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**Decay estimates for wave equation  
with a potential on exterior domains**

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# DECAY ESTIMATES FOR WAVE EQUATION WITH A POTENTIAL ON EXTERIOR DOMAINS

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ABSTRACT. The purpose of the present paper is to establish the local energy decay estimates and dispersive estimates for 3-dimensional wave equation with a potential to the initial-boundary value problem on exterior domains. The geometrical assumptions on domains are rather general, for example non-trapping condition is not imposed in the local energy decay result. As a by-product, Strichartz estimate is obtained too.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $\Omega$  be an exterior domain in  $\mathbb{R}^3$  such that the obstacle

$$\mathcal{O} := \mathbb{R}^3 \setminus \Omega$$

is compact and its boundary  $\partial\Omega$  is of  $C^{2,1}$ . For the sake of simplicity, we assume that the origin does not belong to  $\overline{\Omega}$ .

In this work we consider the initial-boundary value problem for the wave equations with a potential in the exterior domain  $\Omega$  and our main goals are to study the local energy decay estimates and dispersive estimates for the corresponding evolution flow.

The study of Strichartz estimates for the Cauchy problem to wave equation has its origin in the paper of Strichartz (see [41]). After him, many authors generalized them (see [17, 20, 21, 25] etc) as well as dispersive estimates (see [6, 7, 26, 34]). These estimates for wave equation with potentials  $V(x)$  are also of great interest, and are expressed by

$$(1.1) \quad \left\| e^{it\sqrt{-\Delta+V}} f \right\|_{L^\infty(\mathbb{R}^n)} \leq C |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,2}^{\frac{n+1}{2}}(\mathbb{R}^n)}, \quad (\text{dispersive estimates}),$$

where  $\dot{B}_{1,2}^{\frac{n+1}{2}}(\mathbb{R}^n)$  is the homogeneous Besov space, and

$$(1.2) \quad \left\| e^{it\sqrt{-\Delta+V}} f \right\|_{L^q(\mathbb{R}; \dot{H}^{\frac{1}{p}-\frac{1}{q}-\frac{1}{2}, p}(\mathbb{R}^n))} \leq C \|f\|_{L^2(\mathbb{R}^n)}, \quad (\text{Strichartz estimates}),$$

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where  $\dot{H}^{\frac{1}{p}-\frac{1}{q}-\frac{1}{2},p}(\mathbb{R}^n)$  are the homogeneous Sobolev spaces and  $p, q$  satisfy the admissible condition:

$$\frac{2}{q} + \frac{n-1}{p} = \frac{n-1}{2}, \quad 2 \leq q \leq \infty, \quad 2 \leq p \leq \frac{2(n-1)}{n-3}, \quad p \neq \infty.$$

There are many results on the dispersive estimates (1.1) or Strichartz estimates (1.2) for the short-range type potentials. Let us overview the known results on the estimates (1.1) and (1.2). When  $n = 3$ , Cuccagna considered the potential  $V(x)$  like

$$(1.3) \quad |V(x)| \leq \frac{C}{(1+|x|^2)^{\delta/2}} \quad \text{for some } \delta > 3,$$

and proved the estimates (1.2) together with (1.1) (see [11]). We also find the previous known results on more restrictive assumptions on potentials (see, e.g., Beals [3], Beals and Strauss [4], and also Georgiev, Heiming and Kubo [15]). When  $n \geq 4$  and  $V(x)$  satisfies (1.3) for some  $\delta > (n+1)/2$ , Vodev proved  $L^p$ - $L^{p'}$ -estimates in high frequency (see [42], and also Cardoso and Vodev [10]). Moulin compensated the estimates in low frequency for the potentials of Kato class (see [33]). However, it is assumed in all of the above results that

$$(1.4) \quad \text{zero is neither an eigenvalue nor a resonance for the operator } -\Delta + V(x).$$

After these results, the estimates without appealing to the assumption (1.4) on operator  $-\Delta + V(x)$  were revealed by some authors. Yajima clarified the spectrum for the Schrödinger operators (see [45]), and obtained  $L^p$ - $L^{p'}$ -estimates for the Schrödinger equations. When  $n = 3$  and  $V(x)$  behaves like

$$0 \leq V(x) \leq \frac{C}{|x|^2(|x|^\varepsilon + |x|^{-\varepsilon})} \quad \text{for } \varepsilon > 0,$$

Georgiev and Visciglia established the estimates (1.1) and (1.2) (see [16]). D'Ancona and Pierfelice also established the dispersive estimate (1.1) in the case when  $n = 3$  and  $V(x)$  is a potential of Kato class (see [14]), and D'Ancona and Fannelli proved Strichartz estimates (1.2) for wave equation with the magnetic potentials (see [12]).

Contrary to the Cauchy problem in  $\mathbb{R}^n$ , there are no results on the optimal dispersive estimates for wave equation with potentials on exterior domains. As to wave equation (without potentials) in non-trapping exterior domains, Shibata and Tsutsumi proved  $L^p$ - $L^{p'}$ -estimates with some derivative loss of data, and applied them to get global small amplitude solutions to nonlinear wave equations (see [39]). Besides, there are only a few results on Strichartz estimates in exterior problems. For wave equation with perturbed Laplacian in non-trapping exterior domains, Smith and Sogge studied the corresponding Strichartz estimates in 3-dimensional space (see [40]), and Burq and Metcalfe extended to higher spatial dimensions greater than or equal to 4 independently (see [8, 27]). After them, some authors have investigated Strichartz estimates for wave equation with a potential in an exterior domain outside a star-shaped obstacle; Metcalfe and Tataru proved these estimates for hyperbolic equations with variable coefficients under a certain long-range type of potentials (see [28]).

The present paper is devoted to the investigation of local energy decay estimates and dispersive estimates, or even  $L^p$ - $L^{p'}$ -estimates for wave equation with a potential in exterior domains without appealing to the non-trapping condition on  $\Omega$  (see Theorem 1.1 and Theorem 1.3 below). The strategy of proof is based on spectral analysis. Also, it is not assumed that zero is neither an eigenvalue nor a resonance for the operator  $-\Delta + V(x)$  on the exterior domain. As a by-product of these estimates, Strichartz estimates will be obtained in Theorem 1.4 by using  $TT^*$  argument of Ginibre and Velo (see [17], and also Yajima [46]).

We now formulate the problem more precisely. In this paper we are concerned with the following initial-boundary value problem, for a function  $u = u(t, x)$ :

$$(1.5) \quad \partial_t^2 u - \Delta u + V(x)u = 0, \quad t \neq 0, \quad x \in \Omega,$$

with the initial condition

$$(1.6) \quad u(0, x) = f(x), \quad \partial_t u(0, x) = g(x),$$

and the boundary condition

$$(1.7) \quad u(t, x) = 0, \quad t \in \mathbb{R}, \quad x \in \partial\Omega,$$

where  $V$  is a real-valued measurable function on  $\Omega$  satisfying

$$(1.8) \quad -c_0|x|^{-\delta_0} \leq V(x) \leq c_1|x|^{-\delta_0} \quad \text{for some } 0 < c_0 < \frac{1}{4}, \quad c_1 > 0 \text{ and } \delta_0 > 2.$$

Let us introduce some operators and function spaces. We denote by

$$\mathbb{G}_0 = -\Delta \text{ the free Hamiltonian in } \mathbb{R}^3,$$

and by

$$\mathbb{G}_V \text{ a self-adjoint realization on } L^2(\Omega) \text{ of the operator } -\Delta|_D + V,$$

where

$$\mathbb{G} := -\Delta|_D \text{ is the Dirichlet Laplacian}$$

with domain

$$D(\mathbb{G}) = D(\mathbb{G}_V) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then  $\mathbb{G}_V$  is non-negative on  $L^2(\Omega)$  on account of (1.8). It will be shown in Proposition A.1 that zero is neither an eigenvalue nor a resonance of  $\mathbb{G}_V$  (see appendix A). Also, it is known that no eigenvalues are present on  $(0, \infty)$  (see Mochizuki [29], and also (1.19) below). Hence the continuous spectrum of  $\mathbb{G}_V$  coincides with the interval  $[0, \infty)$ . The main theorem involves the perturbed Besov spaces  $\dot{B}_{p,q}^s(\mathbb{G}_V)$  generated by  $\mathbb{G}_V$ . Following Iwabuchi, Matsuyama and Taniguchi [19], we define these spaces in the following way. Let  $\{\varphi_j(\lambda)\}_{j=-\infty}^{\infty}$  be the Littlewood-Paley partition of unity:  $\varphi(\lambda)$  is a non-negative function having its compact support in  $\{\lambda : 1/2 \leq \lambda \leq 2\}$  such that

$$(1.9) \quad \sum_{j=-\infty}^{+\infty} \varphi(2^{-j}\lambda) = 1 \quad (\lambda \neq 0), \quad \varphi_j(\lambda) = \varphi(2^{-j}\lambda), \quad (j \in \mathbb{Z}).$$

For any  $s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$  we define the homogeneous Besov spaces  $\dot{B}_{p,q}^s(\mathbb{G}_V)$  by letting

$$(1.10) \quad \dot{B}_{p,q}^s(\mathbb{G}_V) := \{f \in \mathcal{Z}'_V(\Omega) : \|f\|_{\dot{B}_{p,q}^s(\mathbb{G}_V)} < \infty\},$$

where

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{G}_V)} := \left\| \left\{ 2^{sj} \|\varphi_j(\sqrt{\mathbb{G}_V})f\|_{L^p(\Omega)} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})}.$$

Here  $\mathcal{Z}'_V(\Omega)$  is the dual space of a linear topological space  $\mathcal{Z}_V(\Omega)$  which is defined by letting

$$\mathcal{Z}_V(\Omega) := \left\{ f \in L^1(\Omega) \cap D(\mathbb{G}_V) : \mathbb{G}_V^M f \in L^1(\Omega) \cap D(\mathbb{G}_V) \quad \text{and} \right. \\ \left. \sup_{j \leq 0} 2^{M|j|} \|\varphi_j(\sqrt{\mathbb{G}_V})f\|_{L^1(\Omega)} < \infty \text{ for all } M \in \mathbb{N} \right\}$$

equipped with the family of semi-norms  $\{q_{V,M}(\cdot)\}_{M=1}^\infty$  given by

$$q_{V,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j \in \mathbb{Z}} 2^{M|j|} \|\varphi_j(\sqrt{\mathbb{G}_V})f\|_{L^1(\Omega)}.$$

It is proved in Theorem 2.5 from [19] that the norms  $\|f\|_{\dot{B}_{p,q}^s(\mathbb{G}_V)}$  are independent of the choice of  $\varphi_j$ . We shall also use the perturbed Sobolev spaces over  $\Omega$ :

$$\dot{H}_V^s(\Omega) := \dot{B}_{2,2}^s(\mathbb{G}_V).$$

In particular case  $V = 0$ , replacing  $\varphi_j(\sqrt{\mathbb{G}_V})$  by  $\varphi_j(\sqrt{\mathbb{G}})$  in the definition, we define

$$\dot{B}_{p,q}^s(\mathbb{G}) \quad \text{and} \quad \dot{H}^s(\Omega) = \dot{B}_{2,2}^s(\mathbb{G}),$$

where we recall

$$\mathbb{G} = -\Delta|_D$$

with domain

$$D(\mathbb{G}) = H^2(\Omega) \cap H_0^1(\Omega).$$

We shall use the inhomogeneous Sobolev spaces  $H_V^s(\Omega)$  for  $s > 0$ . We say that  $f \in H_V^s(\Omega)$  ( $f \in H^s(\Omega)$  resp.) for  $s > 0$  if

$$\|(I + \mathbb{G}_V)^{s/2} f\|_{L^2(\Omega)} < \infty \quad (\|(I + \mathbb{G})^{s/2} f\|_{L^2(\Omega)} < \infty \text{ resp.})$$

Before stating the results, we introduce a class of potentials of generic type in  $L_{-s}^2(\Omega)$ . Here  $L_{-s}^2(\Omega)$  is the weighted  $L^2$ -spaces whose definitions are as follows: For a non-negative integer  $m$  and real number  $\kappa$ , we define the weighted Sobolev spaces  $H_\kappa^m(\Omega)$  by letting

$$H_\kappa^m(\Omega) = \{f : \langle x \rangle^\kappa \partial_x^\alpha f \in L^2(\Omega), |\alpha| \leq m\}, \quad \langle x \rangle = \sqrt{1 + |x|^2},$$

and in particular, we put

$$L_\kappa^2(\Omega) = H_\kappa^0(\Omega).$$

Let  $R(\lambda^2 \pm i0)$  be the resolvent operators of  $\mathbb{G}$ :

$$(1.11) \quad R(\lambda^2 \pm i0) = s - \lim_{\varepsilon \searrow 0} (\mathbb{G} - (\lambda^2 \pm i\varepsilon)I)^{-1} \quad \text{in } \mathcal{B}(L_s^2(\Omega), H_{-s}^2(\Omega))$$

for some  $s > 1/2$  and for any  $\lambda > 0$ . The space  $\mathcal{B}(L_s^2(\Omega), H_{-s}^2(\Omega))$  consists of all bounded linear operators from  $L_s^2(\Omega)$  to  $H_{-s}^2(\Omega)$ . The existence of these limits is called the limiting absorption principle, and the limits (1.11) certainly exist (see, e.g.,

Mochizuki [31] and Wilcox [43]). It should be noted that (1.11) will be established in Lemma 2.6 below without appealing to [31] and [43]. Referring to Yajima [45], we define the null space of  $I + R(0)V$  by letting

$$\mathcal{M} = \{u \in L^2_{-s}(\Omega) : u + R(0)Vu = 0 \text{ in } \Omega\}$$

for some  $1 < s \leq \delta_0/2$ . Now, any  $u \in \mathcal{M}$  satisfies the boundary value problem for the stationary Schrödinger equation:

$$(1.12) \quad \begin{cases} -\Delta u + V(x)u(x) = 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Conversely, any function  $u \in L^2_{-s}(\Omega)$  for some  $1 < s \leq \delta_0/2$  satisfying (1.12) belongs to  $\mathcal{M}$ , since  $Vu$  belongs to  $L^2_s(\Omega)$  for such an  $s$ . Hence the eigenspace, denoted by  $\mathcal{E}$ , of  $\mathbb{G}_V$  with eigenvalue 0 is a subspace of  $\mathcal{M}$ . Elements in  $\mathcal{M} \setminus \mathcal{E}$  are called resonances of  $\mathbb{G}_V$ . Then we define a class of potentials as follows:

**Definition.**  $V$  is said to be of generic type if  $\mathcal{M} = \{0\}$ .

In appendix A we prove that the potential  $V$  satisfying assumption (1.8) is of generic type. Thus, it is understood that zero is neither an eigenvalue nor a resonance of operator  $\mathbb{G}_V$  for such a potential  $V$ .

Local energy for wave equations is defined by letting

$$E_R(u)(t) = \int_{\Omega \cap \{|x| \leq R\}} \{|\nabla u(t, x)|^2 + |\partial_t u(t, x)|^2\} dx,$$

where, here and below,  $R > 0$  is chosen such that

$$\mathcal{O} = \mathbb{R}^3 \setminus \Omega \subseteq \{|x| \leq R\}.$$

The result due to Ralston [36] concerns the case that

$\mathcal{O}$  is a compact and trapping obstacle,

and his result asserts that, given any  $\mu \in (0, 1)$  and any  $T > 0$ , there exist  $f, g \in C_0^\infty(\Omega)$  with

$$\int_{\Omega} \{|\nabla f(x)|^2 + |g(x)|^2\} dx = 1$$

such that the solution to the initial-boundary value problem

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & t \neq 0, \quad x \in \Omega, \\ u(t, x) = 0, & t \in \mathbb{R}, \quad x \in \partial\Omega, \\ u(0, x) = f(x), \quad \partial_t u(0, x) = g(x), & x \in \Omega \end{cases}$$

satisfies the inequality

$$E_R(u)(T) \geq 1 - \mu.$$

On the other hand, the scattering theory developed by Lax and Phillips (see [24], and also Petkov [35]) gives a construction of the scattering operator by using weaker form of local energy decay

$$(1.13) \quad \liminf_{t \rightarrow \infty} E_R(u)(t) = 0.$$

Note that (1.13) follows directly from the RAGE (or simply ergodic type) theorem

$$(1.14) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T E_R(u)(t) dt = 0$$

and the property that zero is not eigenvalue of  $\mathbb{G}$ , i.e.,

$$u \in D(\mathbb{G}), \quad \mathbb{G}u = 0 \implies u = 0.$$

An important consequence of weak energy decay (1.14) is the existence of the wave operators

$$W_{\mp} := s - \lim_{t \rightarrow \pm\infty} e^{it\sqrt{\mathbb{G}}} J_0 e^{-it\sqrt{\mathbb{G}_0}},$$

where  $J_0$  is the orthogonal projection

$$J_0 : L^2(\mathbb{R}^3) \rightarrow L^2(\Omega).$$

This observation implies that scattering theory and existence of wave operators are established without appealing to additional geometric assumption of type

$$(1.15) \quad \mathcal{O} = \mathbb{R}^3 \setminus \Omega \text{ is non-trapping obstacle.}$$

The condition (1.15) is crucial for the strong local energy decay in view of the results of Morawetz, Ralston and Strauss [32] and Ralston [36].

Our main decay estimates (1.16)–(1.18) below are obtained also without appealing to assumption (1.15) and these are probably the main novelty in our work.

We shall prove the following:

**Theorem 1.1.** *Assume that the measurable potential  $V$  satisfies (1.8). Let  $\sigma \geq 2$ . If  $f, g \in C_0^\infty(\Omega)$  and  $R > 0$  is such that*

$$\mathcal{O} \subseteq \{|x| \leq R\},$$

*then the solution  $u$  to the initial-boundary value problem (1.5)–(1.7) satisfies the estimate*

$$(1.16) \quad E_R(u)(t) \leq \frac{C}{t^2} \left( \|f\|_{H_V^{2\sigma+1}(\Omega)}^2 + \|g\|_{H_V^{2\sigma}(\Omega)}^2 \right)$$

*for any  $t \neq 0$ .*

Interpolation between (1.16) and standard energy estimate

$$E_R(u)(t) \leq C \left( \|f\|_{H^1(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2 \right)$$

gives the following:

**Corollary 1.2.** *Assume that the measurable potential  $V$  satisfies (1.8). If  $f, g \in C_0^\infty(\Omega)$  and  $R > 0$  is such that*

$$\mathcal{O} \subseteq \{|x| \leq R\},$$

*then for any  $k \in (0, 1]$ , the solution  $u$  to the initial-boundary value problem (1.5)–(1.7) satisfies the estimate*

$$(1.17) \quad E_R(u)(t) \leq \frac{C}{|t|^{k/2}} \left( \|f\|_{H^{k+1}(\Omega)}^2 + \|g\|_{H^k(\Omega)}^2 \right)$$

for any  $t \neq 0$ .

**Remark 1.1.** If  $V = 0$ , then we are able to prove (1.17) for any  $k > 0$ . In the case of presence of potential satisfying (1.8), we use the fact that

$$D(\mathbb{G}_V^{s/2}) = D(\mathbb{G}^{s/2}), \quad \|f\|_{H_V^s(\Omega)} \sim \|f\|_{H^s(\Omega)}, \quad f \in D(\mathbb{G}^{s/2})$$

for any  $s \in [0, 2]$ . Therefore we need the restriction  $0 < k \leq 1$  in Corollary 1.2, when there is a potential.

**Remark 1.2.** It should be mentioned that the estimate (1.17) is slightly better local energy decay estimate compared with the estimate

$$E_R(u)(t) \leq \frac{C}{\log(2+t)^{2k}} \left( \|f\|_{H^{k+1}(\Omega)}^2 + \|g\|_{H^k(\Omega)}^2 \right),$$

which is obtained by Burq (see [9]).

The second result is concerned with  $L^p$ - $L^{p'}$ -estimates:

**Theorem 1.3.** *Let  $1 \leq p' \leq 2 \leq p \leq \infty$  and  $1/p + 1/p' = 1$ . Suppose that the measurable potential  $V$  satisfies (1.8). Then there exists a constant  $C > 0$  such that*

$$(1.18) \quad \left\| \left( \sqrt{\mathbb{G}_V} \right)^{-1} e^{it\sqrt{\mathbb{G}_V}} g \right\|_{\dot{B}_{p,2}^{-(1/2)+(2/p)}(\mathbb{G}_V)} \leq C|t|^{-1+(2/p)} \|g\|_{\dot{B}_{p',2}^{(1/2)-(2/p)}(\mathbb{G}_V)}$$

for any  $g \in \dot{B}_{p',2}^{(1/2)-(2/p)}(\mathbb{G}_V)$  and any  $t \neq 0$ .

The strategy of proof of Theorems 1.1 and 1.3 is based on the spectral representation of an operator  $\varphi(\sqrt{\mathbb{G}_V})$ . More precisely, given any function  $\varphi \in C_0^\infty(0, \infty)$ , we shall use the identity (see [18, Hörmander, vol. II, Distorted Fourier transform]):

$$\varphi(\sqrt{\mathbb{G}_V}) = \frac{1}{\pi i} \int_0^\infty \varphi(\lambda) [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] \lambda d\lambda,$$

where<sup>1</sup>  $R_V(\lambda^2 \pm i0)$  are the operators induced by the resolvent operator

$$R_V(z) = (\mathbb{G}_V - z)^{-1} \quad \text{for } z \in \mathbb{C},$$

whose existence is assured by the limiting absorption principle in Lemma 2.8 (see also Mochizuki [29, 30, 31]): Let  $\delta_0 > 1$ . Then there exist the limits

$$(1.19) \quad s\text{-}\lim_{\varepsilon \searrow 0} R_V(\lambda^2 \pm i\varepsilon) = R_V(\lambda^2 \pm i0) \quad \text{in } \mathcal{B}(L_s^2(\Omega), H_{-s}^2(\Omega))$$

for some  $s > 1/2$  and for any  $\lambda > 0$ . It should be mentioned that the limiting absorption principle (1.19) is true for an arbitrary exterior domain with a compact boundary. If one considers the uniform resolvent estimates obtained in [22, 29, 30, 31], the geometrical condition (1.15) on  $\Omega$  is imposed. However, the argument in this paper does not require any geometrical condition.

<sup>1</sup>typically we shall choose  $\varphi(\lambda) = \varphi_j(\lambda)\lambda^{-1}e^{i\lambda t}$ , where  $\varphi_j$  are functions from Paley - Littlewood partition of unity, introduced in (1.9)



Once the dispersive estimates are established, Strichartz estimates are obtained by  $TT^*$  argument of [17] (see also Yajima [46]). Our final result reads as follows. We consider

$$(1.20) \quad \begin{cases} \partial_t^2 u - \Delta u + V(x)u = F(t, x), & t \neq 0, \quad x \in \Omega, \\ u(t, x) = 0, & t \in \mathbb{R}, \quad x \in \partial\Omega, \\ u(0, x) = f(x), \quad \partial_t u(0, x) = g(x), & x \in \Omega. \end{cases}$$

Then we have:

**Theorem 1.4.** *Suppose that the measurable potential  $V$  satisfies (1.8). Then for any  $p, q, r, s, \gamma$  that satisfy*

$$\frac{2}{q} + \frac{2}{p} \leq 1, \quad \frac{2}{r} + \frac{2}{s} \leq 1, \quad 2 < q, r \leq \infty, \quad 2 \leq p, s < \infty,$$

$$3 \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{q} = \gamma = 1 - 3 \left( \frac{1}{2} - \frac{1}{s} \right) + \frac{1}{r},$$

there exists a constant  $C > 0$  such that the solution  $u$  to the initial-boundary value problem (1.20) satisfies the following estimate:

$$(1.21) \quad \|u\|_{L^q(\mathbb{R}; L^p(\Omega))} \leq C \left( \|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}(\Omega)} + \|F\|_{L^{r'}(\mathbb{R}; L^{s'}(\Omega))} \right),$$

where  $r', s'$  are the conjugate exponents to  $r, s$ , respectively.

In this paper we denote by  $\mathcal{B}(X, Y)$  the space of all bounded linear operators from a Banach space  $X$  to another one  $Y$ . When  $X = Y$ , we denote  $\mathcal{B}(X) = \mathcal{B}(X, X)$ . We will use the notation  $\mathcal{R}(T)$  for the range of an operator  $T$ .

The plan of the work is the following. The crucial point is the proof of appropriate  $L^2$  and  $L^p$ -estimates for perturbed resolvent of  $\mathbb{G}_V$  together with making a representation formula for the perturbed resolvent via the free one, which will be proved in section 2. In section 3  $L^1$ - $L^\infty$ -resolvent estimates will be proved. Section 4 will be devoted to the proof of Theorems 1.1 and 1.3. In section 4 the proof of Theorem 1.4 will be given.

## 2. $L^2$ AND $L^p$ -ESTIMATES FOR PERTURBED RESOLVENT

In this section we shall derive  $L^2$  and  $L^p$ -estimates for perturbed resolvent

$$R_V(z) = (\mathbb{G}_V - zI)^{-1},$$

and make a representation formula via the free one. These estimates will play an important role in proving the local energy decay estimates in Theorem 1.1 and dispersive estimates in Theorem 1.3.

To begin with, let us overview the known resolvent estimates. The limiting absorption principle for the free resolvent

$$R_0(z) = (\mathbb{G}_0 - z)^{-1}$$

is known as

$$(2.1) \quad s\text{-}\lim_{\varepsilon \searrow 0} R_0(\lambda^2 \pm i\varepsilon) = R_0(\lambda^2 \pm i0) \quad \text{in } \mathcal{B}(L_s^2(\mathbb{R}^3), H_{-s}^2(\mathbb{R}^3))$$

for any  $\lambda > 0$  and  $s > 1/2$  (see, e.g., Agmon [2]), and we have the uniform resolvent estimates

$$(2.2) \quad \|R_0(\lambda^2 \pm i\varepsilon)f\|_{L_{-s}^2(\mathbb{R}^3)} \leq \frac{C}{\lambda} \|f\|_{L_s^2(\mathbb{R}^3)}$$

for any  $\lambda, \varepsilon > 0$  and  $s > 1/2$  (see Mochizuki [31], and also Ben-Artzi and Klainerman [5]). We also refer to the result of the limiting absorption principle in the critical case  $s = 1/2$ , where Ruzhansky and Sugimoto proved (see [38]).

Recall the representation of the free resolvent (see Example 1, Ch.IX.7 in Reed and Simon [37]):

**Lemma 2.1.** *If  $\lambda > 0$ , then we have*

$$R_0(\lambda^2 \pm i\varepsilon)(x, y) = \frac{e^{-\sqrt{-\lambda^2 \mp i\varepsilon}|x-y|}}{4\pi|x-y|},$$

where, here and below, we put

$$\sqrt{z} = e^{(\text{Log}(z))/2}$$

and  $\text{Log}(z)$  is the principle branch of the logarithm.

We often use the well known formula:

$$(2.3) \quad [R_0(\lambda^2 \pm i0)f](x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|} f(y) dy.$$

**2.1. Key resolvent identity.** The next step is to represent the perturbed resolvent

$$R(z) = (\mathbb{G} - zI)^{-1},$$

where

$$\mathbb{G} = -\Delta|_D$$

is the Dirichlet Laplacian. We need identify the Hilbert space  $L^2(\mathbb{R}^3)$  that  $\mathbb{G}_0$  acts on, with  $L^2(\Omega)$  that  $\mathbb{G}$  acts on. To begin with, we define *identification operators*

$$J \in \mathcal{B}(L^2(\mathbb{R}^3), L^2(\Omega)), \quad J_* \in \mathcal{B}(L^2(\Omega), L^2(\mathbb{R}^3))$$

as follows (see also Kuroda [23] and Mochizuki [31]). In a very small neighborhood  $U$  of the obstacle  $\mathcal{O}$  we introduce local coordinates in the following way: Since  $\partial\Omega$  is of  $C^{2,1}$ , there exist a constant  $0 < r_0 \ll 1$  and a  $C^{2,1}$ -diffeomorphism

$$\Omega \cap U \ni x \mapsto (y, r) \in \partial\Omega \times (0, r_0)$$

such that

$$(2.4) \quad x = y + r\nu(y),$$

where  $\nu(y)$  is the unit normal at  $y \in \partial\Omega$  that is inward-pointing unit vector (unit vector pointing towards the interior of the domain  $\Omega$ ). Therefore, we have

$$r = \text{dist}(x, \partial\Omega),$$

where  $\text{dist}(x, \partial\Omega)$  is the distance between the point  $x \in \Omega \cap U$  and the boundary. Then let us choose a function  $j(x) \in C^2(\Omega) \cap W^{2,\infty}(\mathbb{R}^3)$  such that

$$(2.5) \quad j(x) \begin{cases} = 0 & \text{for } x \in \mathcal{O}, \\ = r^2 & \text{for } x \in \Omega \cap U \text{ and } r \leq \frac{r_0}{2}, \\ \geq \frac{r_0^2}{4} & \text{for } x \in \Omega \cap U \text{ and } \frac{r_0}{2} \leq r \leq r_0, \\ = 1 & \text{for } x \in U^c \text{ and } r \geq r_0. \end{cases}$$

In this way we define the operator  $J$  by letting

$$(Jf)(x) = j(x)f(x), \quad x \in \Omega,$$

for  $f \in L^2_{\text{loc}}(\mathbb{R}^3)$ , and define the adjoint operator  $J_*$  by letting

$$(J_*g)(x) = \begin{cases} j(x)g(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^3 \setminus \Omega, \end{cases}$$

for  $g \in L^2_{\text{loc}}(\overline{\Omega})$ . In particular, we have:

$$J \in \mathcal{B}(L^2_s(\mathbb{R}^3), L^2_s(\Omega)), \quad J_* \in \mathcal{B}(L^2_s(\Omega), L^2_s(\mathbb{R}^3))$$

for any  $s \in \mathbb{R}$ .

Next, let us consider the zero extension operator  $\iota$  from  $L^2(\Omega)$  to  $L^2(\mathbb{R}^3)$  by letting

$$\iota(f)(x) = \begin{cases} f(x), & \text{if } x \in \Omega; \\ 0, & \text{otherwise} \end{cases}$$

for any  $f \in L^2(\Omega)$ . We shall introduce a splitting relation involving the operators  $\iota$ ,  $\mathbb{G}_0$  and  $\mathbb{G}$  (see also Lemma 3.27 in p.71 of Adams and Fournier [1]). Here and below, we use the Sobolev space  $H^2_0(\Omega)$  which is defined as the completion of  $C^\infty_0(\Omega)$  in  $H^2(\Omega)$ -norm. Since our boundary is assumed to be compact and of  $C^{2,1}$ , it is clear that the restriction of  $C^\infty_0(\mathbb{R}^3)$  to  $\Omega$  defines a space that is dense in  $H^2(\Omega)$ . Indeed, the density property is guaranteed by classical results under essentially weaker assumptions on the boundary, namely the density is fulfilled for domains having segment property (see Section 3 in [1]). The segment property in turns is obviously true for exterior domains with  $C^{2,1}$ -compact boundaries. We also note that  $u \in H^2_0(\Omega)$  if and only if  $u \in H^2(\Omega)$  and

$$(2.6) \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

provided that  $\partial\Omega$  is of  $C^{2,1}$ , where  $\nu$  is the inward-pointing normal vector on  $\partial\Omega$  (see Theorem 8.9 in p.131 of Wloka [44]). Moreover,  $u$  belongs to  $D(\mathbb{G})$  if and only if

$$u \in H^2(\Omega) \quad \text{and} \quad u(x) = 0 \text{ for } x \in \partial\Omega.$$

Then we have:

**Lemma 2.2.** *We have the following assertions:*

- (i) *Let  $\Omega$  be an exterior domain whose boundary  $\partial\Omega$  is of  $C^{1,1}$ . If  $\phi \in H^1_0(\Omega)$ , then  $\iota(\phi) \in H^1(\mathbb{R}^3)$  and we have the identities*

$$(\iota \circ \partial_{x_j})(\phi) = (\partial_{x_j} \circ \iota)(\phi), \quad j = 1, 2, 3,$$

where  $\partial_{x_j} = \partial/\partial x_j$ .

(ii) Let  $\Omega$  be an exterior domain whose boundary  $\partial\Omega$  is of  $C^{2,1}$ . If  $\phi \in H_0^2(\Omega)$ , then  $\iota(\phi) \in D(\mathbb{G}_0)$  and we have the identity

$$(2.7) \quad (\iota \circ \mathbb{G})(\phi) = (\mathbb{G}_0 \circ \iota)(\phi).$$

(iii) Let  $\Omega$  be as in (ii). Then

$$(2.8) \quad J_*D(\mathbb{G}) \subset D(\mathbb{G}_0).$$

*Proof.* We have only to prove the assertions (ii) and (iii), since the proof of (i) is similar to that of (ii). If  $\psi \in C_0^\infty(\Omega)$ , then  $\iota(\psi) \in C_0^\infty(\mathbb{R}^3)$ , and we have

$$(2.9) \quad (\iota \circ \mathbb{G})(\psi) = (\mathbb{G}_0 \circ \iota)(\psi).$$

If

$$\phi \in H_0^2(\Omega) \subset D(\mathbb{G}),$$

then there exists a sequence  $\{\phi_k\}$  in  $C_0^\infty(\Omega)$  such that

$$\phi_k \xrightarrow[L^2(\Omega)]{\longrightarrow} \phi, \quad \mathbb{G}(\phi_k) \xrightarrow[L^2(\Omega)]{\longrightarrow} \mathbb{G}(\phi);$$

whence we write

$$(2.10) \quad \iota(\phi_k) \xrightarrow[L^2(\mathbb{R}^3)]{\longrightarrow} \iota(\phi), \quad (\iota \circ \mathbb{G})(\phi_k) \xrightarrow[L^2(\mathbb{R}^3)]{\longrightarrow} (\iota \circ \mathbb{G})(\phi).$$

Since

$$(2.11) \quad (\iota \circ \mathbb{G})(\phi_k) = (\mathbb{G}_0 \circ \iota)(\phi_k)$$

by (2.9), it follows from (2.10) that  $\{(\mathbb{G}_0 \circ \iota)(\phi_k)\}$  is a convergent sequence in  $L^2(\mathbb{R}^3)$ . Hence, by the standard argument, we deduce that

$$\iota(\phi) \in D(\mathbb{G}_0)$$

and

$$(2.12) \quad (\mathbb{G}_0 \circ \iota)(\phi_k) \xrightarrow[L^2(\mathbb{R}^3)]{\longrightarrow} (\mathbb{G}_0 \circ \iota)(\phi).$$

Therefore we conclude from (2.10)–(2.12) that

$$(\iota \circ \mathbb{G})(\phi) = (\mathbb{G}_0 \circ \iota)(\phi),$$

which proves the identity (2.7).

We turn to prove (iii). Let  $h \in D(\mathbb{G})$ . First, we show that

$$(2.13) \quad (J_*h)|_\Omega \in H_0^2(\Omega),$$

where  $F(\cdot)|_\Omega$  is denoted by the restriction of a function  $F(x)$  on  $\mathbb{R}^3$  to  $\Omega$ . In fact, since  $j(x) = 0$  and  $h(x) = 0$  on  $\partial\Omega$ , it follows that

$$\begin{aligned} J_*h &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} J_*h &= \frac{\partial j}{\partial\nu} h + j \frac{\partial h}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the existence of  $\frac{\partial h}{\partial\nu}$  is assured by the trace theorem. Hence, thanks to (2.6), we conclude (2.13). Consequently, it follows from the assertion (ii) that

$$J_*h = \iota((J_*h)|_\Omega) \in D(\mathbb{G}_0),$$

which proves (2.8). The proof of Lemma 2.2 is complete.  $\square$

Thanks to part (iii) of Lemma 2.2, we are able to define an operator

$$W_* = J_*\mathbb{G} - \mathbb{G}_0J_*,$$

provided that  $\partial\Omega$  is of  $C^{2,1}$ . Given any  $g \in H_0^1(\Omega)$ , we see from part (i) of Lemma 2.2 that

$$\nabla j \cdot \nabla \iota(g) \in L^2(\mathbb{R}^3),$$

and hence,

$$(2.14) \quad W_*g = (\Delta j)\iota(g) + 2\nabla j \cdot \nabla \iota(g) \in L^2(\mathbb{R}^3),$$

so that the support of  $W_*g$  is compact in  $\Omega$ . Therefore we have

$$W_* \in \mathcal{B}(D(\mathbb{G}), L_s^2(\mathbb{R}^3))$$

for any  $s \in \mathbb{R}$ .

Our goal is to represent the resolvent  $R(z)h$  in terms of the free one  $R_0(z)(J_*h)$  for  $z = \lambda^2 \pm i\varepsilon$ , where  $h \in L^2(\Omega)$ .

**Lemma 2.3.** *Let  $\Omega$  be an exterior domain whose boundary  $\partial\Omega$  is of  $C^{2,1}$ . Then we have the resolvent equation*

$$(2.15) \quad (J_* + R_0(z)W_*)R(z)h = R_0(z)J_*h$$

for any  $h \in L^2(\Omega)$  and  $z = \lambda^2 \pm i\varepsilon$  with  $\lambda, \varepsilon > 0$ . Furthermore, together with

$$(2.16) \quad \mathcal{R}(R_0(z)J_*) = R_0(z)J_*(D(\mathbb{G})), \quad \mathcal{R}(J_* + R_0(z)W_*) = (J_* + R_0(z)W_*)(D(\mathbb{G}))$$

we have

$$(2.17) \quad \mathcal{R}(R_0(z)J_*) \subset \mathcal{R}(J_* + R_0(z)W_*)$$

for any  $z = \lambda^2 \pm i\varepsilon$ .

*Proof.* We put

$$P(z) = J_*R(z) - R_0(z)J_*.$$

Since  $R(z)h \in D(\mathbb{G})$  for any  $h \in L^2(\Omega)$ , it follows from part (iii) of Lemma 2.2 that  $\mathbb{G}_0 \circ J_*R(z)h$  is well-defined as an element of  $L^2(\mathbb{R}^3)$ . Now, by an explicit calculation, we see that

$$\begin{aligned} (\mathbb{G}_0 - zI)P(z)h &= (\mathbb{G}_0 - zI)J_*R(z)h - J_*h \\ &= \{(\mathbb{G}_0 - zI)J_* - J_*(\mathbb{G} - zI)\}R(z)h \\ &= -W_*R(z)h. \end{aligned}$$

Thus we get

$$P(z)h = -R_0(z)W_*R(z)h.$$

This proves (2.15). As a consequence of (2.15), we get the inclusion (2.17). The proof of Lemma 2.3 is complete.  $\square$

In the sequel we always assume that  $\Omega$  is the exterior domain whose boundary  $\partial\Omega$  is of  $C^{2,1}$ .

We shall prove here the following:

**Proposition 2.4.** *Let*

$$\overline{(R_0(\lambda^2 \pm i\varepsilon)J_*)(L^2(\Omega))}$$

*be the closure with respect to  $H^2(\mathbb{R}^3)$ -norm of the image  $(R_0(\lambda^2 \pm i\varepsilon)J_*)(L^2(\Omega))$ . If  $D(\mathbb{G})$  is equipped with induced topology of  $H^2(\Omega)$ , then there exist bounded operators*

$$S(\lambda^2 \pm i\varepsilon) : \overline{(R_0(\lambda^2 \pm i\varepsilon)J_*)(L^2(\Omega))} \rightarrow D(\mathbb{G})$$

*such that*

$$(2.18) \quad R(\lambda^2 \pm i\varepsilon)f = S(\lambda^2 \pm i\varepsilon)R_0(\lambda^2 \pm i\varepsilon)J_*f$$

*for any  $\lambda, \varepsilon > 0$  and  $f \in L^2(\Omega)$ .*

*Proof.* If

$$\overline{(R_0(\lambda^2 \pm i\varepsilon)J_*)(D(\mathbb{G}))}$$

is the closure with respect to  $H^3(\mathbb{R}^3)$ -norm of the image  $(R_0(\lambda^2 \pm i\varepsilon)J_*)(D(\mathbb{G}))$ , then we have only to prove that there exist bounded operators

$$S(\lambda^2 \pm i\varepsilon) : \overline{(R_0(\lambda^2 \pm i\varepsilon)J_*)(D(\mathbb{G}))} \rightarrow D(\mathbb{G})$$

such that (2.18) hold. Indeed, the conclusion in the proposition follows from the fact that

$$\overline{(R_0(\lambda^2 \pm i\varepsilon)J_*)(D(\mathbb{G}))} \text{ is dense in } \overline{(R_0(\lambda^2 \pm i\varepsilon)J_*)(L^2(\Omega))}.$$

Hereafter, we put  $z = \lambda^2 \pm i\varepsilon$  for  $\lambda, \varepsilon > 0$ . We divide the proof into three steps.

*First step.* We claim that the operator  $J_* + R_0(z)W_*$  is injective from  $D(\mathbb{G})$  into  $\mathcal{R}(J_* + R_0(z)W_*)$ .

Let  $f \in D(\mathbb{G})$  be a solution to the following integral equation:

$$(2.19) \quad (J_* + R_0(z)W_*)f = 0 \quad \text{in } H^2(\mathbb{R}^3).$$

Then we deduce from the resolvent equation (2.15) and (2.19) that

$$\begin{aligned} R_0(z)J_*(\mathbb{G} - zI)f &= (J_* + R_0(z)W_*)R(z)(\mathbb{G} - zI)f \\ &= (J_* + R_0(z)W_*)f \\ &= 0, \end{aligned}$$

which implies that

$$J_*(\mathbb{G} - zI)f = 0 \quad \text{in } L^2(\mathbb{R}^3);$$

now by the definition (2.5) of  $J_*$ , we see that  $j(x) > 0$  on  $\Omega$  implies that  $f$  satisfies

$$(\mathbb{G} - zI)f = 0 \quad \text{in } L^2(\Omega).$$

Thus we conclude that  $f = 0$  in  $D(\mathbb{G})$ . This proves the assertion.

*Second step.* We show the formula (2.18).

As a consequence of the first step, if  $\mathcal{R}(J_* + R_0(z)W_*)$  is the image defined in (2.16), then the (left) inverse  $S(z)$  of  $J_* + R_0(z)W_*$  exists and it is a map from  $\mathcal{R}(J_* + R_0(z)W_*)$  into  $D(\mathbb{G})$ . Namely, we have

$$S(z) \circ (J_* + R_0(z)W_*) = I \quad \text{on } \mathcal{B}(D(\mathbb{G})).$$

Hence, combining the above identity and the resolvent equations (2.15), we get the formula (2.18).

*End of the proof. We have to prove that*

$$(2.20) \quad S(z) \text{ is continuous from } (R_0(z)J_*)(D(\mathbb{G})) \text{ into } D(\mathbb{G})$$

for each  $z = \lambda^2 \pm i\varepsilon$ . Moreover,  $S(z)$  is extended as a bounded operator on

$$\overline{(R_0(z)J_*)(D(\mathbb{G}))},$$

the closure of the image  $(R_0(z)J_*)(D(\mathbb{G}))$  with respect to  $H^3(\mathbb{R}^3)$ -norm.

For the proof of (2.20), it is sufficient to prove that

$$(2.21) \quad S(z) \text{ is continuous from } (R_0(z)J_*)(D(\mathbb{G})_\kappa) \text{ into } D(\mathbb{G}) \text{ for some } \kappa > 0,$$

since

$$\bigcup_{\kappa > 0} D(\mathbb{G})_\kappa \text{ is dense in } D(\mathbb{G}) \text{ with respect to } H^1(\Omega)\text{-norm,}$$

where, here and below,  $D(\mathbb{G})_\kappa$  is the space which consists of all  $f \in H^2(\Omega)$  such that

$$(2.22) \quad \text{dist}(\mathcal{O}, \text{supp } f) \geq \kappa,$$

and  $\bigcup_{\kappa > 0} D(\mathbb{G})_\kappa$  is endowed with inductive limit topology. Since  $S(z)$  is the linear operator, we have only to verify that it is continuous at the origin. Indeed, let  $\{f_n\}$  be a sequence in  $D(\mathbb{G})_\kappa$  for some  $\kappa > 0$  such that

$$(2.23) \quad R_0(z)J_*f_n \rightarrow 0 \quad \text{in } H^3(\mathbb{R}^3) \text{ as } n \rightarrow \infty.$$

Our goal is to show that

$$(2.24) \quad R(z)f_n \rightarrow 0 \quad \text{in } H^2(\Omega) \text{ as } n \rightarrow \infty,$$

for, thanks to the identity (2.18), we conclude from (2.24) that

$$S(z)R_0(z)J_*f_n \rightarrow 0 \quad \text{in } H^2(\Omega).$$

This proves the assertion (2.21).

To check (2.24) we use (2.23) to deduce that

$$J_*f_n \rightarrow 0 \quad \text{in } H^1(\mathbb{R}^3) \text{ as } n \rightarrow \infty,$$

which implies that

$$f_n \rightarrow 0 \quad \text{in } L^2(\Omega) \text{ as } n \rightarrow \infty,$$

since  $\{f_n\}$  satisfies (2.22). Thus we conclude (2.24). The proof of Proposition 2.4 is complete.  $\square$

We have the uniform estimates for  $R(\lambda^2 \pm i\varepsilon)$ .

**Proposition 2.5.** *Given  $s > 1/2$ , there exists a constant  $C > 0$  such that*

$$(2.25) \quad \|R(\lambda^2 \pm i\varepsilon)f\|_{L^2_{-s}(\Omega)} \leq C \|R_0(\lambda^2 \pm i\varepsilon)J_*f\|_{L^2_{-s}(\mathbb{R}^3)}$$

for any  $\lambda, \varepsilon > 0$  and  $f \in L^2_s(\Omega)$ .

*Proof.* Keeping in mind with Proposition 2.4, let us prove the uniform estimates (2.25). Estimates (2.25) are equivalent to the following: There exists a constant  $C > 0$  such that

$$(2.26) \quad \|S(z)u_z\|_{L^2_s(\Omega)} \leq C\|u_z\|_{L^2_s(\mathbb{R}^3)}$$

for any  $s > 1/2$  and  $u_z \in (R_0(z)J_*)(L^2_s(\Omega))$ , where we put  $z = \lambda^2 \pm i\varepsilon$  for  $\lambda, \varepsilon > 0$ . Observing the resolvent equation (2.15), we find a function

$$v_z \in \mathcal{R}(R(z)) = (R(z))(L^2_s(\Omega))$$

as

$$u_z = (J_* + R_0(z)W_*)v_z.$$

Then (2.26) is equivalent to the following:

$$\|v_z\|_{L^2_s(\Omega)} \leq C\|(J_* + R_0(z)W_*)v_z\|_{L^2_s(\mathbb{R}^3)}.$$

We note that the space  $(R(z))(L^2_s(\Omega))$  is invariant with respect to  $z = \lambda^2 \pm i\varepsilon$ , since

$$(R(z))(L^2_s(\Omega)) = H^2_{-s}(\Omega) \cap H^1_0(\Omega)$$

for any  $z = \lambda^2 \pm i\varepsilon$ . Hence it is sufficient to show that

$$\|v\|_{H^1(\Omega)} \leq C\|(J_* + R_0(z)W_*)v\|_{L^2_s(\mathbb{R}^3)}$$

for any  $v \in H^2_{-s}(\Omega) \cap H^1_0(\Omega)$ . Therefore, all we have to do is to show that there exists a constant  $C > 0$  such that

$$(2.27) \quad \|J_* + R_0(\lambda^2 \pm i\varepsilon)W_*\|_{\mathcal{B}(H^1(\Omega), L^2_s(\mathbb{R}^3))} \geq C$$

for any  $\lambda, \varepsilon > 0$  and  $s > 1/2$ .

By the definition (2.14) of  $W_*$ , and by the resolvent estimates (2.2), we have

$$(2.28) \quad \begin{aligned} \|R_0(\lambda^2 \pm i\varepsilon)W_*f\|_{L^2_s(\mathbb{R}^3)} &\leq \frac{C}{\lambda} \{ \|(\Delta j)f\|_{L^2(\Omega)} + \|\nabla j \cdot \nabla f\|_{L^2(\Omega)} \} \\ &\leq \frac{C}{\lambda} \|f\|_{H^1(\Omega)} \end{aligned}$$

for any  $\lambda, \varepsilon > 0$  and  $f \in H^1_0(\Omega)$ . By using (2.28), there exists  $\lambda_0 > 1$  such that

$$\|R_0(\lambda^2 \pm i\varepsilon)W_*\|_{\mathcal{B}(H^1(\Omega), L^2_s(\mathbb{R}^3))} \leq \frac{1}{2} \|J_*\|_{\mathcal{B}(H^1(\Omega), L^2_s(\mathbb{R}^3))}$$

for any  $\lambda > \lambda_0$ , and hence, we get

$$\|J_* + R_0(\lambda^2 \pm i\varepsilon)W_*\|_{\mathcal{B}(H^1(\Omega), L^2_s(\mathbb{R}^3))} \geq \frac{1}{2} \|J_*\|_{\mathcal{B}(H^1(\Omega), L^2_s(\mathbb{R}^3))}$$

for any  $\lambda > \lambda_0$ . This proves (2.27) for  $\lambda > \lambda_0$ . For  $\lambda \in [a, \lambda_0]$  with a sufficiently small  $a > 0$ , we suppose that

$$(2.29) \quad \inf_{\lambda \in [a, \lambda_0]} \|J_* + R_0(\lambda^2 \pm i\varepsilon)W_*\|_{\mathcal{B}(H^1(\Omega), L^2_s(\mathbb{R}^3))} = 0.$$

Since  $\|J_* + R_0(\lambda^2 \pm i\varepsilon)W_*\|_{\mathcal{B}(H^1(\Omega), L^2_s(\mathbb{R}^3))}$  are continuous on  $[a, \lambda_0]$ , the compactness of  $[a, \lambda_0]$  and assumption (2.29) imply that there exists  $\lambda_* \in [a, \lambda_0]$  such that

$$\|J_* + R_0(\lambda_*^2 \pm i\varepsilon)W_*\|_{\mathcal{B}(H^1(\Omega), L^2_s(\mathbb{R}^3))} = 0.$$



This contradicts the injective property of  $J_* + R_0(\lambda_*^2 \pm i\varepsilon)W_*$ . Thus we must have (2.27) for  $\lambda \in [a, \lambda_0]$ . Since  $a$  is arbitrarily small, we complete the proof of Proposition 2.5.  $\square$

The uniform resolvent estimate (2.2) for  $R_0(z)$  and the inequality (2.25) imply now the following:

**Lemma 2.6.** *We have the following properties of the operators*

$$R(\lambda^2 \pm i\varepsilon) = S(\lambda^2 \pm i\varepsilon)R_0(\lambda^2 \pm i\varepsilon)J_*$$

defined on  $L^2(\Omega)$ :

(i) *given any  $s > 1/2$ , there exists a constant  $C > 0$  such that*

$$(2.30) \quad \|R(\lambda^2 \pm i\varepsilon)f\|_{L^2_{-s}(\Omega)} \leq \frac{C}{\lambda} \|f\|_{L^2_s(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_s(\Omega)$ ;*

(ii) *given any  $s > 1$ , there exists a constant  $C > 0$  such that*

$$\|R(\lambda^2 \pm i\varepsilon)f\|_{L^2_{-s}(\Omega)} \leq C \|f\|_{L^2_s(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_s(\Omega)$ ;*

(iii) *given any  $s > 1/2$ ,  $\lambda > 0$  and any  $f \in L^2_s(\Omega)$  the following limits*

$$(2.31) \quad s - \lim_{\varepsilon \searrow 0} R(\lambda^2 \pm i\varepsilon)f = g \in L^2_{-s}(\Omega)$$

*exist in  $L^2_{-s}(\Omega)$ .*

We have also  $L^p$ -estimates for  $R(\lambda^2 \pm i\varepsilon)$ .

**Lemma 2.7.** *We have the following properties of the operators*

$$R(\lambda^2 \pm i\varepsilon) = S(\lambda^2 \pm i\varepsilon)R_0(\lambda^2 \pm i\varepsilon)J_*$$

defined on  $L^2(\Omega)$ : *Given  $s > 1/2$ , there exists a real  $p_0 > 5$  such that if  $p$  satisfies  $p_0 < p \leq \infty$ , then*

(i) *there exists a constant  $C > 0$  such that*

$$(2.32) \quad \|R(\lambda^2 \pm i\varepsilon)f\|_{L^p(\Omega)} \leq C \|R_0(\lambda^2 \pm i\varepsilon)(J_*f)\|_{L^p(\mathbb{R}^3)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_s(\Omega)$ ;*

(ii) *there exists a constant  $C > 0$  such that*

$$(2.33) \quad \|R(\lambda^2 \pm i\varepsilon)f\|_{L^p(\Omega)} \leq \frac{C}{\lambda^{2/p}} \|f\|_{L^2_s(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_s(\Omega)$ ;*

(iii) *the following limits*

$$(2.34) \quad s - \lim_{\varepsilon \searrow 0} R(\lambda^2 \pm i\varepsilon)f = g \in L^p(\Omega)$$

*exist in  $L^p(\Omega)$  for any  $\lambda > 0$  and any  $f \in L^2_s(\Omega)$ .*

*Proof.* It will be proved in the course of the proof of Lemma B.1 that for given  $s > 1/2$  there exists a real  $p_0 > 5$  such that if  $p$  satisfies  $p_0 < p \leq \infty$ , then

$$(2.35) \quad \|R_0(\lambda^2 \pm i\varepsilon)g\|_{L^p(\mathbb{R}^3)} \leq \frac{C}{\lambda^{2/p}} \|g\|_{L_s^2(\mathbb{R}^3)}$$

for any  $\lambda, \varepsilon > 0$  and for  $g \in L_s^2(\mathbb{R}^3)$  (see (B.8)). Hence we are able to endow the space

$$(R_0(\lambda^2 \pm i\varepsilon)J_*)(L_s^2(\Omega))$$

with the induced topology of  $L^p(\mathbb{R}^3)$ . Then, by using the Sobolev embedding theorem

$$(2.36) \quad H^2(\Omega) \subset L^p(\Omega) \quad \text{for any } p \in [2, \infty],$$

we deduce from Proposition 2.4 that  $S(\lambda^2 \pm i\varepsilon)$  are bounded from

$$(R_0(\lambda^2 \pm i\varepsilon)J_*)(L_s^2(\Omega))$$

into  $L^p(\Omega)$ . We have to find the uniform  $L^p$ -estimates for  $S(\lambda^2 \pm i\varepsilon)$ . Thanks to (2.35), along the argument in the proof of Proposition 2.5, there exists a constant  $C > 0$  such that

$$\|J_* + R_0(\lambda^2 \pm i\varepsilon)W_*\|_{\mathcal{B}(H^2(\Omega), L^p(\mathbb{R}^3))} \geq C$$

for any  $\lambda, \varepsilon > 0$  and  $p \in (p_0, \infty]$ . Hence, noting the embedding (2.36), we see that the operators  $S(\lambda^2 \pm i\varepsilon)$  are bounded from  $L^p(\mathbb{R}^3)$  into  $L^p(\Omega)$ , and further, we have

$$\sup_{\lambda > 0} \|S(\lambda^2 \pm i\varepsilon)\|_{\mathcal{B}(L^p(\mathbb{R}^3), L^p(\Omega))} \leq C.$$

Therefore, by using the key identity (2.18), we get the estimates (2.32). Thus, estimates (2.33) follow from (2.35) and (2.32). The limits (2.34) are a consequence of (B.2) in Lemma B.1. This ends the proof of Lemma 2.7.  $\square$

**2.2. Potential perturbation resolvent identity.** If we consider the perturbed resolvent

$$R_V(z) = (\mathbb{G}_V - zI)^{-1},$$

then the standard resolvent identity

$$R_V(z) - R(z) = -R(z)V R_V(z)$$

implies that

$$(2.37) \quad (I + R(z)V)R_V(z) = R(z).$$

If  $V$  satisfies assumption (1.8), then, given  $s \in (1/2, \delta_0/2]$ , we readily see that

$$Vf \in L_s^2(\Omega)$$

for any  $f \in L_{-s}^2(\Omega)$ , and there exists a constant  $C > 0$  such that

$$\|Vf\|_{L_s^2(\Omega)} \leq C\|f\|_{L_{-s}^2(\Omega)}.$$

Furthermore, given  $s \in (1/2, \delta_0 - 3/2]$  and  $p \in (2, \infty]$ , there exists a constant  $C > 0$  such that

$$\|Vg\|_{L_s^2(\Omega)} \leq C\|g\|_{L^p(\Omega)}.$$

Then the resolvent estimates for  $R(z)$  in Lemmas 2.6 and 2.7 imply now the following:

**Lemma 2.8.** *Assume that the measurable potential  $V$  satisfies (1.8). Then the operators*

$$R(\lambda^2 \pm i\varepsilon)V$$

*satisfy the following properties:*

(i) *for any  $s \in (1/2, \delta_0/2]$  there exists a constant  $C > 0$  such that*

$$(2.38) \quad \|R(\lambda^2 \pm i\varepsilon)Vf\|_{L^2_{-s}(\Omega)} \leq \frac{C}{\lambda} \|f\|_{L^2_{-s}(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_{-s}(\Omega)$ ;*

(ii) *for any  $s \in (1, \delta_0/2]$  there exists a constant  $C > 0$  such that*

$$\|R(\lambda^2 \pm i\varepsilon)Vf\|_{L^2_{-s}(\Omega)} \leq C \|f\|_{L^2_{-s}(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_{-s}(\Omega)$ ;*

(iii) *for any  $s \in (1/2, \delta_0/2]$  and  $\lambda > 0$  the following limits*

$$s - \lim_{\varepsilon \searrow 0} R(\lambda^2 \pm i\varepsilon)Vf = g \in L^2_{-s}(\Omega)$$

*exist in  $L^2_{-s}(\Omega)$ ;*

(iv) *there exist a real  $p_0 > 5$  and a constant  $C > 0$  such that for given any  $p \in (p_0, \infty]$ , we have*

$$\|R(\lambda^2 \pm i\varepsilon)Vf\|_{L^p(\Omega)} \leq \frac{C}{\lambda^{2/p}} \|f\|_{L^p(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^p(\Omega)$ .*

As a consequence of Lemma 2.8, we have the following:

**Theorem 2.9.** *Assume that the measurable potential  $V$  satisfies (1.8). Then the operators*

$$I + R(\lambda^2 \pm i0)V$$

*are well-defined and they satisfy the following properties:*

(i) *they are invertible ones in  $L^2_{-s}(\Omega)$  for some  $s \in (1/2, \delta_0/2]$ , and there exists a constant  $C > 0$  such that*

$$(2.39) \quad \|(I + R(\lambda^2 \pm i0)V)^{-1}\|_{\mathcal{B}(L^2_{-s}(\Omega))} \leq C$$

*for any  $\lambda > 0$ . In particular, if  $1 < s \leq \delta_0/2$ , then the estimates (2.39) is valid for  $\lambda = 0$ ;*

(ii) *there exists a real  $p_0 > 5$  such that if  $p$  satisfies  $p \in (p_0, \infty]$ , then they are invertible ones in  $L^p(\Omega)$ , and there exists a constant  $C > 0$  satisfying*

$$\|(I + R(\lambda^2 \pm i0)V)^{-1}\|_{\mathcal{B}(L^p(\Omega))} \leq C$$

*for any  $\lambda > 0$ .*

The proof of Theorem 2.9 is rather long and will be postponed in appendix C.

Define the operators by letting

$$(2.40) \quad S^\pm(\lambda) = (I + R(\lambda^2 \pm i0)V)^{-1}.$$

Then the identity (2.37) is rewritten now as

$$(2.41) \quad R_V(\lambda^2 \pm i0) = S^\pm(\lambda)R(\lambda^2 \pm i0).$$

The resolvent estimates for  $R_V(z)$  follow directly now:

**Corollary 2.10.** *Assume that the measurable potential  $V$  satisfies (1.8). Then the operators*

$$R_V(\lambda^2 \pm i\varepsilon) = (\mathbb{G}_V - (\lambda^2 \pm i\varepsilon)I)^{-1} = (\mathbb{G} + V - (\lambda^2 \pm i\varepsilon)I)^{-1}$$

*satisfy the following properties:*

(i) *given any  $s \in (1/2, \delta_0/2]$ , there exists a constant  $C > 0$  such that*

$$(2.42) \quad \|R_V(\lambda^2 \pm i\varepsilon)f\|_{L^2_{-s}(\Omega)} \leq \frac{C}{\lambda} \|f\|_{L^2_s(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_s(\Omega)$ ;*

(ii) *given any  $s \in (1, \delta_0/2]$ , there exists a constant  $C > 0$  such that*

$$(2.43) \quad \|R_V(\lambda^2 \pm i\varepsilon)f\|_{L^2_{-s}(\Omega)} \leq C \|f\|_{L^2_s(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_s(\Omega)$ ;*

(iii) *there exists a real  $p_0 > 5$  such that if  $p$  satisfies  $p \in (p_0, \infty]$ , then there exists a constant  $C > 0$  such that*

$$\|R_V(\lambda^2 \pm i\varepsilon)f\|_{L^p(\Omega)} \leq \frac{C}{\lambda^{2/p}} \|f\|_{L^2_s(\Omega)}$$

*for any  $\lambda, \varepsilon > 0$  and any  $f \in L^2_s(\Omega)$ ;*

(iv) *given any  $s \in (1/2, \delta_0/2]$ ,  $\lambda > 0$  and any  $f \in L^2_s(\Omega)$ , the following limits*

$$s - \lim_{\varepsilon \searrow 0} R_V(\lambda^2 \pm i\varepsilon)f = g \in L^2_{-s}(\mathbb{R}^3)$$

*exist in  $L^2_{-s}(\Omega)$ ;*

(v) *there exists a real  $p_0 > 5$  such that if  $p$  satisfies  $p_0 < p \leq \infty$ , then for any  $\lambda > 0$  and any  $f \in L^2_s(\Omega)$ , the following limits*

$$s - \lim_{\varepsilon \searrow 0} R_V(\lambda^2 \pm i\varepsilon)f = g \in L^p(\mathbb{R}^3)$$

*exist in  $L^p(\Omega)$ .*

Let us mention a few remarks on Corollary 2.10. When the obstacle  $\mathcal{O}$  is star-shaped with respect to the origin, the uniform resolvent estimates (2.42) is proved by Mochizuki (see [31]). Therefore, the estimates (2.42) cover [31].

### 3. $L^1$ - $L^\infty$ -RESOLVENT ESTIMATES

In this section we shall derive  $L^1$ - $L^\infty$ -estimates for perturbed resolvent  $R_V(\lambda^2 \pm i0)$ , which are useful to prove the theorems. We start with proving the following:

**Lemma 3.1.** *Assume that the measurable potential  $V$  satisfies (1.8). Then there exists a constant  $C > 0$  such that*

$$(3.1) \quad \|W_* R(\lambda^2 \pm i0)f\|_{L^1(\mathbb{R}^3)} \leq C \|f\|_{L^1(\Omega)},$$

$$(3.2) \quad \|R_V(\lambda^2 \pm i0)^2 f\|_{L^\infty(\Omega)} \leq \frac{C}{\lambda} \|f\|_{L^1(\Omega)}$$

for any  $\lambda > 0$  and  $f \in L^1(\Omega)$ .

*Proof.* First, we prove (3.1). By a density argument, it is sufficient to take  $f \in L^1(\Omega) \cap L_s^2(\Omega)$  for  $s > 1/2$ . We denote by  ${}_{X^*}\langle g, h \rangle_X$  the duality pair of  $g \in X^*$  and  $h \in X$  for a Banach space  $X$  and its dual space  $X^*$ . If we write  $(W_*)^*$  as the adjoint operator of  $W_*$ , then given  $q \in [1, \infty]$ , we have

$$(W_*)^*\psi = (\Delta j)\psi - 2 \operatorname{div}(\psi(\nabla j)) \in L^q(\Omega),$$

provided that  $\psi \in W^{1,q}(\Omega)$ . It is proved in Lemma B.1 of appendix B that given  $s > 1/2$ , there exists a constant  $C > 0$  and a real  $p_0 > 5$  such that if  $p \in (p_0, \infty]$ , then

$$(3.3) \quad \|R_0(\lambda^2 \pm i0)(J_*f)\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^1(\Omega) \cap L_s^2(\Omega)}$$

for any  $\lambda > 0$  (see (B.2)). Now, thanks to the inequality (2.32) in Lemma 2.7 and (3.3), there exists a constant  $C_0 > 0$  such that

$$\begin{aligned} \left| \int_{L^p(\mathbb{R}^3)} \langle W_*R(\lambda^2 \pm i0)f, \varphi \rangle_{L^{p'}(\mathbb{R}^3)} \right| &= \left| \int_{L^p(\Omega)} \langle R(\lambda^2 \pm i0)f, (W_*)^*\varphi \rangle_{L^{p'}(\Omega)} \right| \\ &\leq \|R(\lambda^2 \pm i0)f\|_{L^p(\Omega)} \|(W_*)^*\varphi\|_{L^{p'}(\Omega)} \\ &\leq C \|R_0(\lambda^2 \pm i0)(J_*f)\|_{L^p(\mathbb{R}^3)} \|\varphi\|_{W^{1,p'}(\mathbb{R}^3)} \\ &\leq C_0 \|f\|_{L^1(\Omega) \cap L_s^2(\Omega)} \|\varphi\|_{W^{1,p'}(\mathbb{R}^3)} \end{aligned}$$

for any  $\lambda > 0$ ,  $s > 1/2$  and  $\varphi \in W^{1,p'}(\mathbb{R}^3)$ . Since  $W^{1,p'}(\mathbb{R}^3)$  is dense in  $L^{p'}(\mathbb{R}^3)$ , we get

$$(3.4) \quad \|W_*R(\lambda^2 \pm i0)f\|_{L^p(\mathbb{R}^3)} \leq C_0 \|f\|_{L^1(\Omega) \cap L_s^2(\Omega)}$$

for any  $\lambda > 0$  and  $f \in L^1(\Omega) \cap L_s^2(\Omega)$ . Furthermore, since  $L^1(\Omega) \cap L_s^2(\Omega)$  is dense in  $L^1(\Omega)$ , we conclude from (3.4) that

$$\|W_*R(\lambda^2 \pm i0)f\|_{L^p(\mathbb{R}^3)} \leq C_0 \|f\|_{L^1(\Omega)}$$

for any  $\lambda > 0$  and  $f \in L^1(\Omega)$ . Thus, combining the above estimates with the following inequality:

$$\|W_*R(\lambda^2 \pm i0)f\|_{L^1(\mathbb{R}^3)} \leq C \|W_*R(\lambda^2 \pm i0)f\|_{L^p(\mathbb{R}^3)},$$

we arrive at the required estimate (3.1).

As a preliminary of proof of (3.2), we prove two estimates. In a similar way to the above argument, it is sufficient to take  $f \in L^1(\Omega) \cap L_s^2(\Omega)$  for  $s > 1/2$ .

*Step 1.* The first one we have to prove is that

$$(3.5) \quad \|VR_V(\lambda^2 \pm i0)f\|_{L^1(\Omega)} \leq C \|f\|_{L^1(\Omega)}$$

for any  $\lambda > 0$ . By using the decay assumption (1.8) on  $V$ , we are able to take  $s$  such that

$$\frac{1}{2} < s < \delta_0 - \frac{3}{2},$$

and apply this inequality to deduce that

$$\begin{aligned}
 (3.6) \quad \|VR_V(\lambda^2 \pm i0)f\|_{L^1(\Omega)} &\leq c_1 \left\| |\cdot|^{-\delta_0} R_V(\lambda^2 \pm i0)f \right\|_{L^1(\Omega)} \\
 &\leq c_1 \left\| |\cdot|^{s-\delta_0} \right\|_{L^2(\Omega)} \left\| R_V(\lambda^2 \pm i0)f \right\|_{L^2_{-s}(\Omega)} \\
 &\leq C \left\| R_V(\lambda^2 \pm i0)f \right\|_{L^2_{-s}(\Omega)},
 \end{aligned}$$

where we used the uniform bound;

$$\left\| |\cdot|^{s-\delta_0} \right\|_{L^2(\Omega)} < \infty.$$

Recalling the identities (2.41):

$$R_V(\lambda^2 \pm i0) = S^\pm(\lambda)R(\lambda^2 \pm i0)$$

and the fact from Theorem 2.9 that  $S^\pm(\lambda)$  are bounded on  $L^2_{-s}(\Omega)$ , we are able to write (3.6) as

$$\begin{aligned}
 (3.7) \quad \|VR_V(\lambda^2 \pm i0)f\|_{L^1(\Omega)} &\leq C \left\| R(\lambda^2 \pm i0)f \right\|_{L^2_{-s}(\Omega)} \\
 &\leq C \left\| R_0(\lambda^2 \pm i0)(J_*f) \right\|_{L^2_{-s}(\mathbb{R}^3)},
 \end{aligned}$$

due to the property

$$R(\lambda^2 \pm i\varepsilon)f = S(\lambda^2 \pm i\varepsilon)R_0(\lambda^2 \pm i\varepsilon)(J_*f),$$

Proposition 2.5 and the limiting absorption principle. To estimate the right member of (3.7), we use (2.3) to conclude that

$$\left\| R_0(\lambda^2 \pm i0)(J_*f) \right\|_{L^2_{-s}(\mathbb{R}^3)} \leq \int_{\Omega} \left( \int_{\mathbb{R}^3} \frac{dx}{|x-y|^2 \langle x \rangle^{2s}} \right)^{1/2} j(y) |f(y)| dy.$$

Since  $2 + 2s > 3$ , the integral in the right member is finite; thus we find that

$$(3.8) \quad \left\| R_0(\lambda^2 \pm i0)(J_*f) \right\|_{L^2_{-s}(\mathbb{R}^3)} \leq C \|f\|_{L^1(\Omega)}.$$

Therefore, combining (3.7) and the above estimate we get the required estimate (3.5).

*Step 2.* We prove the second type estimate:

$$(3.9) \quad \left\| R(\lambda^2 \pm i0)^2 VR_V(\lambda^2 \pm i0)f \right\|_{L^\infty(\Omega)} \leq \frac{C}{\lambda} \|f\|_{L^1(\Omega)}$$

for any  $\lambda > 0$ . Indeed, we note the identities:

$$(3.10) \quad [R_0(\lambda^2 \pm i0)^2 g](x) = \frac{\pm i}{8\pi\lambda} \int_{\mathbb{R}^3} e^{\pm i\lambda|x-y|} g(y) dy.$$

Then operators  $R_0(\lambda^2 \pm i0)^2$  map  $L^1(\mathbb{R}^3)$  to  $L^\infty(\mathbb{R}^3)$  and we have the following estimates:

$$(3.11) \quad \left\| R_0(\lambda^2 \pm i0)^2 g \right\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{8\pi\lambda} \|g\|_{L^1(\mathbb{R}^3)}$$

for any  $g \in L^1(\mathbb{R}^3)$  and  $\lambda > 0$ , which implies that

$$\left\| R_0(\lambda^2 \pm i0)^2 W_* R(\lambda^2 \pm i0)f \right\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{8\pi\lambda} \|W_* R(\lambda^2 \pm i0)f\|_{L^1(\Omega)}$$

for any  $\lambda > 0$ . Hence, we deduce from (3.1) and the above estimates that

$$(3.12) \quad \left\| R_0(\lambda^2 \pm i0)^2 W_* R(\lambda^2 \pm i0) f \right\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{\lambda} \|f\|_{L^1(\Omega)}$$

for any  $\lambda > 0$ . Here, differentiating the resolvent equations (2.15), we have

$$(J_* + R_0(\lambda^2 \pm i0)W_*)R(\lambda^2 \pm i0)^2 = R_0(\lambda^2 \pm i0)^2 J_* - R_0(\lambda^2 \pm i0)^2 W_* R(\lambda^2 \pm i0).$$

Since operators

$$J_* + R_0(\lambda^2 \pm i0)W_*$$

have the bounded inverses on  $L^\infty(\Omega)$  due to Lemma 2.7, it follows from (3.11) and (3.12) that

$$(3.13) \quad \left\| R(\lambda^2 \pm i0)^2 f \right\|_{L^\infty(\Omega)} \leq \frac{C}{\lambda} \|f\|_{L^1(\Omega)}$$

for any  $\lambda > 0$ . Hence, (3.9) are immediate consequences of the above estimates and (3.5).

We are now in a position to prove (3.2). Differentiating (2.37), we are able to write

$$(I + R(\lambda^2 \pm i\varepsilon)V)R_V(\lambda^2 \pm i\varepsilon)^2 = R(\lambda^2 \pm i\varepsilon)^2 - R(\lambda^2 \pm i\varepsilon)^2 V R_V(\lambda^2 \pm i\varepsilon).$$

Applying the estimates (3.9) and (3.13), and taking into account the fact that the operators

$$I + R(\lambda^2 \pm i0)V$$

have the bounded inverses in  $L^\infty(\Omega)$  due to Theorem 2.9, we complete the proof of (3.2). The proof of Lemma 3.1 is complete.  $\square$

Based on Lemma 3.1, we prove the following estimates, which are crucial to derive the dispersive estimates.

**Lemma 3.2.** *Assume that the measurable potential  $V$  satisfies (1.8). Then there exists a constant  $C > 0$  such that*

$$(3.14) \quad \left\| \partial_\lambda R_V(\lambda^2 \pm i0) f \right\|_{L^\infty(\Omega)} \leq C \|f\|_{L^1(\Omega)},$$

$$(3.15) \quad \left\| [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] f \right\|_{L^\infty(\Omega)} \leq C \lambda \|f\|_{L^1(\Omega)}$$

for any  $\lambda > 0$ .

*Proof.* Thanks to the identities

$$\partial_\lambda R_V(\lambda^2 \pm i0) = 2\lambda R_V(\lambda^2 \pm i0)^2,$$

and resolvent estimates (3.2) from Lemma 3.1, we get (3.14).

As to (3.15), we use the following identities:

$$(3.16) \quad [R_0(\lambda^2 + i0)g - R_0(\lambda^2 - i0)g](x) = \frac{i}{2\pi} \int_{\mathbb{R}^3} \frac{\sin(\lambda|x-y|)}{|x-y|} g(y) dy;$$

$$(3.17) \quad \begin{aligned} & R(\lambda^2 + i0) - R(\lambda^2 - i0) \\ &= S(\lambda^2 - i0) [R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)] [J_* - W_* R(\lambda^2 + i0)]. \end{aligned}$$

The identity (3.16) is obvious from (2.3). The identity (3.17) is proved as follows: By using resolvent equations (2.15) we easily show that

$$\begin{aligned} & [J_* + R_0(\lambda^2 - i0)W_*] [R(\lambda^2 + i0) - R(\lambda^2 - i0)] \\ &= [R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)] [J_* - W_*R(\lambda^2 + i0)]. \end{aligned}$$

Operating  $S(\lambda^2 - i0)$  to the both sides, we have (3.17). It follows from (2.32) in the course of proof of Lemma 2.7 that  $S(\lambda^2 - i0)$  is the bounded operator from  $\mathcal{R}(R(\lambda^2 \pm i0)J_*)$  into  $L^\infty(\Omega)$ . Hence, combining (3.16) and (3.17) with the estimates (3.1) from Lemma 3.1, we deduce that

$$\begin{aligned} (3.18) \quad & \|[R(\lambda^2 + i0) - R(\lambda^2 - i0)] f\|_{L^\infty(\Omega)} \\ & \leq C \|[R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)] [J_* - W_*R(\lambda^2 + i0)] f\|_{L^\infty(\Omega)} \\ & \leq C\lambda \|[J_* - W_*R(\lambda^2 + i0)] f\|_{L^1(\Omega)} \\ & \leq C\lambda \|f\|_{L^1(\Omega)} \end{aligned}$$

for any  $\lambda > 0$ . Using further the relation

$$\begin{aligned} & [I + R(\lambda^2 + i0)V][R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] \\ &= [R(\lambda^2 + i0) - R(\lambda^2 - i0)][I - VR_V(\lambda^2 - i0)], \end{aligned}$$

and (2.40), we find that

$$(3.19) \quad R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0) = S^+(\lambda)[R(\lambda^2 + i0) - R(\lambda^2 - i0)][I - VR_V(\lambda^2 - i0)].$$

The operators  $S^+(\lambda)$  and  $I - VR_V(\lambda^2 - i0)$  are bounded operators on  $L^\infty(\Omega)$  and  $L^1(\Omega)$ , respectively, and hence, (3.18) and (3.19) imply (3.15). The proof of Lemma 3.2 is complete.  $\square$

#### 4. PROOFS OF THEOREMS 1.1 AND 1.3

Applying estimates obtained in section 3 through integration by parts, we prove Theorems 1.1 and 1.3. Let us start with the following:

**Theorem 4.1.** *Assume that the measurable potential  $V$  satisfies (1.8). For any  $\psi \in C_0^\infty(0, \infty)$  there exists a constant  $C > 0$  such that*

$$(4.1) \quad \left\| \left( \sqrt{\mathbb{G}_V} \right)^{-1} e^{it\sqrt{\mathbb{G}_V}} \psi_j(\sqrt{\mathbb{G}_V}) f \right\|_{L^\infty(\Omega)} \leq \frac{C2^j}{t} \|f\|_{L^1(\Omega)}$$

for all  $j \in \mathbb{Z}$  and any  $t > 0$ , where  $\psi_j(\lambda) = \psi(2^{-j}\lambda)$ .

*Proof.* Consider the integrals of the form:

$$\left( \sqrt{\mathbb{G}_V} \right)^{-1} e^{it\sqrt{\mathbb{G}_V}} \psi_j(\sqrt{\mathbb{G}_V}) f = \frac{1}{\pi i} \int_0^\infty e^{i\lambda t} \psi_j(\lambda) [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] f d\lambda,$$

and after integrating by parts, we get

$$(4.2) \quad \pi i \left( \sqrt{\mathbb{G}_V} \right)^{-1} e^{it\sqrt{\mathbb{G}_V}} \psi_j(\sqrt{\mathbb{G}_V}) f = -\frac{1}{it} (I_1 + I_2),$$

where

$$I_1 = \int_0^\infty e^{i\lambda t} \psi_j(\lambda) [\partial_\lambda R_V(\lambda^2 + i0) - \partial_\lambda R_V(\lambda^2 - i0)] f d\lambda,$$



$$I_2 = \int_0^\infty e^{i\lambda t} \partial_\lambda \psi_j(\lambda) [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] f d\lambda.$$

By using (3.14) from Lemma 3.2 we estimate the integral  $I_1$ :

$$(4.3) \quad \begin{aligned} |I_1| &\leq C \int_{\text{supp } \psi_j} \left\| [\partial_\lambda R_V(\lambda^2 + i0) - \partial_\lambda R_V(\lambda^2 - i0)] f \right\|_{L^\infty(\Omega)} d\lambda \\ &\leq C \left( \int_{\text{supp } \psi_j} d\lambda \right) \|f\|_{L^1(\Omega)} \\ &\leq C 2^j \|f\|_{L^1(\Omega)}. \end{aligned}$$

As to the integral  $I_2$ , we use (3.15) from Lemma 3.2 to deduce that

$$(4.4) \quad \begin{aligned} |I_2| &\leq C 2^{-j} \int_{\text{supp } \psi_j} \left\| [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] f \right\|_{L^\infty(\Omega)} d\lambda \\ &\leq C 2^{-j} \left( \int_{\text{supp } \psi_j} \lambda d\lambda \right) \|f\|_{L^1(\Omega)} \\ &\leq C 2^j \|f\|_{L^1(\Omega)}. \end{aligned}$$

Summarizing (4.2)–(4.4), we arrive at the required estimate (4.1). The proof of Theorem 4.1 is complete.  $\square$

We are now in a position to prove Theorem 1.3.

*Proof of Theorem 1.3.* Let  $\{\varphi_j(\lambda)\}$  be the Littlewood-Paley partition of unity. We put

$$\psi_j(\lambda) = \varphi_{j-1}(\lambda) + \varphi_j(\lambda) + \varphi_{j+1}(\lambda)$$

in Theorem 4.1. As is well known,  $\varphi_j(\lambda)$  are written as

$$\varphi_j(\lambda) = \varphi_j(\lambda) \{\varphi_{j-1}(\lambda) + \varphi_j(\lambda) + \varphi_{j+1}(\lambda)\}.$$

Replacing  $f$  by  $\varphi_j(\lambda)f$ , we then conclude from Theorem 4.1 that

$$2^{-j} \left\| \varphi_j(\sqrt{\mathbb{G}_V}) \left(\sqrt{\mathbb{G}_V}\right)^{-1} e^{it\sqrt{\mathbb{G}_V}} f \right\|_{L^\infty(\Omega)}^2 \leq \frac{C}{t^2} 2^j \|\varphi_j(\sqrt{\mathbb{G}_V})f\|_{L^1(\Omega)}^2.$$

Taking the sum over  $j \in \mathbb{Z}$ , we obtain

$$(4.5) \quad \left\| \left(\sqrt{\mathbb{G}_V}\right)^{-1} e^{it\sqrt{\mathbb{G}_V}} f \right\|_{\dot{B}_{\infty,2}^{-1/2}(\mathbb{G}_V)} \leq \frac{C}{t} \|f\|_{\dot{B}_{1,2}^{1/2}(\mathbb{G}_V)}.$$

As to  $L^2$ -estimate, the functional calculus implies that

$$(4.6) \quad \left\| \left(\sqrt{\mathbb{G}_V}\right)^{-1} e^{it\sqrt{\mathbb{G}_V}} f \right\|_{\dot{B}_{2,2}^{-1/2}(\Omega)} \leq C \|f\|_{\dot{B}_{2,2}^{-1/2}(\mathbb{G}_V)}$$

(see Lemma 4.1 from [19]). Interpolating between (4.5) and (4.6), we get

$$\left\| \left(\sqrt{\mathbb{G}_V}\right)^{-1} e^{it\sqrt{\mathbb{G}_V}} f \right\|_{\dot{B}_{p,2}^{-(1/2)+(2/p)}(\mathbb{G}_V)} \leq C t^{-1+(2/p)} \|f\|_{\dot{B}_{p',2}^{(1/2)-(2/p)}(\mathbb{G}_V)}$$

for  $2 \leq p \leq \infty$ , where  $(1/p) + (1/p') = 1$ . This proves the the required estimate (1.18). The proof of Theorem 1.3 is now complete.  $\square$

By the same spirit of the proof of Theorem 1.3, we prove Theorem 1.1.

*Proof of Theorem 1.1.* It is sufficient to prove the theorem when  $\sigma$  is an integer with  $\sigma \geq 2$ . The non-integer case is proved by the complex interpolation argument. For simplicity, we consider solution  $u(t)$  to the initial-boundary value problem (1.5)–(1.7) with  $f = 0$  and  $g \in C_0^\infty(\Omega)$ . Then

$$u(t) = \left(\sqrt{\mathbb{G}_V}\right)^{-1} \sin(t\sqrt{\mathbb{G}_V}) g.$$

This implies

$$u(t) \in D(\mathbb{G}_V), \quad \chi u(t) \in D(\mathbb{G}_V) \quad (t \geq 0)$$

for any smooth compactly supported function  $\chi(x)$  such that  $\chi(x) = 1$  for  $x$  in small neighborhood of the obstacle  $\mathcal{O}$ .

Again consider the integrals of the form:

$$\begin{aligned} \chi e^{it\sqrt{\mathbb{G}_V}} g &= \chi(1 + \mathbb{G}_V)^\sigma (1 + \mathbb{G}_V)^{-\sigma} e^{it\sqrt{\mathbb{G}_V}} g \\ &= \frac{1}{\pi i} \int_0^\infty e^{i\lambda t} \chi [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] \underbrace{(I + \mathbb{G}_V)^\sigma g}_h \frac{\lambda d\lambda}{(1 + \lambda^2)^\sigma}. \end{aligned}$$

We note from Corollary 2.10 and the compactness of the support of  $\chi$  that

$$e^{i\lambda t} \chi [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] h \frac{\lambda}{(1 + \lambda^2)^\sigma} \rightarrow 0 \quad \text{in } L^2(\Omega)$$

both as  $\lambda \rightarrow 0$  and as  $\lambda \rightarrow \infty$ . Then, after integrating by parts, we get

$$(4.7) \quad \pi i \chi e^{it\sqrt{\mathbb{G}_V}} g = -\frac{1}{it} (I_1 + I_2) \quad \text{in } L^2(\Omega),$$

where

$$\begin{aligned} I_1 &= \int_0^\infty e^{i\lambda t} \chi [\partial_\lambda R_V(\lambda^2 + i0) - \partial_\lambda R_V(\lambda^2 - i0)] h \frac{\lambda d\lambda}{(1 + \lambda^2)^\sigma}, \\ I_2 &= \int_0^\infty e^{i\lambda t} \partial_\lambda \left\{ \frac{\lambda}{(1 + \lambda^2)^\sigma} \right\} \chi [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] h d\lambda. \end{aligned}$$

By using (3.14) from Lemma 3.2 we estimate the integral  $I_1$  as

$$\begin{aligned} (4.8) \quad \|I_1\|_{L^2(\Omega)} &\leq C \int_0^\infty \|[\partial_\lambda R_V(\lambda^2 + i0) - \partial_\lambda R_V(\lambda^2 - i0)] h\|_{L^\infty(\Omega)} \frac{\lambda d\lambda}{(1 + \lambda^2)^\sigma} \\ &\leq C \left\{ \int_0^\infty \frac{\lambda d\lambda}{(1 + \lambda^2)^\sigma} \right\} \|h\|_{L^1(\Omega)} \\ &\leq C \|h\|_{L^2(\Omega)} \end{aligned}$$

due to the fact that

$$I_1(x) \quad \text{and} \quad h(x) = (I + \mathbb{G}_V)^\sigma g(x) \quad \text{are compactly supported,}$$

since  $\sigma$  is the integer with  $\sigma \geq 2$ . As to the integral  $I_2$ , we use the uniform resolvent estimates (2.42) and (2.43) from Corollary 2.10 to deduce that for  $s > 1$ ,

$$\begin{aligned}
(4.9) \quad \|I_2\|_{L^2(\Omega)} &\leq C \|I_2\|_{L^2_{-s}(\Omega)} \\
&\leq C \int_0^\infty \left\| [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] h \right\|_{L^2_{-s}(\Omega)} \frac{d\lambda}{(1 + \lambda^2)^\sigma} \\
&\leq C \left\{ \int_0^\infty \frac{d\lambda}{(1 + \lambda^2)^\sigma} \right\} \|h\|_{L^2_s(\Omega)} \\
&\leq C \|h\|_{L^2(\Omega)},
\end{aligned}$$

since

$$I_2(x) \quad \text{and} \quad h(x) = (I + \mathbb{G}_V)^\sigma g(x) \quad \text{are compactly supported.}$$

Summarizing (4.7)–(4.9), we arrive at the estimates

$$\left\| \chi \sqrt{\mathbb{G}_V} u(t) \right\|_{L^2(\Omega)} \leq \frac{C}{t} \|h\|_{L^2(\Omega)}$$

for any  $t > 0$ . Thus we conclude that

$$\begin{aligned}
E_R(u)(t) &\leq \left\| \chi \sqrt{\mathbb{G}_V} u(t) \right\|_{L^2(\Omega)}^2 \\
&\leq \frac{C}{t^2} \left( \|g\|_{L^2(\Omega)}^2 + \|\mathbb{G}_V^\sigma g\|_{L^2(\Omega)}^2 \right) \\
&\sim \frac{C}{t^2} \|g\|_{H^{2\sigma}_V(\Omega)}^2
\end{aligned}$$

for any  $t > 0$ . This completes the proof of Theorem 1.1.  $\square$

## 5. STRICHARTZ ESTIMATES; PROOF OF THEOREM 1.4

Some perturbed Besov spaces  $\dot{B}_{p,q}^s(\mathbb{G}_V)$  have been introduced in (1.10). In this section we consider two generators. The self-adjoint generators

$$\mathbb{G} = -\Delta|_D \quad \text{and} \quad \mathbb{G}_V = \mathbb{G} + V$$

with respective domains

$$D(\mathbb{G}) = D(\mathbb{G}_V) = H^2(\Omega) \cap H_0^1(\Omega)$$

have been introduced in the previous sections. Then Theorem 2.4 in [19] ensures to define the homogeneous Besov spaces

$$\dot{B}_{p,q}^s(\mathbb{G}) \quad \text{and} \quad \dot{B}_{p,q}^s(\mathbb{G}_V).$$

For the proof of Theorem 1.4, we need a result showing the equivalence between the perturbed and the unperturbed Besov spaces. The following theorem is proved in [19].

**Theorem 5.1** (Proposition 3.5 from [19]). *Assume that the measurable potential  $V$  satisfies (1.8). Let  $s, p$  and  $q$  be such that*

$$-\min \left\{ 2, 3 \left( 1 - \frac{1}{p} \right) \right\} < s < \min \left\{ \frac{3}{p}, 2 \right\}, \quad 1 \leq p, q \leq \infty.$$

Then

$$(5.1) \quad \dot{B}_{p,q}^s(\mathbb{G}_V) \cong \dot{B}_{p,q}^s(\mathbb{G}).$$

In particular, for any  $s$  satisfying  $|s| < 3/2$ , we have

$$(5.2) \quad \dot{H}_V^s(\Omega) \cong \dot{H}^s(\Omega).$$

When  $\Omega = \mathbb{R}^3$  and  $p = 1$ , D'Ancona and Pierfelice investigated the isomorphism among the (inhomogeneous) perturbed Besov spaces  $B_{1,q,V}^s(\mathbb{R}^3)$  and classical ones  $B_{1,q}^s(\mathbb{R}^3)$  for all  $q \in [1, \infty]$  and  $0 < s < 2$  (see [14]). Georgiev and Visciglia obtained the equivalence relation for a very small  $s$  if  $V$  is *non-negative* on  $\mathbb{R}^3$  and belongs to the class  $C_{\text{loc}}^{0,\alpha}(\mathbb{R}^3)$  ( $0 < \alpha < 1$ ) (see [16]). Thus, (5.1) and (5.2) cover the results of [14] and [16].

We are now in a position to prove Theorem 1.4.

*Proof of Theorem 1.4.* We established the following embedding relations between the Besov and Lebesgue spaces on open sets in Proposition 3.3 from [19]:

$$(5.3) \quad \begin{cases} \dot{B}_{p,2}^0(\mathbb{G}) \subset L^p(\Omega), & \text{if } p \geq 2, \\ L^q(\Omega) \subset \dot{B}_{q,2}^0(\mathbb{G}), & \text{if } q \leq 2. \end{cases}$$

The Strichartz estimates (1.21) is proved by using the argument of [17] and the embeddings (5.3). It is sufficient to show only the case that  $f = 0$  and  $F = 0$ . Combining  $L^p$ - $L^{p'}$ -estimates (1.18) with  $TT^*$  argument of [17, 21] we have:

$$(5.4) \quad \|u\|_{L_t^{\frac{2p}{p-2}} \dot{B}_{p,2}^{-1/2+2/p+s}(\mathbb{G}_V)} \leq C \|g\|_{\dot{H}_V^{-1/2+s}(\Omega)}, \quad s \in \mathbb{R}.$$

Since we have established the equivalence relation between the perturbed Besov spaces and the free ones in Theorem 5.1, the required Strichartz estimates are proved by the routine work of [17]. For example, if  $(1/q, 1/p) = (0, 1/2)$  and  $s = -1/2$  in (5.4), we have

$$\|u\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{G}_V)} \leq C \|g\|_{\dot{H}_V^{-1}(\Omega)}.$$

Hence, by using (5.1) in Theorem 5.1 and the continuous embedding (5.3), we have:

$$\dot{B}_{p,2}^0(\mathbb{G}_V) \cong \dot{B}_{p,2}^0(\mathbb{G}) \subset L^p(\Omega) \quad \text{for } p \geq 2,$$

and hence, we conclude from (5.2) that

$$\|u\|_{L_t^\infty L^2(\Omega)} \leq C \|g\|_{\dot{H}^{-1}(\Omega)}.$$

As to the other estimates, one consults with the argument of [16] and we get the required estimates by interpolation. The proof of Theorem 1.4 is finished.  $\square$

#### APPENDIX A. (ZERO IS NOT A RESONANCE POINT)

In this appendix we prove that the assumption (1.8) on  $V$  assures that zero is neither an eigenvalue nor a resonance of  $\mathbb{G}_V$ , i.e.,  $\mathcal{M} = \{0\}$ .

Our concern in this appendix is the following:

**Proposition A.1.** *Assume that the measurable potential  $V$  satisfies (1.8). If  $u \in L^2_{-s}(\Omega)$  for some  $s \in (1, \delta_0/2]$  is a solution to the equation*

$$(A.1) \quad (I + R(0)V)u = 0,$$

then  $u = 0$ .

To prove Proposition A.1, we prepare the following:

**Lemma A.2.** *If  $f \in L^2_\delta(\mathbb{R}^3)$  for some  $\delta > 1/2$  and  $\tilde{u} \in L^2_{-s}(\mathbb{R}^3)$  for some  $s > 1$  is a solution to the equation  $\Delta \tilde{u} = f$ , then there exists a constant  $C > 0$  such that*

$$(A.2) \quad |\tilde{u}(x)| \leq \begin{cases} \frac{C}{\langle x \rangle^{\delta - \frac{1}{2}}} \|f\|_{L^2_\delta(\mathbb{R}^3)} & \text{if } \frac{1}{2} < \delta < \frac{3}{2}, \\ \frac{C \log^{1/2}(2 + |x|)}{\langle x \rangle} \|f\|_{L^2_\delta(\mathbb{R}^3)} & \text{if } \delta = \frac{3}{2}, \\ \frac{C}{\langle x \rangle} \|f\|_{L^2_\delta(\mathbb{R}^3)} & \text{if } \delta > \frac{3}{2}. \end{cases}$$

*Proof.* The estimate (A.2) follows from the representation

$$\tilde{u}(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x - y|} dy$$

and a simple estimate

$$\left( \int_{\mathbb{R}^3} \frac{dy}{|x - y|^2 \langle y \rangle^{2\delta}} \right)^{1/2} \leq \begin{cases} \frac{C}{\langle x \rangle^{\delta - \frac{1}{2}}} & \text{if } \frac{1}{2} < \delta < \frac{3}{2}, \\ \frac{C \log^{1/2}(2 + |x|)}{\langle x \rangle} & \text{if } \delta = \frac{3}{2}, \\ \frac{C}{\langle x \rangle} & \text{if } \delta > \frac{3}{2}. \end{cases}$$

The proof of Lemma A.2 is complete.  $\square$

We are now in a position to prove Proposition A.1.

*Proof of Proposition A.1.* Let  $u \in L^2_{-s}(\Omega)$  for some  $s \in (1, \delta_0/2]$  be a solution to the integral equation (A.1). We have to prove that

$$u(x) = 0 \quad \text{in } \Omega.$$

The solution  $u$  to (A.1) satisfies the boundary value problem for the stationary Schrödinger equation:

$$(A.3) \quad -\Delta u + V(x)u = 0 \quad \text{in } \Omega$$

with homogeneous Dirichlet boundary condition. This equation implies

$$(A.4) \quad u \in L^2_{-s}(\Omega) \cap H^2_{\text{loc}}(\overline{\Omega}).$$

We claim that given any  $\delta_0 \in (2, 3)$  we have

$$(A.5) \quad |\nabla u(x)| \leq \begin{cases} \frac{C}{\langle x \rangle^{2\delta_0 - s - \frac{3}{2}}} \|u\|_{H^1_{-s}(\Omega)}, & \text{if } 2\delta_0 - s - \frac{1}{2} < 3, \\ \frac{C \log^{1/2}(2 + |x|)}{\langle x \rangle^2} \|u\|_{H^1_{-s}(\Omega)}, & \text{if } 2\delta_0 - s - \frac{1}{2} = 3, \\ \frac{C}{\langle x \rangle^2} \|u\|_{H^1_{-s}(\Omega)}, & \text{if } 2\delta_0 - s - \frac{1}{2} > 3. \end{cases}$$

When  $\delta_0 \geq 3$ ,  $\nabla u$  decays faster than the case  $2 < \delta_0 < 3$ , and the proof of the proposition is easier. So we may omit the proof in this case. We note from the assumption  $(1 <) s \leq \delta_0/2$  that

$$(A.6) \quad 2\delta_0 - s - \frac{3}{2} \geq \frac{3(\delta_0 - 1)}{2}.$$

To show the asymptotic behaviour (A.5), let us consider an extension  $\tilde{u}$  of  $u$  to  $\mathbb{R}^3$ . More precisely, let  $\tilde{\mathcal{O}}$  be a bounded domain containing  $\mathcal{O} (= \mathbb{R}^3 \setminus \Omega)$ , and we define  $\tilde{u}$  by letting

$$\tilde{u}(x) = \psi(x)u(x),$$

where  $\psi(x) \in C^\infty(\mathbb{R}^3)$  equals 0 in  $\mathcal{O}$  and 1 in  $\Omega \setminus \tilde{\mathcal{O}}$ . This  $\tilde{u} \in L^2_{-s}(\mathbb{R}^3)$  satisfies the Poisson equation

$$-\Delta \tilde{u} = f \quad \text{in } \mathbb{R}^3,$$

where

$$(A.7) \quad f = -\psi V u - 2\nabla \psi \cdot \nabla u - (\Delta \psi)u.$$

It is well known that Poisson equation has a unique solution in  $L^2_{-s}(\mathbb{R}^3)$  for  $s > 1/2$  and  $\tilde{u}$  is represented as

$$\tilde{u}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy.$$

Thanks to (A.7) and  $1 < s \leq \delta_0/2$ , it is readily checked that  $f \in L^2_\delta(\mathbb{R}^3)$  provided

$$(A.8) \quad \delta = -s + \delta_0 \geq \frac{\delta_0}{2}.$$

In fact, by using the decay assumption (1.8) on  $V$ , we have

$$\begin{aligned} \|f\|_{L^2_\delta(\mathbb{R}^3)} &\leq C \left\| \langle \cdot \rangle^{\delta - \delta_0} u \right\|_{L^2(\Omega)} + \|2\nabla \psi \cdot \nabla u - (\Delta \psi)u\|_{L^2_\delta(\mathbb{R}^3)} \\ &\leq C \|u\|_{H^1_{-s}(\Omega)} < \infty. \end{aligned}$$

Hence we see from (A.7) that

$$\begin{aligned} |\nabla \tilde{u}(x)| &\leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|^2} dy \\ &\leq \frac{1}{4\pi} \left( \int_{\mathbb{R}^3} \frac{|\psi(y)V(y)u(y)|}{|x-y|^2} dy + \int_{\mathbb{R}^3} \frac{2|\nabla \psi(y)||\nabla u(y)|}{|x-y|^2} dy + \int_{\mathbb{R}^3} \frac{|\Delta \psi(y)||u(y)|}{|x-y|^2} dy \right). \end{aligned}$$

We shall estimate each term in the right side of the above estimates. The second and third terms bounded by  $C\langle x \rangle^{-2}$ , since  $\nabla \psi$  and  $\Delta \psi$  are compactly supported. The

first term is handled by using (A.2) in Lemma A.2 and the decay assumption (1.8) on  $V$  and (A.8), where  $\delta_0/2 \leq \delta < 3/2$ . Then, noting (A.8), we have

$$\delta_0 + \delta - \frac{1}{2} = 2\delta_0 - s - \frac{1}{2},$$

and hence, we estimate

$$\begin{aligned} & \int_{\mathbb{R}^3} \frac{|\psi(y)V(y)u(y)|}{|x-y|^2} dy \\ & \leq \int_{\mathbb{R}^3} \frac{C\|f\|_{L^2_\delta(\mathbb{R}^3)}}{|x-y|^2\langle y \rangle^{2\delta_0-s-\frac{1}{2}}} dy \\ & \leq \begin{cases} \frac{C}{\langle x \rangle^{2\delta_0-s-\frac{3}{2}}} \|u\|_{H^1_{-s}(\Omega)}, & \text{if } 2\delta_0 - s - \frac{1}{2} < 3, \\ \frac{C \log^{1/2}(2+|x|)}{\langle x \rangle^2} \|u\|_{H^1_{-s}(\Omega)}, & \text{if } 2\delta_0 - s - \frac{1}{2} = 3, \\ \frac{C}{\langle x \rangle^2} \|u\|_{H^1_{-s}(\Omega)}, & \text{if } 2\delta_0 - s - \frac{1}{2} > 3. \end{cases} \end{aligned}$$

This proves (A.5).

Once (A.5) is checked, we use (A.4) and integrate by parts in (A.3), so we have

$$\int_{\Omega \cap \{|x| < R\}} \{|\nabla u(x)|^2 + V(x)|u(x)|^2\} dx = \int_{|x|=R} u_r(x) \overline{u(x)} dS_R,$$

where  $u_r = \partial u / \partial r$  ( $r = |x|$ ) and  $dS_R$  is the 2-dimensional surface element. The pointwise estimates (A.2) and (A.5) guarantee that taking the limit  $R \rightarrow \infty$ , and noting from (A.6) and (A.8) that

$$\left(2\delta_0 - s - \frac{3}{2}\right) + \left(\delta - \frac{1}{2}\right) \geq \frac{3(\delta_0 - 1)}{2} + \frac{\delta_0 - 1}{2} = 2(\delta_0 - 1) > 2,$$

we find that

$$\int_{\Omega} \{|\nabla u(x)|^2 + V(x)|u(x)|^2\} dx = 0.$$

Here, by using the assumption (1.8) on  $V$ :  $V(x) \geq -c_0|x|^{-2}$ , where  $0 < c_0 < 1/4$ , we estimate

$$\int_{\Omega} V(x)|u(x)|^2 dx \geq - \int_{\Omega} c_0 \frac{|u(x)|^2}{|x|^2} dx,$$

and hence, resorting to the Hardy inequality, we get

$$\begin{aligned} \int_{\Omega} \{|\nabla u(x)|^2 + V(x)|u(x)|^2\} dx & \geq \int_{\Omega} \left\{ |\nabla u(x)|^2 - c_0 \frac{|u(x)|^2}{|x|^2} \right\} dx \\ & \geq (1 - 4c_0) \int_{\Omega} |\nabla u(x)|^2 dx. \end{aligned}$$

Therefore we arrive at

$$\int_{\Omega} |\nabla u(x)|^2 dx = 0,$$

which implies that  $u$  is constant in  $\Omega$ . Thus we conclude that  $u = 0$  in  $\Omega$ . The proof of Proposition A.1 is complete.  $\square$

## APPENDIX B. (COMPACTNESS ARGUMENT)

In this appendix we discuss the compactness of the operators  $R(\lambda^2 \pm i0)V$ . We start with the following observation:

**Lemma B.1.** *Assume that the measurable potential  $V$  satisfies (1.8). Then, given  $s > 1/2$ , there exists a real  $p_0 > 5$  satisfying the following properties; for any  $p \in (p_0, \infty]$ , there exist constants  $C > 0$ ,  $c > 0$  and a continuous function  $\mu_p(x) = \mu_p(|x|)$  on  $\mathbb{R}^3$  such that*

$$\inf_{x \in \mathbb{R}^3} \mu_p(x) \geq c,$$

$$\mu_p(x) \rightarrow \infty \quad \text{as } |x| \rightarrow \infty,$$

$$(B.1) \quad \|\mu_p(\cdot)R_0(\lambda^2 \pm i0)J_*(Vf)\|_{L^p(\mathbb{R}^3)} \leq \frac{C}{\lambda^{2/p}} \|f\|_{L^p(\Omega)},$$

$$(B.2) \quad \|\mu_p(\cdot)R_0(\lambda^2 \pm i0)(J_*f)\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^1(\Omega) \cap L_s^2(\Omega)}$$

for any  $\lambda > 0$ . Furthermore, we have

$$(B.3) \quad \lim_{\lambda \rightarrow \infty} |[R_0(\lambda^2 \pm i0)J_*(Vf)](x)| = 0 \quad \text{for each } x \in \mathbb{R}^3 \text{ and } f \in L^\infty(\Omega).$$

*Proof.* First, we prove that

$$(B.4) \quad \|\mu(\cdot)R_0(\lambda^2 \pm i0)g\|_{L^\infty(\mathbb{R}^3)} \leq C \|g\|_{L_s^2(\mathbb{R}^3)}$$

for any  $g \in L_s^2(\mathbb{R}^3)$  and  $\lambda > 0$ , where we put

$$\mu(x) = \begin{cases} \langle x \rangle^{s-\frac{1}{2}}, & \text{if } \frac{1}{2} < s < \frac{3}{2}, \\ \langle x \rangle \log^{-\frac{1}{2}}(2 + |x|), & \text{if } s = \frac{3}{2}, \\ \langle x \rangle, & \text{if } s > \frac{3}{2}. \end{cases}$$

Thanks to the formula (2.3), we write

$$(B.5) \quad [R_0(\lambda^2 \pm i0)g](x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|} g(y) dy.$$

Then we estimate the right member as

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|} g(y) dy \right| &\leq \int_{\mathbb{R}^3} \frac{1}{|x-y| \langle y \rangle^s} \langle y \rangle^s |g(y)| dy \\ &\leq \left( \int_{\mathbb{R}^3} \frac{dy}{|x-y|^2 \langle y \rangle^{2s}} \right)^{1/2} \|g\|_{L_s^2(\mathbb{R}^3)}. \end{aligned}$$

Hence, combining the above estimates and the following inequality

$$\int_{\mathbb{R}^3} \frac{dy}{|x-y|^2 \langle y \rangle^{2s}} \leq C \begin{cases} \langle x \rangle^{-(2s-1)}, & \text{if } \frac{1}{2} < s < \frac{3}{2}, \\ \langle x \rangle^{-2} \log(2 + |x|), & \text{if } s = \frac{3}{2}, \\ \langle x \rangle^{-2}, & \text{if } s > \frac{3}{2}, \end{cases}$$



we get the required estimates (B.4). Furthermore, we observe from the above argument that the function  $\frac{g(y)}{|x-y|}$  is integrable on  $\mathbb{R}_y^3$ . Hence, applying Riemann-Lebesgue's lemma to the formula (B.5), we conclude that

$$(B.6) \quad \lim_{\lambda \rightarrow \infty} |[R_0(\lambda^2 \pm i0)g](x)| = 0$$

for each  $x \in \mathbb{R}^3$  and  $g \in L_s^2(\mathbb{R}^3)$  for  $s > 1/2$ .

Next, by using the uniform resolvent estimates (2.2), we have

$$(B.7) \quad \|R_0(\lambda^2 \pm i0)g\|_{L_s^2(\mathbb{R}^3)} \leq \frac{C}{\lambda} \|g\|_{L_s^2(\mathbb{R}^3)}$$

for any  $s > 1/2$  and  $\lambda > 0$ , and hence, the interpolation between (B.4) and (B.7) implies that

$$(B.8) \quad \|\mu_p(\cdot)R_0(\lambda^2 \pm i0)g\|_{L^p(\mathbb{R}^3)} \leq \frac{C}{\lambda^{2/p}} \|g\|_{L_s^2(\mathbb{R}^3)}$$

for any  $\lambda > 0$ , where we put

$$(B.9) \quad \mu_p(x) = \begin{cases} \langle x \rangle^{(s-\frac{1}{2})(1-\frac{2}{p})-\frac{2s}{p}}, & \text{if } \frac{1}{2} < s < \frac{3}{2}, \\ \langle x \rangle^{(1-\frac{2}{p})-\frac{2s}{p}} \log^{-\frac{1}{2}}(1-\frac{2}{p})(2+|x|), & \text{if } s = \frac{3}{2}, \\ \langle x \rangle^{(1-\frac{2}{p})-\frac{2s}{p}}, & \text{if } s > \frac{3}{2}. \end{cases}$$

Let  $p_0$  be the root of the following equations:

$$\begin{aligned} \left(s - \frac{1}{2}\right) \left(1 - \frac{2}{p}\right) - \frac{2s}{p} &= 0 \quad \text{with } \frac{1}{2} < s < \frac{3}{2}, \\ \left(1 - \frac{2}{p}\right) - \frac{2s}{p} &= 0 \quad \text{with } s \geq \frac{3}{2}. \end{aligned}$$

Then the explicit calculation implies that

$$\mu_p(x) \rightarrow \infty \quad \text{as } |x| \rightarrow \infty,$$

provided  $p \in (p_0, \infty]$ . We see from the decay assumption (1.8) on  $V$  that

$$J_* \circ V : L^p(\Omega) \rightarrow L_s^2(\mathbb{R}^3)$$

is a bounded operator for

$$(B.10) \quad s < \delta_0 - \frac{3}{2} + \frac{3}{p},$$

and we have

$$(B.11) \quad \|J_*(Vf)\|_{L_s^2(\mathbb{R}^3)} \leq C \|f\|_{L^p(\Omega)}.$$

Here, if

$$s < \delta_0 - \frac{3}{2},$$

then (B.10) is valid for any  $p \in (p_0, \infty]$ . Thus, under the following restriction:

$$(B.12) \quad \frac{1}{2} < s < \delta_0 - \frac{3}{2},$$

letting  $g = J_*(Vf)$  with  $f \in L^p(\Omega)$ , we conclude from (B.8) and (B.11) that the estimates (B.1) hold.

Estimates (B.2) are an immediate consequence of the interpolation between (B.4) and (3.8) in the course of proof of Lemma 3.1:

$$\|R_0(\lambda^2 \pm i0)(J_*f)\|_{L^2_{-s}(\mathbb{R}^3)} \leq C\|f\|_{L^1(\Omega)}.$$

Finally, the limits (B.3) are the consequence of (B.6), since  $g = J_*(Vf) \in L^2_s(\Omega)$  provided that  $f \in L^\infty(\Omega)$  and  $s$  satisfies (B.12). The proof of Lemma B.1 is now complete.  $\square$

We are now in a position to prove the compactness of operators  $R(\lambda^2 \pm i0)V$ .

**Lemma B.2.** *Assume that the measurable potential  $V$  satisfies assumption (1.8). Let  $1/2 < s \leq \delta_0/2$ . Then the operators*

$$R(\lambda^2 \pm i0)V = S(\lambda^2 \pm i0) \circ R_0(\lambda^2 \pm i0) \circ J_* \circ V$$

*are well-defined and compact on  $L^2_{-s}(\Omega)$  for any  $\lambda > 0$ . In particular, if  $1 < s \leq \delta_0/2$ , then the operator  $R(0)V$  is compact on  $L^2_{-s}(\Omega)$ .*

*Proof.* First, we consider the case  $\lambda = 0$  and  $1 < s \leq \delta_0/2$ . In this case, the argument of Proposition 2.4 works well, and hence,  $S(0)$  is continuous from  $\mathcal{R}(R_0(0)J_*)$  into  $L^2_{-s}(\Omega)$  for any  $s \in (1, \delta_0/2]$ . Next, we see that  $R_0(0)$  is the compact operator from  $L^2_s(\mathbb{R}^3)$  into  $L^2_{-s}(\mathbb{R}^3)$  for any  $s > 1$ , since the kernel of  $R_0(0)$  is Hilbert-Schmidt type:

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle x \rangle^{-2s} \frac{1}{|x-y|^2} \langle y \rangle^{-2s} dx dy \leq C.$$

Finally, the decay assumption (1.8) on  $V$  implies that

$$J_* \circ V : L^2_{-s}(\Omega) \rightarrow L^2_s(\mathbb{R}^3)$$

is a bounded operator for

$$\frac{1}{2} < s \leq \frac{\delta_0}{2}.$$

Therefore, the operator  $S(0) \circ R_0(0) \circ J_* \circ V$  is compact from  $L^2_{-s}(\Omega)$  into itself for any  $s \in (1, \delta_0/2]$ . This proves the compactness in the case  $\lambda = 0$ .

We now turn to the case  $\lambda > 0$ . Let  $\{f_n\}$  be a sequence such that

$$\sup_{n \in \mathbb{N}} \|f_n\|_{L^2_{-s}(\Omega)} \leq M$$

for any  $s \in (1/2, \delta_0/2]$ , where  $M$  is a positive constant. We prove the compactness of the following sequence of functions

$$g_n = R(\lambda^2 \pm i0)Vf_n.$$

We may assume that the obstacle  $\mathcal{O}$  is contained in the unit ball  $\{x \in \mathbb{R}^3 : |x| < 1\}$  without loss of generality. We divide the proof into three steps.

*First step. Compactness on the sets  $A_r := \{x \in \Omega : |x| \leq r\}$  ( $r > 1$ ).*

By using the uniform resolvent estimates (2.30) and the limiting absorption principle (2.31) from Lemma 2.6, the decay assumption (1.8) on  $V$  and the assumption  $s \in (1/2, \delta_0/2]$ , we obtain

$$(B.13) \quad \begin{aligned} \|g_n\|_{L^2_{-s}(\Omega)} &\leq \frac{C}{\lambda} \|V f_n\|_{L^2_s(\Omega)} \\ &\leq \frac{C}{\lambda} \|f_n\|_{L^2_{-s}(\Omega)} \\ &\leq \frac{CM}{\lambda} \end{aligned}$$

for any  $\lambda > 0$ , while the definition of operators  $R(\lambda^2 \pm i0)$  implies that the sequence  $\{g_n\}$  satisfies the elliptic equation

$$-\Delta g_n = h_n := \lambda^2 g_n + V f_n.$$

Since  $V \in L^\infty(\Omega)$ , it follows that

$$\|V f_n\|_{L^2(A_r)} \leq CM$$

for any  $r > 1$ . Hence combining (B.13) and the previous estimate, we get

$$\|h_n\|_{L^2(A_r)} \leq C(\lambda)M.$$

Then, resorting to the ellipticity of the operator  $-\Delta$ , we find from the previous estimate that

$$\|g_n\|_{H^2(A_r)} \leq C(\lambda)M$$

for any  $r > 1$ . Since  $H^2(A_r)$  is compact in  $L^2(A_r)$ , there exists a subsequence  $\{g_{n'}\}$  which converges to some  $g_{1,r} \in L^2(A_r)$  strongly in  $L^2(A_r)$  for any  $r > 1$ .

*Second step. Weak compactness on the sets  $\Omega \setminus A_r$ .*

Let  $r > 1$  be fixed. The estimate (B.13) implies that  $\{g_{n'}\}$  is uniformly bounded in  $L^2_{-s}(\Omega \setminus A_r)$ . As a consequence, there exists a subsequence  $\{g_{n''}\}$  of  $\{g_{n'}\}$  which converges to some  $g_{2,r} \in L^2_{-s}(\Omega \setminus A_r)$  weakly in  $L^2_{-s}(\Omega \setminus A_r)$ . Furthermore, for any  $\varepsilon > 0$  there exists  $r(\varepsilon) > 1$  such that

$$(B.14) \quad \int_{\Omega \setminus A_{r(\varepsilon)}} \langle x \rangle^{-2s} |g_{n''}|^2 dx < \varepsilon \quad \text{and} \quad \int_{\Omega \setminus A_{r(\varepsilon)}} \langle x \rangle^{-2s} |g_{2,r(\varepsilon)}|^2 dx < \varepsilon$$

for all  $n''$ .

*End of the proof.* Put

$$g = \chi_{A_{r(\varepsilon)}} g_{1,r(\varepsilon)} + \{1 - \chi_{A_{r(\varepsilon)}}\} g_{2,r(\varepsilon)},$$

where  $\chi_{A_{r(\varepsilon)}}$  is the characteristic function on  $A_{r(\varepsilon)}$ . By using (B.14), we have

$$\begin{aligned} &\int_{\Omega} \langle x \rangle^{-2s} |g_{n''} - g|^2 dx \\ &= \int_{A_{r(\varepsilon)}} \langle x \rangle^{-2s} |g_{n''} - g_{1,r(\varepsilon)}|^2 dx + \int_{\Omega \setminus A_{r(\varepsilon)}} \langle x \rangle^{-2s} |g_{n''} - g_{2,r(\varepsilon)}|^2 dx \\ &\leq \int_{A_{r(\varepsilon)}} \langle x \rangle^{-2s} |g_{n''} - g_{1,r(\varepsilon)}|^2 dx + 2\varepsilon. \end{aligned}$$

Finally, letting  $n'' \rightarrow \infty$ , we conclude from the first step that

$$\overline{\lim}_{n'' \rightarrow \infty} \int_{\Omega} \langle x \rangle^{-2s} |g_{n''} - g|^2 dx \leq 2\varepsilon.$$

Thus  $\{g_n\}$  is compact, since  $\varepsilon > 0$  is arbitrary. Hence  $R(\lambda^2 \pm i0)V$  are compact operators from  $L^2_s(\Omega)$  into itself. The proof of Lemma B.2 is complete.  $\square$

We have also the compactness of the resolvent operators on  $L^p(\Omega)$ .

**Lemma B.3.** *Assume that the measurable potential  $V$  satisfies assumption (1.8). Let  $p_0$  be a real as in Lemma B.1. Then the operators*

$$R(\lambda^2 \pm i0)V = S(\lambda^2 \pm i0)R_0(\lambda^2 \pm i0) \circ J_* \circ V$$

are compact on  $L^p(\Omega)$  for any  $p \in (p_0, \infty]$  and any  $\lambda > 0$ .

*Proof.* It is sufficient to prove the compactness of operators

$$R_0(\lambda^2 \pm i0) \circ J_* \circ V$$

from  $L^p(\Omega)$  into  $L^p(\mathbb{R}^3)$  for any  $p \in (p_0, \infty]$ , since  $S(\lambda^2 \pm i0)$  are the bounded operators from  $\mathcal{R}(R_0(\lambda^2 \pm i0)J_*)$  into  $L^p(\Omega)$  by Lemma 2.7 or Proposition 2.4. Let  $\{f_n\}$  be a sequence such that

$$\sup_{n \in \mathbb{N}} \|f_n\|_{L^p(\Omega)} \leq M,$$

where  $M$  is a positive constant. We shall prove the compactness of the following sequence of functions

$$g_n = R_0(\lambda^2 \pm i0)J_*(Vf_n).$$

We may assume that the obstacle  $\mathcal{O}$  is contained in  $\{x \in \mathbb{R}^3 : |x| < 1\}$  without loss of generality. We divide the proof into three steps.

*First step. Compactness on the sets  $B_r := \{x \in \mathbb{R}^3 : |x| \leq r\}$  ( $r > 1$ ).*

By using the estimates (B.1), we have

$$(B.15) \quad \begin{aligned} \|\mu_p(\cdot)R_0(\lambda^2 \pm i0)J_*(Vf_n)\|_{L^p(\mathbb{R}^3)} &\leq \frac{C}{\lambda^{2/p}} \|f_n\|_{L^p(\Omega)} \\ &\leq \frac{CM}{\lambda^{2/p}} \end{aligned}$$

for any  $\lambda > 0$ , and hence, we get

$$(B.16) \quad \|g_n\|_{L^p(B_r)} \leq C(\lambda)M,$$

while the definition of operators  $R_0(\lambda^2 \pm i0)$  implies that the sequence  $\{g_n\}$  satisfies the elliptic equation

$$-\Delta g_n = h_n := \lambda^2 g_n + J_*(Vf_n).$$

Since  $V \in L^\infty(\Omega)$ , it follows that

$$\|J_*(Vf_n)\|_{L^p(B_r)} \leq CM$$

for any  $r > 1$ . Hence combining (B.16) and the previous estimate, we get

$$\|h_n\|_{L^p(B_r)} \leq C(\lambda)M.$$

Then, resorting to the ellipticity of the operator  $-\Delta$  and the previous estimate, we find that

$$\|g_n\|_{W^{2,p}(B_r)} \leq C(\lambda)$$

for any  $r > 1$ . Since  $W^{2,p}(B_r)$  is compact in  $L^p(B_r)$  for  $p \in (3, \infty]$ , there exists a subsequence  $\{g_{n'}\}$  which converges to some  $g_{1,r} \in L^p(B_r)$  strongly in  $L^p(B_r)$  for any  $r > 1$ .

*Second step. Weak compactness on the sets  $\mathbb{R}^3 \setminus B_r$ .*

Let  $r > 1$  be a fixed real number. The estimate (B.15) implies that

$$\begin{aligned} \|g_{n'}\|_{L^p(\mathbb{R}^3 \setminus B_r)} &\leq \mu_p(r)^{-1} \|\mu_p(\cdot) g_{n'}\|_{L^p(\mathbb{R}^3 \setminus B_r)} \\ &\leq CM \mu_p(r)^{-1}. \end{aligned}$$

As a consequence of this estimate, there exists a subsequence  $\{g_{n''}\}$  of  $\{g_{n'}\}$  which converges to some  $g_{2,r} \in L^p(\mathbb{R}^3 \setminus B_r)$  weakly in  $L^p(\mathbb{R}^3 \setminus B_r)$ . When  $p = \infty$ , the convergence is weakly\*. Furthermore, for any  $\varepsilon > 0$  there exists  $r(\varepsilon) > 1$  such that

$$(B.17) \quad \|g_{n''}\|_{L^p(\mathbb{R}^3 \setminus B_{r(\varepsilon)})} < \varepsilon \quad \text{and} \quad \|g_{2,r(\varepsilon)}\|_{L^p(\mathbb{R}^3 \setminus B_{r(\varepsilon)})} < \varepsilon$$

for all  $n''$ .

*End of the proof.* Putting

$$g = \chi_{B_{r(\varepsilon)}} g_{1,r(\varepsilon)} + \{1 - \chi_{B_{r(\varepsilon)}}\} g_{2,r(\varepsilon)},$$

where  $\chi_{B_{r(\varepsilon)}}$  is the characteristic function on  $B_{r(\varepsilon)}$ , by using (B.17), we have

$$\begin{aligned} \|g_{n''} - g\|_{L^p(\mathbb{R}^3)} &= \|g_{n''} - g_{1,r(\varepsilon)}\|_{L^p(B_{r(\varepsilon)})} + \|g_{n''} - g_{2,r(\varepsilon)}\|_{L^p(\mathbb{R}^3 \setminus B_{r(\varepsilon)})} \\ &\leq \|g_{n''} - g_{1,r(\varepsilon)}\|_{L^p(B_{r(\varepsilon)})} + 2\varepsilon. \end{aligned}$$

Finally, letting  $n'' \rightarrow \infty$ , we conclude from the first step that

$$\overline{\lim}_{n'' \rightarrow \infty} \|g_{n''} - g\|_{L^p(\mathbb{R}^3)} \leq 2\varepsilon.$$

Thus  $\{g_n\}$  is compact, since  $\varepsilon > 0$  is arbitrary. This proves that  $R_0(\lambda^2 \pm i0) \circ J_* \circ V$  are compact operators from  $L^p(\Omega)$  to  $L^p(\mathbb{R}^3)$ . The proof of Lemma B.3 is complete.  $\square$

### APPENDIX C. (PROOF OF THEOREM 2.9)

This appendix is devoted to the proof of Theorem 2.9. We start with the invertibility of the operators  $I + R(\lambda^2 \pm i0)V$  for  $\lambda > 0$ .

**Lemma C.1.** *Assume that the measurable potential  $V$  satisfies (1.8). Then we have the following:*

- (i) *Let  $1/2 < s \leq \delta_0/2$ . If the function  $u \in L^2_{-s}(\Omega)$  is a solution of the integral equation*

$$(C.1) \quad u + R(\lambda^2 \pm i0)Vu = 0 \quad \text{in } \Omega$$

*for  $\lambda > 0$ , then  $u = 0$  in  $\Omega$ .*

- (ii) *Let  $p_0 > 5$  be as in Lemma B.1. If the function  $u \in L^p(\Omega)$  for some  $p \in (p_0, \infty]$  is a solution of the integral equation (C.1) for  $\lambda > 0$ , then  $u = 0$  in  $\Omega$ .*

*Proof.* The solution  $u \in L^2_{-s}(\Omega)$  to (C.1) solves the following boundary value problem:

$$(C.2) \quad \begin{cases} \{-\Delta + V(x) - \lambda^2\} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that operator  $-\Delta + V(x)$  has no positive eigenvalues (see, e.g., Theorem 2.2 in [31, Mochizuki], and also Theorem 10.2 from [30, Mochizuki]). Namely, the boundary value problem (C.2) admits only a trivial solution. Hence we conclude that  $u = 0$  in  $\Omega$ , which proves the assertion (i).

As to (ii), applying Proposition 2.5 for any  $s > 1/2$  and Hölder's inequality, we write

$$(C.3) \quad \begin{aligned} \|R(\lambda^2 \pm i0)(Vu)\|_{L^2_{-s}(\Omega)} &\leq C \|R_0(\lambda^2 \pm i0)J_*(Vu)\|_{L^2_{-s}(\mathbb{R}^3)} \\ &\leq C \|\mu_p(\cdot)R_0(\lambda^2 \pm i0)J_*(Vu)\|_{L^p(\mathbb{R}^3)} \|\langle \cdot \rangle^{-2s} \mu_p(\cdot)^{-2}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^3)}^{1/2} \end{aligned}$$

for any  $\lambda > 0$  and  $p \in (p_0, \infty]$ , where  $\mu_p(x)$  is the function appearing in (B.1) (see also (B.9)). When  $p = \infty$ , we used the convention  $\frac{p}{p-2} = 1$ . Since

$$-2s - 2 \left(s - \frac{1}{2}\right) \left(1 - \frac{2}{p}\right) + \frac{4s}{p} = -\frac{(4s-1)(p-2)}{p},$$

it follows that

$$\|\langle \cdot \rangle^{-2s} \mu_p(\cdot)^{-2}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^3)} = \left\{ \int_{\mathbb{R}^3} \langle x \rangle^{-(4s-1)} dx \right\}^{\frac{p-2}{p}} < \infty,$$

provided that

$$1 < s < \frac{3}{2}.$$

Hence, combining (C.3) and estimate (B.1) from Lemma B.1, we deduce that

$$\|R(\lambda^2 \pm i0)(Vu)\|_{L^2_{-s}(\Omega)} \leq \frac{C}{\lambda^{2/p}} \|u\|_{L^p(\Omega)}$$

for any  $\lambda > 0$  and  $1 < s < \min\{3/2, \delta_0/2\}$ . Thus these estimates together with equation (C.1) imply that

$$u \in L^p(\Omega) \quad \text{for } p \in (p_0, \infty] \quad \implies \quad u \in L^2_{-s}(\Omega) \quad \text{for } 1 < s < \min\left\{\frac{3}{2}, \frac{\delta_0}{2}\right\}.$$

Hence we apply the previous result (i) to conclude that  $u = 0$  in  $\Omega$ . The proof of Lemma C.1 is complete.  $\square$

We are now in a position to prove Theorem 2.9.

*Proof of Theorem 2.9.* It is sufficient to prove the assertion (i) for  $\lambda > 0$ , since the case  $\lambda = 0$  is proved by using Proposition A.1, the compactness on  $L^2_{-s}(\Omega)$  of  $R(0)V$  and the Fredholm alternative theorem that we just develop below.

Lemma C.1 implies that the operators

$$I + R(\lambda^2 \pm i0)V$$

are injective both in  $\mathcal{B}(L^2_{-s}(\Omega))$  for any  $s \in (1/2, \delta_0/2]$ , and in  $\mathcal{B}(L^p(\Omega))$  for any  $p \in (p_0, \infty]$ . Thanks to Lemmas B.2 and B.3, the operators  $R(\lambda^2 \pm i0)V$  are compact

perturbations of the identity operator  $I$ , thus we now apply the Fredholm alternative theorem to deduce the existence of the operators

$$S^\pm(\lambda) = (I + R(\lambda^2 \pm i0)V)^{-1} \quad \text{both in } \mathcal{B}(L^2_{-s}(\Omega)) \text{ and in } \mathcal{B}(L^p(\Omega)).$$

It remains to prove the uniform estimates (2.39) for the operators  $S^\pm(\lambda)$  with respect to  $\lambda > 0$ . It is sufficient to prove that there exists a constant  $C > 0$  such that

$$(C.4) \quad \|I + R(\lambda^2 \pm i0)V\|_{\mathcal{B}(L^2_{-s}(\Omega))} \geq C$$

for any  $\lambda > 0$ . By using the uniform resolvent estimates (2.38) from Lemma 2.8, there exists  $\lambda_0 > 1$  such that

$$\|R(\lambda^2 \pm i0)V\|_{\mathcal{B}(L^2_{-s}(\Omega))} \leq \frac{1}{2}$$

for any  $\lambda > \lambda_0$ , and hence, we get

$$\|I + R(\lambda^2 \pm i0)V\|_{\mathcal{B}(L^2_{-s}(\Omega))} \geq \frac{1}{2}$$

for any  $\lambda > \lambda_0$ . This proves (C.4) for  $\lambda > \lambda_0$ . Next, we prove (C.4) on any subinterval  $[\varepsilon, \lambda_0]$  of  $(0, \lambda_1]$ . We suppose that

$$(C.5) \quad \inf_{\lambda \in [\varepsilon, \lambda_0]} \|I + R(\lambda^2 \pm i0)V\|_{\mathcal{B}(L^2_{-s}(\Omega))} = 0,$$

and lead to a contradiction. Since  $\|I + R(\lambda^2 \pm i0)V\|_{\mathcal{B}(L^2_{-s}(\Omega))}$  are continuous on  $[\varepsilon, \lambda_0]$ , the compactness of  $[\varepsilon, \lambda_0]$  and assumption (C.5) imply that there exists a real  $\lambda_* \in [\varepsilon, \lambda_0]$  such that

$$\|I + R(\lambda_*^2 \pm i0)V\|_{\mathcal{B}(L^2_{-s}(\Omega))} = 0.$$

This contradicts the injective property of  $I + R(\lambda_*^2 \pm i0)V$ . Thus we must have (C.4) for any  $\lambda \in [\varepsilon, \lambda_0]$ . This proves the assertion (i) in Theorem 2.9.

In a similar way, we prove that

$$\|I + R(\lambda^2 \pm i0)V\|_{\mathcal{B}(L^p(\Omega))} \geq C$$

for any  $p \in (p_0, \infty]$  and  $\lambda > 0$ . In fact, since we obtained the decay estimates (B.1) with respect to  $\lambda > 0$  in Lemma B.1, the above argument enables us to conclude that there exist constants  $C > 0$  and  $\lambda_2 > 1$  such that

$$(C.6) \quad \|I + R(\lambda^2 \pm i0)Vf\|_{\mathcal{B}(L^p(\Omega))} \geq C$$

for any  $\lambda > \lambda_2$  and  $p \in (p_0, \infty)$ . When  $p = \infty$ , we suppose that (C.6) is not true for large  $\lambda > 0$ , and lead to a contradiction. Then we have

$$\varliminf_{\lambda \rightarrow \infty} \|I + R(\lambda^2 \pm i0)V\|_{\mathcal{B}(L^\infty(\Omega))} = 0,$$

which implies that

$$(C.7) \quad \varliminf_{\lambda \rightarrow \infty} |[\{I + R(\lambda^2 \pm i0)V\}f](x)| = 0$$

for any  $x \in \Omega$  and for any  $f \in L^\infty(\Omega)$ . On the other hand, we find from (B.3) in Lemma B.1 that for each  $x \in \Omega$  and  $f \in L^\infty(\Omega)$ ,

$$\varliminf_{\lambda \rightarrow \infty} |[\{I + R(\lambda^2 \pm i0)V\}f](x)| = |f(x)| \neq 0,$$

unless  $f(x) = 0$  on  $\Omega$ . This contradicts (C.7). Thus, when  $p = \infty$ , we obtain the estimate (C.6) for large  $\lambda$ . Finally, as to the lower bound (C.6) for any compact subinterval  $[\varepsilon, \lambda_2]$  of  $(0, \lambda_2]$ , the proof is identical to that in  $L^2_{-s}(\Omega)$ -case. So we may omit the detail, and we conclude the proof of the assertion (ii) in Theorem 2.9. The proof of Theorem 2.9 is finished.  $\square$

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