present paper do not require the geometrical condition that the compliment of an exterior domain is star-shaped with respect to the origin.

As we mentioned above, Theorem 1.1 is an immediate consequence of the definition (1.3) and of the following:

Theorem 1.2. *Let $s > 1$. Then for any $u_0, u_1 \in \gamma^s_{L^2}$ the Cauchy problem (1.2) admits a unique solution $u \in C^1\left([0,\infty);\gamma^{s'}_{L^2}\right)$ for every $s' > s$.*

The statement of Theorem 1.2 will be refined further in Theorem 3.5 with respect to the regularity within the classes $\gamma^s_{L^2}$.

This paper is organised as follows: in §2 energy estimates for linear equations with time-dependent coefficients will be derived, and these estimates will be applied to get a priori estimates. In §3 we state the known result on local existence theorem as Theorem A and continue to prove Theorem 1.2. In §4 global existence for (1.2) with Gevrey data to the initial-boundary value problems will be discussed.

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2. Energy estimate for linear equation

In this section we shall derive energy estimates for solutions of the linear Cauchy problem with time-dependent coefficients. These estimates will be fundamental tools in the proof of the theorems.

Let us consider the linear Cauchy problem

$$\begin{cases} \partial_t^2 u - c(t)^2 \Delta u = 0, & t \in (0,T), \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

The assumptions for the following estimates are related with Theorem 2 from Colombini, Del Santo and Kinoshita [10]. However, here we need more precise conclusions on the behaviour of constants.

Proposition 2.1. *Let $\sigma \geq 1$ and $1 \leq s < q/(q-1)$ for some $q > 1$. Assume that $c = c(t) \in \mathrm{Lip}_{\mathrm{loc}}([0,T))$ satisfies*

$$m_0 \leq c(t) \leq M, \quad t \in [0,T], \tag{2.2}$$

$$|c'(t)| \leq \frac{K}{(T-t)^q}, \quad a.e.\, t \in [0,T), \tag{2.3}$$

for some $0 < m_0 < M$ and $K > 0$. If $((-\Delta)^{\sigma/2}u_0, (-\Delta)^{(\sigma-1)/2}u_1) \in \gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}$ for some η satisfying

$$\eta > \frac{2Km_0^{-1}}{q-1} + 4M^2 m_0^{-1}, \tag{2.4}$$

then the Cauchy problem (2.1) admits a unique solution

$$u \in \bigcap_{j=0}^{1} C^j\left([0,T]; (-\Delta)^{-(\sigma-j)/2}\gamma^s_{\eta',L^2}\right)$$

such that

$$\begin{aligned} &m_0^2 \|(-\Delta)^{\sigma/2}u(t)\|^2_{\gamma^s_{\eta',L^2}} + \|(-\Delta)^{(\sigma-1)/2}\partial_t u(t)\|^2_{\gamma^s_{\eta',L^2}} \\ &\quad \leq \max\{M^2, 1\} e^{4M^2 m_0^{-1}\max\{1, T^{1-(qs-s)}\}} \|((-\Delta)^{\sigma/2}u_0, (-\Delta)^{(\sigma-1)/2}u_1)\|^2_{\gamma^s_{\eta,L^2}\times\gamma^s_{\eta,L^2}} \end{aligned} \tag{2.5}$$

for $t \in [0,T]$, where

$$\eta' = \eta - \left(\frac{2Km_0^{-1}}{q-1} + 4M^2 m_0^{-1}\right) > 0.$$

Proof. Let $v = v(t,\xi)$ be a solution of the Cauchy problem

$$\begin{cases} \partial_t^2 v + c(t)^2|\xi|^2 v = 0, & t \in (0,T), \\ v(0,\xi) = \widehat{u}_0(\xi), \quad \partial_t v(0,\xi) = \widehat{u}_1(\xi). \end{cases}$$

We define

$$c_*(t,\xi) = \begin{cases} c(T) & \text{if } T|\xi|^{1/(qs-s)} \leq 1, \\ c(t) & \text{if } T|\xi|^{1/(qs-s)} > 1 \text{ and } 0 \leq t \leq T - |\xi|^{-1/(qs-s)}, \\ c(T - |\xi|^{-1/(qs-s)}) & \text{if } T|\xi|^{1/(qs-s)} > 1 \text{ and } T - |\xi|^{-1/(qs-s)} < t \leq T, \end{cases}$$

and

$$\alpha(t,\xi) = 2Mm_0^{-1}|c_*(t,\xi) - c(t)||\xi| + \frac{2|c_*'(t,\xi)|}{c_*(t,\xi)}.$$

We adopt an energy for v as

$$E(t,\xi) = \left\{|v'(t)|^2 + c_*(t,\xi)^2|\xi|^2|v(t)|^2\right\} k(t,\xi),$$

where

$$k(t,\xi) = |\xi|^{2(\sigma-1)} \exp\left(-\int_0^t \alpha(\tau,\xi)\, d\tau + \eta|\xi|^{1/s}\right)$$

and η is as in (2.4). We put

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} E(t,\xi)\, d\xi.$$

Hereafter we concentrate on estimating the integral of $\alpha(t,\xi)$. When $T|\xi|^{1/(qs-s)} \le 1$, we can estimate, by using assumption (2.2) on $c(t)$,

$$\begin{aligned}\int_0^t \alpha(\tau,\xi)\,d\tau =& \int_0^T 2Mm_0^{-1}|c_*(\tau,\xi)-c(\tau)||\xi|\,d\tau \\ \le& 4M^2m_0^{-1}T|\xi| \\ \le& 4M^2m_0^{-1}T^{1-(qs-s)},\end{aligned} \tag{2.6}$$

and when $T|\xi|^{1/(qs-s)} > 1$, we can estimate

$$\begin{aligned}\int_0^t \alpha(\tau,\xi)\,d\tau \le& \int_0^{T-|\xi|^{-1/(qs-s)}} \frac{2|c'(\tau)|}{c(\tau)}\,d\tau + \int_{T-|\xi|^{-1/(qs-s)}}^T 2Mm_0^{-1}|c_*(\tau,\xi)-c(\tau)||\xi|\,d\tau \\ \le& \int_0^{T-|\xi|^{-1/(qs-s)}} \frac{2Km_0^{-1}}{(T-\tau)^q}\,d\tau + 4M^2m_0^{-1}|\xi|^{1-1/(qs-s)} \\ \le& \frac{2Km_0^{-1}|\xi|^{1/s}}{q-1} + 4M^2m_0^{-1}|\xi|^{1-1/(qs-s)}.\end{aligned} \tag{2.7}$$

Since $1-1/(qs-s) < 1/s$, it follows that

$$|\xi|^{1-1/(qs-s)} \le 1 + |\xi|^{1/s}.$$

Consequently, we get

$$k(t,\xi) \ge e^{-4M^2m_0^{-1}\max\{1,T^{1-(qs-s)}\}}|\xi|^{2(\sigma-1)}e^{\left(\eta-\frac{2Km_0^{-1}}{q-1}-4M^2m_0^{-1}\right)|\xi|^{1/s}},$$

and hence,

$$\begin{aligned}\mathcal{E}(t) \ge& e^{-4M^2m_0^{-1}\max\{1,T^{1-(qs-s)}\}}\times \\ & \int_{\mathbb{R}^n} e^{\left(\eta-\frac{2Km_0^{-1}}{q-1}-4M^2m_0^{-1}\right)|\xi|^{1/s}}|\xi|^{2(\sigma-1)}\{m_0^2|\xi|^2|v(t)|^2+|v'(t)|^2\}\,d\xi.\end{aligned} \tag{2.8}$$

We compute the derivative of $E(t,\xi)$:

$$\begin{aligned}& E'(t,\xi) = \\ & \left[2\mathrm{Re}\{v''(t)\overline{v'(t)}\} + 2c_*(t,\xi)c_*'(t,\xi)|\xi|^2|v(t)|^2 + 2c_*(t,\xi)^2|\xi|^2\mathrm{Re}\{v'(t)\overline{v(t)}\}\right]k(t,\xi) \\ & -\{c_*(t,\xi)^2|\xi|^2|v(t)|^2+|v'(t)|^2\}\alpha(t,\xi)k(t,\xi) \\ =& \left[2\{c_*(t,\xi)^2-c(t)^2\}|\xi|^2\mathrm{Re}\{v'(t)\overline{v(t)}\} + 2c_*(t,\xi)c_*'(t,\xi)|\xi|^2|v(t)|^2\right]k(t,\xi) \\ & -\alpha(t,\xi)E(t,\xi).\end{aligned}$$

Then we can estimate the right hand side as

$$\left[\frac{2|c_*(t,\xi)^2-c(t)^2||\xi|}{c_*(t,\xi)}|v'(t)|\cdot c_*(t,\xi)|\xi|\,|v(t)|+2\frac{|c_*'(t,\xi)|}{c_*(t,\xi)}c_*(t,\xi)^2|\xi|^2|v(t)|^2\right]k(t,\xi)$$
$$-\alpha(t,\xi)E(t,\xi)$$
$$\le 2Mm_0^{-1}|c_*(t,\xi)-c(t)||\xi|E(t,\xi)+\frac{2|c_*'(t,\xi)|}{c_*(t,\xi)}E(t,\xi)-\alpha(t,\xi)E(t,\xi)$$
$$=0,$$

which implies that $E'(t,\xi)\le 0$ for $t\in(0,T)$, and we find that

$$\mathcal{E}(t)\le\mathcal{E}(0).$$

Thus the required estimate (2.5) follows from this estimate and (2.8). The proof of Proposition 2.1 is now finished. □

3. Proof of Theorem 1.2

First we discuss possible properties of the life span of local solutions to (1.2). Among other things, let us introduce the local existence theorem for the Cauchy problem (1.2). We often use the norm of solution $u(t)$ as follows:

$$\mathscr{E}_{3/2}(u;t):=\left(1+\|\nabla u(t)\|_{L^2}^2\right)\|u(t)\|_{\dot H^{3/2}}^2+\|\partial_t u(t)\|_{\dot H^{1/2}}^2.$$

Here we denote by

$$H^\sigma=H^\sigma(\mathbb{R}^n)=\langle D\rangle^{-\sigma}L^2(\mathbb{R}^n)$$

for $\sigma\in\mathbb{R}$ the standard Sobolev spaces, and $\langle D\rangle=(1-\Delta)^{1/2}$. Their homogeneous version is

$$\dot H^\sigma=\dot H^\sigma(\mathbb{R}^n)=(-\Delta)^{-\sigma/2}L^2(\mathbb{R}^n).$$

The following result is proved by Arosio and Garavaldi (see [2], and also [3]).

Theorem A (Arosio and Garavaldi [2]). *Let $(u_0,u_1)\in H^\sigma\times H^{\sigma-1}$ for some $\sigma\ge 3/2$. Then there exists a life span $T_u=T_u(u_0,u_1)>0$ such that the Cauchy problem (1.2) admits a unique maximal solution $u\in\bigcap_{j=0}^{1}C^j([0,T_u);H^{\sigma-j})$, and at least one of the following statements is valid:*

(a) $T_u=+\infty$;

(b) $T_u<\infty$ *and* $\limsup_{t\nearrow T_u}\mathscr{E}_{3/2}(u;t)=\infty$. *In this case the life span T_u is estimated from below as*

$$T_u\ge T_0:=\frac{1}{2\mathscr{E}_{3/2}(u;0)}. \tag{3.1}$$

We remark that the life span $T_u=T_u(u_0,u_1)$ is to be understood as follows:

$$T_u=\sup\left\{t:\ H^{3/2}\text{-solution } u(\tau,\cdot)\text{ to (1.2) with data }(u_0,u_1)\text{ exists for }0\le\tau<t\right\}.$$

Let us mention also a known result on the local existence of solutions. Before Theorem A was proved, Medeiros and Miranda exhibited the local solvability for Eq. (1.2) in $H^{3/2}\times H^{1/2}$ (see [35]). It should be noted that, however large the regularity of data is, T_u depends only on the norm of the data in $H^{3/2} \times H^{1/2}$. This means that when one would show the global existence of solutions to (1.2), it suffices to obtain that the norm of solutions in $H^{3/2} \times H^{1/2}$ is bounded on $[0, T_u)$.

Kirchhoff equation has a Hamiltonian structure. Namely, we have:

Lemma 3.1. *Let* $u \in \bigcap_{j=0}^{1} C^j([0, T_u); H^{(3/2)-j})$ *be the solution of* (1.2). *Then we have*

$$\mathscr{H}(u; t) = \mathscr{H}(u; 0), \quad \forall t \in [0, T_u),$$

where we recall that

$$\mathscr{H}(u; t) = \frac{1}{2}\left\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\right\} + \frac{1}{4}\|\nabla u(t)\|_{L^2}^4.$$

Proof. The proof is elementary. Multiplying equation (1.2) by $\partial_t u$ and integrating, we get

$$\frac{d}{dt}\mathscr{H}(u; t) = 0,$$

as desired. □

We now establish a local existence theorem for (1.2) in Gevrey classes (see also [16, 18, 22, 23]).

Proposition 3.2. *Let* $s > 1$ *and* $\eta > 0$. *Let* $(u_0, u_1) \in \langle D\rangle^{-3/2}\gamma^s_{\eta,L^2} \times \langle D\rangle^{-1/2}\gamma^s_{\eta,L^2}$. *Then there exists a life span* $T_u = T_u(u_0, u_1) > 0$ *depending only on* $\mathscr{H}(u; 0)$ *and* $\mathscr{E}_{3/2}(u; 0)$ *such that the Cauchy problem* (1.2) *admits a unique solution* u *in the class*

$$\bigcap_{j=0}^{1} C^j\left([0, T_u); \langle D\rangle^{-(3/2)+j}\gamma^s_{\eta,L^2}\right),$$

and one of the following statements is valid:

(a) $T_u = \infty$;
(b) $T_u < \infty$ *and* $\limsup_{t \nearrow T_u} \mathscr{E}_{3/2}(u; t) = \infty$. *Furthermore, we have*

$$T_u \geq \frac{1}{2\mathscr{E}_{3/2}(u; 0)}. \tag{3.2}$$

Proof. Let us interpret the initial data as elements of the space $H^{3/2} \times H^{1/2}$. From Theorem A of the case for $(u_0, u_1) \in H^{3/2} \times H^{1/2}$, we know that the Cauchy problem (1.2) admits a unique maximal solution $u \in \bigcap_{j=0}^{1} C^j([0, T_u); H^{(3/2)-j})$, where the life span T_u satisfies either the assertion (a) or (b). We adopt an energy for

$$v(t) = |\xi|^{1/2}\widehat{u}(t, \xi)$$

as

$$E(t, \xi) = \left\{|v'(t)|^2 + \tilde{c}(t)^2|\xi|^2|v(t)|^2\right\} e^{\eta|\xi|^{1/s}},$$

where

$$\tilde{c}(t)=\sqrt{1+\int_{\mathbb{R}^n}|\nabla u(t,x)|^2\,dx}.$$

We notice that $v(t)$ satisfies the equation

$$v''(t)+\tilde{c}(t)^2|\xi|^2v(t)=0. \tag{3.3}$$

We put

$$\mathcal{E}(t)=\int_{\mathbb{R}^n}E(t,\xi)\,d\xi.$$

By using (3.3), we compute the derivative of $E(t,\xi)$ with respect to t;

$$\begin{aligned}E'(t,\xi)=&\Big[2\mathrm{Re}\big\{v''(t)\overline{v'(t)}\big\}+2\tilde{c}(t)\tilde{c}'(t)|\xi|^2|v(t)|^2+2\tilde{c}(t)^2|\xi|^2\mathrm{Re}\big\{v'(t)\overline{v(t)}\big\}\Big]\,e^{\eta|\xi|^{1/s}}\\ =&2\tilde{c}(t)\tilde{c}'(t)|\xi|^2|v(t)|^2e^{\eta|\xi|^{1/s}}\\ \le&\frac{2|\tilde{c}'(t)|}{\tilde{c}(t)}E(t,\xi);\end{aligned}$$

thus we find from Gronwall's lemma that

$$\mathcal{E}(t)\le\mathcal{E}(0)\exp\left(\int_0^t\frac{2|\tilde{c}'(\tau)|}{\tilde{c}(\tau)}\,d\tau\right)$$

for any $t\in[0,T_u)$. Hence on any subinterval $[0,t]\subset[0,T_u)$ we have the energy estimate with the same η, and so the Cauchy problem (1.2) admits a unique solution u in the class

$$\bigcap_{j=0}^{1}C^j\left([0,T_u);\langle D\rangle^{-(3/2)+j}\gamma^s_{\eta,L^2}\right).$$

Finally, inequality (3.2) comes from (3.1). The proof of Proposition 3.2 is now complete. □

We introduce the class of blow-up data in the Gevrey class. The final aim will be to show that this set is empty by arriving at a contradiction. Let us define the class of blow-up data:

$$\mathcal{B}^s_\eta:=\Big\{(u_0,u_1)\in\langle D\rangle^{-3/2}\gamma^s_{\eta,L^2}\times\langle D\rangle^{-1/2}\gamma^s_{\eta,L^2}:T_u(u_0,u_1)<\infty\Big\}$$

and also

$$\mathcal{B}^s:=\bigcup_{\eta>0}\mathcal{B}^s_\eta.$$

In the lemma below the number $\frac{M^2}{4}-1$ can be replaced by any positive number but we write it in this form to be able to apply the statement directly in the sequel.

Lemma 3.3. *Let $M>2$ and $\eta>0$. If*

$$\left\{(u_0,u_1)\in\mathcal{B}^s_\eta:\ 2\mathscr{H}(u;0)\le\frac{M^2}{4}-1\right\}\ne\emptyset,$$

then

$$\inf_{(u_0,u_1)\in\mathcal{B}^s_\eta:\,2\mathscr{H}(u;0)\le\frac{M^2}{4}-1}T_u(u_0,u_1)=0. \tag{3.4}$$

Proof. By virtue of Lemma 3.1, we have

$$2\mathscr{H}(u;t) = 2\mathscr{H}(u;0) \le \frac{M^2}{4} - 1 \tag{3.5}$$

for all $t \in [0, T_u(u_0, u_1))$. Also, thanks to Proposition 3.2, the solution $u(t,x)$ to (1.2) enjoys the property that

$$(u(t), \partial_t u(t)) \in \langle D\rangle^{-3/2}\gamma^s_{\eta,L^2} \times \langle D\rangle^{-1/2}\gamma^s_{\eta,L^2} \tag{3.6}$$

for all $t \in [0, T_u(u_0, u_1))$ with the same value for η. To prove (3.4), we put

$$\alpha := \inf_{(u_0,u_1)\in\mathcal{B}^s_\eta:\, 2\mathscr{H}(u;0)\le \frac{M^2}{4}-1} T_u(u_0, u_1). \tag{3.7}$$

Let $u = u(t,x) \in \bigcap_{k=0}^{1} C^k([0, T_u(u_0, u_1)); \langle D\rangle^{-(3/2)+k}\gamma^s_{\eta,L^2})$ be the solution to (1.2) with data $(u_0, u_1) \in \mathcal{B}^s_\eta$ satisfying $2\mathscr{H}(u;0) \le \frac{M^2}{4} - 1$. For every $\varepsilon > 0$ this $u(t) = u(t,\cdot)$ at $t = T_u(u_0, u_1) - \varepsilon$ satisfies

$$(u(T_u(u_0, u_1) - \varepsilon), \partial_t u(T_u(u_0, u_1) - \varepsilon)) \in \mathcal{B}^s_\eta$$

in view of (3.6). It also satisfies (3.5). Hence it follows from (3.7) that

$$T_u(u(T_u(u_0, u_1) - \varepsilon), \partial_t u(T_u(u_0, u_1) - \varepsilon)) \ge \alpha.$$

On the other hand, we have

$$T_u(u(T_u(u_0, u_1) - \varepsilon), \partial_t u(T_u(u_0, u_1) - \varepsilon)) = \varepsilon.$$

Thus we must have $\alpha = 0$. This completes the proof of Lemma 3.3. □

Though we can define $\mathcal{B}^s_\eta$ for any $\eta > 0$, we will encounter with the restriction of η to be $\eta > 4M^2$ in Proposition 3.4 below.

Let us now turn to prove the theorem. Given exponents s and q related by

$$1 < s < q/(q-1) \quad \text{and} \quad q > 1, \tag{3.8}$$

let us take a pair of data $v_0, v_1 \in \gamma^s_{L^2}$. For this pair (v_0, v_1) we consider the *linear* Cauchy problem in the strip $(0, T_u(v_0, v_1)) \times \mathbb{R}^n$:

$$\partial_t^2 v - c(t)^2 \Delta v = 0, \quad t \in (0, T_u(v_0, v_1)), \quad x \in \mathbb{R}^n, \tag{3.9}$$

with initial condition

$$v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x). \tag{3.10}$$

Here $c = c(t)$ belongs to a class $\mathscr{K}$ defined as follows:

Class $\mathscr{K}(T)$. *Let $0 < T \le T_u(v_0, v_1)$. Given constants $q > 1$, $M > 2$ and $K_0 > 0$, we say that $c(t)$ belongs to $\mathscr{K}(T) = \mathscr{K}(T, K_0) = \mathscr{K}(q, M, K_0, T, T_u(v_0, v_1))$ if $c = c(t)$ belongs to* $\mathrm{Lip}_{\mathrm{loc}}([0, T_u(v_0, v_1)))$ *and satisfies*

$$1 \le c(t) \le M, \quad t \in [0, T],$$

$$|c'(t)| \le \frac{K_0}{\{T_u(v_0, v_1) - t\}^q}, \quad a.e.\, t \in [0, T_u(v_0, v_1)).$$

Sometimes we may abbreviate the notation by simply writing $\mathscr{K}(T)$ or $\mathscr{K}$.

By the energy estimate (2.5) from Proposition 2.1, if $c(t) \in \mathcal{K}(T_u(v_0,v_1))$ and (v_0,v_1) belongs to the class

$$(-\Delta)^{-3/4}\gamma^s_{\eta,L^2} \times (-\Delta)^{-1/4}\gamma^s_{\eta,L^2}$$

for some η satisfying

$$\eta > \frac{2K_0}{q-1} + 4M^2,$$

then the Cauchy problem (3.9)–(3.10) admits a unique solution $v(t,x)$ in the class

$$\bigcap_{j=0}^{1} C^j\left([0,T_u(v_0,v_1)]; (-\Delta)^{-(3/4)+(j/2)}\gamma^s_{\eta',L^2}\right), \tag{3.11}$$

provided that s and q satisfy (3.8), where η' is the real number such that

$$\eta' = \eta - \left(\frac{2K_0}{q-1} + 4M^2\right) > 0. \tag{3.12}$$

If we define the functional

$$\tilde{c}(t) = \sqrt{1 + \int_{\mathbb{R}^n} |\nabla v(t,x)|^2\,dx},$$

this defines the mapping

$$\Theta : c(t) \mapsto \tilde{c}(t).$$

We will show the convexity and compactness of $\mathcal{K}(T_u(v_0,v_1))$ in $L^\infty_{\text{loc}}([0,T_u(v_0,v_1)))$. If we show that Θ maps continuously $\mathcal{K}(T_u(v_0,v_1))$ into itself, the Schauder-Tychonoff fixed point theorem will allow us to progress with the proof. But first we look into the structure of the blow up.

Keeping Lemma 3.3 in mind, let us introduce a subset of the blow-up class $\mathcal{B}^s$ for $s>1$ as follows:

$$\mathcal{B}^s_{\eta,C_1} := \left\{(u_0,u_1) \in \langle D\rangle^{-3/2}\gamma^s_{\eta,L^2} \times \langle D\rangle^{-1/2}\gamma^s_{\eta,L^2} : T_u(u_0,u_1) \le C_1\right\}$$

for $C_1 > 0$. Let us give some remarks on the properties of $\mathcal{B}^s_{\eta,C_1}$.

(i) If $u_0, u_1 \in \gamma^1_{L^2}$, then the known results of [5, 39] imply that

$$T_u(u_0,u_1) = \infty.$$

Thus, for the analytic class we have $\mathcal{B}^1_{\eta,C_1} = \emptyset$ for all η and C_1, so we may restrict considering $s > 1$. However, we note that the argument works also well for $s = 1$.

(ii) Observing from (3.1) in Theorem A, one can see that if $\mathscr{E}_{3/2}(u;0) \to 0$, then $T_u(u_0,u_1) \to \infty$. This means that $T_u(u_0,u_1)$ is unbounded near $(0,0)$ in $\dot{H}^{3/2} \times \dot{H}^{1/2}$. But, one could not exclude the case that $T_u(u_0,u_1)$ would be bounded for some small data (u_0,u_1) in $\dot{H}^{3/2} \times \dot{H}^{1/2}$. Also, one does not know whether $\mathcal{B}^s$ is empty or not. So, there is a possibility that $T_u(u_0,u_1) = \infty$ even if data (u_0,u_1) are very large. From this point of view, assuming $\mathcal{B}^s \neq \emptyset$, we will employ the fixed point argument.

We shall prove here the following:

Proposition 3.4. *Let s be related to the exponent q as follows:*

$$s = \frac{1}{q-1} \quad \text{and} \quad 1 < q < 2. \tag{3.13}$$

Assume that $M > 2$ and $\eta > 4M^2$, and set

$$\varepsilon_0 := \left\{ \frac{(q-1)(\eta - 4M^2)}{4e^{4M^2}} \right\}^{\frac{1}{q-1}}. \tag{3.14}$$

Let (v_0, v_1) belong to

$$(-\Delta)^{-3/4}\gamma^s_{\eta,L^2} \times (-\Delta)^{-1/4}\gamma^s_{\eta,L^2}$$

and satisfy

$$2\mathscr{H}(v;0) \le \frac{M^2}{4} - 1. \tag{3.15}$$

If there exists a constant $C_1 \in (0, \varepsilon_0)$ such that

$$(v_0, v_1) \text{ belong to } \mathcal{B}^s_{\eta,C_1} \tag{3.16}$$

and satisfy

$$\left\| \left((-\Delta)^{3/4} v_0, (-\Delta)^{1/4} v_1 \right) \right\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} < \frac{(q-1)(\eta - 4M^2)}{2C_1^q M^2 e^{4M^2}}, \tag{3.17}$$

then, setting

$$K_0 := C_1^q M^2 e^{4M^2} \left\| \left((-\Delta)^{3/4} v_0, (-\Delta)^{1/4} v_1 \right) \right\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}}, \tag{3.18}$$

we have the following statement: For any $c(t) \in \mathscr{K}(T_u(v_0, v_1), K_0)$, let v be the solution to the Cauchy problem (3.9)–(3.10) satisfying (3.11). Then

$$1 \le \tilde{c}(t) \le M, \quad t \in [0, T_u(v_0, v_1)], \tag{3.19}$$

$$|\tilde{c}'(t)| \le \frac{K_0}{\{T_u(v_0, v_1) - t\}^q}, \quad t \in [0, T_u(v_0, v_1)). \tag{3.20}$$

Let us give a few remarks on assumptions in Proposition 3.4.

(i) Observing (3.2) in Proposition 3.2, if (v_0, v_1) belong to $\mathcal{B}^s_{\eta,C_1}$, we have

$$\mathscr{E}_{3/2}(v;0) \ge \frac{1}{2C_1}.$$

Also, assumption (3.15) on the Hamiltonian implies that

$$1 + \|\nabla v_0\|^2_{L^2} \le \frac{M^2}{4}.$$

Hence, going back to the definition of $\mathscr{E}_{3/2}(v;0)$ and combining the previous inequalities, we get

$$\frac{1}{2C_1} \le \frac{M^2}{4} \left(\|v_0\|^2_{\dot{H}^{3/2}} + \|v_1\|^2_{\dot{H}^{1/2}} \right).$$

Obviously, we have

$$\left\| \left((-\Delta)^{3/4} v_0, (-\Delta)^{1/4} v_1 \right) \right\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} \ge \|v_0\|^2_{\dot{H}^{3/2}} + \|v_1\|^2_{\dot{H}^{1/2}}.$$

Then, combining this inequality with (3.17), we see that C_1 must satisfy

$$\frac{1}{2C_1} < \frac{M^2}{4} \cdot \frac{(q-1)(\eta - 4M^2)}{2C_1^q M^2 e^{4M^2}},$$

which means that

$$0 < C_1 < \varepsilon_0,$$

where ε_0 is the constant given in (3.14).

(ii) It is not restrictive to assume that $C_1 > 0$ is small also from the point of view of Lemma 3.3.

(iii) One does not know whether or not, (3.15) and (3.16)–(3.17) are compatible. Certainly, we can specify some data in the previous remark (i) that the two conditions (3.15) and (3.17) are compatible. But we have no knowledge to prove this incompatibility for any data. To make it clear, assuming that any data satisfies both (3.16) and (3.17), we will lead to a contradiction in the argument after the proof of this proposition.

We now turn to prove Proposition 3.4.

Proof of Proposition 3.4. For the sake of simplicity, we write

$$T_u = T_u(v_0, v_1).$$

Notice that, by virtue of $(v_0, v_1) \in \mathcal{B}^s_{\eta, C_1}$, it follows that

$$T_u \le C_1. \tag{3.21}$$

First, we prove (3.20). One can readily see that

$$2\tilde{c}(t)\tilde{c}'(t) = 2\mathrm{Re}\left((-\Delta)^{3/4} v(t), (-\Delta)^{1/4}\partial_t v(t)\right)_{L^2},$$

and hence, we have

$$\begin{aligned} |\tilde{c}'(t)| &\le \|v(t)\|_{\dot{H}^{3/2}} \|\partial_t v(t)\|_{\dot{H}^{1/2}} \\ &\le \left\|(-\Delta)^{3/4} v(t)\right\|_{\gamma^s_{\eta', L^2}} \left\|(-\Delta)^{1/4}\partial_t v(t)\right\|_{\gamma^s_{\eta', L^2}} \end{aligned}$$

for any $\eta' > 0$, since $\tilde{c}(t) \ge 1$. Notice that (v_0, v_1) satisfy (3.17). Then, by the definition (3.18) of K_0 and (3.17), we have the following inequality:

$$\eta > \frac{2K_0}{q-1} + 4M^2.$$

Hence, if η' is chosen as in (3.12), then, applying the energy estimate (2.5) from Proposition 2.1 to the right hand side of the previous estimate, we can write

$$|\tilde{c}'(t)| \le M^2 e^{4M^2 \max\{1, T_u^{1-(qs-s)}\}} \left\|\left((-\Delta)^{3/4} v_0, (-\Delta)^{1/4} v_1\right)\right\|^2_{\gamma^s_{\eta, L^2} \times \gamma^s_{\eta, L^2}} \tag{3.22}$$

for $t \in [0, T_u]$. Since $1 - (qs - s) = 0$ by assumption (3.13), it follows that

$$e^{4M^2 \max\{1, T_u^{1-(qs-s)}\}} = e^{4M^2}. \tag{3.23}$$

Hence, by using inequality (3.21) and recalling the definition (3.18) of K_0, we conclude from (3.22)–(3.23) that

$$\begin{aligned}|\tilde{c}'(t)| \le & M^2 e^{4M^2} T_u^q \left\| \left((-\Delta)^{3/4} v_0, (-\Delta)^{1/4} v_1 \right) \right\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} \cdot \frac{1}{T_u^q} \\ \le & M^2 e^{4M^2} C_1^q \left\| \left((-\Delta)^{3/4} v_0, (-\Delta)^{1/4} v_1 \right) \right\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} \cdot \frac{1}{(T_u - t)^q} \\ = & \frac{K_0}{(T_u - t)^q}\end{aligned}$$

for $t \in [0, T_u)$. This proves (3.20).

Finally we prove (3.19). In this case, we will not use the energy estimate (2.5) from Proposition 2.1. Our assumption (3.15) implies that

$$1 \le \tilde{c}(0) \le \sqrt{1 + 2\mathscr{H}(v;0)} \le \frac{M}{2}.$$

Since $\tilde{c}(t)$ is continuous, there exists a time $t_1 < T_u$ such that

$$1 \le \tilde{c}(t) \le M$$

for $0 \le t \le t_1$. Fixing data (v_0, v_1) satisfying (3.13)–(3.17), we can show that the class $\mathcal{K}(t_1, K_0)$ is the convex and compact subset of the Banach space $L^\infty([0, t_1])$, and resorting to (3.20), we can also prove that Θ is continuous from $\mathcal{K}(t_1, K_0)$ into itself. This argument will be also done in the whole interval $[0, T_u]$ in the last step, where we give its details. Then Schauder's fixed point theorem allows us to conclude that Θ has a fixed point in $\mathcal{K}(t_1, K_0)$:

$$c(t) = \Theta(c(t)) = \tilde{c}(t)$$

for $0 \le t \le t_1$. This means that solution $v(t,x)$ to the linear Cauchy problem (3.9)–(3.10) is also a solution to the nonlinear Cauchy problem (1.2) with data (v_0, v_1) on $[0, t_1]$. Hence it follows from Lemma 3.1 and assumption (3.15) that

$$2\mathscr{H}(v;t) = 2\mathscr{H}(v;0) \le \frac{M^2}{4} - 1, \quad t \in [0, t_1],$$

and as a result, we deduce that

$$1 \le \tilde{c}(t) \le \sqrt{1 + 2\mathscr{H}(v;t)} \le \frac{M}{2}$$

for $0 \le t \le t_1$. Therefore, by the continuity of $\tilde{c}(t)$, there exists a time $t_2 \in (t_1, T_u)$ such that

$$1 \le \tilde{c}(t) \le M$$

for $0 \le t \le t_2$. Hence, we can develop the previous fixed point argument; the solution $v(t,x)$ to the linear Cauchy problem (3.9)–(3.10) is also a solution to the nonlinear Cauchy problem (1.2) with data (v_0, v_1) on $[0, t_2]$ satisfying

$$2\mathscr{H}(v;t) = 2\mathscr{H}(v;0) \le \frac{M^2}{4} - 1, \quad t \in [0, t_2],$$

where we have used assumption (3.15) in the last step. Now, we define a time t_* by the maximal time such that

$$1 \leq \tilde{c}(t) \leq M$$

for $0 \leq t \leq t_*$. Suppose that $t_* < T_u$. Then, after employing the fixed point argument on the interval $[0, t_*]$, we deduce from Lemma 3.1 and assumption (3.15) that

$$2\mathscr{H}(v; t_*) = 2\mathscr{H}(v; 0) \leq \frac{M^2}{4} - 1,$$

and hence, we get

$$1 \leq \tilde{c}(t_*) \leq \frac{M}{2}.$$

Therefore, the fixed point argument will be also applicable, and $v(t, x)$ coincides with the solution to (1.2) with data (v_0, v_1) on some interval $[0, t_{**}]$ strictly containing $[0, t_*]$. This implies that $\tilde{c}(t)$ is bounded by M on $[0, t_{**}]$. But this contradicts the maximality of t_*. Thus we must have the required estimate (3.19). The proof of Proposition 3.4 is now complete. □

Based on Proposition 3.4, let us prove the following theorem.

Theorem 3.5. *Let $s > 1$, $M > 2$ and $\eta > 4M^2$. Let (u_0, u_1) belong to $\langle D\rangle^{-3/2}\gamma^s_{\eta,L^2} \times \langle D\rangle^{-1/2}\gamma^s_{\eta,L^2}$ and satisfy*

$$2\mathscr{H}(u; 0) \leq \frac{M^2}{4} - 1. \tag{3.24}$$

Then the Cauchy problem (1.2) admits a unique solution u in the class

$$\bigcap_{j=0}^{1} C^j\left([0, \infty); \langle D\rangle^{-(3/2)+j}\gamma^s_{\eta,L^2}\right).$$

One problem that we will encounter later in the proof of Theorem 1.2 (in particular leading to the appearance of $s' > s$ there) is that we need η to be sufficiently large in Theorem 3.5, namely, to satisfy $\eta > 4M^2$ for $M > 2$ also satisfying (3.24). Writing these conditions as $\eta > 4M^2 > 16$ and $\eta > 4M^2 \geq 4(8\mathscr{H}(u; 0) + 4)$, Theorem 3.5 gives the following

Corollary 3.6. *Let $s > 1$. Let us write*

$$\gamma^s_{L^2} = \Gamma^s_-(\nu) \cup \Gamma^s_+(\nu), \text{ where } \Gamma^s_-(\nu) = \bigcup_{0<\eta\leq\nu} \gamma^s_{\eta,L^2},\ \Gamma^s_+(\nu) = \bigcup_{\eta>\nu} \gamma^s_{\eta,L^2}.$$

If $\langle D\rangle^{3/2}u_0, \langle D\rangle^{1/2}u_1 \in \Gamma^s_+(\nu)$ for $\nu > \max\{16, 4(8\mathscr{H}(u; 0) + 4)\}$, then the Cauchy problem (1.2) admits a unique solution

$$u \in C^1\left([0, \infty); \gamma^s_{L^2}\right).$$

Proof of Theorem 3.5. Assuming that (u_0, u_1) satisfy (3.16)–(3.17) from Proposition 3.4, we lead to a contradiction. Hereafter, we write

$$T_u = T_u(u_0, u_1) \quad \text{and} \quad \mathcal{K} = \mathcal{K}(T_u, K_0).$$

First, we notice that if $\{(u_0,u_1)\in\mathcal{B}^s_\eta:\ 2\mathscr{H}(u;0)\le\frac{M^2}{4}-1\}\neq\emptyset$ for some $\eta>4M^2$, then we have (3.4) from Lemma 3.3:

$$\inf_{(u_0,u_1)\in\mathcal{B}^s_\eta:\ 2\mathscr{H}(u;0)\le\frac{M^2}{4}-1} T_u(u_0,u_1)=0.$$

Letting $c=c(t)\in\mathscr{K}$, where s and q are related by (3.13), we consider the linear Cauchy problem (3.9) with data (u_0,u_1), and we put

$$\tilde{c}(t)=\sqrt{1+\int_{\mathbb{R}^n}|\nabla v(t,x)|^2\,dx}.$$

Then it follows from Proposition 3.4 that the mapping

$$\Theta: c(t)\mapsto\tilde{c}(t)$$

maps from $\mathscr{K}$ into itself. Now $\mathscr{K}$ may be regarded as the convex subset of the Fréchet space $L^\infty_{\mathrm{loc}}([0,T_u))$, and we endow $\mathscr{K}$ with the induced topology. We shall prove the compactness of $\mathscr{K}$ and continuity of the mapping Θ. Then the Schauder-Tychonoff theorem allows us to conclude the proof.

Compactness of $\mathscr{K}$. We show that $\mathscr{K}$ is uniformly bounded and equicontinuous on every compact interval of $[0,T_u)$. Let $\{c_k(t)\}_{k=1}^\infty$ be a sequence in $\mathscr{K}$ so that

$$1\le c_k(t)\le M,\quad t\in[0,T_u], \tag{3.25}$$

$$|c_k'(t)|\le\frac{K_0}{(T_u-t)^q},\quad a.e.\, t\in[0,T_u). \tag{3.26}$$

Observing

$$c_k(t)-c_k(t')=\int_{t'}^t c_k'(\tau)\,d\tau,$$

we obtain from (3.26) that

$$|c_k(t)-c_k(t')|\le\frac{K_0}{q-1}\left\{\frac{1}{(T_u-t)^{q-1}}-\frac{1}{(T_u-t')^{q-1}}\right\}$$

for $0\le t'<t<T_u$. Since $1/(T_u-t)^{q-1}$ is uniformly continuous on every compact interval of $[0,T_u)$, the sequence $\{c_k(t)\}_{k=1}^\infty$ is equicontinuous on that interval. Thus one can deduce from the Ascoli-Arzelà theorem that $\mathscr{K}$ is compact in $L^\infty_{\mathrm{loc}}([0,T_u))$, and hence, every sequence $\{c_k(t)\}_{k=1}^\infty$ in $\mathscr{K}$ has a subsequence, denoted by the same, converging to some $c(\cdot)\in L^\infty_{\mathrm{loc}}([0,T_u))$:

$$\begin{cases} c_k(\cdot)\underset{(k\to\infty)}{\to}c(\cdot)\quad\text{in } L^\infty_{\mathrm{loc}}([0,T_u));\\ 1\le c(t)\le M\quad\text{for every compact interval in } [0,T_u);\\ |c(t)-c(t')|\le\dfrac{K_0}{q-1}\left\{\dfrac{1}{(T_u-t)^{q-1}}-\dfrac{1}{(T_u-t')^{q-1}}\right\},\quad 0\le t'<t<T_u.\end{cases} \tag{3.27}$$

The last statement of (3.27) implies that $c(t)$ is in $\mathrm{Lip}_{\mathrm{loc}}([0,T_u))$, since the function $(T_u-t)^{-(q-1)}$ is in $\mathrm{Lip}_{\mathrm{loc}}([0,T_u))$. Furthermore, $c(t)$ must be bounded by M even at $t=T_u$,

$$1\le c(t)\le M,\quad t\in[0,T_u]. \tag{3.28}$$

Indeed, if

$$\limsup_{t \nearrow T_u} c(t) > M,$$

there exists a sequence $\{t_j\}$ such that

$$t_j \nearrow T_u \quad \text{and} \quad c(t_j) > M, \quad (j = 1, 2, \ldots). \tag{3.29}$$

Going back to (3.25), and resorting to the first statement of (3.27), we have

$$c(t_j) = \lim_{k \to \infty} c_k(t_j) \leq M, \quad (\forall\, j).$$

This contradicts (3.29). Thus we conclude that $c(t)$ satisfies (3.28) and

$$c(\cdot) \in \mathrm{Lip}_{\mathrm{loc}}([0, T_u)),$$

and the derivative $c'(t)$ exists for a.e. $t \in [0, T_u)$. Now, for the derivative $c'(t)$, if we prove that

$$|c'(t)| \leq \frac{K_0}{(T_u - t)^q}, \quad a.e.\, t \in [0, T_u), \tag{3.30}$$

then $c(t) \in \mathscr{K}$, which proves the compactness of $\mathscr{K}$. We prove (3.30). Let $t_0 \in (0, T_u)$ be an arbitrary point where $c(t)$ is differentiable. Since we have, by using (3.26),

$$\begin{aligned} \left| \frac{c_k(t_0 + h) - c_k(t_0 - h)}{2h} \right| &= \left| \frac{1}{2h} \int_{t_0 - h}^{t_0 + h} c_k'(t)\, dt \right| \\ &\leq \frac{K_0}{2h(q-1)} \left\{ \frac{1}{(T_u - (t_0 - h))^{q-1}} - \frac{1}{(T_u - (t_0 + h))^{q-1}} \right\} \end{aligned}$$

for $h > 0$, we can take the limit in this equation with respect to k, so that

$$\left| \frac{c(t_0 + h) - c(t_0 - h)}{2h} \right| \leq \frac{K_0}{2h(q-1)} \left\{ \frac{1}{(T_u - (t_0 - h))^{q-1}} - \frac{1}{(T_u - (t_0 + h))^{q-1}} \right\}.$$

Then, letting $h \to +0$, we conclude that

$$|c'(t_0)| \leq \frac{K_0}{(T_u - t_0)^q}.$$

Since t_0 is arbitrary, we get (3.30).

Continuity of Θ on $\mathscr{K}$. Let us take a sequence $\{c_k(\cdot)\}$ in $\mathscr{K}$ such that

$$c_k(\cdot) \to c(\cdot) \in \mathscr{K} \quad \text{in } L^\infty_{\mathrm{loc}}([0, T_u)) \quad (k \to \infty),$$

and let $v_k(t, x)$ and $v(t, x)$ be the corresponding solutions to the linear Cauchy problem (3.9)–(3.10) with coefficients $c_k(t)$ and $c(t)$, respectively, with fixed data (u_0, u_1). Then it is sufficient to prove that the images $\tilde{c}_k(t) := \Theta(c_k(t))$ and $\tilde{c}(t) := \Theta(c(t))$ satisfy

$$\tilde{c}_k(\cdot) \to \tilde{c}(\cdot) \quad \text{in } L^\infty_{\mathrm{loc}}([0, T_u)) \quad (k \to \infty). \tag{3.31}$$

The functions $w_k := v_k - v$, $k = 1, 2, \ldots$, solve the following Cauchy problem:

$$\begin{cases} \partial_t^2 w_k - c(t)^2 \Delta w_k = \{c_k(t)^2 - c(t)^2\}\, \Delta v_k, & (t, x) \in (0, T_u) \times \mathbb{R}^n, \\ w_k(0, x) = 0, \quad \partial_t w_k(0, x) = 0, & x \in \mathbb{R}^n. \end{cases}$$

Differentiate the energy $\mathscr{E}(w_k(t))$ for w_k with respect to t, where

$$\mathscr{E}(w_k(t)) = \|\partial_t w_k(t)\|_{L^2}^2 + c(t)^2\|\nabla w_k(t)\|_{L^2}^2.$$

Then we get

$$\begin{aligned}\mathscr{E}'(w_k(t)) =& -2\left\{c_k(t)^2 - c(t)^2\right\} \mathrm{Re}\,(\Delta v_k(t), \partial_t w_k(t))_{L^2} \\ &+ 2c(t)c'(t)\,\|\nabla w_k(t)\|_{L^2}^2 \\ \le& 2\left|c_k(t)^2 - c(t)^2\right| \|v_k(t)\|_{\dot H^{3/2}}\|\partial_t w_k(t)\|_{\dot H^{1/2}} + 2\frac{|c'(t)|}{c(t)}\mathscr{E}(w_k(t)).\end{aligned} \tag{3.32}$$

Here, we see from (2.5) in Proposition 2.1 and assumption (3.13) on s and q that

$$\begin{aligned}&\|v_k(t)\|_{\dot H^{3/2}}\|\partial_t w_k(t)\|_{\dot H^{1/2}} \\ \le& M^2 e^{4M^2\max\{1,T_u^{1-(qs-s)}\}}\|((-\Delta)^{3/2}u_0, (-\Delta)^{1/2}u_1)\|^2_{\gamma^s_{\eta,L^2}\times\gamma^s_{\eta,L^2}} \\ =& M^2e^{4M^2}\|((-\Delta)^{3/2}u_0, (-\Delta)^{1/2}u_1)\|^2_{\gamma^s_{\eta,L^2}\times\gamma^s_{\eta,L^2}}\end{aligned}$$

for $0 \le t \le T_u$. Then we integrate (3.32) and apply Gronwall's lemma to obtain

$$\begin{aligned}\mathscr{E}(w_k(t)) \le 2M^2e^{4M^2}&\|((-\Delta)^{3/2}u_0, (-\Delta)^{1/2}u_1)\|^2_{\gamma^s_{\eta,L^2}\times\gamma^s_{\eta,L^2}} \times \\ &\left(\int_0^t \left|c_k(\tau)^2 - c(\tau)^2\right| d\tau\right) \exp\left(2\int_0^t \frac{|c'(\tau)|}{c(\tau)}\,d\tau\right)\end{aligned}$$

for $t \in [0, T_u)$, which implies that

$$\left.\begin{aligned}\nabla v_k(t) &\to \nabla v(t)\\ \partial_t v_k(t) &\to \partial_t v(t)\end{aligned}\right\} \quad \text{in } L^\infty_{\mathrm{loc}}([0,T_u);L^2) \text{ as } k\to\infty.$$

Hence we get (3.31), which proves the continuity of Θ.

We are now in a position to conclude the proof.

Let $(u_0, u_1) \in \langle D\rangle^{-3/2}\gamma^s_{\eta,L^2} \times \langle D\rangle^{-1/2}\gamma^s_{\eta,L^2}$ satisfy (3.13) and (3.24) for some $\eta > 4M^2$. Then the previous results assure that Θ is continuous from $\mathscr{K}$ into itself, provided that the data (u_0, u_1) is assumed to satisfy (3.16)–(3.17) from Proposition 3.4. Since $\mathscr{K}$ is the convex and compact subset of the Fréchet space $L^\infty_{\mathrm{loc}}([0,T_u))$, the Schauder-Tychonoff theorem implies that Θ has a fixed point in $\mathscr{K}$, and hence, we conclude that solution $v(t,x)$ to the linear Cauchy problem (3.9)–(3.10) with data (u_0,u_1) is also a solution $u(t,x)$ to the nonlinear Cauchy problem (1.2) with data (u_0,u_1) on $[0,T_u]$ satisfying

$$\limsup_{t\nearrow T_u} \mathscr{E}_{3/2}(u;t) < \infty.$$

Therefore, resorting to Proposition 3.2, we must have $T_u(u_0,u_1) = \infty$. This contradicts the assumption (3.16) in Proposition 3.4 that $(u_0,u_1) \in \mathcal{B}^s_{\eta,C_1}$. Therefore, for $(u_0, u_1) \in \langle D\rangle^{-3/2}\gamma^s_{\eta,L^2} \times \langle D\rangle^{-1/2}\gamma^s_{\eta,L^2}$ satisfying (3.13) and (3.24), we conclude that either assumption (3.16) or (3.17) is false. Namely, the class of

$$(u_0, u_1) \in \langle D\rangle^{-3/2}\gamma^s_{\eta,L^2} \times \langle D\rangle^{-1/2}\gamma^s_{\eta,L^2}$$

satisfying (3.13) and (3.24) leads to the following two alternatives: for any $\tilde{C} \in (0, \varepsilon_0)$, either

(i) $(u_0, u_1) \notin \mathcal{B}^s_{\eta,\tilde{C}}$,

or

(ii) $\left\| \left((-\Delta)^{3/4} u_0, (-\Delta)^{1/4} u_1 \right) \right\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} \geq \dfrac{(q-1)(\eta - 4M^2)}{2\tilde{C}^q M^2 e^{4M^2}}$,

where ε_0 is the constant given in (3.14). If the assertion (ii) is satisfied for a sequence of $\tilde{C}$ accumulating to 0, by letting $\tilde{C} \to 0$,

$$\left\| \left((-\Delta)^{3/4} u_0, (-\Delta)^{1/4} u_1 \right) \right\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} = \infty,$$

and hence, we conclude that $(u_0, u_1) \notin (-\Delta)^{-3/4} \gamma^s_{\eta,L^2} \times (-\Delta)^{-1/4} \gamma^s_{\eta,L^2}$. Therefore, (ii) never occurs in some interval $[0, \varepsilon_1)$ for some $0 < \varepsilon_1 < \varepsilon_0$. As a result, any pair of data $(u_0, u_1) \in \langle D \rangle^{-3/2} \gamma^s_{\eta,L^2} \times \langle D \rangle^{-1/2} \gamma^s_{\eta,L^2}$ fulfilling (3.13) and (3.24) satisfies only the assertion (i). Now, the assertion (i) implies that for $\varepsilon_1 > 0$, we have

$$T(u_0, u_1) > \varepsilon_1$$

for any $(u_0, u_1) \in \langle D \rangle^{-3/2} \gamma^s_{\eta,L^2} \times \langle D \rangle^{-1/2} \gamma^s_{\eta,L^2}$ satisfying (3.13) and (3.24). But this contradicts (3.4), and hence, we must have

$$\mathcal{B}^s_\eta \cap \left\{ 2\mathscr{H}(u; 0) \leq \frac{M^2}{4} - 1 \right\} = \emptyset. \tag{3.33}$$

We finally remove the condition $2\mathscr{H}(u; 0) \leq \frac{M^2}{4} - 1$. We note that throughout the argument, M was a fixed constant > 2. Suppose $\mathcal{B}^s_\eta \neq \emptyset$ so that there exists some $(u_0, u_1) \in \mathcal{B}^s_\eta$. Setting, for example, $M := \sqrt{8\mathscr{H}(u; 0) + 4} + 1$, we have $M > 2$ and also $2\mathscr{H}(u; 0) \leq \frac{M^2}{4} - 1$. Hence we also have $(u_0, u_1) \in \mathcal{B}^s_\eta \cap \{2\mathscr{H}(u; 0) \leq \frac{M^2}{4} - 1\}$ which contradicts (3.33). Thus we arrive at $\mathcal{B}^s_\eta = \emptyset$.

In conclusion, for any data $(u_0, u_1) \in \langle D \rangle^{-3/2} \gamma^s_{\eta,L^2} \times \langle D \rangle^{-1/2} \gamma^s_{\eta,L^2}$ satisfying (3.13) and (3.24), with $\eta > 4M^2$, $M > 2$, we have $T_u(u_0, u_1) = \infty$. Therefore, we conclude from Proposition 3.2 that the Cauchy problem (1.2) admits a unique solution u in the class

$$\bigcap_{j=0}^{1} C^j \left([0, \infty); \langle D \rangle^{-(3/2)+j} \gamma^s_{\eta,L^2} \right).$$

The proof of Theorem 3.5 is now finished. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Theorem 1.2 is an immediate consequence of Theorem 3.5 and Lemma A.1 in Appendix A. In fact, let $s > 1$, and suppose that there exists $\eta' > 0$ such that $(u_0, u_1) \in \langle D \rangle^{-3/2} \gamma^{s-\varepsilon}_{\eta',L^2} \times \langle D \rangle^{-1/2} \gamma^{s-\varepsilon}_{\eta',L^2}$ for some $0 < \varepsilon < s - 1$. Assume that

$$2\mathscr{H}(u; 0) \leq \frac{M^2}{4} - 1.$$

Then Lemma A.1 assures that $(u_0, u_1) \in \langle D\rangle^{-3/2}\gamma^s_{\eta,L^2} \times \langle D\rangle^{-1/2}\gamma^s_{\eta,L^2}$ for some $\eta > 4M^2$. Therefore, thanks to Theorem 3.5, the corresponding Gevrey class solution to (1.2) with data (u_0, u_1) exists globally on $[0,\infty)$, and belongs to the class

$$\bigcap_{j=0}^{1} C^j \left([0,\infty); \langle D\rangle^{-(3/2)+j}\gamma^s_{\eta,L^2}\right).$$

The uniqueness follows from Proposition 3.2. The proof of Theorem 1.2 is now complete. □

4. Initial-boundary value problem for the Kirchhoff equation

The argument in the proof of Theorem 1.2 is also applicable for the initial-boundary value problems in an open set Ω in $\mathbb{R}^n$. In this section we discuss the global existence for the initial-boundary value problem to the Kirchhoff equation in exterior domains and in bounded domains. The results in this section can be proved by the generalised Fourier transform method in exterior domains, and the Fourier series expansion method in bounded domains.

It is known from the spectral theorem that a self-adjoint operator on a separable Hilbert space is unitarily equivalent to a multiplication operator on some $L^2(\mathcal{M},\mu)$, where $(\mathcal{M},\mu)$ is a measure space. Then $L^2(\Omega)$ is unitarily equivalent to $L^2(\mathbb{R}^n)$. This means that the Fourier transform method in $\mathbb{R}^n$ is available for L^2 space on an open set Ω in $\mathbb{R}^n$; any multiplier acting on $L^2(\mathbb{R}^n)$ is unitarily transformed into an multiplier acting on $L^2(\Omega)$.

4.1. **The case: Ω is an exterior domain.** Replacing the Fourier transform over $\mathbb{R}^n$ by the generalised Fourier transform over exterior domains and applying exactly the same argument of Theorems 1.2, we can also prove a similar result for the initial-boundary value problem in exterior domains. More precisely, we consider the following problem:

$$\begin{cases} \partial_t^2 u - \left(1 + \displaystyle\int_\Omega |\nabla u(t,y)|^2\, dy\right)\Delta u = 0, & t>0, \quad x\in\Omega, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x\in\Omega, \\ u(t,x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Here, Ω is a domain of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is compact and its boundary $\partial\Omega$ is analytic. The latter assumption may be in principle relaxed but this would require an extension of known analytic solvability results to the Gevrey setting, so we omit it for this moment, and refer to [1] and [27] for further details.

Following Wilcox [43], let us define the generalised Fourier transforms in an arbitrary exterior domain Ω. Let A be a self-adjoint realisation of the Dirichlet Laplacian $-\Delta$ with domain $H^2(\Omega)\cap H^1_0(\Omega)$. Then A is non-negative on $L^2(\Omega)$, and we can define the square root $A^{1/2}$ of A. We recall the resolvent operators $R(|\xi|^2 \pm i0)$:

$$R(|\xi|^2 \pm i0) = \lim_{\varepsilon\to+0} (A - (|\xi|^2 \pm i\varepsilon))^{-1},$$

and $R(|\xi|^2 \pm i0)$ are bounded from $L^2(\Omega, \langle x\rangle^s dx)$ to $H^2(\Omega, \langle x\rangle^{-s} dx)$ for each $\xi \in \mathbb{R}^n$ and some $s > 1/2$, where $\langle x\rangle = (1+|x|)^{1/2}$ (see, e.g., Mochizuki [36]; note that this result does not require $\mathbb{R}^n \setminus \Omega$ to be star-shaped). Introducing a function $j = j(x) \in C^\infty(\mathbb{R}^n)$ vanishing in a neighbourhood of $\mathbb{R}^n \setminus \Omega$ and equal to one for large $|x|$, we define the generalised Fourier transforms as follows:

$$(\mathscr{F}_\pm f)(\xi) := \lim_{L\to\infty} (2\pi)^{-n/2} \int_{\Omega\cap\{|x|<L\}} \overline{\psi_\pm(x,\xi)} f(x)\, dx \quad \text{in} \quad L^2(\mathbb{R}^n),$$

where we put

$$\psi_\pm(x,\xi) = j(x)e^{ix\cdot\xi} + [R(|\xi|^2 \pm i0)M_\xi(\cdot)](x)$$
$$\text{with} \quad M_\xi(x) = -(A - |\xi|^2)(j(x)e^{ix\cdot\xi}).$$

Notice that we can write formally

$$M_\xi(x) = \{\Delta j(x) + 2i\xi\cdot\nabla j(x)\}e^{ix\cdot\xi}.$$

The kernels $\psi_\pm(x,\xi)$ are called eigenfunctions of the operator A with eigenvalue $|\xi|^2$ in the sense that, formally,

$$(A - |\xi|^2)\psi_\pm(x,\xi) = 0,$$

but $\psi_\pm(x,\xi) \notin L^2(\Omega)$. Similarly, the inverse transforms are defined by

$$(\mathscr{F}^*_\pm g)(x) = \lim_{L\to\infty} (2\pi)^{-n/2} \int_{\{|\xi|<L\}} \psi_\pm(x,\xi) g(\xi)\, d\xi \quad \text{in } L^2(\Omega).$$

We treat $\mathscr{F}_+ f$ only and drop the subscript $+$, since $\mathscr{F}_- f$ can be dealt with by essentially the same method. The transform $\mathscr{F} f$ thus defined obeys the following properties (see, e.g., Shenk II [42, Theorem 1 and Corollary 5.1]):

(i) $\mathscr{F}$ is a unitary mapping

$$\mathscr{F} : L^2(\Omega) \to L^2(\mathbb{R}^n).$$

Hence

$$\mathscr{F}\mathscr{F}^* = I.$$

(ii) $\mathscr{F}$ satisfies the generalised Parseval identity:

$$(\mathscr{F} f, \mathscr{F} g)_{L^2(\mathbb{R}^n)} = (f,g)_{L^2(\Omega)}, \quad f,g \in L^2(\Omega).$$

(iii) $\mathscr{F}$ diagonalizes the operator A in the sense that

$$\mathscr{F}(\varphi(A)f)(\xi) = \varphi(|\xi|^2)(\mathscr{F} f)(\xi),$$

where $\varphi(A)$ is the operator defined by the spectral representation theorem for self-adjoint operators.

We say that $f \in \gamma^s_{L^2}(\Omega)$ for $s \geq 1$ if and only if there exists $\eta > 0$ such that

$$\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} |(\mathscr{F} f)(\xi)|^2\, d\xi < \infty. \tag{4.2}$$

We denote by

$$H^\sigma(\Omega) = \langle D\rangle^{-\sigma} L^2(\Omega)$$

for $\sigma \in \mathbb{R}$ the Sobolev spaces over Ω, and $\langle D\rangle = (1 + A)^{1/2}$. Their homogeneous version is

$$\dot{H}^\sigma(\Omega) = A^{-\sigma/2} L^2(\Omega).$$

To state the results, we need to introduce the analytic compatibility condition.

The Gevrey compatibility condition. f satisfies the Gevrey compatibility condition if and only if $f \in \gamma^s_{L^2}(\Omega)$ satisfies

$$A^k f \in H^1_0(\Omega), \quad k = 0, 1, \cdots .$$

Based on the properties (i)–(iii) of the generalised Fourier transform, we have:

Theorem 4.1. *Assume that Ω is an exterior domain of $\mathbb{R}^n$ with analytic boundary such that $\mathbb{R}^n \setminus \Omega$ is compact. Let $s > 1$. Then for any $u_0, u_1 \in \gamma^s_{L^2}(\Omega)$ satisfying the Gevrey compatibility condition, the initial-boundary value problem (4.1) admits a unique solution*

$$u \in C^1\left([0,\infty); \gamma^{s'}_{L^2}(\Omega)\right)$$

for every $s' > s$.

In the previous known results of Sobolev well-posedness (for small data) on exterior problems, the differentiation of generalised Fourier transform with respect to spectral parameter is a powerful tool (see [20, 30, 31, 40]). There, some geometrical condition, say, star-shaped assumption on the set of obstacles, would be necessary for getting appropriate a priori estimates for solutions (see [20, 30, 31, 40]). Compared with the previous literature, Theorem 4.1 does not require any geometrical condition on Ω. This is because we need not differentiate the generalised Fourier transform of the data.

4.2. **The case: Ω is a bounded domain.** Replacing Fourier transform by Fourier series expansion and applying exactly the same argument of the proof of Theorem 1.2, we can prove a similar result for the initial-boundary value problem in $[0,\infty) \times \Omega$, where Ω is a bounded domain in $\mathbb{R}^n$ with analytic boundary $\partial\Omega$.

Let $\{w_k\}_{k=1}^\infty$ be a complete orthonormal system of eigenfunctions of the operator $-\Delta$ whose domain is $H^2(\Omega) \cap H^1_0(\Omega)$, and let λ_k be eigenvalues corresponding to w_k. Namely, $\{w_k, \lambda_k\}$ satisfy the elliptic equations:

$$\begin{cases} -\Delta w_k = \lambda_k w_k & \text{in } \Omega, \\ \quad w_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $(w_k, w_\ell)_{L^2(\Omega)} = \delta_{k\ell}$ and

$$0 < \lambda_1 \le \lambda_2 \le \cdots \lambda_k \le \cdots \quad \text{and } \lambda_k \to \infty,$$

where $(\phi, \psi)_{L^2(\Omega)}$ stands for the inner product of ϕ and ψ in $L^2(\Omega)$. We say that $f \in H^\sigma(\Omega)$ for real σ if

$$\sum_{k=1}^\infty \lambda_k^{2\sigma} \left|(f, w_k)_{L^2(\Omega)}\right|^2 < \infty,$$

and $f \in \gamma^s_{L^2}(\Omega)$ for $s \ge 1$ if and only if there exists $\eta > 0$ such that

$$\sum_{k=1}^\infty e^{\eta \lambda_k^{1/s}} \left|(f, w_k)_{L^2(\Omega)}\right|^2 < \infty.$$

Then we have:

Theorem 4.2. *Assume that Ω is a bounded domain with analytic boundary. Let $s > 1$. Then for any $u_0, u_1 \in \gamma^s_{L^2}(\Omega)$ satisfying the Gevrey compatibility condition, the initial-boundary value problem (4.1) admits a unique solution u in the class*

$$C^1\left([0,\infty); \gamma^{s'}_{L^2}(\Omega)\right)$$

for every $s' > s$.

Appendix A.

Here we briefly recall the inclusion relation among the classes $\gamma^s_{L^2}$.

Lemma A.1. *Let $s > s' \geq 1$. Then*

$$\gamma^{s'}_{\eta',L^2} \subsetneq \gamma^s_{\eta,L^2} \tag{A.1}$$

for every $\eta, \eta' > 0$.

Although this property is well known, we give a short proof for completeness.

Proof. Let us consider the characteristic function $\chi(\xi)$ on the set

$$\{\xi \in \mathbb{R}^n : |\xi| \geq (\eta/\eta')^\sigma\},$$

where σ is defined as $\frac{1}{\sigma} = \frac{1}{s'} - \frac{1}{s}$. Then we can estimate

$$\int_{\mathbb{R}^n} \chi(\xi) e^{\eta|\xi|^{1/s}} |\widehat{f}(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}^n} \chi(\xi) e^{\eta'|\xi|^{1/\sigma}|\xi|^{1/s}} |\widehat{f}(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}^n} e^{\eta'|\xi|^{1/s'}} |\widehat{f}(\xi)|^2 \, d\xi,$$

since $\eta \leq \eta'|\xi|^{1/\sigma}$ on the support of $\chi(\xi)$. On the support of $1 - \chi(\xi)$, we have $|\xi| \leq (\eta/\eta')^\sigma$, and hence,

$$\begin{aligned}\int_{\mathbb{R}^n} [1-\chi(\xi)] e^{\eta|\xi|^{1/s}} |\widehat{f}(\xi)|^2 \, d\xi \leq & \int_{\mathbb{R}^n} [1-\chi(\xi)] e^{\eta(\eta/\eta')^{(\sigma/s)}} |\widehat{f}(\xi)|^2 \, d\xi \\ \leq & e^{\eta(\eta/\eta')^{(\sigma/s)}} \int_{\mathbb{R}^n} e^{\eta'|\xi|^{1/s'}} |\widehat{f}(\xi)|^2 \, d\xi.\end{aligned}$$

Summarising the above estimates, we get

$$\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} |\widehat{f}(\xi)|^2 \, d\xi \leq \left\{1 + e^{\eta(\eta/\eta')^{(\sigma/s)}}\right\} \int_{\mathbb{R}^n} e^{\eta'|\xi|^{1/s'}} |\widehat{f}(\xi)|^2 \, d\xi,$$

which proves (A.1). ∎

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