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Abstract

We remark that there is no smooth flat function $f(x)$ on $[0, \infty)$ such that for a fixed interval $(0, \alpha)$ ($\alpha > 0$) on which the derivative of any higher order is positive.

We consider smooth functions on the half line $[0, \infty)$ which are flat at the origin, namely of class C^∞ and any derivative $f^{(n)}(x)$ converges to 0 when $x \to 0 + 0$. Eventually it is equivalent to say that f extends to the whole real line as a smooth function by defining $f(x) = 0$ for $x < 0$. In this tiny note we make a small remark on the asymptotics of higher derivatives around the origin.

Among non-tirivial flat functions the most well-known might be the one which is defined as follows.

$$f(0) = 0 \quad \text{and} \quad f(x) = e^{-\frac{1}{x}} \text{ for } x > 0$$

If we imagine its graph, of course it seems smooth enough, and it can be extended as constantly 0 on $(-\infty, 0]$ as a smooth function on the real line $\mathbb{R}$. Its first derivative is positive on $(0, \infty)$, but the second derivative vanishes at $x = \frac{1}{2} = x_2$ and the third vanishes at $x_3 = \frac{1-1/\sqrt{3}}{2} < x_2$, and so on. That is, setting $x_n = \min\{x\,;\, f^{(n)}(x) = 0,\ x > 0\}$ for $n = 2, 3, 4, \ldots$, it is clear that $\{x_n\}$ is strictly decreasing, and in fact $\lim_{n\to\infty} x_n = 0$. More over, if we fix any interval $[0, \alpha)$ ($\alpha > 0$), $f^{(n)}(x)$ tend to behave more and more wildly when $n \to \infty$ on the interval.

Also, if we take $g_0(x) = f(x)(\sin(\frac{1}{x}) + 1)$ and

$$g_n(x) = \int_0^x \int_0^{t_{n-1}} \cdots \int_0^{t_1} g_0(t_0) dt_0 \cdots dt_{n-2} dt_{n-1}\,,$$

then for $n = 1, 2, 3, \cdots$, $g_n(x)$ is positive on $(0, \infty)$ and is flat at $x = 0$, and apparently $g_n^{(k)}(x) > 0$ when $x > 0$ for $0 \leq k \leq n-1$ but there is no interval $(0, \alpha)$ on which $g_n^{(n)}(x)$ is positive.

They present some features of a general property of positive flat functions, which is formulated as follows.

Theorem 1 Let $f(x)$ be a smooth function on $[0, \infty)$ which is flat at $x = 0$. Then there does not exist positive real number α such that for any higher derivatives (*i.e.*, for $\forall n \in \mathbb{N}$), $f^{(n)}(x) > 0$ holds on $(0, \alpha]$.

Lemma 2 Let n be an integer and $g(x)$ be a function on $[0, 1]$ of class C^{n+1} with the following properties.

(1) $g^{(k)}(0) = 0$ for $k = 0, \dots, n$, and $g(1) = 1$,

(2) $g^{(n+1)}(x) > 0$ for $x > 0$.

Then $g(x) < x^n$ holds on $(0, 1)$.

Proof of Theorem. The theorem is easily deduced from the lemma by contradiction. Assume for some $\alpha > 0$ that $f(x)$ is smooth on $[0, \alpha]$, is flat at $x = 0$, and that its n-th derivative is positive on $(0, \alpha]$ for any $n \in \mathbb{N}$. We adjust the function f into $g(x) = f(\alpha)^{-1} f(\alpha x)$. Then $g(x)$ satisfies the condition of the lemma for any $n \in \mathbb{N}$. Therefore $g(x) \equiv 0$ on $[0, 1)$, and we obtained a contradiction. □

Proof of Lemma. It is enough to show that $g(x)/x^n$ is increasing on $[0, 1]$. As $\dfrac{d}{dx}\left(\dfrac{g(x)}{x^n}\right) = \dfrac{xg'(x) - ng(x)}{x^{n+1}}$, it is also sufficient to show that the numerator $xg'(x) - ng(x)$ is positive on $(0, 1)$.

Then because $(xg'(x) - ng(x))^{(n)} = xg^{(n+1)}(x)$ is positive on $(0, 1]$ from our condition, we see successively that each k-th derivative $(xg'(x) - ng(x))^{(k)} = xg^{(k+1)} - (n - k)g^{(k)}(x)$ vanishes at $x = 0$ and therefore is positive on $(0, 1]$ for $k = n - 1, n - 2, \dots, k = 0$. This completes the proof. □

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