CHUO MATH NO.117(2016)

Besov spaces on open sets

by
Tsukasa Iwabuchi, Tokio Matsuyama
and Koichi Taniguchi

DEPARTMENT OF MATHEMATICS
CHUO UNIVERSITY
BUNKYOKU TOKYO JAPAN

Feb. 18 , 2016

BESOV SPACES ON OPEN SETS

TSUKASA IWABUCHI, TOKIO MATSUYAMA AND KOICHI TANIGUCHI

Abstract. This paper is devoted to giving definitions of Besov spaces on an arbitrary open set of $\mathbb{R}^n$ via the spectral theorem for the Schrödinger operator with the Dirichlet boundary condition. The crucial point is to introduce some test function spaces on Ω. The fundamental properties of Besov spaces are also shown, such as embedding relations and duality, etc. Furthermore, the isomorphism relations are established among the Besov spaces in which regularity of functions is measured by the Dirichlet Laplacian and the Schrödinger operators.

1. Introduction

In 1959–61 Besov introduced the Besov spaces in his papers [2, 3]. There are a lot of literatures on characterization of Besov spaces, and we refer to the books of Triebel [37, 38, 40] for history of Besov spaces. It was by Peetre that the Fourier transform was employed to study the Besov spaces on $\mathbb{R}^n$ (see [27–29], and also Frazier and Jawerth [11, 12]). On a general domain, if the boundary is bounded and smooth, the theory of Besov spaces is well established by extending functions on the domain to those on $\mathbb{R}^n$. Otherwise, the situation is quite different as is indicated in previous studies (see e.g. [37, 39]), and there appear to be considerable difficulties to construct such theory.

Let Ω be an open set of $\mathbb{R}^n$ with $n \geq 1$. Our aim is to define the Besov spaces on Ω based on the spectral theory by referring to Peetre's idea. If the boundary $\partial\Omega$ of Ω is smooth, then some basic notions are available; the restriction method of the function on $\mathbb{R}^n$ to Ω, the zero extension to the outside of Ω, and certain intrinsic characterization (see [25, 32, 33, 36–41]). Recently, Bui, Duong and Yan introduced some test function spaces to define the Besov spaces $\dot{B}^s_{p,q}$ generated by the Dirichlet Laplacian on an arbitrary open set, where s, p and q satisfy $|s| < 1$ and $1 \leq p, q \leq \infty$ (see [4]). They also proved the equivalence relation among the Besov spaces generated by the Laplacian and some operators, including the Schrödinger operators, on the whole space $\mathbb{R}^n$, $n \geq 3$ with some additional conditions such as Hölder continuity for the kernel of semi-group generated by them. As to the results on the Besov spaces generated by the elliptic operators on manifolds, or Hermite operators, we refer to [1, 4–7, 10, 23] and the references therein. To the best of our knowledge, it is necessary to impose some smoothness assumptions on the boundary $\partial\Omega$ in order to define the

1991 *Mathematics Subject Classification.* Primary 30H25; Secondary 81Q10, 46F05;

Key words and phrases. Besov spaces, Schrödinger operators, potential of Kato class.

The first author was supported by Grant-in-Aid for Young Scientists Research (B) (No. 25800069), Japan Society for the Promotion of Science. The second author was supported by Grant-in-Aid for Scientific Research (C) (No. 15K04967), Japan Society for the Promotion of Science.

Besov spaces $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ with all indices s, p, q satisfying $s \in \mathbb{R}$ and $1 \le p, q \le \infty$.

In this paper we shall define the Besov spaces $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ generated by the Schrödinger operator $-\Delta + V$ with the Dirichlet boundary condition for all indices s, p, q *without any geometrical and smoothness assumption on the boundary* $\partial\Omega$, and shall prove the fundamental properties such as embedding relations and lifting, etc. Furthermore, regarding the Besov spaces generated by the Dirichlet Laplacian as the standard one, and adopting the potential V belonging to the Lorentz space $L^{\frac{n}{2},\infty}(\Omega)$, we shall establish the equivalence relation between the Besov spaces generated by the Dirichlet Laplacian $-\Delta_{|D}$ and Schrödinger operator $-\Delta_{|D} + V$. The motivation of the study of such properties and equivalence relation comes from their applications to partial differential equations, and one can consult the papers of D'Ancona and Pierfelice (see [9]), Georgiev and Visciglia (see [14]) and Jensen and Nakamura (see [21, 22]).

Let us consider the Schrödinger operator

$$-\Delta + V(x) = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + V(x)$$

on an arbitrary open set Ω with the Dirichlet boundary condition, where $V(x)$ is a real-valued measurable function on Ω. In this paper we adopt potentials whose negative parts belong to the Kato class. More precisely, let us assume that the potential V satisfies

$$V = V_+ - V_-, \quad V_\pm \ge 0, \quad V_+ \in L^1_{\mathrm{loc}}(\Omega) \text{ and } V_- \in K_n(\Omega). \tag{1.1}$$

Here, the negative part V_- of V is said to belong to the Kato class $K_n(\Omega)$ if V_- satisfies

$$\begin{cases} \displaystyle\lim_{r\to 0} \sup_{x\in\Omega} \int_{\Omega\cap\{|x-y|<r\}} \frac{|V_-(y)|}{|x-y|^{n-2}}\, dy = 0, & n \ge 3, \\ \displaystyle\lim_{r\to 0} \sup_{x\in\Omega} \int_{\Omega\cap\{|x-y|<r\}} \log(|x-y|^{-1})|V_-(y)|\, dy = 0, & n = 2, \\ \displaystyle\sup_{x\in\Omega} \int_{\Omega\cap\{|x-y|<1\}} |V_-(y)|\, dy < \infty, & n = 1. \end{cases}$$

Then $-\Delta + V$ has a self-adjoint realization on $L^2(\Omega)$ (see Lemma A.2 in appendix A). Throughout this paper, we use the following notation:

Notation. *We denote by A_V the self-adjoint realization of $-\Delta + V$ with the domain*

$$\mathcal{D}(A_V) = \big\{ f \in H^1_0(\Omega) \,\big|\, \sqrt{V_+} f,\ A_V f \in L^2(\Omega) \big\}, \tag{1.2}$$

where $H^1_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$\|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)}.$$

By the spectral theorem there exists a spectral resolution $\{E_{A_V}(\lambda)\}_{\lambda\in\mathbb{R}}$ of the identity and A_V is written as

$$A_V = \int_{-\infty}^{\infty} \lambda \, dE_{A_V}(\lambda).$$

For a Borel measurable function $\phi(\lambda)$ on $\mathbb{R}$, $\phi(A_V)$ is defined by letting

$$\phi(A_V) = \int_{-\infty}^{\infty} \phi(\lambda) \, dE_{A_V}(\lambda)$$

with the domain

$$\mathcal{D}(\phi(A_V)) = \left\{ f \in L^2(\Omega) \,\Big|\, \int_{-\infty}^{\infty} |\phi(\lambda)|^2 d\|E_{A_V}(\lambda)(f)\|^2_{L^2(\Omega)} < \infty \right\}.$$

Due to such a spectral resolution, we can define the Sobolev spaces $H^s(A_V)$ by letting

$$H^s(A_V) = \big\{ f \in L^2(\Omega) \,\big|\, (I + A_V)^{\frac{s}{2}} f \in L^2(\Omega) \big\} \quad \text{for } s \geq 0. \tag{1.3}$$

Then, the regularity and boundary value of functions in $H^s(A_V)$ are determined by A_V. General approach to $H^s(A_V)$ on bounded open sets is discussed in Ruzhansky and Tokmagambetov [31], where the operator does not have to be self-adjoint. Hereafter, we call $H^s(A_V)$ *the Sobolev spaces by A_V-regularity.* In particular case $\Omega = \mathbb{R}^n$ and $V = 0$, the Sobolev spaces defined in (1.3) coincide with the Bessel-potential spaces defined via the Fourier transform.

We shall apply the above characterization of $H^s(A_V)$ to those of the inhomogeneous and homogeneous Besov spaces (see Theorem 2.5 below). For the Besov spaces by this characterization, we obtain fundamental properties of the spaces (see Propositions 3.1–3.4 below) and find a sufficient condition on the integrability of V such that the isomorphism holds between the Besov spaces by A_0 and A_V-regularity (see Proposition 3.5 below). *It should be noted that our framework on open sets Ω of $\mathbb{R}^n$ is the most general setting. The crucial point is to introduce test function spaces on Ω.*

Let us recall the definitions of the test function spaces on $\mathbb{R}^n$ and the classical Besov spaces, i.e., spaces when $\Omega = \mathbb{R}^n$ and $V = 0$. It is well known that the inhomogeneous Besov spaces and homogeneous ones are characterized as subspaces of $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{Z}'(\mathbb{R}^n)$ by the Littlewood-Paley dyadic decomposition of the spectrum of $\sqrt{-\Delta}$, namely, $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ consist of all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}} = \big\|\mathcal{F}^{-1}\psi(|\xi|)\mathcal{F}f\big\|_{L^p(\mathbb{R}^n)} + \Big\|\big\{2^{sj}\|\mathcal{F}^{-1}\phi(2^{-j}|\xi|)\mathcal{F}f\|_{L^p(\mathbb{R}^n)}\big\}_{j\in\mathbb{N}}\Big\|_{\ell^q(\mathbb{N})} < \infty,$$

$$\|f\|_{\dot{B}^s_{p,q}} = \Big\|\big\{2^{sj}\|\mathcal{F}^{-1}\phi(2^{-j}|\xi|)\mathcal{F}f\|_{L^p(\mathbb{R}^n)}\big\}_{j\in\mathbb{Z}}\Big\|_{\ell^q(\mathbb{Z})} < \infty,$$

respectively, for some smooth functions ψ, ϕ with compact supports. Here $\mathcal{S}'(\mathbb{R}^n)$ is the space of the tempered distributions on $\mathbb{R}^n$, which is the topological dual of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The space $\mathcal{S}(\mathbb{R}^n)$ consists of rapidly decreasing functions equipped with the family of semi-norms

$$\sup_{x\in\mathbb{R}^n} (1 + |x|^2)^{\frac{M}{2}} \sum_{|\alpha|\leq M} |\partial_x^\alpha f(x)|, \quad M = 1, 2, \cdots. \tag{1.4}$$

$\mathcal{Z}'(\mathbb{R}^n)$ is the dual space of $\mathcal{Z}(\mathbb{R}^n)$, which is the subspace of $\mathcal{S}(\mathbb{R}^n)$ defined by letting

$$\mathcal{Z}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} x^\alpha f(x)dx = 0 \ \text{ for all } \alpha \in (\mathbb{N} \cup \{0\})^n \Big\} \tag{1.5}$$

endowed with the induced topology of $\mathcal{S}(\mathbb{R}^n)$. It is known that $\mathcal{Z}'(\mathbb{R}^n)$ is characterized by the quotient space of $\mathcal{S}'(\mathbb{R}^n)$ modulo polynomials, i.e.,

$$\mathcal{Z}'(\mathbb{R}^n) \simeq \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n),$$

where $\mathcal{P}(\mathbb{R}^n)$ is the set of all polynomials of n real variables (see e.g. [16, 37]).

Now, when $\Omega \neq \mathbb{R}^n$, a question naturally arises what the spaces corresponding to $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{Z}'(\mathbb{R}^n)$ are. We introduce such a kind of spaces as $\mathcal{X}'_V(\Omega)$ and $\mathcal{Z}'_V(\Omega)$ in §§2.1. There we will encounter with two problems in the formulations:

(a) To handle the neighborhood of zero spectrum in the definition of the homogeneous Besov spaces;
(b) To develop the dyadic resolution of identity operators on our spaces $\mathcal{X}'_V(\Omega)$ and $\mathcal{Z}'_V(\Omega)$; dyadic resolution lifted from $L^2(\Omega)$.

Let us explain the problem (a). Looking at the definition (1.5) of $\mathcal{Z}(\mathbb{R}^n)$, one understands that the low frequency part of f is treated by

$$\int_{\mathbb{R}^n} x^\alpha f(x)dx = 0 \quad \text{for any } \alpha \in (\mathbb{N} \cup \{0\})^n. \tag{1.6}$$

However, when $\Omega \neq \mathbb{R}^n$ it seems difficult to get an idea corresponding to (1.6). To overcome this difficulty, instead of (1.6), we propose

$$\sup_{j \le 0} 2^{M|j|} \|\phi_j(\sqrt{A_V})f\|_{L^1(\Omega)} < \infty, \quad M = 1, 2, \cdots \tag{1.7}$$

in (2.4) below, where we put $\phi_j(\sqrt{A_V}) := \phi(2^{-j}\sqrt{A_V})$. This is probably a main novelty in our work. The condition (1.7) seems one of important ingredients to introduce test function spaces for not only Besov spaces but also other ones of homogeneous type.

We turn to explain the problem (b). For the sake of simplicity, let us consider the case when $V = 0$. Clearly, in this case, A_V becomes the Dirichlet Laplacian A_0. As is well-known, the identity operator is resolved by the dyadic decomposition of the spectrum for the Dirichlet Laplacian in $L^2(\Omega)$, namely,

$$I = \psi(A_0) + \sum_{j \in \mathbb{N}} \phi_j(\sqrt{A_0}), \tag{1.8}$$

which is assured by the spectral theorem, where ψ is a smooth function such that

$$\psi(\lambda^2) + \sum_{j \in \mathbb{N}} \phi_j(\lambda) = 1 \quad \text{for any } \lambda \ge 0.$$

Initially, the resolution (1.8) holds in $L^2(\Omega)$, and then, it is lifted to the space $\mathcal{X}'_0(\Omega)$. This argument is accomplished in Lemma 4.5 below. When one considers $\mathcal{Z}'_0(\Omega)$, (1.8) is replaced by

$$I = \sum_{j\in\mathbb{Z}} \phi_j(\sqrt{A_0}). \tag{1.9}$$

Thanks to these resolutions (1.8) and (1.9), the well known methods in the classical Besov spaces on $\mathbb{R}^n$ work well also in the present case. The starting point of this argument is to extend the spectral restriction operators $\phi_j(\sqrt{A_0})$ on $L^2(\Omega)$ to those on $L^1(\Omega)$. There, the uniform boundedness on $L^1(\Omega)$ of $\{\phi_j(\sqrt{A_0})\}_j$, i.e.,

$$\sup_j \|\phi_j(\sqrt{A_0})\|_{L^1(\Omega)\to L^1(\Omega)} < \infty \tag{1.10}$$

plays a crucial role in proving (1.8) in $\mathcal{X}_0'(\Omega)$ and (1.9) in $\mathcal{Z}_0'(\Omega)$, respectively. For the proof, see Proposition A.1 in appendix A (see also [19,20]). Furthermore, (1.10) guarantees the independence of the choice of $\{\phi_j\}_{j\in\mathbb{Z}} \cup \{\psi\}$, when we define spaces $\mathcal{X}_0(\Omega)$, $\mathcal{X}_0'(\Omega)$, $\mathcal{Z}_0(\Omega)$, $\mathcal{Z}_0'(\Omega)$ and Besov spaces defined in §2.

This paper is organized as follows. In §2, we state a main result on the Besov spaces by A_V-regularity. §3 is devoted to stating some fundamental properties of Besov spaces. In §4, we introduce key lemmas and fundamental properties of test function spaces on Ω, which are essential for our theory. §5–§9 are devoted to the proof of our results. In appendix A, we show the uniform L^p-boundedness of $\phi(\theta A_V)$, the self-adjointness of A_V and the pointwise estimate for the kernel of e^{-tA_V} which are verified with some modifications of our previous work [19]. Finally, we prove in appendix B that zero is not an eigenvalue of A_V under some smallness condition on the negative part of V.

2. Statement of results

In this section we shall state several results on the Besov spaces by A_V-regularity. We divide this section into two subsections: the introduction of test function spaces, and statement of the result.

2.1. Definitions of test function spaces on Ω.

In this subsection we shall define "test function spaces" consisting of functions smooth and integrable on Ω and spaces of a kind of "tempered distributions" as follows:

Let $\phi_0(\cdot) \in C_0^\infty(\mathbb{R})$ be a non-negative function on $\mathbb{R}$ such that

$$\operatorname{supp}\phi_0 \subset \{\,\lambda\in\mathbb{R} \,|\, 2^{-1} \le \lambda \le 2\,\}, \quad \sum_{j\in\mathbb{Z}} \phi_0(2^{-j}\lambda) = 1 \quad \text{for } \lambda > 0, \tag{2.1}$$

and $\{\phi_j\}_{j\in\mathbb{Z}}$ is defined by letting

$$\phi_j(\lambda) = \phi_0(2^{-j}\lambda) \quad \text{for } \lambda\in\mathbb{R}. \tag{2.2}$$

Definition 2.1. (i) (*Linear topological spaces $\mathcal{X}_V(\Omega)$ and $\mathcal{X}_V'(\Omega)$*). *Assume that the measurable potential V satisfies* (1.1). *Then a linear topological space $\mathcal{X}_V(\Omega)$ is defined by letting*

$$\mathcal{X}_V(\Omega) := \big\{ f \in L^1(\Omega)\cap\mathcal{D}(A_V) \,\big|\, A_V^M f \in L^1(\Omega)\cap\mathcal{D}(A_V) \text{ for all } M\in\mathbb{N} \big\}$$

equipped with the family of semi-norms $\{p_{V,M}(\cdot)\}_{M=1}^\infty$ *given by*

$$p_{V,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{N}} 2^{Mj}\|\phi_j(\sqrt{A_V})f\|_{L^1(\Omega)}.$$

$\mathcal{X}'_V(\Omega)$ *denotes the topological dual of* $\mathcal{X}_V(\Omega)$ *and* ${}_{\mathcal{X}'_V}\langle f,g\rangle_{\mathcal{X}_V}$ *is the duality pair of* $f\in\mathcal{X}'_V(\Omega)$ *and* $g\in\mathcal{X}_V(\Omega)$. *A sequence* $\{f_N\}_{N=1}^\infty$ *in* $\mathcal{X}'_V(\Omega)$ *is said to converge to* $f\in\mathcal{X}'_V(\Omega)$ *if*

$${}_{\mathcal{X}'_V}\langle f_N,g\rangle_{\mathcal{X}_V} \to {}_{\mathcal{X}'_V}\langle f,g\rangle_{\mathcal{X}_V} \quad \text{as } N\to\infty \quad \text{for any } g\in\mathcal{X}_V(\Omega).$$

(ii) (*Linear topological spaces* $\mathcal{Z}_V(\Omega)$ *and* $\mathcal{Z}'_V(\Omega)$). *Assume that the measurable potential* V *satisfies* (1.1) *and*

$$\begin{cases} V_- = 0 & \text{if } n=1,2,\\ \displaystyle\sup_{x\in\Omega}\int_\Omega \frac{|V_-(y)|}{|x-y|^{n-2}}\,dy < \frac{\pi^{\frac{n}{2}}}{\Gamma(n/2-1)} & \text{if } n\geq 3. \end{cases} \tag{2.3}$$

Then a linear topological space $\mathcal{Z}_V(\Omega)$ *is defined by letting*

$$\mathcal{Z}_V(\Omega) := \Big\{ f\in\mathcal{X}_V(\Omega)\ \Big|\ \sup_{j\leq 0} 2^{M|j|}\big\|\phi_j\big(\sqrt{A_V}\big)f\big\|_{L^1(\Omega)} < \infty \text{ for all } M\in\mathbb{N}\Big\}$$

equipped with the family of semi-norms $\{q_{V,M}(\cdot)\}_{M=1}^\infty$ *given by*

$$q_{V,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{Z}} 2^{M|j|}\|\phi_j(\sqrt{A_V})f\|_{L^1(\Omega)}. \tag{2.4}$$

$\mathcal{Z}'_V(\Omega)$ *denotes the topological dual of* $\mathcal{Z}_V(\Omega)$ *and* ${}_{\mathcal{Z}'_V}\langle f,g\rangle_{\mathcal{Z}_V}$ *is the duality pair of* $f\in\mathcal{Z}'_V(\Omega)$ *and* $g\in\mathcal{Z}_V(\Omega)$. *A sequence* $\{f_N\}_{N=1}^\infty$ *in* $\mathcal{Z}'_V(\Omega)$ *is said to converge to* $f\in\mathcal{Z}'_V(\Omega)$ *if*

$${}_{\mathcal{Z}'_V}\langle f_N,g\rangle_{\mathcal{Z}_V} \to {}_{\mathcal{Z}'_V}\langle f,g\rangle_{\mathcal{Z}_V} \quad \text{as } N\to\infty \quad \text{for any } g\in\mathcal{Z}_V(\Omega).$$

(iii) (*Spaces generated by the Dirichlet Laplacian*). *In particular case* $V=0$, *we write* $\mathcal{X}_V(\Omega)$, $\mathcal{X}'_V(\Omega)$, $\mathcal{Z}_V(\Omega)$ *and* $\mathcal{Z}'_V(\Omega)$ *as*

$$\mathcal{X}_0(\Omega),\quad \mathcal{X}'_0(\Omega),\quad \mathcal{Z}_0(\Omega)\quad \text{and}\quad \mathcal{Z}'_0(\Omega),$$

respectively.

We notice from assumption (2.3) that A_V is non-negative on $L^2(\Omega)$ and that zero is not an eigenvalue of A_V as well as the Dirichlet Laplacian. In fact, these results hold for a more general assumption (B.1) in appendix B. We also note that assumption (2.3) excludes the potential V like

$$V(x) = -c|x|^{-2}, \quad c>0.$$

For more details, see the remark after the statement of Proposition A.1 in appendix A.

Functions in the Lebesgue spaces are regarded as elements in $\mathcal{X}'_V(\Omega)$ and $\mathcal{Z}'_V(\Omega)$ analogously to the case for $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{Z}'(\mathbb{R}^n)$, respectively. Lemma 4.6 below assures that

$$\int_\Omega \big|f(x)\overline{g(x)}\big|\,dx < \infty$$

for any $f\in L^p(\Omega)$, $1\leq p\leq\infty$, and $g\in\mathcal{X}_V(\Omega)$ ($g\in\mathcal{Z}_V(\Omega)$ resp.). So, we define:

Definition 2.2. *For $f \in L^1(\Omega) + L^\infty(\Omega)$, we identify f as an element in $\mathcal{X}_V'(\Omega)$ ($\mathcal{Z}_V'(\Omega)$ resp.) by letting*

$$
{}_{\mathcal{X}_V'}\langle f, g\rangle_{\mathcal{X}_V} = \int_\Omega f(x)\overline{g(x)}\,dx \quad \left({}_{\mathcal{Z}_V'}\langle f, g\rangle_{\mathcal{Z}_V} = \int_\Omega f(x)\overline{g(x)}\,dx \quad \text{resp.}\right)
$$

for any $g \in \mathcal{X}_V(\Omega)$ ($\mathcal{Z}_V(\Omega)$ resp.).

For a mapping $\phi(A_V)$ on $\mathcal{X}_V(\Omega)$ ($\mathcal{Z}_V(\Omega)$ resp.), we define the dual operator of $\phi(A_V)$ on $\mathcal{X}_V'(\Omega)$ ($\mathcal{Z}_V'(\Omega)$ resp.) induced naturally from that on $L^2(\Omega)$.

Definition 2.3. (i) *For a mapping $\phi(A_V) : \mathcal{X}_V(\Omega) \to \mathcal{X}_V(\Omega)$, we define $\phi(A_V) : \mathcal{X}_V'(\Omega) \to \mathcal{X}_V'(\Omega)$ by letting*

$$
{}_{\mathcal{X}_V'}\big\langle \phi(A_V)f, g\big\rangle_{\mathcal{X}_V} := {}_{\mathcal{X}_V'}\big\langle f, \phi(A_V)g\big\rangle_{\mathcal{X}_V} \quad \textit{for all } g \in \mathcal{X}_V(\Omega). \tag{2.5}
$$

(ii) *For a mapping $\phi(A_V) : \mathcal{Z}_V(\Omega) \to \mathcal{Z}_V(\Omega)$, we define $\phi(A_V) : \mathcal{Z}_V'(\Omega) \to \mathcal{Z}_V'(\Omega)$ by letting*

$$
{}_{\mathcal{Z}_V'}\big\langle \phi(A_V)f, g\big\rangle_{\mathcal{Z}_V} := {}_{\mathcal{Z}_V'}\big\langle f, \phi(A_V)g\big\rangle_{\mathcal{Z}_V} \quad \textit{for all } g \in \mathcal{Z}_V(\Omega). \tag{2.6}
$$

It is shown in Lemma 4.2 below that $\mathcal{X}_V(\Omega)$ and $\mathcal{Z}_V(\Omega)$ are complete, and hence, they are Fréchet spaces. Needless to say, it is not possible to define an operator $\sqrt{A_V}$ if the spectrum of A_V contains negative real numbers. However, since $\phi_j(\lambda) = 0$ for $\lambda \le 0$, we define $\phi_j(\sqrt{A_V})$ as

$$
\phi_j(\sqrt{A_V}) = \int_0^\infty \phi_j(\sqrt{\lambda})dE_{A_V}(\lambda).
$$

Let us give a few remarks on properties of $\mathcal{X}_0(\Omega)$ and $\mathcal{Z}_0(\Omega)$ as follows:

- When $\Omega = \mathbb{R}^n$ and $V = 0$, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is contained in $\mathcal{X}_0(\mathbb{R}^n)$, and the inclusion for tempered distributions are just opposite. Namely, it can be readily checked from Definition 2.1 that

$$
\begin{gathered}
\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{X}_0(\mathbb{R}^n) \hookrightarrow \mathcal{X}_0'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \\
\mathcal{Z}(\mathbb{R}^n) \hookrightarrow \mathcal{Z}_0(\mathbb{R}^n) \hookrightarrow \mathcal{Z}_0'(\mathbb{R}^n) \hookrightarrow \mathcal{Z}'(\mathbb{R}^n), \\
C_0^\infty(\mathbb{R}^n) \subset \mathcal{X}_0(\mathbb{R}^n), \quad C_0^\infty(\mathbb{R}^n) \not\subset \mathcal{Z}_0(\mathbb{R}^n).
\end{gathered} \tag{2.7}
$$

- When $\Omega = \mathbb{R}^n$ and $V = 0$, the restriction of low frequency in the definition (2.4) of $q_{0,M}(f)$ is natural, since one can show that any element $f \in \mathcal{S}(\mathbb{R}^n)$ belongs to $\mathcal{Z}(\mathbb{R}^n)$ if and only if $q_{0,M}(f) < \infty$ for $M = 1, 2, \dots$.
- When $\Omega \ne \mathbb{R}^n$, any $f \in \mathcal{X}_0(\Omega)$ or $\mathcal{Z}_0(\Omega)$ satisfies

$$
f \equiv 0 \quad \text{on } \partial\Omega,
$$

since $f \in H_0^1(\Omega)$. Hence, the condition $p_{0,M}(f) < \infty$ not only determines smoothness and integrability of f but also assures the Dirichlet boundary condition. Also, such an f contacts with $\partial\Omega$ of order infinity in the following way:

$$
A_0^M f \equiv 0 \quad \text{on } \partial\Omega, \quad M = 0, 1, 2, \cdots.
$$

The same assertion holds for $\mathcal{X}_V(\Omega)$, $\mathcal{Z}_V(\Omega)$ and A_V.
- In order to simplify the argument, instead of the polynomial weights appearing on semi-norms (1.4) in $\mathcal{S}(\mathbb{R}^n)$, we adopted the integrability condition on f.

Based on Definitions 2.1–2.3, we establish the definition of Besov spaces on an arbitrary open set of $\mathbb{R}^n$ in §§2.2.

2.2. **Statement of the main result.** We are in this subsection to state the result. Let $\{\phi_j\}_{j\in\mathbb{Z}} \cup \{\psi\}$ be the Littlewood-Paley dyadic decomposition, namely, the sequence $\{\phi_j\}_{j\in\mathbb{Z}}$ is defined by (2.2), and ψ is a smooth function with compact support around the origin. Here, we note that if V satisfies assumption (1.1), then the spectrum of A_V may admit to be negative. It is shown in Lemma A.2 in appendix A that there exists a positive constant λ_0 such that

$$A_V \geq -\lambda_0^2 I. \tag{2.8}$$

Then we need to choose the function ψ such that

$$\psi(\lambda) = 1 \quad \text{for } \lambda \in [-\lambda_0^2, 0], \qquad \psi(\lambda^2) + \sum_{j\in\mathbb{N}} \phi_j(\lambda) = 1 \quad \text{for } \lambda \geq 0. \tag{2.9}$$

Based on this choice of ψ, let us introduce the definition of the Besov spaces by A_V-regularity.

Definition 2.4. *For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we define the inhomogeneous and homogeneous Besov spaces as follows*:

(i) $B^s_{p,q}(A_V)$ *is defined by letting*

$$B^s_{p,q}(A_V) := \{f \in \mathcal{X}_V'(\Omega) \mid \|f\|_{B^s_{p,q}(A_V)} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}(A_V)} := \|\psi(A_V)f\|_{L^p(\Omega)} + \big\|\{2^{sj}\|\phi_j(\sqrt{A_V})f\|_{L^p(\Omega)}\}_{j\in\mathbb{N}}\big\|_{\ell^q(\mathbb{N})}.$$

(ii) $\dot{B}^s_{p,q}(A_V)$ *is defined by letting*

$$\dot{B}^s_{p,q}(A_V) := \{f \in \mathcal{Z}_V'(\Omega) \mid \|f\|_{\dot{B}^s_{p,q}(A_V)} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(A_V)} := \big\|\{2^{sj}\|\phi_j(\sqrt{A_V})f\|_{L^p(\Omega)}\}_{j\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})}.$$

Our main result can now be formulated in the following way:

Theorem 2.5. *For any s, p, q with $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the following assertions hold*:

(i) (*Inhomogeneous Besov spaces*) *Assume that the measurable potential V satisfies* (1.1). *Then*:

(i-a) $B^s_{p,q}(A_V)$ *is independent of the choice of $\{\psi\} \cup \{\phi_j\}_{j\in\mathbb{N}}$ satisfying* (2.1), (2.2) *and* (2.9), *and enjoys the following*:

$$\mathcal{X}_V(\Omega) \hookrightarrow B^s_{p,q}(A_V) \hookrightarrow \mathcal{X}_V'(\Omega). \tag{2.10}$$

(i-b) $B^s_{p,q}(A_V)$ *is the Banach space.*

(ii) (*Homogeneous Besov spaces*) *Assume that the measurable potential V satisfies* (1.1) *and* (2.3). *Then*:

(ii-a) $\dot{B}^s_{p,q}(A_V)$ *is independent of the choice of* $\{\phi_j\}_{j\in\mathbb{Z}}$ *satisfying* (2.1) *and* (2.2), *and enjoys the following*:

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^s_{p,q}(A_V) \hookrightarrow \mathcal{Z}'_V(\Omega). \tag{2.11}$$

(ii-b) $\dot{B}^s_{p,q}(A_V)$ *is the Banach space.*

Let us give a remark on the theorem. It is meaningful to consider the space $\mathcal{X}'_V(\Omega)$ ($\mathcal{Z}'_V(\Omega)$ resp.), when one defines the spaces $B^s_{p,q}(A_V)$ ($\dot{B}^s_{p,q}(A_V)$ resp.). In fact, when $\Omega = \mathbb{R}^n$ and $V = 0$, we see from (2.7) that

$$C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{X}_0(\mathbb{R}^n) \hookrightarrow \mathcal{X}'_0(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in the classical Besov spaces $B^s_{p,q}$ for $s \in \mathbb{R}$ and $1 \le p, q < \infty$, $B^s_{p,q}$ as subspaces of $\mathcal{X}'_0(\mathbb{R}^n)$ are isomorphic to those as subspaces of $\mathcal{S}'(\mathbb{R}^n)$. Similarly, $\dot{B}^s_{p,q}$ as subspaces of $\mathcal{Z}'_0(\mathbb{R}^n)$ are isomorphic to those as subspaces of $\mathcal{Z}'(\mathbb{R}^n)$.

3. Dual spaces, embedding relations, lifting properties and isomorphic properties

In this section, we shall introduce important properties of Besov spaces. Let us consider the dual spaces of Besov spaces, lifting properties and embedding relations.

The following proposition is concerned with the dual spaces.

Proposition 3.1. *Assume that V satisfies the same assumptions as in Theorem 2.5. Let $s \in \mathbb{R}$, $1 \le p, q < \infty$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Then the dual spaces of $B^s_{p,q}(A_V)$ and $\dot{B}^s_{p,q}(A_V)$ are $B^{-s}_{p',q'}(A_V)$ and $\dot{B}^{-s}_{p',q'}(A_V)$, respectively.*

We have the lifting properties and embedding relations of our Besov spaces.

Proposition 3.2. *Assume that V satisfies the same assumptions as in Theorem 2.5. Let λ_0 be the constant as in* (2.8), *i.e., $A_V \ge -\lambda_0^2 I$. Let $s, s_0 \in \mathbb{R}$ and $1 \le p, q, q_0, r \le \infty$. Then the following assertions hold*:

(i) *The inhomogeneous Besov spaces enjoy the following properties*:

$$\begin{aligned}
&\big\{(\lambda_0^2+1)I + A_V\big\}^{s_0/2} f \in B^{s-s_0}_{p,q}(A_V) \quad \textit{for any } f \in B^s_{p,q}(A_V);\\
&B^{s+\varepsilon}_{p,q}(A_V) \hookrightarrow B^s_{p,q_0}(A_V) \quad \textit{for any } \varepsilon > 0;\\
&B^s_{p,q}(A_V) \hookrightarrow B^{s_0}_{p,q}(A_V) \quad \textit{if } s \ge s_0;\\
&B^{s+n(\frac{1}{r}-\frac{1}{p})}_{r,q}(A_V) \hookrightarrow B^s_{p,q_0}(A_V) \quad \textit{if } 1 \le r \le p \le \infty \textit{ and } q \le q_0.
\end{aligned}$$

(ii) *The homogeneous Besov spaces enjoy the following properties*:

$$\begin{aligned}
&A_V^{s_0/2} f \in \dot{B}^{s-s_0}_{p,q}(A_V) \quad \textit{for any } f \in \dot{B}^s_{p,q}(A_V);\\
&\dot{B}^{s+n(\frac{1}{r}-\frac{1}{p})}_{r,q}(A_V) \hookrightarrow \dot{B}^s_{p,q_0}(A_V) \quad \textit{if} \quad 1 \le r \le p \le \infty \textit{ and } q \le q_0.
\end{aligned}$$

The Besov and Lebesgue spaces have the inclusion relation with each other.

Proposition 3.3. *Assume that V satisfies the same assumptions as in Theorem 2.5. Then the following continuous embeddings hold*:

(i) $L^p(\Omega) \hookrightarrow B^0_{p,2}(A_V), \dot{B}^0_{p,2}(A_V)$ *if* $1 < p \le 2$.

(ii) $B^0_{p,2}(A_V), \dot{B}^0_{p,2}(A_V) \hookrightarrow L^p(\Omega)$ *if* $2 \le p < \infty$.

As was stated in §1, the classical homogeneous Besov spaces are considered as subspaces of quotient space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. The following proposition states that the homogeneous Besov spaces with some indices s, p, q are characterized by subspaces of $\mathcal{X}'_V(\Omega)$ which is not a quotient space. Such characterization is known in the case of $\Omega = \mathbb{R}^n$ (see, e.g. [24]).

Proposition 3.4. *Assume that* V *satisfies* (1.1) *and* (2.3). *Let* $s \in \mathbb{R}$ *and* $1 \le p, q \le \infty$. *If either* $s < n/p$ *or* $(s, q) = (n/p, 1)$, *then the homogeneous Besov spaces* $\dot{B}^s_{p,q}(A_V)$ *are regarded as subspaces of* $\mathcal{X}'_V(\Omega)$ *according to the following isomorphism:*

$$\dot{B}^s_{p,q}(A_V) \simeq \Big\{ f \in \mathcal{X}'_V(\Omega) \,\Big|\, \|f\|_{\dot{B}^s_{p,q}(A_V)} < \infty,\ f = \sum_{j \in \mathbb{Z}} \phi_j(\sqrt{A_V}\,) f \ \text{ in } \mathcal{X}'_V(\Omega) \Big\}.$$

We conclude this section by stating a result on the equivalence relation among the Besov spaces by A_0 and A_V-regularity with $V \in L^{\frac{n}{2},\infty}(\Omega)$. For the definition of the Lorentz space $L^{\frac{n}{2},\infty}(\Omega)$, see §9.

Proposition 3.5. *Let* n, s, p, q *be such that*

$$n \ge 2, \quad 1 \le p, q \le \infty, \quad -\min\Big\{2, n\Big(1 - \frac{1}{p}\Big)\Big\} < s < \min\Big\{\frac{n}{p}, 2\Big\}.$$

In addition to the same assumption on V *as in Theorem 2.5, we further assume that*

$$\begin{cases} V \in L^1(\Omega) & \text{if } n = 2, \\ V \in L^{\frac{n}{2},\infty}(\Omega) & \text{if } n \ge 3. \end{cases} \tag{3.1}$$

Then

$$B^s_{p,q}(A_V) \simeq B^s_{p,q}(A_0),$$
$$\dot{B}^s_{p,q}(A_V) \simeq \dot{B}^s_{p,q}(A_0).$$

Let us give some remarks on Proposition 3.5.

(i) Proposition 3.5 implies not only the equivalence of norms, but also that of the following two approximations of the identity

$$f = \sum_{j \in \mathbb{Z}} \phi_j(\sqrt{A_0}) f \quad \text{in } \mathcal{Z}'_0(\Omega), \qquad f = \sum_{j \in \mathbb{Z}} \phi_j(\sqrt{A_V}) f \quad \text{in } \mathcal{Z}'_V(\Omega),$$

for f belonging to the homogeneous Besov spaces. Analogous approximations in $\mathcal{X}'_0(\Omega)$ and $\mathcal{X}'_V(\Omega)$ are also equivalent for the inhomogeneous Besov spaces.

(ii) By considering the Lorentz spaces, it is possible to treat the potential V like

$$V(x) = c|x|^{-2}, \qquad c > 0,$$

which, in fact, $V \in L^{\frac{n}{2},\infty}(\Omega)$.

(iii) If V is smooth more and more, then, s can be taken bigger and bigger so that the isomorphism holds. For instance, this comes from the following identity:

$$(-\Delta + V)^2 f = (-\Delta)^2 f + (-\Delta)(Vf) + V(-\Delta)f + V^2 f$$

when we consider the case $s = 4$. In fact, the term $(-\Delta)(Vf)$ requires the differentiability of V.

4. Key lemmas

In this section we introduce some tools and prove fundamental properties of $\mathcal{X}_V(\Omega)$ and $\mathcal{Z}_V(\Omega)$, which are important in later arguments. Here and below, we denote by $\|\cdot\|_{L^p}$ the norm of $L^p(\Omega)$ and $\|\cdot\|_{L^p(\mathbb{R}^n)}$ the norm of $L^p(\mathbb{R}^n)$.

We start with the functional calculus; L^p-boundedness of operators $\psi(A_V)$ and $\phi_j(\sqrt{A_V})$ for $1 \le p \le \infty$. In the previous work [19] we have established such kind of estimates for some potential V when $n \ge 3$. We improve them by some slight modifications, and obtain L^p-estimates under more general conditions on our potential V in all space dimensions (see Proposition A.1 in appendix A).

Based on Proposition A.1, we have the following useful lemma.

Lemma 4.1. *Let $1 \le r \le p \le \infty$. Assume that the measurable potential V satisfies* (1.1)*. Then we have the following assertions:*

(i) *For any $\phi \in C_0^\infty(\mathbb{R})$ and $m \in \mathbb{N} \cup \{0\}$ there exists a constant $C > 0$ such that*

$$\|A_V^m \phi(A_V) f\|_{L^p} \le C\|f\|_{L^r} \tag{4.1}$$

for all $f \in L^r(\Omega)$.

(ii) *For any $\phi \in C_0^\infty((0,\infty))$ and $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$\|A_V^\alpha \phi(2^{-j}\sqrt{A_V}) f\|_{L^p} \le C 2^{n(\frac{1}{r}-\frac{1}{p})j + 2\alpha j}\|f\|_{L^r} \tag{4.2}$$

for all $j \in \mathbb{N}$ and $f \in L^r(\Omega)$.

(iii) *Assume further that V satisfies* (2.3)*. Then for any $\phi \in C_0^\infty((0,\infty))$ and $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$\|A_V^\alpha \phi(2^{-j}\sqrt{A_V}) f\|_{L^p} \le C 2^{n(\frac{1}{r}-\frac{1}{p})j + 2\alpha j}\|f\|_{L^r} \tag{4.3}$$

for all $j \in \mathbb{Z}$ and $f \in L^r(\Omega)$.

Proof. Let $m \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$. To begin with, we note that the following inequality

$$\|A_V^m \phi(A_V) g\|_{L^p} \le C\|g\|_{L^p} \tag{4.4}$$

holds for any $g \in L^p(\Omega)$. In fact, writing

$$A_V^m \phi(A_V) = \{A_V^m e^{A_V}\phi(A_V)\} e^{-A_V},$$

and noting

$$\lambda^m e^{t\lambda}\phi(\lambda) \in C_0^\infty(\mathbb{R}),$$

we conclude from Proposition A.1 that (4.4) holds. In a similar way, we get

$$\|A_V^\alpha \phi(2^{-j}\sqrt{A_V}) g\|_{L^p} \le C\|g\|_{L^p} \tag{4.5}$$

for any $j \in \mathbb{N}$ and $g \in L^p(\Omega)$, provided that $\phi \in C_0^\infty((0,\infty))$.

Taking account of these considerations, we show (4.1). Let $G_t(x)$ be the function of Gaussian type appearing in the pointwise estimate (A.7) of kernel of e^{-tA_V} from Lemma A.4, i.e.,

$$G_t(x) = Ct^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{Ct}\right), \qquad t > 0, \quad x \in \mathbb{R}^n,$$

where C is a certain positive constant. We write

$$\phi(2^{-j}\sqrt{A_V})f = e^{-2^{-2j}A_V}\big\{e^{2^{-2j}A_V}\phi(2^{-j}\sqrt{A_V})\big\}f.$$

By using pointwise estimate (A.7) for e^{-tA_V}, we have

$$\big|e^{-2^{-2j}A_V}f(x)\big| \le \int_{\mathbb{R}^n} G_{2^{-2j}}(x-y)\big|\tilde{f}(y)\big|\,dy, \quad j\in\mathbb{N}, \quad x\in\Omega, \tag{4.6}$$

where $\tilde{f}$ is a zero extension of f outside of Ω. Let r_0 be such that $1/p = 1/r_0 + 1/r - 1$. Then we conclude from the estimates (4.4), (4.6) and Young's inequality that

$$\begin{aligned}\|A_V^m\phi(A_V)f\|_{L^p} &= \big\|\big\{A_V^m e^{A_V}\phi(A_V)\big\}e^{-A_V}f\big\|_{L^p}\\ &\le C\|e^{-A_V}f\|_{L^p}\\ &\le C\|G_1 * |\tilde{f}|\|_{L^p(\mathbb{R}^n)}\\ &\le C\|G_1\|_{L^{r_0}(\mathbb{R}^n)}\|\tilde{f}\|_{L^r(\mathbb{R}^n)}\\ &\le C\|f\|_{L^r}.\end{aligned}$$

This proves (4.1).

As to (4.2), again by using (4.5), (4.6) and Young's inequality, we get

$$\begin{aligned}\|A_V^\alpha\phi(2^{-j}\sqrt{A_V})f\|_{L^p} &= 2^{2\alpha j}\big\|\big\{(2^{-2j}A_V)^\alpha e^{2^{-2j}A_V}\phi(2^{-j}\sqrt{A_V})\big\}e^{-2^{-2j}A_V}f\big\|_{L^p}\\ &\le C2^{2\alpha j}\|e^{-2^{-2j}A_V}f\|_{L^p}\\ &\le C2^{2\alpha j}\|G_{2^{-2j}} * |\tilde{f}|\|_{L^p(\mathbb{R}^n)}\\ &\le C2^{2\alpha j}\|G_{2^{-2j}}\|_{L^{r_0}(\mathbb{R}^n)}\|\tilde{f}\|_{L^r(\mathbb{R}^n)}\\ &\le C2^{2\alpha j}2^{n(\frac{1}{r}-\frac{1}{p})j}\|f\|_{L^r}\end{aligned}$$

for any $j\in\mathbb{N}$, which proves (4.2). The estimate (4.3) is also proved in the analogous way to the above argument by applying (A.2) in Proposition A.1 and (A.8) in Lemma A.4 instead of (A.1) in Proposition A.1 and (A.7) in Lemma A.4, respectively. The proof of Lemma 4.1 is finished. □

The second lemma concerns with the completeness of test function spaces.

Lemma 4.2. *Assume that the measurable potential V satisfies* (1.1)*. Then $\mathcal{X}_V(\Omega)$ is complete. In addition to the assumption* (1.1)*, if V satisfies* (2.3)*, then $\mathcal{Z}_V(\Omega)$ is complete.*

Proof. We first show the completeness of $\mathcal{X}_V(\Omega)$. Let $\{f_N\}_{N=1}^\infty$ be a Cauchy sequence in $\mathcal{X}_V(\Omega)$. Then, for $M = 1, 2, \dots,$ there exists $C_M > 0$ such that

$$p_{V,M}(f_N) \le C_M \quad \text{for all } N\in\mathbb{N}. \tag{4.7}$$

Since $\{f_N\}$ is a Cauchy sequence in $L^1(\Omega)$, there exists a function $f\in L^1(\Omega)$ such that

$$f_N \to f \quad \text{ in } L^1(\Omega) \text{ as } N\to\infty.$$

Combining this convergence with the boundedness of $2^{Mj}\phi_j(\sqrt{A_V})$ from $L^1(\Omega)$ to itself, which is assured by (4.2) for $\alpha = 0$ and (4.7), we have

$$2^{Mj}\|\phi_j(\sqrt{A_V})f\|_{L^1} = \lim_{N\to\infty} 2^{Mj}\|\phi_j(\sqrt{A_V})f_N\|_{L^1},$$

and hence,

$$p_{V,M}(f) \le C_M$$

for $M = 1, 2, \ldots$. Hence we get $f \in \mathcal{X}_V(\Omega)$. We next show the convergence of f_N to f in $\mathcal{X}_V(\Omega)$. For each M, let us take a subsequence $\{f_{N(k)}\}_{k=1}^{\infty}$ such that

$$p_{V,M}(f_{N(k)} - f_{N(k-1)}) \le 2^{-k},$$

where we put $f_{N(0)} = 0$. Hence we have

$$\sum_{k=1}^{\infty} p_{V,M}(f_{N(k)} - f_{N(k-1)}) < \infty. \tag{4.8}$$

Since $\{f_{N(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^1(\Omega)$, f is written by

$$f = \lim_{L\to\infty} f_{N(L)} = \lim_{L\to\infty} \sum_{k=1}^{L} (f_{N(k)} - f_{N(k-1)}) \quad \text{in } L^1(\Omega). \tag{4.9}$$

Then (4.8) and (4.9) yield the convergence of $p_{V,M}(f_{N(L)} - f)$ to zero as $L \to \infty$, and hence,

$$p_{V,M}(f_N - f) \to 0 \quad \text{as } N \to \infty \quad \text{for } M = 1, 2, \cdots.$$

Therefore, $\mathcal{X}_V(\Omega)$ is complete.

We next show the completeness of $\mathcal{Z}_V(\Omega)$. Let $\{f_N\}_{N=1}^{\infty}$ be a Cauchy sequence in $\mathcal{Z}_V(\Omega)$. Since $\mathcal{Z}_V(\Omega)$ is a subspace of $\mathcal{X}_V(\Omega)$ and $\mathcal{X}_V(\Omega)$ is complete, $\{f_N\}_{N=1}^{\infty}$ is also a Cauchy sequence in $\mathcal{X}_V(\Omega)$ and there exists an element $f \in \mathcal{X}_V(\Omega)$ such that f_N converges to f in $\mathcal{X}_V(\Omega)$ as $N \to \infty$. In order to prove $f \in \mathcal{Z}_V(\Omega)$, we show that

$$\sup_{j\le 0} 2^{M|j|} \|\phi_j(\sqrt{A_V})f\|_{L^1} < \infty \quad \text{for } M = 1, 2, \cdots. \tag{4.10}$$

Since f_N converges to f in $L^1(\Omega)$ as $N \to \infty$ and $\phi_j(\sqrt{A_V})$ is bounded on $L^1(\Omega)$ for each $j \in \mathbb{Z}$ by (4.3) for $\alpha = 0$, it follows that

$$\lim_{N\to\infty} \|\phi_j(\sqrt{A_V})f_N\|_{L^1} = \|\phi_j(\sqrt{A_V})f\|_{L^1} \quad \text{for any } j \in \mathbb{Z}.$$

Since $\{f_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathcal{Z}_V(\Omega)$, $\{q_{V,M}(f_N)\}_{N=1}^{\infty}$ is a bounded sequence for each M and there exists a constant $C_M > 0$ depending only on M such that

$$2^{M|j|} \|\phi_j(\sqrt{-\Delta})f_N\|_{L^1} \le C_M \quad \text{for all } j \le 0 \text{ and } N = 1, 2, \cdots.$$

By taking the limit as $N \to \infty$ in the above inequality, we conclude that f satisfies (4.10), and hence, $f \in \mathcal{Z}_V(\Omega)$. Finally, the convergence of f_N to f in $\mathcal{Z}_V(\Omega)$ follows from the analogous argument to (4.8) and (4.9):

$$\sum_{k=1}^{\infty} q_{V,M}(f_{N(k)} - f_{N(k-1)}) < \infty,$$

$$f = \lim_{L\to\infty} \sum_{k=1}^{L} (f_{N(k)} - f_{N(k-1)}) \quad \text{in } L^1(\Omega),$$

which imply that

$$q_{V,M}(f_N - f) \to 0 \quad \text{as } N \to \infty \quad \text{for } M = 1, 2, \cdots.$$

Thus we conclude that $\mathcal{Z}_V(\Omega)$ is complete. The proof of Lemma 4.2 is finished. □

The third lemma is useful in proving Lemma 4.5.

Lemma 4.3. *Assume that the measurable potential V satisfies* (1.1). *Then the following assertions hold*:

(i) *For any $f \in \mathcal{X}_V'(\Omega)$, there exist a number $M_0 \in \mathbb{N}$ and a constant $C_f > 0$ such that*

$$\big|{}_{\mathcal{X}_V'}\langle f, g\rangle_{\mathcal{X}_V}\big| \le C_f\, p_{V,M_0}(g) \quad \textit{for any } g \in \mathcal{X}_V(\Omega).$$

(ii) *In addition to the assumption* (1.1), *if V satisfies* (2.3), *then for any $f \in \mathcal{Z}_V'(\Omega)$, there exist a number $M_1 \in \mathbb{N}$ and a constant $C_f > 0$ such that*

$$\big|{}_{\mathcal{Z}_V'}\langle f, g\rangle_{\mathcal{Z}_V}\big| \le C_f q_{V,M_1}(g) \quad \textit{for any } g \in \mathcal{Z}_V(\Omega).$$

Proof. Suppose that (i) is not true. Then for any $m \in \mathbb{N}$ there exists $g_m \in \mathcal{X}_V(\Omega)$ such that

$$\big|{}_{\mathcal{X}_V'}\langle f, g_m\rangle_{\mathcal{X}_V}\big| > m p_{V,m}(g_m). \tag{4.11}$$

Put

$$\widetilde{g}_m := \frac{g_m}{m p_{V,m}(g_m)}.$$

Noting that $p_{V,k}(\widetilde{g}_m)$ is monotonically increasing in $k \in \{1, 2, \dots, m\}$, we have

$$p_{V,k}(\widetilde{g}_m) \le p_{V,m}(\widetilde{g}_m) = \frac{1}{m} \quad \text{for } k = 1, 2, \cdots, m.$$

Hence it follows that for any fixed $k \in \mathbb{N}$

$$p_{V,k}(\widetilde{g}_m) \to 0 \quad \text{as } m \to \infty;$$

thus we find that

$$\widetilde{g}_m \to 0 \quad \text{in } \mathcal{X}_V(\Omega) \text{ as } m \to \infty.$$

The above convergence yields that

$$\big|{}_{\mathcal{X}_V'}\langle f, \widetilde{g}_m\rangle_{\mathcal{X}_V}\big| \to 0 \quad \text{as } m \to \infty. \tag{4.12}$$

However, the assumption (4.11) implies that

$$\big|{}_{\mathcal{X}_V'}\langle f, \widetilde{g}_m\rangle_{\mathcal{X}_V}\big| > 1 \quad \text{for all } m \in \mathbb{N};$$

therefore this inequality contradicts (4.12). Thus the assertion (i) holds. The assertion (ii) follows analogously. This ends the proof of Lemma 4.3. □

The following lemma states that the mapping $\phi(A_V)$ is well-defined on $\mathcal{X}_V(\Omega)$, $\mathcal{Z}_V(\Omega)$ and their duals.

Lemma 4.4. *Assume that the measurable potential V satisfies* (1.1). *Then the following assertions hold*:

(i) *For any $\phi \in C_0^\infty(\mathbb{R})$, $\phi(A_V)$ maps continuously from $\mathcal{X}_V(\Omega)$ into itself, and maps continuously from $\mathcal{X}_V'(\Omega)$ into itself.*

(ii) *In addition to the assumption* (1.1), *if V satisfies* (2.3), *then for any $\phi \in C_0^\infty((0,\infty))$, $\phi(A_V)$ maps continuously from $\mathcal{Z}_V(\Omega)$ into itself, and maps continuously from $\mathcal{Z}_V'(\Omega)$ into itself.*

Proof. First we prove the assertion (i). Let $f \in \mathcal{X}_V(\Omega)$. It follows from (4.1) in Lemma 4.1 that

$$A_V^m \phi(A_V) f \in \mathcal{D}(A_V), \quad p_{V,M}(\phi(A_V)f) \le C p_{V,M}(f) \tag{4.13}$$

for $m = 0, 1, 2, \ldots$; $M = 1, 2, \ldots$. This proves that $\phi(A_V)$ is continuous from $\mathcal{X}_V(\Omega)$ into itself. The continuity of $\phi(A_V)$ from $\mathcal{X}_V'(\Omega)$ into itself follows from the definition (2.5).

As to the assertion (ii), since V satisfies (1.1), $\phi(A_V)$ enjoys the assertion (i), and hence, we conclude that

$$\phi(A_V)f \in \mathcal{X}_V(\Omega) \quad \text{for any } f \in \mathcal{Z}_V(\Omega).$$

We show that

$$q_{V,M}(\phi(A_V)f) \le C q_{V,M}(f) \tag{4.14}$$

for $M = 1, 2, \ldots$. Indeed, recalling the definition (2.4) of $q_{V,M}(f)$ and noting that

$$q_{V,M}(\phi(A_V)f) \le p_{V,M}(\phi(A_V)f) + \sup_{j \le 0} 2^{M|j|} \|\phi_j(\sqrt{A_V})\phi(A_V)f\|_{L^1},$$

we apply (4.13) to the first term to obtain

$$p_{V,M}(\phi(A_V)f) \le C p_{V,M}(f) \le C q_{V,M}(f).$$

For the second term in $q_{V,M}(\phi(A_V)f)$, again applying (4.1) for $m = 0$, we estimate

$$\begin{aligned} \sup_{j \le 0} 2^{M|j|} \|\phi_j(\sqrt{A_V})\phi(A_V)f\|_{L^1} &\le C \sup_{j \le 0} 2^{M|j|} \|\phi_j(\sqrt{A_V})f\|_{L^1} \\ &\le C q_{V,M}(f) \end{aligned}$$

for $M = 1, 2, \ldots$. Therefore, the above two estimates imply (4.14), which concludes the continuity of $\phi(A_V)$ from $\mathcal{Z}_V(\Omega)$ into itself. Finally, the continuity of $\phi(A_V)$ from $\mathcal{Z}_V'(\Omega)$ into itself follows from the definition (2.6). The proof of Lemma 4.4 is finished. □

The approximation of identity is established by the following lemma.

Lemma 4.5. *Assume that the measurable potential V satisfies* (1.1). *Then the following assertions hold*:

(i) *For any $f \in \mathcal{X}_V(\Omega)$, we have*

$$f = \psi(A_V)f + \sum_{j \in \mathbb{N}} \phi_j(\sqrt{A_V})f \quad \text{in} \quad \mathcal{X}_V(\Omega). \tag{4.15}$$

Furthermore, for any $f \in \mathcal{X}_V'(\Omega)$, we have also the identity (4.15) *in $\mathcal{X}_V'(\Omega)$, and $\psi(A_V)f$ and $\phi_j(\sqrt{A_V})f$ are regarded as elements in $L^\infty(\Omega)$.*

(ii) *In addition to the assumption* (1.1), *if V satisfies* (2.3), *then for any $f \in \mathcal{Z}_V(\Omega)$, we have*

$$f = \sum_{j \in \mathbb{Z}} \phi_j(\sqrt{A_V})f \quad \text{in} \quad \mathcal{Z}_V(\Omega). \tag{4.16}$$

Furthermore, for $f \in \mathcal{Z}_V'(\Omega)$, we have also the identity (4.16) *in $\mathcal{Z}_V'(\Omega)$, and $\phi_j(\sqrt{A_V})f$ are regarded as elements in $L^\infty(\Omega)$.*

Proof. First we prove the assertion (i). Let $f \in \mathcal{X}_V(\Omega)$. Then we have $f \in L^2(\Omega)$, and f is written as

$$f = \psi(A_V)f + \sum_{j\in\mathbb{N}} \phi_j(\sqrt{A_V})f \quad \text{in} \quad L^2(\Omega).$$

It is sufficient to verify that the series in the right member is absolutely convergent in $\mathcal{X}_V(\Omega)$. Let $M \in \mathbb{N}$ be arbitrarily fixed. Applying (4.2) for $\alpha = 0, 1$ from Lemma 4.1, we have

$$p_{V,M}\big(\psi(A_V)f\big) \le Cp_{V,M}(f),$$

$$\begin{aligned} p_{V,M}\big(\phi_j(\sqrt{A_V})f\big) \le & C2^{-2j}p_{V,M}\big(A_V\phi_j(\sqrt{A_V})f\big) \\ \le & C2^{-2j}p_{V,M+2}(f), \end{aligned}$$

which imply that

$$\sum_{j\in\mathbb{N}} p_{V,M}\big(\phi_j(\sqrt{A_V})f\big) \le Cp_{V,M+2}(f)\sum_{j\in\mathbb{N}} 2^{-2j} < \infty. \tag{4.17}$$

Hence (4.15) holds for $f \in \mathcal{X}_V(\Omega)$. As to the expansion (4.15) for $f \in \mathcal{X}_V'(\Omega)$, applying the identity (4.15) for $g \in \mathcal{X}_V(\Omega)$, we have formally the following identity:

$$\begin{aligned} {}_{\mathcal{X}_V'}\langle f, g\rangle_{\mathcal{X}_V} &= {}_{\mathcal{X}_V'}\langle f, \psi(A_V)g\rangle_{\mathcal{X}_V} + \sum_{j\in\mathbb{N}} {}_{\mathcal{X}_V'}\langle f, \phi_j(\sqrt{A_V})g\rangle_{\mathcal{X}_V} \\ &= {}_{\mathcal{X}_V'}\langle \psi(A_V)f, g\rangle_{\mathcal{X}_V} + \sum_{j\in\mathbb{N}} {}_{\mathcal{X}_V'}\langle \phi_j(\sqrt{A_V})f, g\rangle_{\mathcal{X}_V}, \end{aligned} \tag{4.18}$$

where the second equality is valid due to the definition (2.5). We must prove the absolute convergence of the series in (4.18). By Lemma 4.3 (i), there exist $M_0 \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} \big|{}_{\mathcal{X}_V'}\langle \phi_j(\sqrt{A_V})f, g\rangle_{\mathcal{X}_V}\big| =& \big|{}_{\mathcal{X}_V'}\langle f, \phi_j(\sqrt{A_V})g\rangle_{\mathcal{X}_V}\big| \\ \le & C_f p_{V,M_0}(\phi_j(\sqrt{A_V})g). \end{aligned}$$

Then, the above estimate and (4.17) yield the absolute convergence of the series in (4.18).

For the proof of $\psi(A_V)f \in L^\infty(\Omega)$, we begin by proving that

$$\big|{}_{\mathcal{X}_V'}\langle \psi(A_V)f, g\rangle_{\mathcal{X}_V}\big| \le C\|g\|_{L^1} \quad \text{for all } g \in \mathcal{X}_V(\Omega). \tag{4.19}$$

By the definition (2.5), Lemma 4.3 (i) and (4.1) for $m = 0$, there exist $M_0 \in \mathbb{N}$ and $C_f, C_{f,\psi} > 0$ such that

$$\begin{aligned} \big|{}_{\mathcal{X}_V'}\langle \psi(A_V)f, g\rangle_{\mathcal{X}_V}\big| =& \big|{}_{\mathcal{X}_V'}\langle f, \psi(A_V)g\rangle_{\mathcal{X}_V}\big| \\ \le & C_f p_{V,M_0}(\psi(A_V)g) \\ \le & C_{f,\psi}\|g\|_{L^1}, \end{aligned}$$

which proves (4.19). Thanks to (4.19), the Hahn-Banach theorem allows us to deduce that the mapping

$${}_{\mathcal{X}_V'}\langle \psi(A_V)f, \cdot\rangle_{\mathcal{X}_V} : \mathcal{X}_V(\Omega) \to \mathbb{C}$$

is extended as a mapping from $L^1(\Omega)$ to $\mathbb{C}$. Since $L^1(\Omega)^* = L^\infty(\Omega)$, there exists a function $F \in L^\infty(\Omega)$ such that

$$ {}_{\mathcal{X}_V'}\langle \psi(A_V)f, g\rangle_{\mathcal{X}_V} = \int_\Omega F(x)\overline{g(x)}\,dx \quad \text{for all } g \in \mathcal{X}_V(\Omega). $$

Then we conclude that $\psi(A_V)f \in L^\infty(\Omega)$. In a similar way, it is possible to prove that $\phi_j(\sqrt{A_V})f \in L^\infty(\Omega)$. The proof of (i) is now complete.

As to the assertion (ii), noting that any $f \in \mathcal{Z}_V(\Omega)$ is in $L^2(\Omega)$, we first prove that

$$ f = \sum_{j\in\mathbb{Z}} \phi_j(\sqrt{A_V})f \quad \text{in } L^2(\Omega) \tag{4.20} $$

for any $f \in L^2(\Omega)$. Put

$$ g_L := \int_{-\infty}^{\infty} \Big(1 - \sum_{j\ge L} \phi_j(\sqrt{\lambda})\Big) dE_{A_V}(\lambda)f. \tag{4.21} $$

It is readily checked that $\{g_L\}$ is a Cauchy sequence in $L^2(\Omega)$, so we put

$$ g := \lim_{L\to-\infty} g_L \quad \text{ in } L^2(\Omega). $$

Noting that A_V is non-negative on $L^2(\Omega)$ and that the support of $1 - \sum_{j\ge L}\phi_j(\sqrt{\lambda})$ is contained in the interval $(-\infty, 2^{2L}]$, we find that

$$ \begin{aligned} \|A_V g_L\|_{L^2}^2 = &\int_{-\infty}^{2^{2L}} \Big|\lambda\Big(1 - \sum_{j\ge L}\phi_j(\sqrt{\lambda})\Big)\Big|^2 d\|E_{A_V}(\lambda)f\|_{L^2}^2 \\ \le & C2^{4L}\|f\|_{L^2}^2 \to 0 \quad \text{as } L \to -\infty. \end{aligned} $$

Hence we deduce that

$$ g \in \mathcal{D}(A_V), \qquad A_V g = 0 \quad \text{in } L^2(\Omega) $$

by the fact that $g_L \in \mathcal{D}(A_V)$, the definition of g, and the closeness of A_V on $L^2(\Omega)$. Since zero is not an eigenvalue of A_V by Lemma B.1, we conclude that $g = 0$, which proves (4.20) for any $f \in L^2(\Omega)$.

Now, as in the previous argument, it is sufficient to show that the series in the right member of (4.20) is absolutely convergent in $\mathcal{Z}_V(\Omega)$. For the series (4.20) with $j \ge 1$, the absolute convergence is obtained by the same argument as (4.17). For the case $j \le 0$, it follows from (4.3) for $\alpha = \pm 1$ that

$$ q_{V,M}\big(\phi_j(\sqrt{A_V})f\big) \le C2^{2j} q_{V,M}\big(A_V^{-1}\phi_j(\sqrt{A_V})f\big) \le C2^{2j} q_{V,M+2}(f), $$

which imply that

$$ \sum_{j\le 0} q_{V,M}\big(\phi_j(\sqrt{A_V})f\big) \le C q_{V,M+2}(f) \sum_{j\le 0} 2^{2j} < \infty $$

for all $M \in \mathbb{N}$. Therefore, (4.16) is verified for $f \in \mathcal{Z}_V(\Omega)$.

Finally, as to the identity (4.16) for $f \in \mathcal{Z}_V'(\Omega)$, we proceed the analogous argument to that with replacing the assertion (i) for $p_{V,M}$ and Lemma 4.3 (i) by $q_{V,M}$ and Lemma 4.3 (ii), respectively. The proof of $\phi_j(\sqrt{A_V})f \in L^\infty(\Omega)$ also follows from the analogous argument to that of the assertion (i) as above. So we may omit the details. The proof of Lemma 4.5 is complete. □

As a consequence of Lemmas 4.1 and 4.5, we have:

Lemma 4.6. *The following inclusion relations hold*:

$$\mathcal{X}_V(\Omega) \subset L^1(\Omega) \cap L^\infty(\Omega), \tag{4.22}$$

$$L^p(\Omega) \subset \mathcal{X}_V'(\Omega) \quad \text{for any } 1 \le p \le \infty. \tag{4.23}$$

As a consequence, we have

$$\mathcal{Z}_V(\Omega) \subset L^1(\Omega) \cap L^\infty(\Omega), \tag{4.24}$$

$$L^p(\Omega) \subset \mathcal{Z}_V'(\Omega) \quad \text{for any } 1 \le p \le \infty. \tag{4.25}$$

Proof. Once (4.22) and (4.23) are proved, (4.24) and (4.25) hold, since

$$\mathcal{Z}_V(\Omega) \subset \mathcal{X}_V(\Omega) \quad \text{and} \quad \mathcal{X}_V'(\Omega) \subset \mathcal{Z}_V'(\Omega).$$

We show the inclusion relation (4.22). Put

$$\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}.$$

Let $f \in \mathcal{X}_V(\Omega)$. Then it follows from the definition of semi-norms $p_{V,M}(\cdot)$ that

$$\|f\|_{L^1} \le p_{V,0}(f).$$

As to the L^∞-norm, we deduce from the identities (4.15), $\phi_j = \Phi_j \phi_j$ and the estimate (4.2) for $\alpha = 0$ that

$$\begin{aligned}
\|f\|_{L^\infty} \le& \|\psi(A_V)f\|_{L^\infty} + \sum_{j\in\mathbb{N}} \|\Phi_j(\sqrt{A_V})\phi_j(\sqrt{A_V})f\|_{L^\infty} \\
\le& C\|f\|_{L^1} + C\sum_{j\in\mathbb{N}} 2^{-j}\cdot 2^j 2^{nj}\|\phi_j(\sqrt{A_V})f\|_{L^1} \\
\le& Cp_{V,0}(f) + C\sum_{j\in\mathbb{N}} 2^{-j}\sup_{k\in\mathbb{N}} 2^{(n+1)k}\|\phi_k(\sqrt{A_V})f\|_{L^1} \\
\le& Cp_{V,n+1}(f).
\end{aligned}$$

Summarizing the above estimates now, we conclude the inclusion relation (4.22).

Finally, we prove the inclusion relation (4.23). Let $f \in L^p(\Omega)$ and $g \in \mathcal{X}_V(\Omega)$. Then it follows from Hölder's inequality and the above two estimates that

$$\begin{aligned}
\int_\Omega |f(x)g(x)|dx \le& \|f\|_{L^p}\|g\|_{L^{p'}} \\
\le& \|f\|_{L^p}\|g\|_{L^1\cap L^\infty} \\
\le& C\|f\|_{L^p} p_{V,n+1}(g),
\end{aligned}$$

where p' is the conjugate exponent of p. This estimate means that $f \in L^p(\Omega)$ belongs to $\mathcal{X}_V'(\Omega)$. Hence we conclude (4.23). The proof of Lemma 4.6 is complete. □

5. Proof of Theorem 2.5

In this section we prove Theorem 2.5.

Proof of independence of the choice of ψ and $\{\phi_j\}$. The proof of the independence in (i-a) and (ii-a) is similar to that of Triebel [37]. As to (i-a), let us take $\psi = \psi^{(k)}$, $\phi_j = \phi_j^{(k)}$ $(k = 1, 2)$ satisfying (2.1), (2.2) and (2.9). Since $\psi^{(1)}$ and $\phi_j^{(1)}$ satisfy

$$\begin{aligned} \psi^{(1)} = \psi^{(1)}\big(\psi^{(2)} + \phi_1^{(2)}\big), \quad \phi_1^{(1)} = \phi_1^{(1)}\big(\psi^{(2)} + \phi_1^{(2)} + \phi_2^{(2)}\big), \\ \phi_j^{(1)} = \phi_j^{(1)}\big(\phi_{j-1}^{(2)} + \phi_j^{(2)} + \phi_{j+1}^{(2)}\big) \quad \text{for } j = 2, 3, \cdots, \end{aligned} \tag{5.1}$$

it follows from (4.1) and (4.2) in Lemma 4.1 that

$$\|\psi^{(1)}(A_V)f\|_{L^p} + \|\phi_1^{(1)}(\sqrt{A_V})f\|_{L^p} \le C\Big\{\|\psi^{(2)}(A_V)f\|_{L^p} + \sum_{k=1}^{2}\|\phi_k^{(2)}(\sqrt{A_V})f\|_{L^p}\Big\},$$

$$\|\phi_j^{(1)}(\sqrt{A_V})f\|_{L^p} \le C\sum_{k=-1}^{1}\|\phi_{j+k}^{(2)}(\sqrt{A_V})f\|_{L^p} \quad \text{for } j = 2, 3, \cdots,$$

which imply that

$$\begin{aligned} &\|\psi^{(1)}(A_V)f\|_{L^p} + \big\|\big\{2^{sj}\|\phi_j^{(1)}(\sqrt{A_V})f\|_{L^p}\big\}_{j\in\mathbb{N}}\big\|_{\ell^q(\mathbb{N})} \\ \le & C\Big\{\|\psi^{(2)}(A_V)f\|_{L^p} + \big\|\big\{2^{sj}\|\phi_j^{(2)}(\sqrt{A_V})f\|_{L^p}\big\}_{j\in\mathbb{N}}\big\|_{\ell^q(\mathbb{N})}\Big\}. \end{aligned}$$

This proves the independence in (i-1) for the inhomogeneous Besov spaces.

As to (ii-a), we use the identity (5.1) for all $j \in \mathbb{Z}$ and apply (4.3) for $\alpha = 0$ in Lemma 4.1 to get

$$\big\|\big\{2^{sj}\|\phi_j^{(1)}(\sqrt{A_V})f\|_{L^p}\big\}_{j\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \le C\Big\{\big\|\big\{2^{sj}\|\phi_j^{(2)}(\sqrt{A_V})f\|_{L^p}\big\}_{j\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})}\Big\}.$$

This ends the proof of the required independence of the choice of ψ and $\{\phi_j\}$. □

Proof of inclusion relations (2.10) **and** (2.11)**.** Let p' and q' be such that $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. First we prove the embedding (2.10), namely,

$$\mathcal{X}_V(\Omega) \hookrightarrow B^s_{p,q}(A_V) \hookrightarrow \mathcal{X}'_V(\Omega).$$

Take Ψ and Φ_j such that

$$\Psi := \psi + \phi_1, \quad \Phi_1 := \psi + \phi_1 + \phi_2, \quad \Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \text{ for } j = 2, 3, \cdots.$$

Let $M \in \mathbb{N}$ be such that $M > s + n(1 - 1/p)$. Then, for any $f \in \mathcal{X}_V(\Omega)$, we deduce from the identities $\phi_j = \Phi_j\phi_j$ and the estimate (4.2) for $\alpha = 0$ in Lemma 4.1 that

$$\begin{aligned} \|f\|_{B^s_{p,q}(A_V)} &= \|\psi(A_V)f\|_{L^p} + \Big\{\sum_{j\in\mathbb{N}}\Big(2^{sj}\|\Phi_j(\sqrt{A_V})\phi_j(\sqrt{A_V})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \\ &\le C\|f\|_{L^1} + C\Big\{\sum_{j\in\mathbb{N}}\Big(2^{sj}2^{n(1-\frac{1}{p})j}2^{-Mj}\cdot 2^{Mj}\|\phi_j(\sqrt{A_V})f\|_{L^1}\Big)^q\Big\}^{\frac{1}{q}} \\ &\le Cp_{V,M}(f) + C\Big\{\sum_{j\in\mathbb{N}}\Big(2^{sj}2^{n(1-\frac{1}{p})j}2^{-Mj}\Big)^q\Big\}^{\frac{1}{q}}p_{V,M}(f) \\ &\le Cp_{V,M}(f) \end{aligned}$$

for any $f \in \mathcal{X}_V(\Omega)$. Thus we get the first embedding:

$$\mathcal{X}_V(\Omega) \hookrightarrow B^s_{p,q}(A_V). \tag{5.2}$$

To prove the second embedding

$$B^s_{p,q}(A_V) \hookrightarrow \mathcal{X}'_V(\Omega), \tag{5.3}$$

we take $M' \in \mathbb{N}$ such that $M' > -s + n(1-1/p')$. Applying Lemma 4.5 (i), the identities $\psi = \Psi\psi$, $\phi_j = \Phi_j\phi_j$, Hölder's inequality and the embedding (5.2) for s, p, q replaced by $-s, p', q'$, i.e.,

$$\mathcal{X}_V(\Omega) \hookrightarrow B^{-s}_{p',q'}(A_V),$$

we have, for $f \in B^s_{p,q}(A_V)$ and $g \in \mathcal{X}_V(\Omega)$

$$\begin{aligned}
|{}_{\mathcal{X}'_V}\langle f, g\rangle_{\mathcal{X}_V}| &= \Big| {}_{\mathcal{X}'_V}\big\langle \psi(A_V)f, \Psi(A_V)g\big\rangle_{\mathcal{X}_V} + \sum_{j\ge 1} {}_{\mathcal{X}'_V}\big\langle \phi_j(\sqrt{A_V})f, \Phi_j(\sqrt{A_V})g\big\rangle_{\mathcal{X}_V} \Big| \\
&\le \|\psi(A_V)f\|_{L^p}\|\Psi(A_V)g\|_{L^{p'}} \\
&\quad + \big\|\big\{2^{sj}\|\phi_j(\sqrt{A_V})f\|_{L^p}\big\}_{j\in\mathbb{N}}\big\|_{\ell^q(\mathbb{N})}\big\|\big\{2^{-sj}\|\Phi_j(\sqrt{A_V})g\|_{L^{p'}}\big\}_{j\in\mathbb{N}}\big\|_{\ell^{q'}(\mathbb{N})} \\
&\le C\|f\|_{B^s_{p,q}(A_V)}\|g\|_{B^{-s}_{p',q'}(A_V)} \\
&\le C\|f\|_{B^s_{p,q}(A_V)}p_{M'}(g).
\end{aligned}$$

Therefore, (5.3) is proved, and as a result, we get the embedding (2.10).

Next we show the embedding (2.11), namely,

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^s_{p,q}(A_V) \hookrightarrow \mathcal{Z}'_V(\Omega).$$

Put

$$\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \quad \text{for all } j \in \mathbb{Z}.$$

Let $L \in \mathbb{N}$ be such that $L > |s| + n(1-1/p)$. For any $f \in \mathcal{Z}(\Omega)$, we deduce from the identity $\phi_j = \Phi_j\phi_j$ and the estimate (4.3) for $\alpha = 0$ that

$$\begin{aligned}
\|f\|_{\dot{B}^s_{p,q}(A_V)} &= \Big\{\sum_{j\in\mathbb{Z}} \Big(2^{sj}\|\Phi_j(\sqrt{A_V})\phi_j(\sqrt{A_V})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\Big\{\Big(\sum_{j\le 0} + \sum_{j\ge 1}\Big)\Big(2^{sj}2^{n(1-\frac{1}{p})j}\|\phi_j(\sqrt{A_V})f\|_{L^1}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\Big(\sup_{j\le 0} 2^{-Lj}\|\phi_j(\sqrt{A_V})f\|_{L^1}\Big)\Big\{\sum_{j\le 0}\Big(2^{sj}2^{n(1-\frac{1}{p})j}2^{Lj}\Big)^q\Big\}^{\frac{1}{q}} \\
&\quad + C\Big(\sup_{j\ge 1} 2^{Lj}\|\phi_j(\sqrt{A_V})f\|_{L^1}\Big)\Big\{\sum_{j\ge 1}\Big(2^{sj}2^{n(1-\frac{1}{p})j}2^{-Lj}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le Cq_{V,L}(f),
\end{aligned}$$

which implies that

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^s_{p,q}(A_V). \tag{5.4}$$

To prove the second embedding

$$\dot{B}^s_{p,q}(A_V) \hookrightarrow \mathcal{Z}'_V(\Omega),$$

we take $L' \in \mathbb{N}$ such that $L' > |s| + n(1-1/p')$. For any $f \in \dot{B}^s_{p,q}(A_V)$ and $g \in \mathcal{Z}_V(\Omega)$, using the identities $\phi_j = \Phi_j\phi_j$, Hölder's inequality and the embedding (5.4) for s, p, q replaced by $-s, p', q'$, i.e.,

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^{-s}_{p',q'}(A_V),$$

we estimate

$$\begin{aligned}
|{}_{\mathcal{Z}'_V}\langle f, g\rangle_{\mathcal{Z}_V}| &= \Big| \sum_{j\in\mathbb{Z}} {}_{\mathcal{Z}'_V}\big\langle \phi_j(\sqrt{A_V})f, \Phi_j(\sqrt{A_V})g\big\rangle_{\mathcal{Z}_V} \Big| \\
&\le \big\| \big\{ 2^{sj}\|\phi_j(\sqrt{A_V})f\|_{L^p} \big\}_{j\in\mathbb{Z}} \big\|_{\ell^q(\mathbb{Z})} \big\| \big\{ 2^{-sj}\|\Phi_j(\sqrt{A_V})g\|_{L^{p'}} \big\}_{j\in\mathbb{Z}} \big\|_{\ell^{q'}(\mathbb{Z})} \\
&\le C\|f\|_{\dot{B}^s_{p,q}(A_V)}\|g\|_{\dot{B}^{-s}_{p',q'}(A_V)} \\
&\le C\|f\|_{\dot{B}^s_{p,q}(A_V)} q_{L'}(g).
\end{aligned}$$

Thus we conclude (2.11). □

It remains to show that $B^s_{p,q}(A_V)$ and $\dot{B}^s_{p,q}(A_V)$ are Banach spaces. It is easy to check that they are normed vector spaces, and hence, it suffices to prove the completeness.

Proof of the completeness of $B^s_{p,q}(A_V)$ and $\dot{B}^s_{p,q}(A_V)$. We have only to prove the completeness of the homogeneous Besov spaces $\dot{B}^s_{p,q}(A_V)$, since the inhomogeneous case is similar. The proof is done by the analogous argument to that by Triebel [37]. Indeed, let $\{f_N\}_{N=1}^\infty$ be a Cauchy sequence in $\dot{B}^s_{p,q}(A_V)$. We may assume that

$$\|f_{N+1} - f_N\|_{\dot{B}^s_{p,q}(A_V)} \le 2^{-N} \tag{5.5}$$

without loss of generality. Then $\{f_N\}_{N=1}^\infty$ is also a Cauchy sequence in $\mathcal{Z}'_V(\Omega)$ by the inclusion relation (2.11), and hence, there exists an element $f \in \mathcal{Z}'_V(\Omega)$ with the property that

$$f_N \to f \quad \text{in } \mathcal{Z}'_V(\Omega) \quad \text{as } N \to \infty,$$

since $\mathcal{Z}'_V(\Omega)$ is complete. This together with the boundedness of $\phi_j(\sqrt{A_V})$ on $\mathcal{Z}'_V(\Omega)$ imply that

$$\phi_j(\sqrt{A_V})f_N \to \phi_j(\sqrt{A_V})f \quad \text{ in } \mathcal{Z}'_V(\Omega) \text{ as } N \to \infty, \tag{5.6}$$

and we have $\phi_j(\sqrt{A_V})f \in L^\infty(\Omega)$ by Lemma 4.5 (ii). Furthermore, fixing $j \in \mathbb{Z}$, we see that $\{\phi_j(\sqrt{A_V})f_N\}_{N=1}^\infty$ is also a Cauchy sequence in $L^p(\Omega)$, and there exists $F_j \in L^p(\Omega)$ such that

$$\phi_j(\sqrt{A_V})f_N \to F_j \quad \text{in } L^p(\Omega) \text{ as } N \to \infty,$$

which implies that

$$F_j(x) = \phi_j(\sqrt{A_V})f(x) \quad \text{almost every } x \in \Omega,$$

and the convergence (5.6) also holds in the topology of $L^p(\Omega)$.

It remains to show that $f \in \dot{B}^s_{p,q}(A_V)$ and f_N tends to f in $\dot{B}^s_{p,q}(A_V)$ for the above $f \in \mathcal{Z}'_V(\Omega)$. Since $\big\{\{2^{sj}\|\phi_j(\sqrt{A_V})f_N\|_{L^p}\}_{j\in\mathbb{Z}}\big\}_{N=1}^\infty$ is a Cauchy sequence in $\ell^q(\mathbb{Z})$ and

$$2^{sj}\|\phi_j(\sqrt{A_V})f_N\|_{L^p} \to 2^{sj}\|\phi_j(\sqrt{A_V})f\|_{L^p} \quad \text{as } N \to \infty,$$

we get

$$\|f\|_{\dot{B}^s_{p,q}(A_V)} < \infty,$$

and hence,

$$f \in \dot{B}^s_{p,q}(A_V).$$

For the convergence of f_N to f, writing

$$f = \sum_{k=1}^{\infty} (f_k - f_{k-1}) = \lim_{N\to\infty} f_N \quad \text{in } \mathcal{Z}'_V(\Omega),$$

where $f_0 = 0$, we conclude from (5.5) that the above series converges absolutely in the topology of $\dot{B}^s_{p,q}(A_V)$. Thus the completeness of $\dot{B}^s_{p,q}(A_V)$ is proved. The proof of Theorem 2.5 is now finished. □

6. Proof of Proposition 3.1

In this section we prove Proposition 3.1. We treat only the homogeneous Besov spaces $\dot{B}^s_{p,q}(A_V)$, since the inhomogeneous case follows analogously. We prove that

$$\dot{B}^s_{p,q}(A_V)^* = \dot{B}^{-s}_{p',q'}(A_V) \tag{6.1}$$

for any $s \in \mathbb{R}$ and $1 \le p, q < \infty$. Let us first show that

$$\dot{B}^{-s}_{p',q'}(A_V) \hookrightarrow \dot{B}^s_{p,q}(A_V)^*. \tag{6.2}$$

Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be as in (2.2) and put

$$\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \quad \text{for } j \in \mathbb{Z}.$$

For any $f \in \dot{B}^{-s}_{p',q'}(A_V)$, we define an operator T_f as

$$T_f g := \sum_{j\in\mathbb{Z}} \int_\Omega \Big(\phi_j(\sqrt{A_V}) f\Big) \, \overline{\Phi_j(\sqrt{A_V}) g} \, dx \quad \text{for } g \in \dot{B}^s_{p,q}(A_V).$$

Then

$$\begin{aligned} |T_f g| &\le \big\| \{2^{-sj} \|\phi_j(\sqrt{A_V}) f\|_{L^{p'}}\}_{j\in\mathbb{Z}} \big\|_{\ell^{q'}(\mathbb{Z})} \big\| \{2^{sj} \|\Phi_j(\sqrt{A_V}) g\|_{L^p}\}_{j\in\mathbb{Z}} \big\|_{\ell^q(\mathbb{Z})} \\ &\le C \|f\|_{\dot{B}^{-s}_{p',q'}(A_V)} \|g\|_{\dot{B}^s_{p,q}(A_V)}, \end{aligned}$$

which implies that the operator norm $\|T_f\|_{\dot{B}^s_{p,q}(A_V)^*}$ is bounded by $C\|f\|_{\dot{B}^{-s}_{p',q'}(A_V)}$. This proves the inclusion (6.2).

We prove the converse inclusion:

$$\dot{B}^s_{p,q}(A_V)^* \hookrightarrow \dot{B}^{-s}_{p',q'}(A_V). \tag{6.3}$$

Let $F \in \dot{B}^s_{p,q}(A_V)^*$. We define an operator

$$T : \ell^q(\mathbb{Z}; L^p(\Omega)) \to \mathbb{C}$$

as follows. For $G = \{G_j\}_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}; L^p(\Omega))$, we put

$$T(G) := F\Big(\sum_{j\in\mathbb{Z}} 2^{-sj} \phi_j(\sqrt{A_V}) G_j \Big).$$

Here we estimate

$$
\begin{aligned}
&\Big\| \sum_{j\in\mathbb{Z}} 2^{-sj}\phi_j(\sqrt{A_V})G_j \Big\|_{\dot{B}^s_{p,q}(A_V)} \\
=&\Big\{ \sum_{k\in\mathbb{Z}} \Big(2^{sk} \Big\| \phi_k(\sqrt{A_V}) \sum_{j=k-1}^{k+1} 2^{-sj}\phi_j(\sqrt{A_V})G_j \Big\|_{L^p} \Big)^q \Big\}^{\frac{1}{q}} \\
=&\Big\{ \sum_{k\in\mathbb{Z}} \Big(2^{sk} \Big\| \phi_k(\sqrt{A_V}) \sum_{r=-1}^{1} 2^{-s(k+r)}\phi_{k+r}(\sqrt{A_V})G_{k+r} \Big\|_{L^p} \Big)^q \Big\}^{\frac{1}{q}} \\
\le& C \sum_{r=-1}^{1} 2^{-sr} \Big\{ \sum_{k\in\mathbb{Z}} \|G_k\|_{L^p}^q \Big\}^{\frac{1}{q}} \\
\le& C\|G\|_{\ell^q L^p},
\end{aligned}
$$

where we used the estimate (4.3) for $\alpha = 0$ in Lemma 4.1. Hence we deduce that

$$
\begin{aligned}
|T(G)| \le& \|F\|_{\dot{B}^s_{p,q}(A_V)^*} \Big\| \sum_{j\in\mathbb{Z}} 2^{-sj}\phi_j(\sqrt{A_V})G_j \Big\|_{\dot{B}^s_{p,q}(A_V)} \\
\le& C\|F\|_{\dot{B}^s_{p,q}(A_V)^*} \|G\|_{\ell^q L^p}.
\end{aligned}
$$

Since $(\ell^q L^p)^* = \ell^{q'} L^{p'}$, there exists $\{F_j\}_{j\in\mathbb{Z}} \in \ell^{q'} L^{p'}$ such that

$$
T(G) = \sum_{j\in\mathbb{Z}} \int_\Omega F_j(x)\overline{G_j(x)}\,dx \quad \text{and} \quad \|\{F_j\}_{j\in\mathbb{Z}}\|_{\ell^{q'}L^{p'}} \le C\|F\|_{\dot{B}^s_{p,q}(A_V)^*}. \tag{6.4}
$$

Then for any $g \in \dot{B}^s_{p,q}(A_V)$, let us take $G = \{G_j\}_{j\in\mathbb{Z}}$ as

$$
G_j = 2^{sj}\Phi_j(\sqrt{A_V})g.
$$

It follows from $g \in \mathcal{Z}'_V(\Omega)$, Lemma 4.5 (ii) and the identities $\phi_j = \phi_j\Phi_j$ that

$$
\begin{aligned}
F(g) &= F\Big(\sum_{j\in\mathbb{Z}} 2^{-sj}\phi_j(\sqrt{A_V})\big(2^{sj}\Phi_j(\sqrt{A_V})g\big) \Big) \\
&= T(G) \\
&= \sum_{j\in\mathbb{Z}} \int_\Omega F_j(x)\overline{G_j(x)}\,dx \\
&= \sum_{j\in\mathbb{Z}} \int_\Omega F_j(x)\overline{2^{sj}\Phi_j(\sqrt{A_V})g}\,dx \\
&= \sum_{j\in\mathbb{Z}} \int_\Omega \Big(2^{sj}\Phi_j(\sqrt{A_V})F_j(x) \Big)\overline{g}\,dx.
\end{aligned}
$$

Taking f as

$$
f = \sum_{j\in\mathbb{Z}} 2^{sj}\Phi_j(\sqrt{A_V})F_j,
$$

we deduce from (6.4) that

$$\begin{aligned}\|f\|_{\dot{B}^{-s}_{p',q'}(A_V)} \le& C\|\{F_j\}_{j\in\mathbb{Z}}\|_{\ell^{q'}L^{p'}}\\ \le& C\|F\|_{\dot{B}^s_{p,q}(A_V)^*},\end{aligned}$$

which implies that $f \in \dot{B}^{-s}_{p',q'}(A_V)$. Hence F is regarded as an element in $\dot{B}^{-s}_{p',q'}(A_V)$, and we get the inclusion (6.3); thus we conclude the isomorphism (6.1). This ends the proof of Proposition 3.1.

7. Proof of Proposition 3.2

In this section we prove Proposition 3.2. The embedding relations are immediate consequences of Lemma 4.1. The main point is to prove the lifting properties.

First we prove the homogeneous case, namely,

$$A_V^{s_0/2} f \in \dot{B}^{s-s_0}_{p,q}(A_V) \quad \text{for any } f \in \dot{B}^s_{p,q}(A_V).$$

To begin with, we show that

$$A_V^{s_0/2} \text{ is a continuous operator from } \mathcal{Z}_V'(\Omega) \text{ to itself.} \tag{7.1}$$

By the definition (2.6), it is sufficient to verify that $A_V^{s_0/2}$ is the continuous operator from $\mathcal{Z}_V(\Omega)$ to itself. Let us take $M_0 \in \mathbb{N}$ such that $M_0 > |s_0|$. It follows from (4.3) for $\alpha = s_0/2$ and (4.16) that

$$q_{V,M}\big(A_V^{s_0/2} g\big) \le C q_{V,M+M_0}(g)$$

for any $g \in \mathcal{Z}_V(\Omega)$, which implies that $A_V^{s_0/2} g \in \mathcal{Z}_V(\Omega)$. This proves (7.1). Hence, all we have to do is to prove that $f \in \dot{B}^s_{p,q}(A_V)$ satisfies

$$\|A_V^{s_0/2} f\|_{\dot{B}^{s-s_0}_{p,q}(A_V)} \le C\|f\|_{\dot{B}^s_{p,q}(A_V)}. \tag{7.2}$$

In fact, let

$$\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}.$$

We note that $\Phi_j(\lambda)\lambda^{s_0} \in C_0^\infty((0,\infty))$. Writing

$$\Phi_j(\lambda)\lambda^{s_0} = 2^{s_0 j}\cdot\Phi_j(\lambda)\cdot(2^{-s_0 j}\lambda^{s_0}),$$

we get

$$\begin{aligned}\|\phi_j(\sqrt{A_V})A_V^{s_0/2} f\|_{L^p} =& 2^{s_0 j}\big\|\big\{\Phi_j(\sqrt{A_V})2^{-s_0 j}A_V^{s_0/2}\big\}\phi_j(\sqrt{A_V})f\big\|_{L^p}\\ \le& C2^{s_0 j}\|\phi_j(\sqrt{A_V})f\|_{L^p}.\end{aligned}$$

Hence, multiplying $2^{(s-s_0)j}$ to the above inequality and taking the $\ell^q(\mathbb{Z})$-norm, we obtain the required inequality (7.2).

As to inhomogeneous case, we have to consider the operators

$$(\lambda_0^2 + 1 + A_V)^{s_0/2}\phi_j(\sqrt{A_V}).$$

The only different point from the homogeneous case is to show the following estimates:

$$\Big\|(\lambda_0^2 + 1 + A_V)^{s_0/2}\phi_j(\sqrt{A_V})f\Big\|_{L^p} \le C2^{s_0 j}\Big\|\phi_j(\sqrt{A_V})f\Big\|_{L^p} \tag{7.3}$$

for any $j \in \mathbb{N}$. We write

$$
\begin{aligned}
(\lambda_0^2 + 1 + A_V)^{s_0/2} =& \Big[2^{s_0 j}\big\{2^{-2j}(\lambda_0^2+1) + 2^{-2j}A_V\big\}^{s_0/2} - 2^{s_0 j}\big(2^{-2j}A_V\big)^{s_0/2}\Big] \\
&+ 2^{s_0 j}\big(2^{-2j}A_V\big)^{s_0/2} \\
=:& T_1 + T_2.
\end{aligned}
$$

As to $T_2\phi_j(\sqrt{A_V})f$, it follows from (4.2) for $\alpha = s_0/2$ in Lemma 4.1 that

$$
\|T_2\phi_j(\sqrt{A_V})f\|_{L^p} \le C2^{s_0 j}\|\phi_j(\sqrt{A_V})f\|_{L^p}.
$$

Writing

$$
\begin{aligned}
T_1 =& 2^{s_0 j}\int_0^{2^{-2j}(\lambda_0^2+1)} \partial_\theta(\theta + 2^{-2j}A_V)^{\frac{s_0}{2}}\,d\theta \\
=& 2^{s_0 j}\int_0^{2^{-2j}(\lambda_0^2+1)} \frac{s_0}{2}(\theta + 2^{-2j}A_V)^{\frac{s_0}{2}-1}\,d\theta,
\end{aligned}
$$

we estimate $T_1\phi_j(\sqrt{A_V})f$ as

$$
\left\|T_1\phi_j(\sqrt{A_V})f\right\|_{L^p} \le C2^{s_0 j}\int_0^{2^{-2j}(\lambda_0^2+1)} \left\|(\theta + 2^{-2j}A_V)^{\frac{s_0}{2}-1}\phi_j(\sqrt{A_V})f\right\|_{L^p}\,d\theta.
$$

When $p = 2$, we use the spectral theorem on the Hilbert space $L^2(\Omega)$ to obtain

$$
\begin{aligned}
\left\|(\theta + 2^{-2j}A_V)^{\frac{s_0}{2}-1}\phi_j(\sqrt{A_V})f\right\|_{L^2}^2 =& \int_{2^{2(j-1)}}^{2^{2(j+1)}} (\theta + 2^{-2j}\lambda)^{s_0-2}\,d\left\|E_{A_V}(\lambda)\phi_j(\sqrt{A_V})f\right\|_{L^2}^2 \\
\le& C\int_{2^{2(j-1)}}^{2^{2(j+1)}} (2^{-2j}\lambda)^{s_0-2}\,d\left\|E_{A_V}(\lambda)\phi_j(\sqrt{A_V})f\right\|_{L^2}^2 \\
\le& C\left\|\phi_j(\sqrt{A_V})f\right\|_{L^2}^2,
\end{aligned}
$$

since $j \in \mathbb{N}$ and $0 \le \theta \le 2^{-2j}(\lambda_0^2+1)$. When $p \neq 2$, we have to obtain the following estimate:

$$
\left\|(\theta + 2^{-2j}A_V)^{\frac{s_0}{2}-1}\phi_j(\sqrt{A_V})f\right\|_{L^p} \le C\left\|\phi_j(\sqrt{A_V})f\right\|_{L^p}. \tag{7.4}
$$

Since θ is small compared with the spectrum of $2^{-2j}A_V\phi_j(\sqrt{A_V})$, θ is able to be neglected. Hence, the proof of estimate (7.4) is done by the argument of our paper [19]. So, we may omit the details. Summarizing the estimates obtained now, we conclude the estimate (7.3). The proof of Proposition 3.2 is finished.

8. Proofs of Propositions 3.3 and 3.4

In this section we prove Propositions 3.3 and 3.4. Let us start by preparing two lemmas.

Lemma 8.1. *Let $1 < p \le 2$. Then there exists a constant $C > 0$ such that*

$$
\|f\|_{B^0_{p,2}(A_V)} \le C\|f\|_{L^p} + C\big\|\big\{\|e^{-2^{-2j}A_V}f\|_{L^p}\big\}_{j\in\mathbb{N}}\big\|_{\ell^2(\mathbb{N})}, \tag{8.1}
$$

$$
\|f\|_{\dot{B}^0_{p,2}(A_V)} \le C\big\|\big\{\|e^{-2^{-2j}A_V}f\|_{L^p}\big\}_{j\in\mathbb{Z}}\big\|_{\ell^2(\mathbb{Z})} \tag{8.2}
$$

for any $f \in C_0^\infty(\Omega)$.

Proof. Since $e^{2^{-2j}\lambda^2}\phi_j(\lambda)$ is in $C_0^\infty((0,\infty))$, it follows from (4.2) for $\alpha = 0$ that

$$\begin{aligned}\|\phi_j(\sqrt{A_V})f\|_{L^p} =& \big\|\big(e^{2^{-2j}A_V}\phi_j(\sqrt{A_V})\big)e^{-2^{-2j}A_V}f\|_{L^p} \\ \le& C\|e^{-2^{-2j}A_V}f\|_{L^p}\end{aligned} \tag{8.3}$$

for any $j \in \mathbb{N}$. Then, taking the $\ell^2(\mathbb{N})$-norm, we obtain (8.1). As to the homogeneous case, thanks to (4.3) for $\alpha = 0$, inequality (8.3) is also valid for any $j \in \mathbb{Z}$, and hence, taking the $\ell^2(\mathbb{Z})$-norm, we conclude (8.2). □

Lemma 8.2 (The Khinchine inequality)**.** *Let $\{r_j(t)\}_{j=1}^\infty$ be a sequence of Rademacher functions, that is,*

$$r_j(t) := \sum_{k=1}^{2^j}(-1)^{k-1}\chi_{[(k-1)2^{-j},k2^{-j})}(t) \quad \textit{for } t \in [0,1],$$

where χ_I denotes the characteristic function on the interval I. Then for any p with $1 < p < \infty$, there exists a constant $C > 0$ such that

$$C^{-1}\|a\|_{\ell^2(\mathbb{N})} \le \Big\|\sum_{j\in\mathbb{N}} a_j r_j\Big\|_{L^p(0,1)} \le C\|a\|_{\ell^2(\mathbb{N})} \tag{8.4}$$

for all $a = \{a_j\}_{j\in\mathbb{Z}} \in \ell^2(\mathbb{N})$.

Proof of Proposition 3.3 (i): The embedding

$$L^p(\Omega) \hookrightarrow \dot{B}^0_{p,2}(A_V) \quad \textbf{for } 1 < p \le 2. \tag{8.5}$$

It is sufficient to show that

$$\|f\|_{\dot{B}^0_{p,2}(A_V)} \le C\|f\|_{L^p} \quad \text{for any } f \in C_0^\infty(\Omega) \tag{8.6}$$

due to the fact that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$. Let $\{r_j(t)\}$ be the sequence of Rademacher functions as in Lemma 8.2. If we show that there exists a constant $C > 0$ such that

$$\Big\|\sum_{j=1}^{N} r_j(t)e^{-2^{-2j}A_V}f\Big\|_{L^p} + \Big\|\sum_{j=-N}^{-1} r_{-j}(t)e^{-2^{-2j}A_V}f\Big\|_{L^p} \le C\|f\|_{L^p} \tag{8.7}$$

for all $t \in [0,1]$ and $N \in \mathbb{N}$, then (8.6) is verified. Indeed, by using the Minkowski inequality, we have

$$\begin{aligned}&\Big(\sum_{|j|\le N}\|e^{-2^{-2j}A_V}f\|_{L^p}^2\Big)^{1/2} \\ \le&\|e^{-A_V}f\|_{L^p} + \Big\|\Big(\sum_{j=1}^{N}|e^{-2^{-2j}A_V}f|^2\Big)^{1/2}\Big\|_{L^p} + \Big\|\Big(\sum_{j=-N}^{-1}|e^{-2^{-2j}A_V}f|^2\Big)^{1/2}\Big\|_{L^p}.\end{aligned}$$

Since $\|e^{-A_V}f\|_{L^p} \le C\|f\|_{L^p}$ by (A.8), and since the third term in the right member of the above estimate is treated analogously to the second one, we may consider only the second term. By using (8.4) and (8.7), we estimate

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{N}|e^{-2^{-2j}A_V}f|^2\Big)^{1/2}\Big\|_{L^p} \le& C\Big\|\Big(\int_0^1\Big|\sum_{j=1}^{N}r_j(t)e^{-2^{-2j}A_V}f\Big|^p\,dt\Big)^{1/p}\Big\|_{L^p}\\ =& C\Big(\int_0^1\Big\|\sum_{j=1}^{N}r_j(t)e^{-2^{-2j}A_V}f\Big\|_{L^p}^p\,dt\Big)^{1/p}\\ \le& C\Big(\int_0^1\|f\|_{L^p}^p\,dt\Big)^{1/p}\\ =& C\|f\|_{L^p},\end{aligned}$$

which implies that

$$\Big(\sum_{|j|\le N}\|e^{-2^{-2j}A_V}f\|_{L^p}^2\Big)^{1/2}\le C\|f\|_{L^p}\quad\text{for any } N\in\mathbb{N}.$$

Taking the limit as $N\to\infty$ in the above inequality, and combining the resultant with the inequality (8.2) in Lemma 8.1, we obtain the required inequality (8.6). Thus, we get the embedding (8.5).

We must show (8.7). Let $\widetilde{f}$ be the zero extension of f to the outside of Ω. Recall that $G_t(x)$ is the function of Gaussian type in the right member of (A.8). Noting that

$$|r_j(t)|\le 1\quad\text{for all } t\in[0,1],$$

we deduce from (A.8) that

$$\Big|\sum_{j=1}^{N}r_j(t)e^{-2^{-2j}A_V}f\Big|+\Big|\sum_{j=-N}^{-1}r_j(t)e^{-2^{-2j}A_V}f\Big|\le C\int_{\mathbb{R}^n}\sum_{j=-N}^{N}G_{2^{-2j}}(x-y)|\widetilde{f}(y)|dy\quad(8.8)$$

for all $t\in[0,1]$. Here, it is certain to check that for each $\alpha\in(\mathbb{N}\cup\{0\})^n$

$$|x|^{n+|\alpha|}\big|\partial_x^\alpha G_{2^{-2j}}(x)\big|\le C|2^jx|^{n+|\alpha|}\big|\partial_x^\alpha G_1(2^jx)\big|,$$

and hence,

$$\begin{aligned}\sup_{t\in[0,1],N\in\mathbb{N},x\in\mathbb{R}^n}|x|^{n+|\alpha|}\Big|\sum_{j=-N}^{N}\partial_x^\alpha G_{2^{-2j}}(x)\Big| &\le C\sum_{j\in\mathbb{Z}}|2^jx|^{n+|\alpha|}\big|\partial_x^\alpha G_1(2^jx)\big|\\ &\le C\sum_{j\in\mathbb{Z}}2^{(n+|\alpha|)j}e^{-c2^{-2j}}<\infty.\end{aligned}$$

Then, applying the L^p-boundedness of the singular integral operators (see e.g. p.29 in [35]), we get

$$\Big\|\int_{\mathbb{R}^n}\sum_{j=-N}^{N}G_{2^{-2j}}(x-y)|\widetilde{f}(y)|dy\Big\|_{L^p(\mathbb{R}^n)}\le C\|\widetilde{f}\|_{L^p(\mathbb{R}^n)}=C\|f\|_{L^p}.$$

Hence the required inequality (8.7) is a consequence of (8.8) and the above estimate. Therefore, the proof of the embedding (8.5) is completed. □

Proof of Proposition 3.3 (ii): The embedding

$$\dot{B}^0_{p,2}(A_V) \hookrightarrow L^p(\Omega) \quad \textbf{for } 2 \le p < \infty. \tag{8.9}$$

Let p' be such that $1/p + 1/p' = 1$. Then the embedding (8.9) is an immediate consequence of $1 < p' \le 2$, $L^{p'}(\Omega) \hookrightarrow \dot{B}^0_{p',2}(A_V)$, $L^{p'}(\Omega)^* = L^p(\Omega)$ and $\dot{B}^0_{p',2}(A_V)^* = \dot{B}^0_{p,2}(A_V)$. □

Proofs of Proposition 3.3 (i) and (ii) for the inhomogeneous Besov spaces. Let $1 < p \le 2$. Then all we have to do is to show that

$$\|f\|_{B^0_{p,2}(A_V)} \le C\|f\|_{L^p} \quad \text{for any } f \in C_0^\infty(\Omega).$$

Referring to the estimate (8.1), we have only to show the corresponding estimate to (8.7), that is,

$$\Big\| \sum_{j=1}^{N} r_j(t) e^{-2^{-2j}A_V} f \Big\|_{L^p} \le C\|f\|_{L^p},$$

which is proved in the same way as in the proof of (8.7) by using the pointwise estimate (A.7) for the kernel of e^{-tA_V}. Hence we have the embedding

$$L^p(\Omega) \hookrightarrow B^0_{p,2}(A_V) \quad \text{for } 1 < p \le 2. \tag{8.10}$$

Finally, referring to the proof of (8.9), we obtain the embedding

$$B^0_{p,2}(A_V) \hookrightarrow L^p(\Omega) \quad \text{for } 2 \le p < \infty$$

by taking the duality of $L^{p'}(\Omega) \hookrightarrow B^0_{p',2}(A_V)$. The proof of Proposition 3.3 is now finished. □

We now turn to the proof of Proposition 3.4.

Proof of Proposition 3.4. Putting

$$\dot{X}^s_{p,q}(A_V) := \Big\{ f \in \mathcal{X}'_V(\Omega) \,\Big|\, \|f\|_{\dot{B}^s_{p,q}(A_V)} < \infty,\ f = \sum_{j\in\mathbb{Z}} \phi_j(\sqrt{A_V}\,)f \text{ in } \mathcal{X}'_V(\Omega) \Big\},$$

we see that

$$\dot{X}^s_{p,q}(A_V) \subset \dot{B}^s_{p,q}(A_V).$$

Hence it is sufficient to prove that

$$\dot{B}^s_{p,q}(A_V) \hookrightarrow \dot{X}^s_{p,q}(A_V). \tag{8.11}$$

Let $f \in \dot{B}^s_{p,q}(A_V)$. Then $f \in \mathcal{Z}'_V(\Omega)$, and thanks to Lemma 4.5 (ii), f is written as

$$\begin{aligned} f &= \sum_{j\le 0} \phi_j(\sqrt{A_V})f + \sum_{j\ge 1} \phi_j(\sqrt{A_V})f \quad \text{in } \mathcal{Z}'_V(\Omega) \\ &=: I + II. \end{aligned} \tag{8.12}$$

For the low frequency part, it follows from (4.3) for $\alpha = 0$ that

$$\begin{aligned}\|I\|_{L^\infty} \le& \sum_{j\le 0} \|\phi_j(\sqrt{A_V})f\|_{L^\infty}\\ \le& C\sum_{j\le 0} 2^{\frac{n}{p}j}\|\phi_j(\sqrt{A_V})f\|_{L^p},\end{aligned}$$

where the right member is finite when $(s, q) = (n/p, 1)$. In the case when $s < n/p$, we estimate

$$\begin{aligned}\|I\|_{L^\infty} \le& C\sum_{j\le 0} 2^{(\frac{n}{p}-s)j} \sup_{k\le 0} 2^{sk}\|\phi_k(\sqrt{A_V})f\|_{L^p}\\ \le& C\|f\|_{\dot{B}^s_{p,\infty}(A_V)}\\ \le& C\|f\|_{\dot{B}^s_{p,q}(A_V)},\end{aligned}$$

where we used the embedding in Proposition 3.2 (ii) in the last step. Hence the above two estimates and Lemma 4.6 imply that I belongs to $\mathcal{X}'_V(\Omega)$. As to II, since the high frequency part of $q_{V,M}(\cdot)$ is equivalent to that of $p_{V,M}(\cdot)$, it follows that $II \in \mathcal{X}'_V(\Omega)$. Hence the identity (8.12) holds in the topology of $\mathcal{X}'_V(\Omega)$. Therefore, we get $f \in \dot{X}^s_{p,q}(A_V)$. Thus we conclude the embedding (8.11). This completes the proof of Proposition 3.4. □

9. Proof of Proposition 3.5

In this section we prove Proposition 3.5. We utilize the theory of Lorentz spaces and introduce the following notations (see e.g. [15, 42]). Let f be a measurable function on Ω. We define the non-increasing rearrangement of f as

$$f^*(t) := \inf\{\ \sigma > 0 \mid m_f(\sigma) \le t\ \},$$

where $m_f(\sigma)$ is the distribution function of f which is defined by the Lebesgue measure of the set $\{\, x \in \mathbb{R}^n \mid |f(x)| > \sigma\,\}$. We define a function $f^{**}(t)$ on $(0,\infty)$ as

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(t')\,dt'.$$

Lorentz spaces $L^{p,q}(\Omega)$ are defined by letting

$$L^{p,q}(\Omega) := \{\ f\ : \text{measurable on } \Omega \mid \|f\|_{L^{p,q}} < \infty\ \},$$

where

$$\|f\|_{L^{p,q}} := \begin{cases} \Big\{ \displaystyle\int_0^\infty \big(t^{\frac{1}{p}} f^{**}(t)\big)^q \frac{dt}{t}\Big\}^{\frac{1}{q}} & \text{if } 1 \le p, q < \infty,\\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^{**}(t). & \text{if } 1 \le p \le \infty, q = \infty.\end{cases}$$

In what follows, we denote by $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ the norm of $L^{p,q}(\mathbb{R}^n)$ only when $\Omega = \mathbb{R}^n$. Note that

$$\begin{gathered} L^{p,1}(\Omega) \hookrightarrow L^{p,q}(\Omega) \quad \text{if } 1 \le p, q \le \infty, \qquad (9.1)\\ L^p(\Omega) = L^{p,\infty}(\Omega) \quad \text{if } p = 1, \infty,\\ L^{p,1}(\Omega) \hookrightarrow L^p(\Omega) = L^{p,p}(\Omega) \hookrightarrow L^{p,\infty}(\Omega) \quad \text{if } 1 < p < \infty.\end{gathered}$$

Let $1 < p < \infty$. We have the Hölder inequality and Young inequality in the Lorentz spaces:

$$\|fg\|_{L^{p,q}} \le \|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}} \quad \text{if } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{9.2}$$

$$\|fg\|_{L^1} \le \|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}} \quad \text{if } 1 = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{9.3}$$

$$\|f * g\|_{L^{p,q}(\mathbb{R}^n)} \le \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,q_2}(\mathbb{R}^n)} \tag{9.4}$$
$$\text{if } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - 1, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

where $1 \le p_1, p_2, q, q_1, q_2 \le \infty$. We often use the estimates in the Lorentz spaces on $\mathbb{R}^n$ for functions on Ω extending them by zero extension to the outside of Ω when the necessity arises.

We prove Proposition 3.5 only for the homogeneous Besov spaces $\dot{B}^s_{p,q}(A_V)$, since the inhomogeneous case is proved in an analogous way.

We prepare the following four lemmas.

Lemma 9.1. *Let $1 \le p_0 < p < \infty$ and $1 \le q \le \infty$. Assume that V satisfies* (1.1) *and* (2.3). *Then there exists a constant $C > 0$ such that*

$$\|\phi_j(\sqrt{A_V})f\|_{L^{p,q}} + \|\phi_j(\sqrt{A_0})f\|_{L^{p,q}} \le C2^{n(\frac{1}{p_0}-\frac{1}{p})j}\|f\|_{L^{p_0}}. \tag{9.5}$$

for all $j \in \mathbb{Z}$ and $f \in L^{p_0}(\Omega)$.

Proof. It is sufficient to consider the case $q = 1$ due to the embedding (9.1). Let p_1 be such that $1/p = 1/p_0 + 1/p_1 - 1$. Then it follows from the Young inequality (9.4) and the same argument as Lemma 4.1 that

$$\begin{aligned}
\|\phi_j(\sqrt{A_V})f\|_{L^{p,1}} &= \big\|e^{-2^{-2j}A_V}\{e^{2^{-2j}A_V}\phi_j(\sqrt{A_V})\}f\big\|_{L^{p,1}} \\
&\le \|G_{2^{-2j}}\|_{L^{p_1,1}(\mathbb{R}^n)}\big\|\{e^{2^{-2j}A_V}\phi_j(\sqrt{A_V})\}f\big\|_{L^{p_0,\infty}} \\
&\le C(p_1)2^{n(\frac{1}{p_0}-\frac{1}{p})j}\big\|\{e^{2^{-2j}\Delta}\phi_j(\sqrt{A_V})\}f\big\|_{L^{p_0}} \\
&\le C(p_1)2^{n(\frac{1}{p_0}-\frac{1}{p})j}\|f\|_{L^{p_0}},
\end{aligned} \tag{9.6}$$

where G_t is the function of Gaussian type appearing in the right member of (A.8), and we used the fact that

$$\|G_{2^{-2j}}\|_{L^{p_1,1}} = C(p_1)2^{n(\frac{1}{p_0}-\frac{1}{p})j} \quad \text{for } p_1 > 1.$$

Here we note that the above constant $C = C(p_1)$ is finite if and only if $p_1 > 1$, and hence, we have to assume that $p_1 > 1$, namely, $p_0 < p$. The estimate for $\phi_j(\sqrt{A_0})f$ is obtained in the same way. Thus the proof of of Lemma 9.1 is completed. □

Lemma 9.2. *Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be defined by* (2.2). *Assume that V satisfies* (1.1), (2.3) *and* (3.1). *Let $1 \le p \le \infty$. Then*

$$A_V^m\phi_j(\sqrt{A_0})f \in \mathcal{Z}_V'(\Omega) \quad \textit{and} \quad A_0^m\phi_j(\sqrt{A_V})f \in \mathcal{Z}_0'(\Omega)$$

for any $j, m \in \mathbb{Z}$ and $f \in L^p(\Omega)$.

Proof. Let $j \in \mathbb{Z}$ be fixed. Since $\phi_j(\sqrt{A_0})f \in L^p(\Omega)$ for any $f \in L^p(\Omega)$ by (4.3) for $\alpha = 0$ in Lemma 4.1, it follows from (4.25) in Lemma 4.6 that $\phi_j(\sqrt{A_0})f \in \mathcal{Z}'_V(\Omega)$. We proved the assertion (7.1) in the proof of Proposition 3.2; A_V^m is the mapping from $\mathcal{Z}'_V(\Omega)$ to itself. This proves the first assertion. In the same way, the second assertion holds. The proof of Lemma 9.2 is complete. □

Lemma 9.3. *Let* $\{\phi_j\}_{j\in\mathbb{Z}}$ *be defined by* (2.2)*, and take* $\Phi_j := \phi_{j-1}+\phi_j+\phi_{j+1}$*. Assume that* V *satisfies* (1.1)*,* (2.3) *and* (3.1)*. Then the following assertions hold*:

(i) *Let* $p = 1$ *for* $n = 2$ *and* $1 \le p < n/2$ *for* $n \ge 3$*. Then we have, for any* $f \in L^p(\Omega)$

$$\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p} \le C2^{-2(j-k)}\|f\|_{L^p}, \tag{9.7}$$

$$\|\phi_k(\sqrt{A_0})\Phi_j(\sqrt{A_V})f\|_{L^p} \le C2^{-2(k-j)}\|f\|_{L^p}. \tag{9.8}$$

(ii) *Let* $p = \infty$ *for* $n = 2$ *and* $n/(n-2) < p \le \infty$ *for* $n \ge 3$*. Then we have, for any* $f \in L^p(\Omega)$

$$\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p} \le C2^{-2(k-j)}\|f\|_{L^p}, \tag{9.9}$$

$$\|\phi_k(\sqrt{A_0})\Phi_j(\sqrt{A_V})f\|_{L^p} \le C2^{-2(j-k)}\|f\|_{L^p}. \tag{9.10}$$

Proof. We prove only (i), since the estimates (9.9) and (9.10) are obtained by the duality argument for (9.8) and (9.7), respectively.

We first consider the case $n = 2$ and $p = 1$. We note from Lemma 9.2 that

$$\Phi_k(\sqrt{A_0})f = A_V^{-1}A_V\Phi_k(\sqrt{A_0})f \quad \text{in } \mathcal{Z}'_V(\Omega).$$

Thanks to the estimate (4.3) for $\alpha = 1$ and the assumption (3.1) on V, a formal calculation implies that

$$\begin{aligned}
\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^1} &=\|\phi_j(\sqrt{A_V})A_V^{-1}A_V\Phi_k(\sqrt{A_0})f\|_{L^1}\\
&\le C2^{-2j}\Big\{\|A_0\Phi_k(\sqrt{A_0})f\|_{L^1} + \|V\Phi_k(\sqrt{A_0})f\|_{L^1}\Big\}\\
&\le C2^{-2j}\Big\{2^{2k}\|f\|_{L^1} + \|V\|_{L^1}\|\phi_k(\sqrt{A_0})f\|_{L^\infty}\Big\}\\
&\le C2^{-2j}2^{2k}\|f\|_{L^1},
\end{aligned} \tag{9.11}$$

which proves (9.7). As to the estimate (9.8), again by using (4.3) and the assumption (3.1) on V, we estimate

$$\begin{aligned}
&\|\phi_k(\sqrt{A_0})\Phi_j(\sqrt{A_V})f\|_{L^1}\\
=&\|\phi_k(\sqrt{A_0})A_0^{-1}(A_V - V)\Phi_j(\sqrt{A_V})f\|_{L^1}\\
\le& C2^{-2k}\Big\{\|A_V\Phi_j(\sqrt{A_V})f\|_{L^1} + \|V\Phi_j(\sqrt{A_V})f\|_{L^1}\Big\}\\
\le& C2^{-2k}\Big\{2^{2j}\|f\|_{L^1} + \|V\|_{L^1}\|\Phi_j(\sqrt{A_V})f\|_{L^\infty}\Big\}\\
\le& C2^{-2k}2^{2j}\|f\|_{L^1}.
\end{aligned} \tag{9.12}$$

This proves (9.8). Thus the estimate (i) for $n = 2$ and $p = 1$ is obtained.

In the case when $n \geq 3$, we estimate by the use of the Lorentz spaces. As to the estimate (9.7), by using the same argument as in (9.11), we get

$$\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p} \leq C2^{-2j}\{2^{2k}\|f\|_{L^p} + \|V\Phi_k(\sqrt{A_0})f\|_{L^p}\}. \tag{9.13}$$

Here, the Hölder inequalities (9.3) and (9.2) together with the estimate (9.5) in Lemma 9.1 imply that for $p = 1$,

$$\begin{aligned}\|V\Phi_k(\sqrt{A_0})f\|_{L^1} \leq & \|V\|_{L^{\frac{n}{2},\infty}}\|\Phi_k(\sqrt{A_0})f\|_{L^{\frac{n}{n-2},1}} \\ \leq & C2^{2k}\|f\|_{L^1},\end{aligned} \tag{9.14}$$

and for $p > 1$,

$$\begin{aligned}\|V\Phi_k(\sqrt{A_0})f\|_{L^p} \leq & \|V\|_{L^{\frac{n}{2},\infty}}\|\Phi_k(\sqrt{A_0})f\|_{L^{p_0,p}} \\ \leq & C2^{2k}\|f\|_{L^p},\end{aligned} \tag{9.15}$$

where p_0 is a real number with $1/p = 2/n + 1/p_0$. Then (9.7) is obtained by estimates (9.13)–(9.15). It remains to prove the estimate (9.8). By the same argument as in (9.12) we estimate

$$\|\phi_k(\sqrt{A_0})\Phi_j(\sqrt{A_V})f\|_{L^p} \leq C2^{-2k}\Big\{2^{2j}\|f\|_{L^p} + \|V\Phi_j(\sqrt{A_V})f\|_{L^p}\Big\}.$$

Here, it follows from the same argument as (9.14) and (9.15) that

$$\|V\Phi_j(\sqrt{A_V})f\|_{L^p} \leq C2^{2j}\|f\|_{L^p}.$$

Then, (9.8) is a consequence of the above two estimates. The proof of Lemma 9.3 is complete. □

Lemma 9.4. *Under the same assumptions as Lemma 9.3, the following assertions hold*:

(i) *Let* $1 \leq p < \infty$ *and* $0 \leq \alpha < \min\{2, n/p\}$. *Then we have*

$$\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p} \leq C2^{-\alpha(j-k)}\|f\|_{L^p}, \tag{9.16}$$

$$\|\phi_k(\sqrt{A_0})\Phi_j(\sqrt{A_V})f\|_{L^p} \leq C2^{-\alpha(k-j)}\|f\|_{L^p}. \tag{9.17}$$

for any $j, k \in \mathbb{Z}$ *and* $f \in L^p(\Omega)$.

(ii) *Let* $1 < p \leq \infty$ *and* $0 \leq \alpha < \min\{2, n(1-1/p)\}$. *Then we have*

$$\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p} \leq C2^{-\alpha(k-j)}\|f\|_{L^p}, \tag{9.18}$$

$$\|\phi_k(\sqrt{A_0})\Phi_j(\sqrt{A_V})f\|_{L^p} \leq C2^{-\alpha(j-k)}\|f\|_{L^p}. \tag{9.19}$$

for any $j, k \in \mathbb{Z}$ *and* $f \in L^p(\Omega)$.

Proof. The strategy of the proof is to apply the Riesz-Thorin interpolation theorem to the estimates in Lemma 9.3 and the following uniform estimates:

$$\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^q} \leq C\|f\|_{L^q}, \tag{9.20}$$

$$\|\phi_k(\sqrt{A_0})\Phi_j(\sqrt{A_V})f\|_{L^q} \leq C\|f\|_{L^q}, \tag{9.21}$$

for all $j, k \in \mathbb{Z}$, which are proved by (4.3) for $\alpha = 0$.

Let $0 \le \alpha < \min\{2, n/p\}$. Then the proof of (9.16) for $1 \le p < n/2$ is performed by combining (9.7) and (9.20) with $q = p$. In fact, we estimate

$$\begin{aligned}
&\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p} \\
=&\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p}^{\frac{\alpha}{2}}\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p}^{1-\frac{\alpha}{2}} \\
\le&C\{2^{-2(j-k)}\}^{\frac{\alpha}{2}}\|f\|_{L^p} \\
=&C2^{-\alpha(j-k)}\|f\|_{L^p}.
\end{aligned} \tag{9.22}$$

This proves (9.16). In a similar way, by using (9.8) and (9.21), we get the estimate (9.17). When $n/2 \le p \le \infty$, we apply the Riesz-Thorin interpolation theorem to (9.20) with $q = \infty$ and the estimate (9.7) together with the argument (9.22).

Finally, estimates (9.18) and (9.19) are proved in analogous way as in (9.16) and (9.17), if we divide the cases into $n/(n-2) < p \le \infty$ and $1 \le p \le n/(n-2)$. The proof of Lemma 9.4 is complete. □

Remark. When (2.3) is not imposed on V, which is the assumption on the inhomogeneous Besov spaces, the same estimates in Lemmas 9.1–9.4 also hold for $j, k \in \mathbb{N}$, since the proof is done analogously by applying (4.2), (A.7) instead of (4.3), (A.8), respectively.

In what follows, we prove the equivalence relation between $\dot{B}^s_{p,q}(A_0)$ and $\dot{B}^s_{p,q}(A_V)$ under the assumption on V in Proposition 3.5.

Proof of the isomorphism:

$$\dot{B}^s_{p,q}(A_0) \cong \dot{B}^s_{p,q}(A_V). \tag{9.23}$$

The case: $s > 0$**.** First we prove that

$$\dot{B}^s_{p,q}(A_0) \hookrightarrow \dot{B}^s_{p,q}(A_V) \tag{9.24}$$

for any $s > 0$. To begin with, for any $f \in \dot{B}^s_{p,q}(A_0)$, we show that

$$f = \sum_{j\in\mathbb{Z}} \phi_j(\sqrt{A_V})f \quad \text{in } \mathcal{Z}'_V(\Omega). \tag{9.25}$$

To see (9.25), we consider the formal identity

$${}_{\mathcal{Z}'_V}\langle f, g\rangle_{\mathcal{Z}_V} = \sum_{j\in\mathbb{Z}} {}_{\mathcal{Z}'_V}\langle f, \phi_j(\sqrt{A_V})g\rangle_{\mathcal{Z}_V} = \sum_{j\in\mathbb{Z}} {}_{\mathcal{Z}'_V}\langle \phi_j(\sqrt{A_V})f, g\rangle_{\mathcal{Z}_V}, \tag{9.26}$$

where the first identity is deduced from Lemma 4.5 (ii). Note that

$$f = \sum_{k\in\mathbb{Z}} \phi_k(\sqrt{A_0})f \quad \text{in } \mathcal{Z}'_0(\Omega) \tag{9.27}$$

by Lemma 4.5 (ii). Plugging (9.27) into (9.26), we can write formally

$${}_{\mathcal{Z}'_V}\langle f, g\rangle_{\mathcal{Z}_V} = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} {}_{\mathcal{Z}'_V}\langle \phi_k(\sqrt{A_0})f, \phi_j(\sqrt{A_V})g\rangle_{\mathcal{Z}_V}.$$

Then it is sufficient to show that for any $g \in \mathcal{Z}_V(\Omega)$

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \big| {}_{\mathcal{Z}_V'}\langle \phi_k(\sqrt{A_0})f, \phi_j(\sqrt{A_V})g\rangle_{\mathcal{Z}_V}\big| \le C\|f\|_{\dot B^s_{p,q}(A_0)}\|g\|_{\dot B^{-s}_{p',q'}(A_V)}, \tag{9.28}$$

since

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot B^{-s}_{p',q'}(A_V).$$

Let $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$. By using $\phi_j = \phi_j\Phi_j$ and Hölder's inequality we estimate

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \big| {}_{\mathcal{Z}_V'}\langle \phi_k(\sqrt{A_0})f, \phi_j(\sqrt{A_V})g\rangle_{\mathcal{Z}_V}\big| \\
=&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \big| {}_{\mathcal{Z}_V'}\langle \phi_j(\sqrt{A_V})\phi_k(\sqrt{A_0})f, \Phi_j(\sqrt{A_V})g\rangle_{\mathcal{Z}_V}\big| \\
\le&\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\in\mathbb{Z}}\big\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})\phi_k(\sqrt{A_0})f\big\|_{L^p}\Big)^q\Big\}^{\frac1q} \\
&\quad\times\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{-sj}\big\|\Phi_j(\sqrt{A_V})g\big\|_{L^{p'}}\Big)^{q'}\Big\}^{\frac1{q'}} \\
=:&I(s,f)\times II(s,g).
\end{aligned} \tag{9.29}$$

The estimate of the second factor $II(s,g)$ is an immediate consequence of the definition of norm of Besov spaces $\dot B^{-s}_{p',q'}(A_V)$, that is, we have

$$II(s,g) \le C\|g\|_{\dot B^{-s}_{p',q'}(A_V)}. \tag{9.30}$$

As to the first factor $I(s,f)$, applying (9.16), we have, for any $j\in\mathbb{Z}$

$$\big\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})\phi_k(\sqrt{A_0})f\big\|_{L^p} \le C\begin{cases} 2^{-\alpha(j-k)}\|\phi_k(\sqrt{A_0})f\|_{L^p} & \text{if } k\le j,\\ \|\phi_k(\sqrt{A_0})f\|_{L^p} & \text{if } k\ge j,\end{cases}$$

where α is a fixed constant such that $s<\alpha<\min\{2,n/p\}$. For the sake of simplicity, we put

$$a_k := \|\phi_k(\sqrt{A_0})f\|_{L^p}. \tag{9.31}$$

When $k\le j$, by using the above estimate, we estimate the first factor $I(s,f)$ in (9.29) as

$$\begin{aligned}
I(s,f) \le& C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\le j}2^{-\alpha(j-k)}a_k\Big)^q\Big\}^{\frac1q} \\
=&C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\ge0}2^{-(\alpha-s)k'}2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac1q} \\
\le&C\sum_{k'\ge0}2^{-(\alpha-s)k'}\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac1q} \\
\le&C\|f\|_{\dot B^s_{p,q}},
\end{aligned} \tag{9.32}$$

and when $k \geq j$, we have

$$
\begin{aligned}
I(s,f) \leq & C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\geq j}a_k\Big)^q\Big\}^{\frac{1}{q}} \\
= & C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\leq 0}2^{sk'}2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}} \\
\leq & C\sum_{k'\leq 0}2^{sk'}\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}} \\
\leq & C\|f\|_{\dot{B}^s_{p,q}}.
\end{aligned}
\tag{9.33}
$$

Summarizing (9.30)–(9.33), we conclude that the series (9.26) is absolutely convergent, and hence, the identity (9.25) is justified. Also, as a consequence of (9.32) and (9.33), we obtain

$$
\begin{aligned}
\|f\|_{\dot{B}^s_{p,q}(A_V)} \leq & \Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\in\mathbb{Z}}\big\|\phi_j(\sqrt{A_V})\phi_k(\sqrt{A_0})f\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \\
\leq & C\|f\|_{\dot{B}^s_{p,q}(A_0)}.
\end{aligned}
$$

Therefore, the embedding (9.24) holds.

It is also possible to show the embedding

$$
\dot{B}^s_{p,q}(A_V) \hookrightarrow \dot{B}^s_{p,q}(A_0)
$$

by the same argument as above, if we apply (9.17) instead of (9.16). The proof of isomorphism (9.23) for $s > 0$ is complete.

The case: $s < 0$**.** In this case, the argument for $s > 0$ works well. The only difference is to obtain estimates corresponding to (9.32) and (9.33), so that we concentrate on proving that

$$
\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\in\mathbb{Z}}\big\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})\phi_k(\sqrt{A_0})f\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} \leq C\|f\|_{\dot{B}^s_{p,q}(A_0)}. \tag{9.34}
$$

It follows from (9.16) that for any $j \in \mathbb{Z}$

$$
\big\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})\phi_k(\sqrt{A_0})f\big\|_{L^p} \leq C\begin{cases}\|\phi_k(\sqrt{A_0})f\|_{L^p} & \text{if } k \leq j,\\ 2^{-\alpha(k-j)}\|\phi_k(\sqrt{A_0})f\|_{L^p} & \text{if } k \geq j,\end{cases}
$$

where α is a fixed constant such that $|s| < \alpha < \min\{2, n(1-1/p)\}$. Then, by using the above estimate and recalling the definition (9.31) of a_k, we have for $k \le j$,

$$\begin{aligned}
&\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\le j}\big\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})\phi_k(\sqrt{A_0})f\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\le j}a_k\Big)^q\Big\}^{\frac{1}{q}}\\
=& C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k'\ge 0}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}}\\
=& C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\ge 0}2^{sk'}2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\sum_{k'\ge 0}2^{sk}\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\|f\|_{\dot{B}^s_{p,q}(A_0)},
\end{aligned}$$

and in the case when $k \ge j$, we estimate

$$\begin{aligned}
&\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\ge j}\big\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})\phi_k(\sqrt{A_0})f\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k\ge j}2^{-\alpha(k-j)}a_k\Big)^q\Big\}^{\frac{1}{q}}\\
=& C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{k'\le 0}2^{\alpha k'}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}}\\
=& C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\le 0}2^{(\alpha+s)k'}2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\sum_{k'\le 0}2^{(\alpha+s)k'}\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\|f\|_{\dot{B}^s_{p,q}(A_0)}.
\end{aligned}$$

Therefore, the estimate (9.34) is verified, and the proof of the isomorphism (9.23) for $s < 0$ is finished.

The case: $s = 0$. In this case we have only to show the corresponding estimates to (9.34). Since $1 < p < \infty$, Lemma 9.4 implies that

$$\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})f\|_{L^p} \le C2^{-\alpha|j-k|}\|f\|_{L^p},$$

$$\|\phi_k(\sqrt{A_0})\Phi_j(\sqrt{A_V})f\|_{L^p} \le C2^{-\alpha|j-k|}\|f\|_{L^p},$$

where $0 < \alpha < \min\{2, n/p, n(1-1/p)\}$. Then it follows from Young's inequality that

$$
\begin{aligned}
&\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\big\|\phi_j(\sqrt{A_V})\Phi_k(\sqrt{A_0})\phi_k(\sqrt{A_0})f\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}2^{-\alpha|j-k|}\big\|\phi_k(\sqrt{A_0})f\big\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big(\sum_{j\in\mathbb{Z}}2^{-\alpha|j|}\Big)\Big\{\sum_{k\in\mathbb{Z}}\big\|\phi_k(\sqrt{A_0})f\big\|_{L^p}^q\Big\}^{\frac{1}{q}}\\
\le& C\|f\|_{\dot{B}^0_{p,q}(A_0)}.
\end{aligned}
$$

Therefore, the case $s = 0$ also holds. Thus the proof of isomorphism (9.23) for homogeneous case is finished. □

Let us now prove the inhomogeneous case.

Proof of the isomorphism:

$$B^s_{p,q}(A_0) \cong B^s_{p,q}(A_V). \tag{9.35}$$

The proof of (9.35) is similar to the homogeneous case. Indeed, as to the proof of the embedding

$$B^s_{p,q}(A_0) \hookrightarrow B^s_{p,q}(A_V),$$

the main point is to show that for any $f \in B^s_{p,q}(A_0)$,

$$f = \psi(A_V)f + \sum_{j\in\mathbb{N}}\phi_j(\sqrt{A_V})f \quad \text{in } \mathcal{X}'_V(\Omega).$$

This identity is obtained by using the following estimate:

$$
\begin{aligned}
&\big|{}_{\mathcal{X}'_V}\langle\psi(A_0)f, \psi(A_V))g\rangle_{\mathcal{X}_V}\big| + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}\big|{}_{\mathcal{X}'_V}\langle\phi_k(\sqrt{A_0})f, \phi_j(\sqrt{A_V})g\rangle_{\mathcal{X}_V}\big|\\
&+\sum_{j\in\mathbb{N}}\big|{}_{\mathcal{X}'_V}\langle\psi_k(A_0)f, \phi_j(\sqrt{A_V})g\rangle_{\mathcal{X}_V}\big| + \sum_{k\in\mathbb{N}}\big|{}_{\mathcal{X}'_V}\langle\phi_k(\sqrt{A_0})f, \psi(A_V))g\rangle_{\mathcal{X}_V}\big|\\
\le& C\|f\|_{B^s_{p,q}(A_V)}\|g\|_{B^{-s}_{p',q'}(A_V)}
\end{aligned}
$$

for any $g \in \mathcal{X}_V(\Omega)$. The proof of the above estimate is analogous to those of (9.32) and (9.33) by taking the sum over $j, k \in \mathbb{N}$. So we may omit the details. Thus we conclude (9.35). □

Appendix A. (L^p-boundedness, self-adjointness and pointwise estimates for e^{-tA_V})

We discuss the uniform L^p-boundedness of $\phi(\theta A_V)$ in this appendix.

Proposition A.1. *Let $\phi \in \mathcal{S}(\mathbb{R})$ and $1 \le p \le \infty$.*

(i) *Assume that V satisfies (1.1). Then*

$$\sup_{0<\theta\le 1}\|\phi(\theta A_V)\|_{L^p\to L^p} < \infty. \tag{A.1}$$

(ii) *Assume that V satisfies* (1.1) *and* (2.3). *Then*

$$\sup_{0<\theta<\infty} \|\phi(\theta A_V)\|_{L^p\to L^p} < \infty. \tag{A.2}$$

Remark. We note that the potential like

$$V(x) \simeq -c|x|^{-2} \quad \text{as } |x| \to \infty, \quad c > 0$$

is very interesting. However, it is excluded from assumption (2.3) on V. The reason is that the uniform boundedness in Proposition A.1 would not be generally obtained, since

$$\lim_{t\to\infty} \|e^{-tA_V}\|_{L^p\to L^p} = \infty$$

for some $p \neq 2$ which was proved in [17, 18].

The proof of Proposition A.1 is similar to that of our previous works [19, 20] by using Lemmas A.2–A.4 below and we may omit the complete proof of Proposition A.1. So, we shall concentrate on the proof of the self-adjointness of A_V and the pointwise estimate of integral kernel of e^{-tA_V}, which need certain adjustment to the method in [20].

Let us prove that A_V is self-adjoint on $L^2(\Omega)$. Following the argument in [30] (see also [19]), we consider the quadratic form q defined by letting

$$q(u,v) := \int_\Omega \nabla u(x)\cdot \overline{\nabla v(x)}\,dx + \int_\Omega V(x)u(x)\overline{v(x)}\,dx, \quad u,v \in \mathcal{Q}(q),$$

where $\mathcal{Q}(q) := \{u \in H^1_0(\Omega) \,|\, \sqrt{V_+}u \in L^2(\Omega)\}$.

Lemma A.2. *Assume that the measurable potential V satisfies* (1.1). *Then there exists a self-adjoint operator A_V on $L^2(\Omega)$ such that*

$$\begin{cases} \mathcal{D}(A_V) = \left\{ u \in \mathcal{Q}(q) \,\middle|\, {}^\exists w_u \in L^2(\Omega) \text{ s.t. } \displaystyle\int_\Omega w_u \overline{v}\,dx = q(u,v) \text{ for all } v \in \mathcal{Q}(q) \right\}, \\ A_V u = w_u \quad \text{for } u \in \mathcal{D}(A_V). \end{cases}$$

Moreover, A_V is semi-bounded, i.e., there exists a constant $\lambda_0 > 0$ such that

$$A_V \geq -\lambda_0^2 I.$$

Remark. We define $A_V u \in L^2(\Omega)$ for $u \in \mathcal{Q}(q)$ if $u \in \mathcal{D}(A_V)$. Then $\mathcal{D}(A_V)$ is simply rewritten as

$$\begin{aligned} \mathcal{D}(A_V) &= \{u \in \mathcal{Q}(q) \,|\, A_V u \in L^2(\Omega)\} \\ &= \{u \in H^1_0(\Omega) \,|\, \sqrt{V_+}u \in L^2(\Omega), A_V u \in L^2(\Omega)\}. \end{aligned}$$

This is nothing but the identity (1.2) given in §1.

To prove Lemma A.2, we need the following lemma.

Lemma A.3. ([9, 19, 34]) *Assume that V_- is in the Kato class $K_n(\Omega)$. Then for any $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that*

$$\int_\Omega V_-(x)|f(x)|^2\,dx \leq \varepsilon\|\nabla f\|^2_{L^2} + \lambda_0^2\|f\|^2_{L^2} \tag{A.3}$$

for any $f \in H_0^1(\Omega)$.

Proof. By the density argument, we may take $f \in C_0^\infty(\Omega)$. Let $\tilde{f}$ and $\tilde{V}_-$ be the zero extension of f and V_- to $\mathbb{R}^n$, respectively. Then (A.3) is equivalent to

$$\int_{\mathbb{R}^n} \tilde{V}_-(x)|\tilde{f}(x)|^2\, dx \le \varepsilon \|\nabla \tilde{f}\|^2_{L^2(\mathbb{R}^n)} + \lambda_0^2 \|\tilde{f}\|^2_{L^2(\mathbb{R}^n)}.$$

Since this inequality is proved in [9, 34], we may omit the details. The proof of Lemma A.3 is finished. □

Proof of Lemma A.2. It suffices to show that the quadratic form q is closed and semi-bounded by Theorem VIII.15 in [30] (see also Lemma 2.3 in [19]). We first show that q is closed. Put

$$q_1(u,v) = \int_\Omega \nabla u(x)\cdot \overline{\nabla v(x)}\, dx - \int_\Omega V_-(x)u(x)\overline{v(x)}\, dx, \quad u,v \in \mathcal{Q}(q_1) := H_0^1(\Omega),$$

$$q_2(u,v) = \int_\Omega V_+(x)u(x)\overline{v(x)}\, dx, \quad u,v \in \mathcal{Q}(q_2) := \{u \in L^2(\Omega) \,|\, \sqrt{V_+}u \in L^2(\Omega)\}.$$

Then we get

$$q(u,v) = q_1(u,v) + q_2(u,v), \quad u,v \in \mathcal{Q}(q) = \mathcal{Q}(q_1)\cap \mathcal{Q}(q_2).$$

Since q_1 is closed (see Proposition 2.1 in [19]) and the sum of two closed quadratic forms is also closed, it is enough to show that q_2 is closed.

We show that q_2 is closed. Put $q_2(u) = q_2(u,u)$ for simplicity. Assume that

$$u \in L^2(\Omega), \quad u_j \in \mathcal{Q}(q_2), \quad q_2(u_j - u_k) \to 0, \quad \|u_j - u\|_{L^2} \to 0 \quad \text{as } j,k \to \infty,$$

and we prove that

$$u \in \mathcal{Q}(q_2) \quad \text{and} \quad q_2(u_j - u) \to 0 \quad \text{as } j \to \infty. \tag{A.4}$$

Since $\{\sqrt{V_+}u_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^2(\Omega)$, there exists $v \in L^2(\Omega)$ such that

$$\sqrt{V_+}u_j \to v \quad \text{in } L^2(\Omega).$$

Hence the sequence $\{\sqrt{V_+}u_j\}_{j=1}^\infty$ converges to v almost everywhere along a subsequence denoted by the same, namely,

$$\sqrt{V_+}u_j(x) \to v(x) \quad \text{for almost every } x \in \Omega \text{ as } j \to \infty.$$

On the other hand, since any convergent sequence in $L^2(\Omega)$ contains a subsequence which converges almost everywhere in Ω, it follows that

$$\sqrt{V_+}u_j(x) \to \sqrt{V_+}u(x) \quad \text{for almost every } x \in \Omega \text{ as } j \to \infty.$$

Summarizing three convergences obtained now, we get $\sqrt{V_+}u = v \in L^2(\Omega)$. This proves (A.4).

Finally we prove that q is semi-bounded. Let $u \in \mathcal{D}(A_V)$. By the estimate (A.3), for any $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that

$$\begin{aligned}\int_\Omega (A_V u)\overline{u}\, dx &\ge \int_\Omega |\nabla u|^2 dx - \int_\Omega V_-|u|^2\, dx \\ &\ge \|\nabla u\|^2_{L^2} - \varepsilon\|\nabla u\|^2_{L^2} - \lambda_0^2\|u\|^2_{L^2}.\end{aligned} \tag{A.5}$$

Taking $\varepsilon = 1$, we obtain

$$\int_\Omega (A_V u)\overline{u}\, dx \geq -\lambda_0^2 \|u\|_{L^2(\Omega)}^2, \tag{A.6}$$

which implies that q is semi-bounded. This ends the proof of Lemma A.2. □

As to the pointwise estimate on the kernel of e^{-tA_V}, we have the following.

Lemma A.4. *The integral kernel $e^{-tA_V}(x,y)$ of the semi-group $\{e^{-tA_V}\}_{t\geq 0}$ enjoys the following estimates*:

(i) *Assume that V satisfies* (1.1)*. Then there exist constants $\omega, C > 0$ such that*

$$|e^{-tA_V}(x,y)| \leq Ce^{\omega t} t^{-\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{Ct}\Big)$$

for any $t > 0$ and $x, y \in \Omega$. In particular, we have

$$|e^{-tA_V}(x,y)| \leq Ct^{-\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{Ct}\Big) \quad \textit{if} \ \ 0 < t \leq 1. \tag{A.7}$$

(ii) *Assume that V satisfies* (1.1) *and* (2.3)*. Then there exists $C > 0$ such that*

$$|e^{-tA_V}(x,y)| \leq Ct^{-\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{Ct}\Big) \tag{A.8}$$

for any $t > 0$ and $x, y \in \Omega$.

Proof. Since the assertions (i) with $n \geq 1$ and (ii) with $n = 1, 2$ were already proved in [26], it is enough to show (ii) in the case when $n \geq 3$. We prove the lemma for $n \geq 3$ in a formal way for the sake of simplicity. For more rigorous argument, see [19, 20].

Put

$$V_* := -V_-.$$

It is proved in Proposition 3.1 from [19] that if V_* satisfies assumption (2.3), then

$$|e^{-tA_{V_*}}(x,y)| \leq Ct^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{Ct}} \quad \text{for } t > 0, \quad x, y \in \Omega.$$

Once the following inequality

$$|e^{-tA_V}(x,y)| \leq |e^{-tA_{V_*}}(x,y)| \tag{A.9}$$

is proved, the proof of the lemma is complete. So, we prove (A.9). Let

$$u^{(1)}(t) := e^{-tA_{V_*}} f, \quad u^{(2)}(t) := e^{-tA_V} f,$$
$$u(t) := u^{(1)}(t) - u^{(2)}(t), \quad u_-(t) := -\min\{u(t), 0\},$$

where $f \in C_0^\infty(\Omega)$ is non-negative. Note that $u^{(1)}(t) \in \mathcal{D}(A_{V_*})$ and $u^{(2)}(t) \in \mathcal{D}(A_V)$ for any $t > 0$. An explicit calculation and non-negativity of A_{V_*} imply that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \big(u_-\big)^2\,dx &= -\int_\Omega (\partial_t u)u_-\,dx \\
&= \int_\Omega \big(A_{V_*}u\big)u_-\,dx - \int_\Omega V_+u^{(2)}(t)u_-\,dx \\
&= -\int_\Omega \big(|\nabla u_-|^2 - V_-(u_-)^2\big)\,dx - \int_\Omega V_+u^{(2)}(t)u_-\,dx \\
&\le -\int_\Omega V_+u^{(2)}(t)u_-\,dx.
\end{aligned}$$

For the negative part of $u^{(2)}$, i.e.,

$$u_-^{(2)}(t) := -\min\{u^{(2)}(t), 0\},$$

it follows from the analogous argument to the above that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \big(u_-^{(2)}\big)^2\,dx &= \int_\Omega \big(A_Vu^{(2)}\big)u_-^{(2)}\,dx \\
&= -\int_\Omega \big(|\nabla u_-^{(2)}|^2 - V_-(u_-^{(2)})^2\big)\,dx - \int_\Omega V_+\big(u_2^{(2)}\big)^2\,dx \\
&\le 0.
\end{aligned}$$

The above two inequalities imply that $\|u_-(t)\|_{L^2}^2$ and $\|u_-^{(2)}(t)\|_{L^2}^2$ do not increase. Hence we conclude that

$$u_-(t) = u_-^{(2)}(t) = 0 \quad \text{in } L^2(\Omega)$$

for all $t \ge 0$, since $u_-(0) = 0$, $u_-^{(2)}(0) = 0$. Therefore, we get

$$0 \le e^{-tA_V}f \le e^{-tA_{V_*}}f.$$

For each point $x_0 \in \Omega$, by taking $f = f_k$ $(k = 1, 2, \cdots)$ which tend to the delta function supported at x_0, we see that the kernel of e^{-tA_V} is bounded by that of $e^{-tA_{V_*}}$. Thus (A.9) is proved. This ends the proof of Lemma A.4. □

Appendix B

In this appendix we prove that zero is not an eigenvalue of A_V.

Lemma B.1. *Assume that V satisfies* (1.1) *and*

$$\begin{cases} V_- = 0 & \text{if } n = 1, 2, \\ \sup\limits_{x\in\Omega}\displaystyle\int_\Omega \frac{|V_-(y)|}{|x-y|^{n-2}}\,dy < \frac{4\pi^{\frac{n}{2}}}{\Gamma(n/2-1)} & \text{if } n \ge 3, \end{cases} \tag{B.1}$$

where $\Gamma(\cdot)$ is the Gamma function. Then A_V is non-negative on $L^2(\Omega)$, and zero is not an eigenvalue of A_V.

For the difference between assumptions (2.3) and (B.1), we refer to Proposition A.1 and Lemma A.4 (ii) (cf. Proposition 3.1 in [19] and Proposition 5.1 in [9]).

To prove Lemma B.1, we need the following lemma, which is proved in [19].

Lemma B.2. ([9, 19]) *Assume that* $n \geq 3$. *Suppose that* V_- *satisfies*

$$\|V_-\|_{K_n(\Omega)} := \sup_{x\in\Omega} \int_\Omega \frac{|V_-(y)|}{|x-y|^{n-2}}\, dy < \infty.$$

Then

$$\int_\Omega V_-(x)|f(x)|^2\, dx \leq \frac{\Gamma(n/2-1)\|V_-\|_{K_n(\Omega)}}{4\pi^{n/2}} \|\nabla f\|^2_{L^2(\Omega)} \tag{B.2}$$

for any $f \in H^1_0(\Omega)$.

Proof. Since the proof is similar to Lemma A.3, we may omit the details. The proof of Lemma B.2 is finished. □

Proof of Lemma B.1. We prove that if f satisfies

$$f \in \mathcal{D}(A_V) \quad \text{and} \quad A_V f = 0 \text{ in } L^2(\Omega), \tag{B.3}$$

then $f = 0$. By Lemma B.2 and assumption (B.1) on V_- we have

$$\int_\Omega V_-|f|^2\, dx \leq \gamma_n \|\nabla f\|^2_{L^2},$$

where γ_n is a constant such that $0 < \gamma_n < 1$ if $n \geq 3$ and $\gamma_n = 0$ if $n = 1, 2$. Then we find from assumption (B.3) that

$$\begin{aligned} 0 = &\int_\Omega (A_V f)\overline{f}\, dx = \int_\Omega \left(|\nabla f|^2 - V_-|f|^2\right) dx + \int_\Omega V_+|f|^2\, dx \\ \geq &(1-\gamma_n)\|\nabla f\|^2_{L^2}, \end{aligned}$$

which implies that $f = 0$, since $f \in \mathcal{D}(A_V) \subset H^1_0(\Omega)$. Finally, the above inequality also implies that A_V is non-negative on $L^2(\Omega)$. This ends the proof of Lemma B.1. □

References

[1] J. J. Benedetto and S. Zheng, *Besov spaces for the Schrödinger operator with barrier potential*, Complex Anal. Oper. Theory **4** (2010), no. 4, 777–811.

[2] O. V. Besov, *On some families of functional spaces. Imbedding and extension theorems*, Dokl. Akad. Nauk SSSR **126** (1959), 1163–1165 (Russian).

[3] O. V. Besov, *Investigation of a class of function spaces in connection with imbedding and extension theorems*, Trudy. Mat. Inst. Steklov. **60** (1961), 42–81 (Russian).

[4] H.-Q. Bui, X. T. Duong, and L. Yan, *Calderón reproducing formulas and new Besov spaces associated with operators*, Adv. Math. **229** (2012), no. 4, 2449–2502.

[5] H.-Q. Bui, M. Paluszyński, and M. H. Taibleson, *A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces*, Studia Math. **119** (1996), no. 3, 219–246.

[6] ———, *Characterization of the Besov-Lipschitz and Triebel-Lizorkin spaces. The case $q < 1$*, Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996), 1997, pp. 837–846.

[7] T. A. Bui and X. T. Duong, *Besov and Triebel-Lizorkin spaces associated to Hermite operators*, J. Fourier Anal. Appl. **21** (2015), no. 2, 405–448.

[8] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, Oxford Lecture Series in Mathematics and its Applications, vol. 13, The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[9] P. D'Ancona and V. Pierfelice, *On the wave equation with a large rough potential*, J. Funct. Anal. **227** (2005), no. 1, 30–77.

[10] R. A. DeVore and R. C. Sharpley, *Besov spaces on domains in* $\mathbf{R}^d$, Trans. Amer. Math. Soc. **335** (1993), no. 2, 843–864.
[11] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), no. 4, 777–799.
[12] ———, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), no. 1, 34–170.
[13] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.
[14] V. Georgiev and N. Visciglia, *Decay estimates for the wave equation with potential*, Comm. Partial Differential Equations **28** (2003), no. 7-8, 1325–1369.
[15] L. Grafakos, *Classical Fourier analysis*, 3rd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014.
[16] ———, *Modern Fourier analysis*, 3rd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014.
[17] N. Ioku, K. Ishige, and E. Yanagida, *Sharp decay estimates of* L^q*-norms for nonnegative Schrödinger heat semigroups*, J. Funct. Anal. **264** (2013), no. 12, 2764–2783.
[18] ———, *Sharp decay estimates in Lorentz spaces for nonnegative Schrödinger heat semigroups*, J. Math. Pures Appl. (9) **103** (2015), no. 4, 900–923 (English, with English and French summaries).
[19] T. Iwabuchi, T. Matsuyama, and K. Taniguchi, L^p*–mapping properties for the Schrödinger operators in open sets of* $\mathbb{R}^d$, preprint.
[20] ———, L^p*-boundedness of functions of Schrödinger operators on an open set of* $\mathbb{R}^d$, preprint.
[21] A. Jensen and S. Nakamura, *Mapping properties of functions of Schrödinger operators between* L^p*-spaces and Besov spaces*, Spectral and scattering theory and applications, Adv. Stud. Pure Math., vol. 23, Math. Soc. Japan, Tokyo.
[22] ———, *Mapping properties of functions of Schrödinger operators between* L^p*-spaces and Besov spaces*, Spectral and scattering theory and applications, Adv. Stud. Pure Math., vol. 23, Math. Soc. Japan, Tokyo, 1994, pp. 187–209.
[23] G. Kerkyacharian and P. Petrushev, *Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces*, Trans. Amer. Math. Soc. **367** (2015), no. 1, 121–189.
[24] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations **19** (1994), no. 5-6, 959–1014.
[25] T. Muramatsu, *On Besov spaces and Sobolev spaces of generalized functions defined on a general region*, Publ. Res. Inst. Math. Sci. **9** (1973/74), 325–396. MR 0341063 (49 #5813)
[26] E. M. Ouhabaz, *Sharp Gaussian bounds and* L^p*-growth of semigroups associated with elliptic and Schrödinger operators*, Proc. Amer. Math. Soc. **134** (2006), no. 12, 3567–3575 (electronic).
[27] J. Peetre, *Sur les espaces de Besov*, C. R. Acad. Sci. Paris Sér. A-B **264** (1967), A281–A283 (French).
[28] ———, *On spaces of Triebel-Lizorkin type*, Ark. Mat. **13** (1975), 123–130.
[29] ———, *New thoughts on Besov spaces*, Mathematics Department, Duke University, Durham, N.C., 1976. Duke University Mathematics Series, No. 1.
[30] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York-London, 1972.
[31] M. Ruzhansky and N. Tokmagambetov, *Nonharmonic analysis of boundary value problems*, Int. Math. Res. Notices, doi:10.1093/imrn/rnv243.
[32] V. S. Rychkov, *Intrinsic characterizations of distribution spaces on domains*, Studia Math. **127** (1998), no. 3, 277–298.
[33] V. S. Rychkov, *On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains*, J. London Math. Soc. (2) **60** (1999), no. 1, 237–257.
[34] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 3, 447–526.

[35] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.
[36] H. Triebel, *Interpolation theory, function spaces, differential operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
[37] ______, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.
[38] ______, *Theory of function spaces. II*, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.
[39] ______, *Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers*, Rev. Mat. Complut. **15** (2002), no. 2, 475–524.
[40] ______, *Theory of function spaces. III*, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006.
[41] H. Triebel and H. Winkelvoß, *Intrinsic atomic characterizations of function spaces on domains*, Math. Z. **221** (1996), no. 4, 647–673.
[42] W. P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

Tsukasa Iwabuchi
Department of Mathematics
Osaka City University
3-3-138 Sugimoto, Sumiyoshi-ku
Osaka 558-8585
Japan
E-mail address: iwabuchi@sci.osaka-cu.ac.jp

Tokio Matsuyama
Department of Mathematics
Chuo University
1-13-27, Kasuga, Bunkyo-ku
Tokyo 112-8551
Japan
E-mail address: tokio@math.chuo-u.ac.jp

Koichi Taniguchi
Department of Mathematics
Chuo University
1-13-27, Kasuga, Bunkyo-ku
Tokyo 112-8551
Japan
E-mail address: koichi-t@gug.math.chuo-u.ac.jp