WELL-POSEDNESS FOR MUTATIONAL EQUATIONS UNDER A GENERAL TYPE OF DISSIPATIVITY CONDITIONS

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Abstract. This paper is concerned with mutational analysis found by Aubin and developed by Lorenz. To extend their results so that they can be applied to quasi-linear evolution equations initiated by Kato, we focus on a mutational framework where for each $r > 0$ there exists $M \geq 1$ such that $d(\vartheta(t, x), \vartheta(t, y)) \leq M d(x, y)$ for $t \in [0, 1]$ and $x, y \in D_r(\phi)$, where ϑ is a transition and $D_r(\phi)$ is the revel set of a proper lower semicontinuous functional ϕ. The setting that the constant M may be larger than 1 plays an important role in applying to quasi-linear evolution equations. In that case, it is difficult to estimate the distance between two approximate solutions to mutational equations. Our strategy is to construct a family of metrics depending on both time and state, with respect to which transitions are contractive in some sense.

1. Introduction

This paper is concerned with mutational analysis initiated by Aubin [3, 2]. He introduced a set of transitions and studied the so-called mutational equations in a metric space. His result extends classical results such as the existence theorems of Cauchy-Lipschitz and Nagumo concerning ordinary differential equations in the Euclidean spaces and is applied to morphological equations.

Recently, Lorenz [11] has introduced a new functional and generalized Aubin's mutational framework in that Lipschitz conditions on transitions do not always have globally bounded Lipschitz constants. His modified mutational analysis ([11, 4]) makes it possible to deal with nonlinear transport equations for finite real-valued Radon measures on the Euclidean spaces. Their results are based on Euler method combined with appropriate compactness assumptions.

We are interested in extending their results so that they can be applied to quasi-linear evolution equations developed by Hughes *et. al.* [5]. Our mutational framework is stated as follows: Let E be a complete metric space with metric d and let ϕ be a proper lower semicontinuous functional from E into $[0, \infty]$ such that $D(\phi)$ is dense in E, where $D(\phi)$ is the effective domain of ϕ. Let $D_r(\phi) = \{x \in E; \phi(x) \leq r\}$ for $r > 0$. A continuous mapping $\vartheta : [0, 1] \times D(\phi) \to D(\phi)$ is called a *transition* on (E, d, ϕ) if the following conditions are satisfied:

(t1) $\vartheta(0, x) = x$ for $x \in D(\phi)$.

(t2) $\vartheta(t + h, x) = \vartheta(h, \vartheta(t, x))$ for $x \in D(\phi)$ and $t, h \in [0, 1]$ with $t + h \in [0, 1]$.

(t3) For each $r > 0$ there exists $M \geq 1$ such that

$$d(\vartheta(t, x), \vartheta(t, y)) \leq M d(x, y) \quad \text{for } t \in [0, 1] \text{ and } x, y \in D_r(\phi).$$

(t4) For each $r > 0$ there exists $\beta > 0$ such that
$$\limsup_{h \downarrow 0} h^{-1} d(\vartheta(h, x), x) \le \beta \quad \text{for } x \in D_r(\phi).$$

(t5) For each $r > 0$ there exists $K > 0$ such that
$$\phi(\vartheta(t, x)) \le K \quad \text{for } t \in [0, 1] \text{ and } x \in D_r(\phi).$$

The setting that the constant M appearing condition (t3) is possibly larger than 1 plays an important role in applying to quasi-linear evolution equations. In other words, it is hard to estimate the distance between two approximate solutions to mutational equations. Our strategy is to construct a family of metrics on E depending on both time and state, with respect to which transitions are contractive in some sense (see Lemmas 2.5 and 2.6). This together with techniques developed in [8] enables us to prove the well-posedness for mutational equations. We also emphasize that no compactness assumption is imposed but a dissipativity condition with respect to a metric-like functional is used in our formulation.

2. Main theorem

Given a set $\Theta(E, d, \phi)$ of transitions, the pair $(E, \Theta(E, d, \phi))$ is called a *mutational space*. For $r > 0$ we define $\beta_r : \Theta(E, d, \phi) \to [0, \infty)$ by
$$\beta_r(\vartheta) = \sup_{x \in D_r(\phi)} \left(\limsup_{h \downarrow 0} h^{-1} d(\vartheta(h, x), x) \right)$$

for $\vartheta \in \Theta(E, d, \phi)$ and define $D_r : \Theta(E, d, \phi) \times \Theta(E, d, \phi) \to [0, \infty)$ by
$$D_r(\vartheta, \hat{\vartheta}) = \sup_{x \in D_r(\phi)} \left(\limsup_{h \downarrow 0} h^{-1} d(\vartheta(h, x), \hat{\vartheta}(h, x)) \right)$$

for $\vartheta, \hat{\vartheta} \in \Theta(E, d, \phi)$. Each function $D_r(\cdot, \cdot)$ is symmetric and satisfies the triangle inequality, although it is not always a metric on $\Theta(E, d, \phi)$. Note that

(2.1) $$|\beta_r(\vartheta) - \beta_r(\hat{\vartheta})| \le D_r(\vartheta, \hat{\vartheta}) \le \beta_r(\vartheta) + \beta_r(\hat{\vartheta})$$

for $\vartheta, \hat{\vartheta} \in \Theta(E, d, \phi)$. To specify the stability of transitions we introduce the mapping $M_r : \Theta(E, d, \phi) \to [0, \infty)$ defined by
$$M_r(\vartheta) = \sup\{d(x, y)^{-1} d(\vartheta(t, x), \vartheta(t, y)); x, y \in D_r(\phi), x \ne y, t \in [0, 1]\}$$

for $r > 0$. Note that $M_r(\vartheta) \ge 1$ for $r > 0$ and $\vartheta \in \Theta(E, d, \phi)$. Moreover, we introduce the mapping $K_r : \Theta(E, d, \phi) \to [0, \infty)$ defined by
$$K_r(\vartheta) = \sup\{\phi(\vartheta(t, x)); t \in [0, 1], x \in D_r(\phi)\}$$

for $r > 0$.

Let $(E, \Theta(E, d, \phi))$ be a mutational space and let u be a function on $[0, \tau)$ such that $u(t) \in D(\phi)$ for $t \in [0, \tau)$ where $\tau \in (0, \infty]$. For each $t \in [0, \tau)$ the set
$$\overset{\circ}{u}(t) = \left\{ \vartheta \in \Theta(E, d, \phi); \lim_{h \downarrow 0} h^{-1} d(u(t + h), \vartheta(h, u(t))) = 0 \right\}$$

is called the *mutation* of u at t. Let D be a subset of E and let f be a mapping from D into $\Theta(E, d, \phi)$. A function $u \in C([0, \tau); E)$ where $\tau \in (0, \infty]$ is called *a solution to the mutational equation*
$$\overset{\circ}{u}(t) \ni f(u(t)) \quad \text{for } t \in [0, \tau)$$

if $u(t) \in D \cap D(\phi)$ for $t \in [0, \tau)$ and

$$\lim_{h \downarrow 0} h^{-1} d(u(t+h), f(u(t))(h, u(t))) = 0 \quad \text{for } t \in [0, \tau).$$

To develop the theory of mutational equations so that it can be applied to quasilinear evolution equations initiated by Kato [7], we consider a mapping f on D to $\Theta(E, d, \phi)$ such that there exists a proper lower semicontinuous functional ψ from E into $[0, \infty]$ with $D(\psi) = D$ satisfying the following conditions:

(S1) For each $\nu > 0$ and $r > 0$, $\sup_{w \in D_\nu(\psi)} M_r(f(w)) < \infty$.

(S2) For each $\nu > 0$ and $r > 0$, $\sup_{w \in D_\nu(\psi)} K_r(f(w)) < \infty$.

These are stability conditions of considerably general type and this setting is motivated by the following example of a mutational space associated with the abstract quasilinear evolution equation

(QE) $$u'(t) = A(u(t))u(t) \quad \text{for } t \in [0, \tau)$$

in a Banach space X, where Y is a reflexive Banach space such that Y is densely and continuously embedded in X and $\{A(w); w \in Y\}$ is a family of closed linear operators in X. Let $Y_r = \{y \in Y; \|y\|_Y \leq r\}$ for $r > 0$, and assume that for each $r > 0$ there exist $\lambda_N(r) \geq 1$, $\mu_N(r) > 0$, $\beta_A(r) \geq 0$, $\lambda_A(r) > 0$, $\mu_A(r) > 0$ and $\lambda_B(r) > 0$ such that the following conditions are satisfied:

(N) There exists a family $\{N_w; w \in Y\}$ of norms in X such that

(2.2) $$\|x\|_X \leq N_w(x) \leq \lambda_N(r)\|x\|_X \quad \text{for } w \in Y_r \text{ and } x \in X,$$

(2.3) $$N_w(x) \leq N_{\hat{w}}(x)(1 + \mu_N(r)\|w - \hat{w}\|_X) \quad \text{for } w, \hat{w} \in Y_r \text{ and } x \in X.$$

(A1) For each $w \in Y$, the operator $A(w)$ generates a semigroup $\{T_w(t); t \geq 0\}$ on X of class (C_0) such that $N_w(T_w(t)x) \leq e^{\beta_A(r)t} N_w(x)$ for $t \geq 0$, $x \in X$ and $w \in Y_r$.

(A2) There exist an isomorphism S of Y onto X and a family $\{B(w); w \in Y\}$ in $B(X)$ such that

$$SA(w)S^{-1} = A(w) + B(w) \quad \text{for } w \in Y,$$
$$\|B(w)\|_{X,X} \leq \lambda_B(r) \quad \text{for } w \in Y_r.$$

(A3) For each $w \in Y$, $D(A(w)) \supset Y$ and $A(w) \in B(Y, X)$, and

$$\|A(w)\|_{Y,X} \leq \lambda_A(r) \quad \text{for } w \in Y_r,$$
$$\|A(w) - A(\hat{w})\|_{Y,X} \leq \mu_A(r)\|w - \hat{w}\|_X \quad \text{for } w, \hat{w} \in Y_r.$$

Without loss of generality, we may assume that $\|y\|_X \leq \|y\|_Y$ for $y \in Y$ and $\|y\|_Y = \|Sy\|_X$ for $y \in Y$ and that λ_N, μ_N, β_A, λ_A, μ_A and λ_B are defined on $[0, \infty)$, and they are nondecreasing and continuous on $[0, \infty)$. This corresponds to the case where $X = Z = Z'$ in the setting considered by Hughes *et. al.* [5] and localizes their conditions to handle the global well-posedness for (QE).

Let $E = X$ and $D = Y$, and consider the metric d on E defined by $d(x, y) = \|x - y\|_X$ for $x, y \in E$ and the functional ϕ on E defined by $\phi(x) = \|Sx\|_X (= \|x\|_Y)$ if $x \in D$, and $\phi(x) = \infty$ otherwise. By the completeness of X and the reflexivity of Y, the lower semicontinuity of ϕ is verified. From condition (A2) we see that for each $w \in Y$, $T_w(t)(Y) \subset Y$ for $t \geq 0$, and

(2.4) $$ST_w(t)y = T_w(t)Sy + \int_0^t T_w(t-s)B(w)ST_w(s)y \, ds$$

for $t \geq 0$ and $y \in Y$. For $w \in D$, define $\vartheta_w(t, x) = T_w(t)x$ for $(t, x) \in [0, 1] \times D(\phi)$. Then we observe that ϑ_w is a continuous mapping from $[0, 1] \times D(\phi)$ into $D(\phi)$ and have the following proposition:

Proposition 2.1. *Define* $\Theta(E, d, \phi) = \{\vartheta_w; w \in D\}$. *Then* $(E, \Theta(E, d, \phi))$ *is a mutational space, and the mapping f from D into $\Theta(E, d, \phi)$ defined by $f(w) = \vartheta_w$ for $w \in D$ satisfies the following two conditions:*

(i) *For each $\nu > 0$ there exists $M \geq 1$ such that*
$$d(f(w)(t, x), f(w)(t, y)) \leq M d(x, y)$$
for $t \in [0, 1]$, $x, y \in D(\phi)$ and $w \in D$ with $N_w(Sw) \leq \nu$.

(ii) *For $r > 0$ and $\nu > 0$ there exists $K > 0$ such that*
$$\phi(f(w)(t, x)) \leq K$$
for $t \in [0, 1]$, $x \in D_r(\phi)$ and $w \in D$ with $N_w(Sw) \leq \nu$.

Proof. Since $A(w)$ is the infinitesimal generator of the semigroup $\{T_w(t); t \geq 0\}$ on X, we infer from condition (A3) that condition (t4) is satisfied and $\beta_r(\vartheta_w) = \|A(w)\|_{Y,X} r$ for $r > 0$ and $w \in D$. To verify conditions (t3) and (t5), it suffices to prove that the mapping f satisfies conditions (i) and (ii). By (2.2) and (A1) we have

$$d(f(w)(t, x), f(w)(t, y)) \leq N_w(T_w(t)(x - y)) \leq e^{\beta_A(r)t} N_w(x - y)$$
$$\leq \lambda_N(r) e^{\beta_A(r)t} d(x, y)$$

for $w \in Y_r$, $x, y \in D(\phi)$ and $t \in [0, 1]$. Since $\|w\|_Y \leq N_w(Sw)$ for $w \in Y$, condition (i) is satisfied with $M = \lambda_N(\nu) e^{\beta_A(\nu)}$ for each $\nu > 0$. To verify condition (ii), let $x \in Y$ and $w \in Y$ with $N_w(Sw) \leq \nu$. By (2.4) we have

$$N_w(ST_w(t)x) \leq e^{\beta_A(\nu)t} N_w(Sx)$$
$$+ \int_0^t e^{\beta_A(\nu)(t-s)} \lambda_N(\nu) \lambda_B(\nu) N_w(ST_w(s)x) \, ds$$

for $t \geq 0$. Application of Gronwall's inequality yields $N_w(ST_w(t)x) \leq e^{a(\nu)t} N_w(Sx)$ for $t \geq 0$, where $a(\nu) = \beta_A(\nu) + \lambda_N(\nu)\lambda_B(\nu)$. This implies that $\phi(f(w)(t, x)) \leq \lambda_N(\nu) e^{a(\nu)t} \phi(x)$ for $t \in [0, 1]$. $\qquad\square$

Let $(E, \Theta(E, d, \phi))$ be a mutational space and consider the mutational equation

$$\overset{\circ}{u}(t) \ni f(u(t))$$

for a mapping f on D to $\Theta(E, d, \phi)$ satisfying conditions (S1) and (S2). Instead of the compactness assumption imposed in [11], we discuss the mutational equation under a general type of dissipativity condition on f. To define a general type of dissipativity condition on f, we make the hypothesis

(H) for each $\nu > 0$ there exists $r > 0$ such that $D_\nu(\psi) \subset D_r(\phi)$,

and use a nonnegative functional Φ on $D(\phi) \times D(\phi)$ satisfying the following conditions:

(Φ1) For each $r > 0$ there exists $L_r > 0$ such that
$$|\Phi(x, y) - \Phi(\hat{x}, \hat{y})| \leq L_r(d(x, \hat{x}) + d(y, \hat{y}))$$
for (x, y), $(\hat{x}, \hat{y}) \in D_r(\phi) \times D_r(\phi)$.

(Φ2) For each $\nu > 0$ there exist $M_\nu \geq m_\nu > 0$ such that

$$m_\nu d(x, y) \leq \Phi(x, y) \leq M_\nu d(x, y)$$

for $(x, y) \in D_\nu(\psi) \times D_\nu(\psi)$.

Assume that f satisfies the following three conditions:

(f1) For each $\nu > 0$, $r > 0$, $\epsilon > 0$, $x \in D_\nu(\psi)$ there exists $\delta > 0$ such that $y \in D_\nu(\psi)$ and $d(y, x) < \delta$ imply $D_r(f(y), f(x)) < \epsilon$.

(f2) There exists $g \in C([0, \infty); \mathbb{R})$ with $g(0) \geq 0$ such that to each $\epsilon > 0$ and $x \in D$ there correspond $h \in (0, \epsilon]$ and $x_h \in D$ such that

$$d(f(x)(h, x), x_h) \leq \epsilon h \quad \text{and} \quad h^{-1}(\psi(x_h) - \psi(x)) \leq g(\psi(x)) + \epsilon.$$

(f3) There exists a nonnegative functional Φ on $D(\phi) \times D(\phi)$ satisfying conditions (Φ1) and (Φ2) such that to each $\nu > 0$ there corresponds $\omega_\nu \geq 0$ satisfying

$$\liminf_{h \downarrow 0} h^{-1}\big(\Phi(f(x)(h, x), f(y)(h, y)) - \Phi(x, y)\big) \leq \omega_\nu \Phi(x, y)$$

for any $x, y \in D_\nu(\psi)$.

Remark 2.2. (i) Condition (f2) is regarded as the so-called subtangential condition combined with a growth condition. A subtangential condition was used by Nagumo [13] to study a viability theorem (see [1]). (ii) Condition (f3) is a general type of dissipativity condition proposed in this paper. A dissipativity condition with respect to a metric-like functional was considered by Okamura [14] to characterize the uniqueness of solutions of ordinary differential equations (see also [15, 10]). The functional ψ is a Liapunov functional and used to localize the dissipativity.

For each $\nu > 0$ we denote by $\tau(\nu)$ the maximal existence time of the noncontinuable maximal solution $m(t; \nu)$ of the Cauchy problem

(2.5) $$p'(t) = g(p(t)) \text{ for } t \geq 0, \quad \text{and} \quad p(0) = \nu.$$

We are now in a position to state the main theorem.

Theorem 2.3. *Under assumptions* (S1), (S2), (H) *and* (f1) *through* (f3), *the following assertions hold:*

(i) *For any $x \in D$, there exists a unique solution $u \in C([0, \tau(\psi(x))); E)$ to*

$$\overset{\circ}{u}(t) \ni f(u(t)) \text{ for } t \geq 0 \quad \text{and} \quad u(0) = x$$

such that $\psi(u(t))$ is locally bounded on $[0, \tau(\psi(x)))$. Moreover, the unique solution u satisfies $\psi(u(t)) \leq m(t; \psi(x))$ for $t \in [0, \tau(\psi(x)))$.

(ii) *Assume that $\tau(\nu) = \infty$ for $\nu > 0$. If $\{S(t); t \geq 0\}$ is defined by $S(t)x = u(t)$ for $t \geq 0$ and $x \in D$, then $\{S(t); t \geq 0\}$ is a semigroup of Lipschitz operators on D such that*

$$\psi(S(t)x) \leq m(t; \psi(x)) \text{ for } t \geq 0 \text{ and } x \in D.$$

Remark 2.4. Theorem 2.3 generalizes the main results in [12] and [8].

The proof Theorem 2.3 will be given in Section 5. From the rest of this section to Section 5 we make the assumptions of Theorem 2.3. The setting that M may be larger than 1 in condition (t3) makes it difficult to estimate the distance between two approximate solutions to mutational equations. To overcome such a difficulty, we construct a family of metrics depending on both time and state, with respect

to which transitions are contractive in some sense, and eliminate an additional structural inequality imposed by Lorenz [11, Section 3.4] in order that contractivity can become dispensable. We conclude this section with the following fundamental estimates.

Lemma 2.5. *There exists a family $\{d_{(t,w)}; (t,w) \in [0,1] \times D\}$ of metrics on E satisfying the following conditions:*

(i) *For $x, y \in E$ and $w \in D$, the function $t \to d_{(t,w)}(x,y)$ is continuous on $[0,1]$.*

(ii) *For $0 \leq s \leq t \leq 1$, $x, y \in D(\phi)$ and $w \in D$,*

$$(2.6) \qquad d_{(t,w)}(f(w)(t-s,x), f(w)(t-s,y)) \leq d_{(s,w)}(x,y).$$

(iii) *For $(t,w) \in [0,1] \times D$ and $x, y \in E$,*

$$(2.7) \qquad\qquad d(x,y) \leq d_{(t,w)}(x,y).$$

(iv) *For $(t,w) \in [0,1] \times D$ and $x, y \in D_r(\phi)$,*

$$(2.8) \qquad\qquad d_{(t,w)}(x,y) \leq M_r(f(w))d(x,y).$$

Proof. For $(t,w) \in [0,1] \times D$ we define a metric $\rho_{(t,w)}$ on $D(\phi)$ by $\rho_{(t,w)}(x,y) = \sup\{d(f(w)(\sigma-t,x), f(w)(\sigma-t,y)); t \leq \sigma \leq 1\}$ for $(x,y) \in D(\phi) \times D(\phi)$. This definition makes sense and the function $t \to \rho_{(t,w)}(x,y)$ is continuous on $[0,1]$ for $x, y \in D(\phi)$ and $w \in D$, since $f(w)(t,x)$ is continuous in $t \in [0,1]$. Since $D(\phi)$ is dense in E and $|\rho_{(t,w)}(x,y) - \rho_{(t,w)}(\hat{x},\hat{y})| \leq \rho_{(t,w)}(x,\hat{x}) + \rho_{(t,w)}(y,\hat{y})$ for $x, \hat{x}, y, \hat{y} \in D(\phi)$ and $(t,w) \in [0,1] \times D$, the metric $\rho_{(t,w)}$ on $D(\phi)$ has the unique extension $d_{(t,w)}$ on E for each $(t,w) \in [0,1] \times D$, and condition (i) is satisfied. The semigroup property proves assertions (ii). Since $d(x,y) \leq \rho_{(t,w)}(x,y)$ for $(t,w) \in [0,1] \times D$ and $x, y \in D(\phi)$, a density argument yields (2.7). Assertion (iv) is verified by condition (S1). $\qquad\square$

To compare the evolution of two arbitrary data along two different transitions, we use the quantity

$$K(r,\nu) = \sup_{w \in D_\nu(\psi)} K_r(f(w))$$

for $r > 0$ and $\nu > 0$. By condition (S2) we have $K(r,\nu) < \infty$ for each $r > 0$ and $\nu > 0$. For $r > 0$ and $\nu > 0$, set

$$\gamma_{r,\nu} = \sup_{w \in D_\nu(\psi)} M_{K(r,\nu)}(f(w)).$$

Then we note that $\gamma_{r,\nu} \geq 1$ for $r > 0$ and $\nu > 0$.

Lemma 2.6. *The following assertions hold:*

(i) *For $0 \leq s \leq t \leq 1$, $w \in D_\nu(\psi)$ and $x \in D_r(\phi)$,*

$$(2.9) \qquad d(f(w)(t,x), f(w)(s,x)) \leq \gamma_{r,\nu}\beta_r(f(w))(t-s).$$

(ii) *For $0 \leq s < 1$, $w, \hat{w} \in D_\nu(\psi)$ and $x, y \in E$ such that*

$$(2.10) \qquad \sup_{t \in [s,1]} \phi(f(w)(t-s,x)) \leq r \quad and \quad \sup_{t \in [s,1]} \phi(f(\hat{w})(t-s,y)) \leq r,$$

$$(2.11) \qquad d_{(t,w)}(f(w)(t-s,x), f(\hat{w})(t-s,y))$$
$$\leq d_{(s,w)}(x,y) + \gamma_{r,\nu}(t-s)D_r(f(w),f(\hat{w}))$$

for $t \in [s,1]$.

Proof. To prove (i), let $0 \le s < 1$, $w \in D_\nu(\psi)$ and $x \in D_r(\phi)$, and define $g_1(t) = d(f(w)(t,x), f(w)(s,x))$ for $t \in [s,1]$. Since $\phi(f(w)(\delta,x)) \le K_r(f(w)) \le K(r,\nu)$ for $\delta \in [0,1]$, we have

$$\delta^{-1}(g_1(t+\delta) - g_1(t)) \le \delta^{-1}d(f(w)(t,f(w)(\delta,x)), f(w)(t,x))$$
$$\le M_{K(r,\nu)}(f(w))\delta^{-1}d(f(w)(\delta,x),x)$$

for $t \in [s,1)$ and $\delta > 0$ with $t + \delta \le 1$. It follows that

$$\limsup_{\delta\downarrow 0} \delta^{-1}(g_1(t+\delta) - g_1(t)) \le \gamma_{r,\nu}\beta_r(f(w))$$

for $t \in [s,1)$. This implies (2.9). To prove (ii), let $0 \le s < 1$, $w, \hat{w} \in D_\nu(\psi)$ and $x, y \in E$ satisfy (2.10). Define $g_2(t) = d_{(t,w)}(f(w)(t-s,x), f(\hat{w})(t-s,y))$ for $t \in [s,1]$. Then we observe from Lemma 2.5 (i) and (iv) that g_2 is continuous on $[s,1]$. By (2.6) we have

$$\delta^{-1}(g_2(t+\delta) - g_2(t))$$
$$\le \delta^{-1}d_{(t+\delta,w)}(f(w)(\delta, f(\hat{w})(t-s,y)), f(\hat{w})(\delta, f(\hat{w})(t-s,y)))$$
$$\le \gamma_{r,\nu}\delta^{-1}d(f(w)(\delta, f(\hat{w})(t-s,y)), f(\hat{w})(\delta, f(\hat{w})(t-s,y)))$$

for $t \in [s,1)$ and $\delta > 0$ with $t + \delta \le 1$, where we have used (2.8) and the fact that $\phi(f(w)(\delta, f(\hat{w})(t-s,y))) \le K_r(f(w)) \le K(r,\nu)$ for $t \in [s,1)$ and $\delta > 0$ with $t + \delta \le 1$. Hence

$$\limsup_{\delta\downarrow 0}(g_2(t+\delta) - g_2(t))/\delta \le \gamma_{r,\nu}D_r(f(w), f(\hat{w}))$$

for $t \in [s,1)$. The desired inequality (2.11) is obtained. $\square$

3. Construction of approximate solutions

To construct approximate solutions we use the well-known fact ([9]) that for each $\epsilon > 0$ and $\nu > 0$, there exists the noncontinuable maximal solution $m^\epsilon(t;\nu)$ of the Cauchy problem

$$p'(t) = g(p(t)) + \epsilon \text{ for } t \ge 0, \quad \text{and} \quad p(0) = \nu,$$

and $\tau^\epsilon(\nu) \to \tau(\nu)$ and $m^\epsilon(t;\nu)$ converges to $m(t;\nu)$ uniformly on any compact subinterval of $[0,\tau(\nu))$ as $\epsilon \downarrow 0$, where $\tau^\epsilon(\nu)$ is the maximal existence time of $m^\epsilon(t;\nu)$. We denote by $B[x,r]$ the closed ball in E of radius $r > 0$ and center $x \in E$.

Lemma 3.1. *Let $x_0 \in D$. Then there exist $\epsilon_0 > 0$, $\nu > 0$, $r_0 > 0$, $r > 0$, $\rho > 0$, $M > 0$ and $\sigma \in (0,1]$ such that $D_\nu(\psi) \subset D_{r_0}(\phi)$, $\tau^\epsilon(\psi(x_0)) > \sigma$, $m^\epsilon(t;\psi(x_0)) \le \nu$ for $t \in [0,\sigma]$ and $\epsilon \in (0,\epsilon_0]$, $\sigma(1 + \gamma_{r,\nu}M) \le \rho$, $K(r_0,\nu) \le r$ and $\beta_r(f(x)) \le M$ for any $x \in D_\nu(\psi) \cap B[x_0,\rho]$.*

Proof. Let $x_0 \in D$. Take $\sigma_0 \in (0, \tau(\psi(x_0)))$ and set $\nu = \sup\{m(t;\psi(x_0)); t \in [0,\sigma_0]\} + 1$. Then there exists $\epsilon_0 > 0$ such that $\tau^\epsilon(\psi(x_0)) > \sigma_0$ and $m^\epsilon(t;\psi(x_0)) \le \nu$ for $t \in [0,\sigma_0]$ and $\epsilon \in (0,\epsilon_0]$. By condition (H) there exists $r_0 > 0$ such that $D_\nu(\psi) \subset D_{r_0}(\phi)$. Set $r = K(r_0,\nu)$. Since $x_0 \in D_\nu(\psi)$, we see from (f1) that there exists $\rho > 0$ satisfying $D_r(f(x), f(x_0)) \le 1$ for $x \in D_\nu(\psi) \cap B[x_0,\rho]$. Set $M = \beta_r(f(x_0)) + 1$. Then we have

$$\beta_r(f(x)) \le \beta_r(f(x_0)) + D_r(f(x), f(x_0)) \le \beta_r(f(x_0)) + 1 = M$$

for $x \in D_\nu(\psi) \cap B[x_0, \rho]$. If we choose $\sigma \in (0,1]$ so that $\sigma \le \sigma_0$ and $\sigma(1+\gamma_{r,\nu}M) \le \rho$, then the conclusion follows. $\qquad\square$

Lemma 3.2. *Let $x_0 \in D$, and let $\epsilon > 0$, $\nu > 0$, $r_0 > 0$, $r > 0$, $\rho > 0$, $M > 0$ and $\sigma \in (0,1]$ be such that $D_\nu(\psi) \subset D_{r_0}(\phi)$, $\tau^\epsilon(\psi(x_0)) > \sigma$, $m^\epsilon(t; \psi(x_0)) \le \nu$ for $t \in [0,\sigma]$, $\sigma(1 + \gamma_{r,\nu}M) \le \rho$, $K(r_0, \nu) \le r$ and $\beta_r(f(x)) \le M$ for any $x \in D_\nu(\psi) \cap B[x_0, \rho]$. Let $\lambda \in (0,1]$. Let $\{(s_j, x_j)\}_{j=0}^N$ be a sequence in $[0,\sigma] \times D$ such that*

  (i) $0 = s_0 < s_1 < \cdots < s_k < \cdots < s_N \le \sigma$,
  (ii) $d(f(x_{j-1})(s_j - s_{j-1}, x_{j-1}), x_j) \le \lambda(s_j - s_{j-1})$ *for $j = 1, 2, \ldots, N$,*
  (iii) $\psi(x_j) \le m^\epsilon(s_j - s_{j-1}; \psi(x_{j-1}))$ *for $j = 1, 2, \ldots, N$.*

Then, for all $j = 0, 1, \ldots, N$,

$$(\mathrm{P}_j) \quad \begin{cases} \text{(a)} & d(x_l, x_j) \le (s_j - s_l)(\lambda + \gamma_{r,\nu}M) \quad \text{for } l = 0, 1, \ldots, j, \\ \text{(b)} & \psi(x_j) \le m^\epsilon(s_j - s_l; \psi(x_l)) \quad \text{for } l = 0, 1, \ldots, j, \\ \text{(c)} & x_j \in B[x_0, \rho] \cap D_\nu(\psi) \cap D_r(\phi), \\ \text{(d)} & \beta_r(f(x_j)) \le M. \end{cases}$$

Proof. Assertion (P_0) clearly holds. Let $j \ge 1$ and assume that (P_{j-1}) holds. Since $\beta_r(f(x_{j-1})) \le M$, we apply (2.9) to find

$$d(x_{j-1}, f(x_{j-1})(s_j - s_{j-1}, x_{j-1})) \le \gamma_{r,\nu}M(s_j - s_{j-1}).$$

It follows that

$$\begin{aligned} d(x_l, x_j) &\le d(x_l, x_{j-1}) + d(x_{j-1}, f(x_{j-1})(s_j - s_{j-1}, x_{j-1})) \\ &\quad + d(f(x_{j-1})(s_j - s_{j-1}, x_{j-1}), x_j) \\ &\le (s_j - s_l)(\lambda + \gamma_{r,\nu}M) \end{aligned}$$

for $l = 0, 1, \ldots, j-1$. In particular, we have $x_j \in B[x_0, \rho]$. Assertion (b) is shown inductively from condition (iii), and so $x_j \in D_\nu(\psi)$ and $\beta_r(f(x_j)) \le M$. Since $D_\nu(\psi) \subset D_{r_0}(\phi)$, we have $\phi(x_j) = \phi(f(x_j)(0, x_j)) \le K_{r_0}(f(x_j)) \le K(r_0, \nu)$. This proves that $x_j \in D_r(\phi)$, since $K(r_0, \nu) \le r$. $\qquad\square$

Proposition 3.3. *Let $x_0 \in D$. Let $\epsilon \in (0,1]$, $\nu > 0$, $r_0 > 0$, $r > 0$, $\rho > 0$, $M > 0$ and $\tau \in (0,1]$ be such that $D_\nu(\psi) \subset D_{r_0}(\phi)$, $\tau^\epsilon(\psi(x_0)) > \tau$, $m^\epsilon(t; \psi(x_0)) \le \nu$ for $t \in [0,\tau]$, $\tau(1 + \gamma_{r,\nu}M) \le \rho$, $K(r_0, \nu) \le r$ and $\beta_r(f(x)) \le M$ for any $x \in D_\nu(\psi) \cap B[x_0, \rho]$. Then there exists a sequence $\{(t_j, x_j)\}_{j=0}^\infty$ in $[0,\tau] \times (D_\nu(\psi) \cap D_r(\phi))$ such that*

  (i) $0 = t_0 < t_1 < \cdots < t_j < \cdots < \tau$,
  (ii) $t_j - t_{j-1} \le \epsilon$ *for $j = 1, 2, \ldots$,*
  (iii) $d(f(x_{j-1})(t_j - t_{j-1}, x_{j-1}), x_j) \le \epsilon(t_j - t_{j-1})$ *for $j = 1, 2, \ldots$,*
  (iv) $\psi(x_j) \le m^\epsilon(t_j - t_l; \psi(x_l))$ *for $l = 0, 1, \ldots, j$ and $j = 0, 1, \ldots$,*
  (v) *if $x \in D_\nu(\psi) \cap B[x_{j-1}, (t_j - t_{j-1})(1 + \gamma_{r,\nu}M)]$, then*

  $$D_r(f(x), f(x_{j-1})) \le \epsilon \quad \text{for } j = 1, 2, \ldots,$$

  (vi) $\lim_{j\to\infty} t_j = \tau$.

Proof. Let $i \ge 1$ and assume that a sequence $\{(t_j, x_j)\}_{j=0}^{i-1}$ in $[0,\tau] \times (D_\nu(\psi) \cap D_r(\phi))$ is defined so that (i) through (v) hold for $j = 0, 1, \ldots, i-1$. Then we define $\bar{h}_i$ by the supremum of $h \in (0,\epsilon]$ such that $t_{i-1} + h < \tau$, $D_r(f(x), f(x_{i-1})) \le \epsilon$ for any $x \in D_\nu(\psi) \cap B[x_{i-1}, h(1 + \gamma_{r,\nu}M)]$ and there exists $u_h \in D$ satisfying

$d(f(x_{i-1})(h, x_{i-1}), u_h) \leq \epsilon h$ and $\psi(u_h) \leq m^\epsilon(h; \psi(x_{i-1}))$. By (f1) and (f2), we see that $\bar{h}_i > 0$ and there exist $h_i \in (0, \epsilon]$ and $x_i \in D$ such that $t_{i-1} + h_i < \tau$, $\bar{h}_i/2 < h_i$, $d(f(x_{i-1})(h_i, x_{i-1}), x_i) \leq \epsilon h_i$, $\psi(x_i) \leq m^\epsilon(h_i; \psi(x_{i-1}))$ and $D_r(f(x), f(x_{i-1})) \leq \epsilon$ for any $x \in D_\nu(\psi) \cap B[x_{i-1}, h_i(1 + \gamma_{r,\nu}M)]$. Set $t_i = t_{i-1} + h_{i-1}$. Then we deduce from Lemma 3.2 that $x_i \in D_\nu(\psi) \cap D_r(\phi)$ and condition (iv) is satisfied. The sequence $\{(t_j, x_j)\}_{j=0}^\infty$ in $[0, \tau] \times (D_\nu(\psi) \cap D_r(\phi))$ obtained inductively satisfies (i) through (v). To verify condition (vi), we assume that $\bar{t} = \lim_{i \to \infty} t_i < \tau$. By Lemma 3.2 we have $\beta_r(f(x_i)) \leq M$ for $i = 0, 1, \dots$ and $d(x_i, x_l) \leq (t_i - t_l)(\epsilon + \gamma_{r,\nu}M)$ for $l = 0, 1, \dots, i$ and $i = 0, 1, \dots$. Hence $\{x_i\}$ is a Cauchy sequence in E, and the limit $\bar{x} = \lim_{i \to \infty} x_i$ exists in E and is in $D_\nu(\psi) \cap D_r(\phi)$. By the lower semicontinuity of ψ, we infer from condition (iv) that $\psi(\bar{x}) \leq m^\epsilon(\bar{t} - t_l; \psi(x_l))$ for $l = 0, 1, \dots$. By (f1) and (f2) there exist $h \in (0, \epsilon]$ and $z_h \in D$ such that $\bar{t} + h < \tau$, $d(f(\bar{x})(h, \bar{x}), z_h) \leq \epsilon h/2$, $\psi(z_h) \leq m^\epsilon(h; \psi(\bar{x}))$ and $D_r(f(x), f(\bar{x})) \leq \epsilon/2$ for any $x \in D_\nu(\psi) \cap B[\bar{x}, 3h(1 + \gamma_{r,\nu}M)]$. Set $\delta_i = \bar{t} + h - t_{i-1}$ for $i \geq 1$. Then we have $t_{i-1} + \delta_i < \tau$ and $\psi(z_h) \leq m^\epsilon(\delta_i; \psi(x_{i-1}))$ for $i \geq 1$. By (2.9) we have

$$(3.1) \qquad d(f(x_{i-1})(\delta_i, x_{i-1}), f(x_{i-1})(h, x_{i-1})) \leq \gamma_{r,\nu}M(\bar{t} - t_{i-1})$$

for $i \geq 1$. Since $x_{i-1} \in D_\nu(\psi) \subset D_{r_0}(\phi)$ for $i \geq 1$, we have

$$\sup_{t \in [0,1]} \phi(f(x_{i-1})(t, x_{i-1})) \leq K_{r_0}(f(x_{i-1})) \leq r$$

for $i \geq 1$. Similarly, we have $\sup_{t \in [0,1]} \phi(f(\bar{x})(t, \bar{x})) \leq r$. By (2.11) we have

$$d_{(h,\bar{x})}(f(x_{i-1})(h, x_{i-1}), f(\bar{x})(h, \bar{x}))$$
$$\leq d_{(0,\bar{x})}(x_{i-1}, \bar{x}) + \gamma_{r,\nu}hD_r(f(x_{i-1}), f(\bar{x}))$$

for $i \geq 1$. This combined with (3.1) implies that $\lim_{i \to \infty} d(f(x_{i-1})(\delta_i, x_{i-1}), z_h) = d(f(\bar{x})(h, \bar{x}), z_h)$, since $d(x_{i-1}, \bar{x}) \to 0$ and $D_r(f(x_{i-1}), f(\bar{x})) \to 0$ as $i \to \infty$. Therefore, there exists an integer $i_1 \geq 1$ such that

$$d(f(x_{i-1})(\delta_i, x_{i-1}), z_h) \leq \epsilon h$$

for $i \geq i_1$. Choose $i_2 \geq i_1$ so that $\delta_i \leq \min\{\epsilon, 2h\}$ and $d(x_{i-1}, \bar{x}) \leq h(1 + \gamma_{r,\nu}M)$ for $i \geq i_2$. Let $i \geq i_2$ and $x \in D_\nu(\psi) \cap B[x_{i-1}, \delta_i(1 + \gamma_{r,\nu}M)]$. Then

$$d(x, \bar{x}) \leq d(x, x_{i-1}) + d(x_{i-1}, \bar{x})$$
$$\leq \delta_i(1 + \gamma_{r,\nu}M) + h(1 + \gamma_{r,\nu}M) \leq 3h(1 + \gamma_{r,\nu}M),$$

and hence $D_r(f(x), f(\bar{x})) \leq \epsilon/2$ for $x \in D_\nu(\psi) \cap B[x_{i-1}, \delta_i(1 + \gamma_{r,\nu}M)]$ and $i \geq i_2$. This implies

$$D_r(f(x), f(x_{i-1})) \leq D_r(f(x), f(\bar{x})) + D_r(f(\bar{x}), f(x_{i-1})) \leq \epsilon/2 + \epsilon/2 = \epsilon$$

for $x \in D_\nu(\psi) \cap B[x_{i-1}, \delta_i(1 + \gamma_{r,\nu}M)]$ and $i \geq i_2$. Thus we have $\bar{h}_i \geq \delta_i$ for $i \geq i_2$, which contradicts the fact that $\delta_i \to h$ and $\bar{h}_i \to 0$ as $i \to \infty$. $\qquad \square$

Lemma 3.4. *Let $x_0 \in D$. Then, there exists $\epsilon_0 > 0$ satisfying the following condition: For any $\epsilon \in (0, \epsilon_0]$ there exists $h_\epsilon > 0$ such that to each $h \in (0, h_\epsilon]$ there corresponds $x_h \in D$ satisfying*

$$d(f(x_0)(h, x_0), x_h) \leq h\epsilon \quad and \quad \psi(x_h) \leq m^\epsilon(h; \psi(x_0)).$$

Proof. Let $x_0 \in D$. By Lemma 3.1, there exist $\epsilon_0 \in (0,1]$, $\nu > 0$, $r_0 > 0$, $r > 0$, $\rho > 0$, $M > 0$ and $\tau_0 \in (0,1]$ such that $D_\nu(\psi) \subset D_{r_0}(\phi)$, $\tau^\epsilon(\psi(x_0)) > \tau_0$, $m^\epsilon(t; \psi(x_0)) \leq \nu$ for $t \in [0, \tau_0]$ and $\epsilon \in (0, \epsilon_0]$, $\tau_0(1 + \gamma_{r,\nu}M) \leq \rho$, $K(r_0, \nu) \leq r$ and $\beta_r(f(x)) \leq M$ for any $x \in D_\nu(\psi) \cap B[x_0, \rho]$. Let $\epsilon \in (0, \epsilon_0]$, and then set $\eta = \epsilon/(2\gamma_{r,\nu})$. By (f1) there exists $h_\epsilon \in (0, \tau_0]$ such that

$$(3.2) \qquad D_r(f(x), f(x_0)) \leq \eta$$

for any $x \in D_\nu(\psi) \cap B[x_0, h_\epsilon(1 + \gamma_{r,\nu}M)]$. Let $h \in (0, h_\epsilon]$. Note that $h < \tau^\epsilon(\psi(x_0))$. From Proposition 3.3 with $\rho = h(1 + \gamma_{r,\nu}M)$ and $\tau = h$ we deduce that there exists a sequence $\{(t_j, x_j)\}_{j=0}^\infty$ in $[0, h] \times (D_\nu(\psi) \cap D_r(\phi))$ such that

(i) $0 = t_0 < t_1 < \cdots < t_j < \cdots < h$,
(ii) $t_j - t_{j-1} \leq \eta$ for $j = 1, 2, \ldots$,
(iii) $d(f(x_{j-1})(t_j - t_{j-1}, x_{j-1}), x_j) \leq \eta(t_j - t_{j-1})$ for $j = 1, 2, \ldots$,
(iv) $\psi(x_j) \leq m^\eta(t_j; \psi(x_0))$ for $j = 0, 1, \ldots$,
(v) if $x \in D_\nu(\psi) \cap B[x_{j-1}, (t_j - t_{j-1})(1 + \gamma_{r,\nu}M)]$, then
$$D_r(f(x), f(x_{j-1})) \leq \eta \quad \text{for } j = 1, 2, \ldots,$$
(vi) $\lim_{j \to \infty} t_j = h$.

By Lemma 3.2 (c), we have $x_j \in D_\nu(\psi) \cap B[x_0, h(1 + \gamma_{r,\nu}M)]$ for $j \geq 0$. It follows from (3.2) that $D_r(f(x_j), f(x_0)) \leq \eta$ for $j \geq 0$. We need to prove that

$$(3.3) \qquad d_{(t_j, x_0)}(f(x_0)(t_j, x_0), x_j) \leq \epsilon t_j$$

for $j \geq 0$. The inequality (3.3) holds for $j = 0$. Let $i \geq 1$ and assume (3.3) holds for $j = i - 1$. Since $x_j \in D_\nu(\psi) \subset D_{r_0}(\phi)$ for $j \geq 0$, we have

$$\phi(f(x_0)(t - t_{i-1}, f(x_0)(t_{i-1}, x_0))) = \phi(f(x_0)(t, x_0)) \leq K_{r_0}(f(x_0)) \leq r$$

for $t \in [t_{i-1}, 1]$. Moreover, we have $\phi(f(x_{i-1})(t - t_{i-1}, x_{i-1})) \leq K_{r_0}(f(x_{i-1})) \leq r$ for $t \in [t_{i-1}, 1]$. Since $f(x_0)(t_i, x_0) = f(x_0)(t_i - t_{i-1}, f(x_0)(t_{i-1}, x_0))$, we see from (2.11) with $t = t_i$, $w = x_0$, $s = t_{i-1}$, $x = f(x_0)(t_{i-1}, x_0)$ and $y = \hat{w} = x_{i-1}$ that

$$(3.4) \quad d_{(t_i, x_0)}(f(x_0)(t_i, x_0), f(x_{i-1})(t_i - t_{i-1}, x_{i-1}))$$
$$\leq d_{(t_{i-1}, x_0)}(f(x_0)(t_{i-1}, x_0), x_{i-1}) + \gamma_{r,\nu}(t_i - t_{i-1})D_r(f(x_0), f(x_{i-1}))$$
$$\leq \epsilon t_{i-1} + \epsilon(t_i - t_{i-1})/2.$$

Since $x_i = f(x_i)(0, x_i)$ we have $\phi(x_i) \leq K_r(f(x_i)) \leq K(r, \nu)$. Similarly, we have $\phi(f(x_{i-1})(t_i - t_{i-1}, x_{i-1})) \leq K(r, \nu)$. By (2.8) and condition (iii) we have

$$(3.5) \qquad d_{(t_i, x_0)}(f(x_{i-1})(t_i - t_{i-1}, x_{i-1}), x_i) \leq M_{K(r,\nu)}(f(x_0))\eta(t_i - t_{i-1})$$
$$\leq \epsilon(t_i - t_{i-1})/2.$$

It follows that $d_{(t_i, x_0)}(f(x_0)(t_i, x_0), x_i) \leq \epsilon t_i$. The inequality (3.3) is proved inductively. By Lemma 3.2 we have

$$d(x_l, x_i) \leq (t_i - t_l)(\eta + \gamma_{r,\nu}M) \quad \text{for } l = 0, 1, \ldots, i \text{ and } i = 0, 1, 2, \ldots.$$

The completeness of E assures the existence of $x \in D_\nu(\psi) \cap D_r(\phi)$ such that $x_j \to x$ in E as $j \to \infty$. Since $\eta < \epsilon$, we have $\tau^\eta(\psi(x_0)) \geq \tau^\epsilon(\psi(x_0))$ and $m^\eta(t; \psi(x_0)) \leq m^\epsilon(t; \psi(x_0))$ for $t \in [0, \tau^\epsilon(\psi(x_0)))$. Since $t_j \leq h < \tau^\epsilon(\psi(x_0))$ for $j = 0, 1, \ldots$, we have $\psi(x_j) \leq m^\epsilon(t_j; \psi(x_0))$ for $j = 0, 1, \ldots$. Taking the limits in (3.3) and the above inequality as $j \to \infty$, we get $d(f(x_0)(h, x_0), x) \leq \epsilon h$ and $\psi(x) \leq m^\epsilon(h; \psi(x_0))$. $\qquad \square$

4. Distance between approximate solutions

Proposition 4.1. *Let* $x_0, \hat{x}_0 \in D$. *Let* $\epsilon, \hat{\epsilon} > 0$, $\nu > 0$, $r_0 > 0$, $r > 0$, $\rho, \hat{\rho} > 0$, $M, \hat{M} > 0$ *and* $\tau \in (0,1]$ *satisfy the following conditions:*

$$D_\nu(\psi) \subset D_{r_0}(\phi).$$

$$3\gamma_{r,\nu}\epsilon \leq 1 \text{ and } 3\gamma_{r,\nu}\hat{\epsilon} \leq 1.$$

$$\tau^\epsilon(\psi(x_0)) > \tau \text{ and } m^\epsilon(t; \psi(x_0)) \leq \nu \text{ for } t \in [0, \tau].$$

$$\tau^{\hat{\epsilon}}(\psi(\hat{x}_0)) > \tau \text{ and } m^{\hat{\epsilon}}(t; \psi(\hat{x}_0)) \leq \nu \text{ for } t \in [0, \tau].$$

$$\tau\big(1 + \gamma_{r,\nu}M\big) \leq \rho \text{ and } \tau\big(1 + \gamma_{r,\nu}\hat{M}\big) \leq \hat{\rho}.$$

$$K(r_0, \nu) \leq r.$$

$$\beta_r(f(x)) \leq M \text{ for } x \in D_\nu(\psi) \cap B[x_0, \rho].$$

$$\beta_r(f(x)) \leq \hat{M} \text{ for } x \in D_\nu(\psi) \cap B[\hat{x}_0, \hat{\rho}].$$

Let $\{(t_j, x_j)\}_{j=0}^{\infty}$ *be a sequence in* $[0, \tau] \times (D_\nu(\psi) \cap D_r(\phi))$ *satisfying the following conditions:*

(i) $0 = t_0 < t_1 < \cdots < t_j < \cdots < \tau$.

(ii) $t_j - t_{j-1} \leq \epsilon/\gamma_{r,\nu}$ *for* $j = 1, 2, \ldots$.

(iii) $d(f(x_{j-1})(t_j - t_{j-1}, x_{j-1}), x_j) \leq \epsilon(t_j - t_{j-1})/\gamma_{r,\nu}$ *for* $j = 1, 2, \ldots$.

(iv) $\psi(x_j) \leq m^\epsilon(t_j - t_l; \psi(x_l))$ *for* $l = 0, 1, \ldots, j$ *and* $j = 0, 1, \ldots$.

(v) *If* $x \in D_\nu(\psi) \cap B[x_{j-1}, (t_j - t_{j-1})(1 + \gamma_{r,\nu}M)]$, *then*

$$D_r(f(x), f(x_{j-1})) \leq \epsilon/\gamma_{r,\nu} \quad \text{for } j = 1, 2, \ldots.$$

(vi) $\lim_{j \to \infty} t_j = \tau$.

Let $\{(\hat{t}_j, \hat{x}_j)\}_{j=0}^{\infty}$ *be a sequence in* $[0, \tau] \times (D_\nu(\psi) \cap D_r(\phi))$ *satisfying the counterparts to the conditions described above. Let* $\{s_j\}_{j=0}^{\infty}$ *be the sequence such that* $0 = s_0 < s_1 < s_2 < \cdots$ *and*

$$\{s_j; j = 0, 1, \ldots\} = \{t_j; j = 0, 1, \ldots\} \cup \{\hat{t}_j; j = 0, 1, \ldots\}.$$

Then there exists a sequence $\{(z_j, \hat{z}_j)\}_{j=0}^{\infty}$ *in* $(D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ *satisfying the following conditions for each* $j = 0, 1, 2, \ldots$:

(a-1) *If* $s_j = t_p$ *for some nonnegative integer* p, *then* $z_j = x_p$; *otherwise*

$$d(f(z_{j-1})(s_j - s_{j-1}, z_{j-1}), z_j) \leq 3\epsilon(s_j - s_{j-1}),$$
$$\psi(z_j) \leq m^\epsilon(s_j - s_{j-1}; \psi(z_{j-1})).$$

(a-2) *If* $s_j = \hat{t}_{\hat{p}}$ *for some nonnegative integer* $\hat{p}$, *then* $\hat{z}_j = \hat{x}_{\hat{p}}$; *otherwise*

$$d(f(\hat{z}_{j-1})(s_j - s_{j-1}, \hat{z}_{j-1}), \hat{z}_j) \leq 3\hat{\epsilon}(s_j - s_{j-1}),$$
$$\psi(\hat{z}_j) \leq m^{\hat{\epsilon}}(s_j - s_{j-1}; \psi(\hat{z}_{j-1})).$$

(b-1) *For* $k = 0, 1, \cdots, j$,

$$d(z_k, z_j) \leq (1 + \gamma_{r,\nu}M)(s_j - s_k) + 5\gamma_{r,\nu}\epsilon \sum_{t_l \in \{s_{k+1}, \ldots, s_j\}} (t_l - t_{l-1}).$$

(b-2) *For* $k = 0, 1, \cdots, j$,

$$d(\hat{z}_k, \hat{z}_j) \leq (1 + \gamma_{r,\nu}\hat{M})(s_j - s_k) + 5\gamma_{r,\nu}\hat{\epsilon} \sum_{\hat{t}_l \in \{s_{k+1}, \ldots, s_j\}} (\hat{t}_l - \hat{t}_{l-1}).$$

(c) $\Phi(z_j, \hat{z}_j) \le e^{s_j \omega_\nu} \left(\Phi(x_0, \hat{x}_0) + L_r(\epsilon + \hat{\epsilon}) s_j + \Gamma_j(\epsilon, \hat{\epsilon}) \right)$, where

$$\Gamma_j(\epsilon, \hat{\epsilon}) = 5 L_r \gamma_{r,\nu} \left(\epsilon \sum_{t_l \in \{s_1, \ldots, s_j\}} (t_l - t_{l-1}) + \hat{\epsilon} \sum_{\hat{t}_l \in \{s_1, \ldots, s_j\}} (\hat{t}_l - \hat{t}_{l-1}) \right).$$

Proof. Set $(z_0, \hat{z}_0) = (x_0, \hat{x}_0)$. Let $i \ge 1$ and assume that a sequence $\{(z_j, \hat{z}_j)\}_{j=0}^{i-1}$ in $(D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ is chosen so that conditions (a) through (c) hold for $j = 0, 1, \ldots, i-1$. We want to construct an element $(z_i, \hat{z}_i) \in (D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ satisfying conditions (a) through (c). Since $s_{i-1} \in [0, \tau)$, there exist two integers $p \ge 1$ and $\hat{p} \ge 1$ such that $t_{p-1} \le s_{i-1} < t_p$ and $\hat{t}_{\hat{p}-1} \le s_{i-1} < \hat{t}_{\hat{p}}$. By the definition of $\{s_j\}$ we have $t_{p-1} \le s_{i-1} < s_i \le t_p$, $\hat{t}_{\hat{p}-1} \le s_{i-1} < s_i \le \hat{t}_{\hat{p}}$, $t_{p-1} = s_q$ and $\hat{t}_{\hat{p}-1} = s_{\hat{q}}$ for some $q \le i-1$ and $\hat{q} \le i-1$. By (a), we have $z_q = x_{p-1}$ and $\hat{z}_{\hat{q}} = \hat{x}_{\hat{p}-1}$. The construction of the desired element $(z_i, \hat{z}_i)$ in $(D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ will be divided into three steps.

Step 1: We construct a sequence $\{(\sigma_j, u_j, \hat{u}_j)\}_{j=0}^\infty$ in $[s_{i-1}, s_i] \times (D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ satisfying the following conditions:

(I) $s_{i-1} = \sigma_0 < \sigma_1 < \cdots < \sigma_j < \cdots < s_i$.
(II-1) $d(f(u_{j-1})(\sigma_j - \sigma_{j-1}, u_{j-1}), u_j) \le \epsilon(\sigma_j - \sigma_{j-1})/\gamma_{r,\nu}$ for $j = 1, 2, \ldots$, where $u_0 = z_{i-1}$.
(II-2) $d(f(\hat{u}_{j-1})(\sigma_j - \sigma_{j-1}, \hat{u}_{j-1}), \hat{u}_j) \le \hat{\epsilon}(\sigma_j - \sigma_{j-1})/\gamma_{r,\nu}$ for $j = 1, 2, \ldots$, where $\hat{u}_0 = \hat{z}_{i-1}$.
(III-1) $\psi(u_j) \le m^\epsilon(\sigma_j - \sigma_{j-1}; \psi(u_{j-1}))$ for $j = 1, 2, \ldots$.
(III-2) $\psi(\hat{u}_j) \le m^{\hat{\epsilon}}(\sigma_j - \sigma_{j-1}; \psi(\hat{u}_{j-1}))$ for $j = 1, 2, \ldots$.
(IV) $(\sigma_j - \sigma_{j-1})^{-1} \left(\Phi(u_j, \hat{u}_j) - \Phi(u_{j-1}, \hat{u}_{j-1}) \right) \le \omega_\nu \Phi(u_{j-1}, \hat{u}_{j-1}) + L_r(\epsilon + \hat{\epsilon})$ for $j = 1, 2, \ldots$.
(V) $\lim_{j \to \infty} \sigma_j = s_i$.

For this purpose, let $k \ge 1$ and assume that a sequence $\{(\sigma_j, u_j, \hat{u}_j)\}_{j=0}^{k-1}$ in $[s_{i-1}, s_i] \times (D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ is chosen so that (I) through (IV) hold for $j = 1, 2, \ldots, k-1$. Let $\bar{h}_k$ be the supremum of $h \in (0, 1]$ such that $\sigma_{k-1} + h < s_i$ and there exist $u, \hat{u} \in D$ satisfying $d(f(u_{k-1})(h, u_{k-1}), u) \le \epsilon h/\gamma_{r,\nu}$, $d(f(\hat{u}_{k-1})(h, \hat{u}_{k-1}), \hat{u}) \le \hat{\epsilon} h/\gamma_{r,\nu}$, $\psi(u) \le m^\epsilon(h; \psi(u_{k-1}))$, $\psi(\hat{u}) \le m^{\hat{\epsilon}}(h; \psi(\hat{u}_{k-1}))$ and

$$h^{-1} \left(\Phi(f(u_{k-1})(h, u_{k-1}), f(\hat{u}_{k-1})(h, \hat{u}_{k-1})) - \Phi(u_{k-1}, \hat{u}_{k-1}) \right)$$
$$\le \omega_\nu \Phi(u_{k-1}, \hat{u}_{k-1}) + L_r(\epsilon + \hat{\epsilon})/2.$$

By Lemma 3.4 and (f3), we have $\bar{h}_k > 0$. This enables us to choose $h_k \in (0, 1]$, $u_k \in D$ and $\hat{u}_k \in D$ such that $\bar{h}_k/2 < h_k$, $\sigma_{k-1} + h_k < s_i$, $d(f(u_{k-1})(h_k, u_{k-1}), u_k) \le \epsilon h_k/\gamma_{r,\nu}$, $d(f(\hat{u}_{k-1})(h_k, \hat{u}_{k-1}), \hat{u}_k) \le \hat{\epsilon} h_k/\gamma_{r,\nu}$, $\psi(u_k) \le m^\epsilon(h_k; \psi(u_{k-1}))$, $\psi(\hat{u}_k) \le m^{\hat{\epsilon}}(h_k; \psi(\hat{u}_{k-1}))$ and

$$h_k^{-1} \left(\Phi(f(u_{k-1})(h_k, u_{k-1}), f(\hat{u}_{k-1})(h_k, \hat{u}_{k-1})) - \Phi(u_{k-1}, \hat{u}_{k-1}) \right)$$
$$\le \omega_\nu \Phi(u_{k-1}, \hat{u}_{k-1}) + L_r(\epsilon + \hat{\epsilon})/2.$$

Set $\sigma_k = \sigma_{k-1} + h_k$. Then we apply Lemma 3.2 to the sequence $\{x_0, \ldots, x_{p-1} = z_q, z_{q+1}, \ldots, z_{i-1} = u_0, \ldots, u_k\}$, so that $u_k \in D_\nu(\psi) \cap D_r(\phi)$ and $\hat{u}_k \in D_\nu(\psi) \cap D_r(\phi)$. The sequence $\{(\sigma_j, u_j, \hat{u}_j)\}_{j=0}^\infty$ in $[s_{i-1}, s_i] \times (D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ is thus constructed so that it satisfies conditions (I) through (IV) for $j = 1, 2, \ldots$. Applying Lemma 3.2 to the sequence $\{x_0, \ldots, x_{p-1} = z_q, z_{q+1}, \ldots, z_{i-1} =$

$u_0, \ldots, u_j\}$, we have

$$(4.1) \qquad d(u_j, u_l) \leq (\sigma_j - \sigma_l)(3\epsilon + \gamma_{r,\nu}M),$$

$$(4.2) \qquad \psi(u_j) \leq m^\epsilon(\sigma_j - \sigma_l; \psi(u_l))$$

for $l = 0, 1, \ldots, j$ and $j = 0, 1, \ldots$, where we have used condition (a) with $j = q+1, \ldots, i-1$. To show that (V) is satisfied, we assume $\lim_{j \to \infty} \sigma_j = \bar{\sigma} < s_i$. Since $s_q = t_{p-1} \leq s_{i-1} < s_i \leq t_p$, we have $\{s_{q+1}, \ldots, s_{i-1}\} \cap \{t_j; j = 0, 1, \ldots\} = \emptyset$. By the completeness of E we see from (4.1) that there exists $u_\infty \in D_\nu(\psi) \cap D_r(\phi)$ such that $d(u_j, u_\infty) \to 0$ as $j \to \infty$. The lower semi-continuity of ψ implies that $\psi(u_\infty) \leq m^\epsilon(\bar{\sigma} - \sigma_{k-1}; \psi(u_{k-1}))$ for $k \geq 1$. Similarly, there exists $\hat{u}_\infty \in D_\nu(\psi) \cap D_r(\phi)$ such that $d(\hat{u}_j, \hat{u}_\infty) \to 0$ as $j \to \infty$ and such that $\psi(\hat{u}_\infty) \leq m^{\hat{\epsilon}}(\bar{\sigma} - \sigma_{k-1}; \psi(\hat{u}_{k-1}))$ for $k \geq 1$. Since $(u_\infty, \hat{u}_\infty) \in D_\nu(\psi) \times D_\nu(\psi)$, we deduce from Lemma 3.4 and condition (f3) that there exist $h \in (0, 1/2]$, $v_h \in D$ and $\hat{v}_h \in D$ satisfying $\bar{\sigma} + h < s_i$, $d(f(u_\infty)(h, u_\infty), v_h) \leq \epsilon h/(2\gamma_{r,\nu})$, $d(f(\hat{u}_\infty)(h, \hat{u}_\infty), \hat{v}_h) \leq \hat{\epsilon}h/(2\gamma_{r,\nu})$, $\psi(v_h) \leq m^\epsilon(h; \psi(u_\infty))$, $\psi(\hat{v}_h) \leq m^{\hat{\epsilon}}(h; \psi(\hat{u}_\infty))$ and

$$(4.3) \qquad h^{-1}\big(\Phi(f(u_\infty)(h, u_\infty), f(\hat{u}_\infty)(h, \hat{u}_\infty)) - \Phi(u_\infty, \hat{u}_\infty)\big)$$
$$\leq \omega_\nu \Phi(u_\infty, \hat{u}_\infty) + L_r(\epsilon + \hat{\epsilon})/3.$$

Set $\delta_k = \bar{\sigma} + h - \sigma_{k-1}$ for $k \geq 1$. Then we have $\sigma_{k-1} + \delta_k < s_i$ and $\psi(v_h) \leq m^\epsilon(\delta_k; \psi(u_{k-1}))$ for $k \geq 1$. We apply Lemma 3.2 to get $\beta_r(f(u_{k-1})) \leq M$ for $k \geq 1$. By (2.9) with $w = u_{k-1}$, $t = \delta_k$, $s = h$ and $x = u_{k-1}$, we have

$$(4.4) \qquad d(f(u_{k-1})(\delta_k, u_{k-1}), f(u_{k-1})(h, u_{k-1})) \leq \gamma_{r,\nu}M(\bar{\sigma} - \sigma_{k-1})$$

for $k \geq 1$, where we have used the fact that $\delta_k - h = \bar{\sigma} - \sigma_{k-1}$. Since $u_\infty \in D_\nu(\psi) \subset D_{r_0}(\phi)$, we have $\phi(f(u_\infty)(t, u_\infty)) \leq K_{r_0}(f(u_\infty)) \leq r$ for $t \in [0, 1]$. Similarly, we have $\phi(f(u_{k-1})(t, u_{k-1})) \leq r$ for $t \in [0, 1]$. By (2.11) we have

$$d_{(h,u_\infty)}(f(u_{k-1})(h, u_{k-1}), f(u_\infty)(h, u_\infty))$$
$$\leq d_{(0,u_\infty)}(u_{k-1}, u_\infty) + \gamma_{r,\nu}hD_r(f(u_{k-1}), f(u_\infty)).$$

This combined with (4.4) yields

$$(4.5) \qquad \lim_{k \to \infty} d(f(u_{k-1})(\delta_k, u_{k-1}), f(u_\infty)(h, u_\infty)) = 0,$$

and so there exists an integer $k_1 \geq 1$ such that $d(f(u_{k-1})(\delta_k, u_{k-1}), v_h) \leq \epsilon h/\gamma_{r,\nu}$ for $k \geq k_1$. Similarly, we find an integer $k_2 \geq 1$ such that $d(f(\hat{u}_{k-1})(\delta_k, \hat{u}_{k-1}), \hat{v}_h) \leq \hat{\epsilon}h/\gamma_{r,\nu}$ for $k \geq k_2$. By (4.5) we apply condition (Φ1) to prove

$$\lim_{k \to \infty} \Phi(f(u_{k-1})(\delta_k, u_{k-1}), f(\hat{u}_{k-1})(\delta_k, \hat{u}_{k-1})) = \Phi(f(u_\infty)(h, u_\infty), f(\hat{u}_\infty)(h, \hat{u}_\infty)).$$

Since $\delta_k \to h$ as $k \to \infty$, we see from (4.3) that there exists an integer $k_3 \geq 1$ satisfying $\delta_k \in (0, 1]$ and

$$\delta_k^{-1}\big(\Phi(f(u_{k-1})(\delta_k, u_{k-1}), f(\hat{u}_{k-1})(\delta_k, \hat{u}_{k-1})) - \Phi(u_{k-1}, \hat{u}_{k-1})\big)$$
$$\leq \omega_\nu \Phi(u_{k-1}, \hat{u}_{k-1}) + L_r(\epsilon + \hat{\epsilon})/2$$

for $k \geq k_3$. By the definition of $\bar{h}_k$ we have $\bar{h}_k \geq \delta_k$ for $k \geq \max\{k_1, k_2, k_3\}$. This contradicts the fact that $\bar{h}_k \to 0$ and $\delta_k \to h$ as $k \to \infty$. The verification of (V) is thus completed.

Step 2: We prove that there exists $(w_i, \hat{w}_i) \in (D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ satisfying the following five conditions:

$$(4.6) \qquad \psi(w_i) \le m^\epsilon(s_i - s_{i-1}; \psi(z_{i-1})).$$

$$(4.7) \qquad \psi(\hat{w}_i) \le m^{\hat{\epsilon}}(s_i - s_{i-1}; \psi(\hat{z}_{i-1})).$$

$$(4.8) \qquad d_{(s_i, z_{i-1})}(f(z_{i-1})(s_i - s_{i-1}, z_{i-1}), w_i) \le 3\epsilon(s_i - s_{i-1}).$$

$$(4.9) \qquad d_{(s_i, \hat{z}_{i-1})}(f(\hat{z}_{i-1})(s_i - s_{i-1}, \hat{z}_{i-1}), \hat{w}_i) \le 3\hat{\epsilon}(s_i - s_{i-1}).$$

$$(4.10) \qquad \Phi(w_i, \hat{w}_i) \le e^{(s_i - s_{i-1})\omega_\nu}\big(\Phi(z_{i-1}, \hat{z}_{i-1}) + L_r(\epsilon + \hat{\epsilon})(s_i - s_{i-1})\big).$$

Let $\{(\sigma_j, u_j, \hat{u}_j)\}_{j=0}^\infty$ be the sequence in $[s_{i-1}, s_i] \times (D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ constructed in Step 1. Since $\{u_j\}$ is a Cauchy sequence in E (see (4.1)), there exists $w_i \in D_\nu(\psi) \cap D_r(\phi)$ such that $\lim_{j\to\infty} d(u_j, w_i) = 0$. Similarly, there exists $\hat{w}_i \in D_\nu(\psi) \cap D_r(\phi)$ such that $\lim_{j\to\infty} d(\hat{u}_j, \hat{w}_i) = 0$. The inequality (4.6) follows from (4.2). Similarly, the inequality (4.7) is obtained. The inequality (4.10) follows by a passage to the limit in the inequality

$$\Phi(u_j, \hat{u}_j) \le e^{(\sigma_j - \sigma_0)\omega_\nu}\big(\Phi(u_0, \hat{u}_0) + L_r(\epsilon + \hat{\epsilon})(\sigma_j - \sigma_0)\big)$$

for $j = 0, 1, \ldots$, which is obtained inductively by (IV). To prove (4.8) it suffices to demonstrate that

$$(4.11) \qquad d_{(\sigma_j, z_{i-1})}(f(z_{i-1})(\sigma_j - s_{i-1}, z_{i-1}), u_j) \le 3\epsilon(\sigma_j - s_{i-1})$$

for $j = 0, 1, \ldots$. The case where $j = 0$ is trivial, since $\sigma_0 = s_{i-1}$ and $u_0 = z_{i-1}$. Let $k \ge 1$ and assume that (4.11) with $j = k - 1$ holds. Applying Lemma 3.2 to the sequence $\{x_0, \ldots, x_{p-1} = z_q, z_{q+1}, \ldots, z_{i-1} = u_0, \ldots, u_l\}$, we find that

$$d(u_l, x_{p-1}) \le (\sigma_l - t_{p-1})(3\epsilon + \gamma_{r,\nu}M) \le (t_p - t_{p-1})(1 + \gamma_{r,\nu}M)$$

for $l = 0, 1, \ldots$, where we have used conditions (iii) and (iv) with $j = 1, \ldots, p-1$, condition (a-1) with $j = q + 1, \ldots, i - 1$ and conditions (II-1) and (III-1) with $j = 1, \ldots, l$. This implies $u_l \in B[x_{p-1}, (t_p - t_{p-1})(1 + \gamma_{r,\nu}M)]$ and we see from condition (v) that $D_r(f(u_l), f(x_{p-1})) \le \epsilon/\gamma_{r,\nu}$ for $l \ge 0$. Since $z_{i-1} = u_0$, it follows that

$$(4.12) \quad D_r(f(u_l), f(z_{i-1})) \le D_r(f(u_l), f(x_{p-1})) + D_r(f(x_{p-1}), f(z_{i-1})) \le 2\epsilon/\gamma_{r,\nu}$$

for $l \ge 0$. Similarly to the derivation of (3.4), we apply Lemma 2.6 (ii) to obtain

$$\begin{aligned}
&d_{(\sigma_k, z_{i-1})}(f(z_{i-1})(\sigma_k - s_{i-1}, z_{i-1}), f(u_{k-1})(\sigma_k - \sigma_{k-1}, u_{k-1})) \\
&\quad \le d_{(\sigma_{k-1}, z_{i-1})}(f(z_{i-1})(\sigma_{k-1} - s_{i-1}, z_{i-1}), u_{k-1}) \\
&\qquad + \gamma_{r,\nu}(\sigma_k - \sigma_{k-1})D_r(f(z_{i-1}), f(u_{k-1})).
\end{aligned}$$

Just as in the proof of (3.5), we have $d_{(\sigma_k, z_{i-1})}(f(u_{k-1})(\sigma_k - \sigma_{k-1}, u_{k-1}), u_k) \le \epsilon(\sigma_k - \sigma_{k-1})$ by condition (II-1) with $j = k$. It follows from (4.12) with $l = k - 1$ and (4.11) with $j = k - 1$ that

$$\begin{aligned}
&d_{(\sigma_k, z_{i-1})}(f(z_{i-1})(\sigma_k - s_{i-1}, z_{i-1}), u_k) \\
&\quad \le 3\epsilon(\sigma_{k-1} - s_{i-1}) + 2\epsilon(\sigma_k - \sigma_{k-1}) + \epsilon(\sigma_k - \sigma_{k-1}) \le 3\epsilon(\sigma_k - s_{i-1}).
\end{aligned}$$

The verification of (4.11) is thus completed. Similarly, the desired inequality (4.9) is obtained.

Step 3: We define $(z_i, \hat{z}_i) \in (D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ by

$$z_i = \begin{cases} x_p & \text{if } s_i = t_p, \\ w_i & \text{otherwise}, \end{cases} \qquad \hat{z}_i = \begin{cases} \hat{x}_{\hat{p}} & \text{if } s_i = \hat{t}_{\hat{p}}, \\ \hat{w}_i & \text{otherwise}. \end{cases}$$

We want to show that the element $(z_i, \hat{z}_i)$ satisfies conditions (a) through (c) with $j = i$. From (4.6) through (4.9) we see that (a) is satisfied for $j = i$. To show that (b) is satisfied for $j = i$, consider the case where $s_i = t_p$, and then $s_q = t_{p-1} \le s_{i-1} < s_i = t_p$. Note that $z_q = x_{p-1}$ and $s_j \ne t_k$ for $q + 1 \le j \le i - 1$ and $k = 0, 1, \ldots$. Then we see from condition (a) that

$$(4.13) \qquad d(f(z_{j-1})(s_j - s_{j-1}, z_{j-1}), z_j) \le 3\epsilon(s_j - s_{j-1}),$$
$$\psi(z_j) \le m^\epsilon(s_j - s_{j-1}; \psi(z_{j-1}))$$

for $q + 1 \le j \le i - 1$ To attain our objective, we first prove that

$$(4.14) \qquad d_{(s_i, z_q)}(f(z_q)(s_i - s_q, z_q), w_i) \le 4\gamma_{r,\nu}\epsilon(s_i - s_q).$$

Obviously, (4.8) implies (4.14) with $q = i - 1$. Let $q < i - 1$. Applying Lemma 3.2 to the sequence $\{x_0, x_1, \ldots, x_{p-1} = z_q, z_{q+1}, \ldots, z_{i-1}\}$, we see that $\beta_r(f(z_j)) \le M$ and that $d(z_j, x_{p-1}) \le (s_j - t_{p-1})(3\epsilon + \gamma_{r,\nu}M) \le (t_p - t_{p-1})(1 + \gamma_{r,\nu}M)$ for $j = q, q+1, \ldots, i - 1$. By (v), we have

$$(4.15) \qquad D_r(f(z_j), f(z_q)) = D_r(f(z_j), f(x_{p-1})) \le \epsilon/\gamma_{r,\nu}$$

for $j = q, q+1, \ldots, i - 1$. By Lemma 2.6 (ii) we find that

$$(4.16) \qquad d_{(s_i, z_q)}(f(z_q)(s_i - s_{i-1}, z_{i-1}), f(z_{i-1})(s_i - s_{i-1}, z_{i-1})) \le \epsilon(s_i - s_{i-1}).$$

Since $w_i = f(w_i)(0, w_i)$, we have $\phi(w_i) \le K_r(f(w_i)) \le K(r, \nu)$. Similarly, we have $\phi(f(z_{i-1})(s_i - s_{i-1}, z_{i-1})) \le K(r, \nu)$. Applying (2.7) and (2.8), we infer from (4.8) that $d_{(s_i, z_q)}(f(z_{i-1})(s_i - s_{i-1}, z_{i-1}), w_i) \le 3\gamma_{r,\nu}\epsilon(s_i - s_{i-1})$. Addition of this inequality and (4.16) gives

$$(4.17) \qquad d_{(s_i, z_q)}(f(z_q)(s_i - s_{i-1}, z_{i-1}), w_i) \le 4\gamma_{r,\nu}\epsilon(s_i - s_{i-1}).$$

To prove (4.14) we need to show that

$$(4.18) \qquad d_{(s_j, z_q)}(f(z_q)(s_j - s_q, z_q), z_j) \le 4\gamma_{r,\nu}\epsilon(s_j - s_q)$$

for $j = q, q+1, \ldots, i - 1$. The inequality (4.18) holds trivially for $j = q$. Let $q + 1 \le k \le i - 1$ and assume that (4.18) with $j = k - 1$ holds. Similarly to the derivation of (3.4), we deduce from Lemma 2.6 (ii) that

$$d_{(s_k, z_q)}(f(z_q)(s_k - s_q, z_q), f(z_{k-1})(s_k - s_{k-1}, z_{k-1}))$$
$$\le d_{(s_{k-1}, z_q)}(f(z_q)(s_{k-1} - s_q, z_q), z_{k-1}) + \gamma_{r,\nu}(s_k - s_{k-1})D_r(f(z_q), f(z_{k-1})).$$

It follows from (4.18) and (4.15) with $j = k - 1$ and (4.13) with $j = k$ that

$$d_{(s_k, z_q)}(f(z_q)(s_k - s_q, z_q), z_k) \le 4\gamma_{r,\nu}\epsilon(s_k - s_q).$$

The inequality (4.18) is inductively proved for $j = q, q+1, \ldots, i - 1$. By (2.6) and (4.18) with $j = i - 1$, we have

$$d_{(s_i, z_q)}(f(z_q)(s_i - s_q, z_q), f(z_q)(s_i - s_{i-1}, z_{i-1}))$$
$$= d_{(s_i, z_q)}(f(z_q)(s_i - s_{i-1}, f(z_q)(s_{i-1} - s_q, z_q)), f(z_q)(s_i - s_{i-1}, z_{i-1}))$$
$$\le d_{(s_{i-1}, z_q)}(f(z_q)(s_{i-1} - s_q, z_q), z_{i-1}) \le 4\gamma_{r,\nu}\epsilon(s_{i-1} - s_q).$$

Combining this inequality and (4.17), we get the desired inequality (4.14). Since $t_{p-1} = s_q$, $t_p = s_i$ and $z_q = x_{p-1}$, it follows from condition (iii) with $j = p$ and (4.14) combined with (2.7) that

$$\begin{aligned} d(x_p, w_i) &\leq d(x_p, f(x_{p-1})(t_p - t_{p-1}, x_{p-1})) + d(f(z_q)(s_i - s_q, z_q), w_i) \\ &\leq \epsilon(t_p - t_{p-1})/\gamma_{r,\nu} + 4\gamma_{r,\nu}\epsilon(s_i - s_q) \\ &\leq 5\gamma_{r,\nu}\epsilon(t_p - t_{p-1}), \end{aligned}$$

hence

$$(4.19) \qquad d(z_i, w_i) \leq 5\gamma_{r,\nu}\epsilon \sum_{t_l = s_i} (t_l - t_{l-1}).$$

By (2.7), (2.9) and (4.8) we have

$$\begin{aligned} d(z_{i-1}, w_i) &\leq d(z_{i-1}, f(z_{i-1})(s_i - s_{i-1}, z_{i-1})) + d(f(z_{i-1})(s_i - s_{i-1}, z_{i-1}), w_i) \\ &\leq \gamma_{r,\nu}\beta_r(f(z_{i-1}))(s_i - s_{i-1}) + 3\epsilon(s_i - s_{i-1}) \\ &\leq (\gamma_{r,\nu}M + 3\epsilon)(s_i - s_{i-1}) \leq (1 + \gamma_{r,\nu}M)(s_i - s_{i-1}), \end{aligned}$$

and hence

$$d(z_i, z_{i-1}) \leq 5\gamma_{r,\nu}\epsilon \sum_{t_j = s_i} (t_j - t_{j-1}) + (1 + \gamma_{r,\nu}M)(s_i - s_{i-1}).$$

From this and (b-1) with $j = i - 1$ we see that (b-1) is satisfied for $j = i$. Similarly, (b-2) is proved to be satisfied for $j = i$, and

$$(4.20) \qquad d(\hat{z}_i, \hat{w}_i) \leq 5\gamma_{r,\nu}\hat{\epsilon} \sum_{\hat{t}_l = s_i} (\hat{t}_l - \hat{t}_{l-1}).$$

By (4.10), (4.19) and (4.20), we have

$$\begin{aligned} \Phi(z_i, \hat{z}_i) \leq e^{(s_i - s_{i-1})\omega_\nu} \Bigg\{ & \Big(\Phi(z_{i-1}, \hat{z}_{i-1}) + L_r(\epsilon + \hat{\epsilon})(s_i - s_{i-1}) \Big) \\ & + 5L_r\gamma_{r,\nu} \Big(\epsilon \sum_{t_l = s_i} (t_l - t_{l-1}) + \hat{\epsilon} \sum_{\hat{t}_l = s_i} (\hat{t}_l - \hat{t}_{l-1}) \Big) \Bigg\}, \end{aligned}$$

where we have used condition (Φ1). From this and (c) with $j = i - 1$ we see that (c) is satisfied for $j = i$. $\qquad\square$

5. Proof of the main theorem

Proposition 5.1. *For any $x_0 \in D$, there exist $\tau > 0$ and a solution $u \in C([0, \tau); X)$ to the mutational equation*

$$(ME) \qquad \overset{\circ}{u}(t) \ni f(u(t)) \text{ for } t \in [0, \tau) \quad \text{and} \quad u(0) = x_0$$

satisfying $\psi(u(t)) \leq m(t; \psi(x_0))$ for $t \in [0, \tau)$.

Proof. Let $x_0 \in D$. Then, by Lemma 3.1 there exist $\epsilon_0 > 0$, $\nu > 0$, $r_0 > 0$, $r > 0$, $\rho > 0$, $M > 0$ and $\tau \in (0, 1]$ such that $D_\nu(\psi) \subset D_{r_0}(\phi)$, $\tau^\epsilon(\psi(x_0)) > \tau$, $m^\epsilon(t; \psi(x_0)) \leq \nu$ for $t \in [0, \tau]$ and $\epsilon \in (0, \epsilon_0]$, $\tau(1 + \gamma_{r,\nu}M) \leq \rho$, $K(r_0, \nu) \leq r$ and $\beta_r(f(x)) \leq M$ for any $x \in D_\nu(\psi) \cap B[x_0, \rho]$. Take a smaller $\epsilon_0 > 0$ so that $3\gamma_{r,\nu}\epsilon_0 \leq 1$. Proposition 3.3 asserts that for any $\epsilon \in (0, \epsilon_0]$ there exists a sequence $\{(t_j^\epsilon, x_j^\epsilon)\}_{j=0}^\infty$ in $[0, \tau] \times (D_\nu(\psi) \cap D_r(\phi))$ satisfying the following conditions:

(i) $0 = t_0^\epsilon < t_1^\epsilon < \cdots < t_j^\epsilon < \cdots < \tau$.

(ii) $t_j^\epsilon - t_{j-1}^\epsilon \le \epsilon/\gamma_{r,\nu}$ for $j = 1, 2, \ldots$.

(iii) $d(f(x_{j-1}^\epsilon)(t_j^\epsilon - t_{j-1}^\epsilon, x_{j-1}^\epsilon), x_j^\epsilon) \le \epsilon(t_j^\epsilon - t_{j-1}^\epsilon)/\gamma_{r,\nu}$ for $j = 1, 2, \ldots$, where $x_0^\epsilon = x_0$.

(iv) $\psi(x_j^\epsilon) \le m^\epsilon(t_j^\epsilon - t_l^\epsilon; \psi(x_l^\epsilon))$ for $l = 0, 1, \ldots, j$ and $j = 1, 2, \ldots$.

(v) If $x \in D_\nu(\psi) \cap B[x_{j-1}^\epsilon, (t_j^\epsilon - t_{j-1}^\epsilon)(1 + \gamma_{r,\nu} M)]$, then
$$D_r(f(x), f(x_{j-1}^\epsilon)) \le \epsilon/\gamma_{r,\nu} \quad \text{for } j = 1, 2, \ldots.$$

(vi) $\lim_{j \to \infty} t_j^\epsilon = \tau$.

For each $\epsilon \in (0, \epsilon_0]$, we define $u^\epsilon : [0, \tau) \to (D_\nu(\psi) \cap D_r(\phi))$ by
$$u^\epsilon(t) = x_{j-1}^\epsilon \quad \text{for } t \in [t_{j-1}^\epsilon, t_j^\epsilon) \text{ and } j = 1, 2, \ldots.$$

By Lemma 3.2 (a) we have

(5.1) $\qquad d(u^\epsilon(t), u^\epsilon(s)) \le (\epsilon + \gamma_{r,\nu} M)(|t - s| + 2\epsilon) \quad \text{for } t, s \in [0, \tau).$

Let $\lambda, \mu \in (0, \epsilon_0]$ and let $\{s_j\}_{j=0}^\infty$ be the sequence such that $\{s_j; j = 0, 1, \ldots\} = \{t_j^\lambda; j = 0, 1, \ldots\} \cup \{t_j^\mu; j = 0, 1, \ldots\}$ and $0 = s_0 < s_1 < s_2 < \ldots$. Then Proposition 4.1 asserts that there exists a sequence $\{(z_j^\lambda, z_j^\mu)\}_{j=0}^\infty$ in $(D_\nu(\psi) \cap D_r(\phi)) \times (D_\nu(\psi) \cap D_r(\phi))$ satisfying the following conditions for each $j = 0, 1, 2, \ldots$:

(a-1) If $s_j = t_p^\lambda$ for some nonnegative integer p, then $z_j^\lambda = x_p^\lambda$; otherwise
$$d(f(z_{j-1}^\lambda)(s_j - s_{j-1}, z_{j-1}^\lambda), z_j^\lambda) \le 3\lambda(s_j - s_{j-1}),$$
$$\psi(z_j^\lambda) \le m^\lambda(s_j - s_{j-1}; \psi(z_{j-1}^\lambda)).$$

(a-2) If $s_j = t_q^\mu$ for some nonnegative integer q, then $z_j^\mu = x_q^\mu$; otherwise
$$d(f(z_{j-1}^\mu)(s_j - s_{j-1}, z_{j-1}^\mu), z_j^\mu) \le 3\mu(s_j - s_{j-1}),$$
$$\psi(z_j^\mu) \le m^\mu(s_j - s_{j-1}; \psi(z_{j-1}^\mu)).$$

(b-1) For $i = 0, 1, \cdots, j$,
$$d(z_i^\lambda, z_j^\lambda) \le (1 + \gamma_{r,\nu} M)(s_j - s_i) + 5\gamma_{r,\nu}\lambda \sum_{t_p^\lambda \in \{s_{i+1}, \ldots, s_j\}} (t_p^\lambda - t_{p-1}^\lambda).$$

(b-2) For $i = 0, 1, \cdots, j$,
$$d(z_i^\mu, z_j^\mu) \le (1 + \gamma_{r,\nu} M)(s_j - s_i) + 5\gamma_{r,\nu}\mu \sum_{t_p^\mu \in \{s_{i+1}, \ldots, s_j\}} (t_p^\mu - t_{p-1}^\mu).$$

(c) $\Phi(z_j^\lambda, z_j^\mu) \le e^{s_j \omega_\nu}(L_r(\lambda + \mu)s_j + \delta_j(\lambda, \mu))$, where
$$\delta_j(\lambda, \mu) = 5L_r\gamma_{r,\nu}\left(\lambda \sum_{t_p^\lambda \in \{s_1, \ldots, s_j\}} (t_p^\lambda - t_{p-1}^\lambda) + \mu \sum_{t_p^\mu \in \{s_1, \ldots, s_j\}} (t_p^\mu - t_{p-1}^\mu)\right).$$

Let $t \in [0, \tau)$. Then there exist positive integers $i, p, q, l(\le i - 1)$ and $k(\le i - 1)$ such that $t_{p-1}^\lambda \le s_{i-1} \le t < s_i \le t_p^\lambda$, $t_{q-1}^\mu \le s_{i-1} \le t < s_i \le t_q^\mu$, $t_{p-1}^\lambda = s_l$ and $t_{q-1}^\mu = s_k$. Since $\{t_j^\lambda; j = 0, 1, \ldots\} \cap \{s_{l+1}, \ldots, s_{i-1}\} = \emptyset$, we see from (b-1) that
$$d(z_{i-1}^\lambda, z_l^\lambda) \le (1 + \gamma_{r,\nu} M)(s_{i-1}^\lambda - s_l^\lambda) \le \lambda(1 + \gamma_{r,\nu} M).$$

Similarly, we have $d(z_{i-1}^\mu, z_k^\mu) \le \mu(1 + \gamma_{r,\nu} M)$. By condition $(\Phi 1)$ we have
$$|\Phi(z_l^\lambda, z_k^\mu) - \Phi(z_{i-1}^\lambda, z_{i-1}^\mu)| \le L_r(d(z_{i-1}^\lambda, z_l^\lambda) + d(z_{i-1}^\mu, z_k^\mu)).$$

Since $u^\lambda(t) = x_{p-1}^\lambda = z_l^\lambda$ and $u^\mu(t) = x_{q-1}^\mu = z_k^\mu$, it follows from (c) that

$$\Phi(u^\lambda(t), u^\mu(t)) \leq L_r(\lambda + \mu)(1 + \gamma_{r,\nu}M)$$
$$+ e^{\tau\omega_\nu}\big(L_r(\lambda + \mu)\tau + 5L_r\gamma_{r,\nu}(\lambda + \mu)\tau\big).$$

By condition ($\Phi2$) we have $\lim_{\lambda,\mu\downarrow0}\sup\{d(u^\lambda(t), u^\mu(t)); t \in [0,\tau)\} = 0$. By the completeness of E, there exists a measurable function u on $[0,\tau)$ to E such that $u(t) \in D_\nu(\psi) \cap D_r(\phi)$ for $t \in [0,\tau)$ and the family $\{u^\epsilon\}$ converges to u uniformly on $[0,\tau)$ as $\epsilon \downarrow 0$. Moreover, we have $\psi(u(t)) \leq m(t; \psi(x_0))$ for $t \in [0,\tau)$. By (5.1) we have $d(u(t), u(s)) \leq \gamma_{r,\nu}M|t - s|$ for $t, s \in [0,\tau)$, and hence $u \in C([0,\tau); E)$.

To show that u is a solution to (ME), let $t \in [0,\tau)$ be fixed. Choose $h_0 \in (0, \tau - t)$ and let $h \in (0, h_0]$. Take $\epsilon \in (0, \epsilon_0]$ so that $t + h + \epsilon < \tau$, and assume that $t_l^\epsilon \leq t < t_{l+1}^\epsilon$ and $t_k^\epsilon \leq t + h < t_{k+1}^\epsilon$. Obviously, we have $l \leq k$ and $0 \leq t_k^\epsilon - t_l^\epsilon \leq (t + h) - (t - \epsilon) = h + \epsilon$. By Lemma 3.2 (a) we have

$$d(u(t), x_j^\epsilon) \leq d(u(t), u^\epsilon(t)) + (1 + \gamma_{r,\nu}M)(t_j^\epsilon - t_l^\epsilon) \leq \delta(\epsilon) + h(1 + \gamma_{r,\nu}M)$$

for $j = l, \ldots, k$, where

$$\delta(\epsilon) = d(u(t), u^\epsilon(t)) + \epsilon(1 + \gamma_{r,\nu}M).$$

It follows that

(5.2) $$D_r(f(x_l^\epsilon), f(x_j^\epsilon)) \leq 2\Lambda\big(\delta(\epsilon) + h(1 + \gamma_{r,\nu}M)\big)$$

for $j = l, \ldots, k$, where

$$\Lambda(\delta) = \sup\{D_r(f(u(t)), f(w)); \ w \in D_\nu(\psi), d(u(t), w) \leq \delta\}$$

for $\delta > 0$. Note that $\lim_{\delta\downarrow0}\Lambda(\delta) = 0$. We shall prove that

(5.3) $$d_{(t_j^\epsilon, x_l^\epsilon)}(f(x_l^\epsilon)(t_j^\epsilon - t_l^\epsilon, x_l^\epsilon), x_j^\epsilon)$$
$$\leq \gamma_{r,\nu}(t_j^\epsilon - t_l^\epsilon)\big\{\epsilon + 2\Lambda\big(\delta(\epsilon) + h(1 + \gamma_{r,\nu}M)\big)\big\}$$

for $j = l, l + 1, \ldots, k$. The inequality (5.3) with $j = l$ holds obviously. Let $l + 1 \leq j \leq k$ and assume that (5.3) with j replaced by $j - 1$ holds. Similarly to the derivation of (3.4), we apply (2.11) to get

$$d_{(t_j^\epsilon, x_l^\epsilon)}(f(x_l^\epsilon)(t_j^\epsilon - t_l^\epsilon, x_l^\epsilon), f(x_{j-1}^\epsilon)(t_j^\epsilon - t_{j-1}^\epsilon, x_{j-1}^\epsilon))$$
$$\leq d_{(t_{j-1}^\epsilon, x_l^\epsilon)}(f(x_l^\epsilon)(t_{j-1}^\epsilon - t_l^\epsilon, x_l^\epsilon), x_{j-1}^\epsilon) + \gamma_{r,\nu}(t_j^\epsilon - t_{j-1}^\epsilon)D_r(f(x_l^\epsilon), f(x_{j-1}^\epsilon))$$
$$\leq \gamma_{r,\nu}(t_{j-1}^\epsilon - t_l^\epsilon)\big(\epsilon + 2\Lambda\big(\delta(\epsilon) + h(1 + \gamma_{r,\nu}M)\big)\big)$$
$$+ 2\gamma_{r,\nu}(t_j^\epsilon - t_{j-1}^\epsilon)\Lambda\big(\delta(\epsilon) + h(1 + \gamma_{r,\nu}M)\big),$$

where we have used (5.2) and (5.3) with j replaced by $j - 1$. Combining this inequality and the inequality that $d_{(t_j^\epsilon, x_l^\epsilon)}(f(x_{j-1}^\epsilon)(t_j^\epsilon - t_{j-1}^\epsilon, x_{j-1}^\epsilon), x_j^\epsilon) \leq \epsilon(t_j^\epsilon - t_{j-1}^\epsilon)$, we obtain (5.3), and so the desired inequality (5.3) is inductively proved. By (2.7) and (5.3) with $j = k$, we have

(5.4) $$d(f(u^\epsilon(t))(t_k^\epsilon - t_l^\epsilon, u^\epsilon(t)), u^\epsilon(t + h))$$
$$\leq \gamma_{r,\nu}(t_k^\epsilon - t_l^\epsilon)\big\{\epsilon + 2\Lambda\big(\delta(\epsilon) + h(1 + \gamma_{r,\nu}M)\big)\big\}.$$

Since $\phi(f(u(t))(t_k^\epsilon - t_l^\epsilon, u(t))) \leq K_{r_0}(f(u(t))) \leq r$, we apply (2.11) to obtain

$$d(f(u(t))(t_k^\epsilon - t_l^\epsilon, u(t)), f(u^\epsilon(t))(t_k^\epsilon - t_l^\epsilon, u^\epsilon(t)))$$
$$\leq \gamma_{r,\nu}\big(d(u^\epsilon(t), u(t)) + (t_k^\epsilon - t_l^\epsilon)D_r(f(u^\epsilon(t)), f(u(t)))\big).$$

A passage to the limit in this inequality combined with (5.4) as $\epsilon \downarrow 0$ yields

$$d(f(u(t))(h, u(t)), u(t + h)) \leq 2\gamma_{r,\nu} h \Lambda\big(h(1 + \gamma_{r,\nu} M)\big).$$

This proves $\limsup_{h \downarrow 0} h^{-1} d(f(u(t))(h, u(t)), u(t + h)) = 0$. $\qquad \square$

Proposition 5.2. *For $i = 1, 2$, let $u_i \in C([0, \tau); X)$ be a solution to the mutational equation $\overset{\circ}{u}_i(t) \ni f(u_i(t))$ for $t \in [0, \tau)$ such that $u_i(t) \in D_\nu(\psi)$ for $t \in [0, \tau)$. Then,*

$$\Phi(u_1(t), u_2(t)) \leq e^{\omega_\nu t} \Phi(u_1(0), u_2(0)) \quad for\ t \in [0, \tau).$$

Proof. Let $t \in [0, \tau)$ and choose $h_0 \in (0, \tau - t)$. Take $r_0 > 0$ so that $D_\nu(\psi) \subset D_{r_0}(\phi)$ by condition (H), and set $r = K(r_0, \nu)$. Since $u_i(s) \in D_\nu(\psi) \subset D_{r_0}(\phi)$ for $s \in [0, \tau)$, we observe that $\phi(f(u_i(s))(h, u_i(s))) \leq r$ for $s \in [0, \tau)$, $h \in [0, h_0]$ and $i = 1, 2$. By condition (Φ1) we have

$$h^{-1}(\Phi(u_1(t + h), u_2(t + h)) - \Phi(u_1(t), u_2(t)))$$
$$\leq L_r\big\{h^{-1} d(u_1(t + h), f(u_1(t))(h, u_1(t))) + h^{-1} d(u_2(t + h), f(u_2(t))(h, u_2(t)))\big\}$$
$$+ h^{-1}(\Phi(f(u_1(t))(h, u_1(t)), f(u_2(t))(h, u_2(t))) - \Phi(u_1(t), u_2(t)))$$

for $h \in (0, h_0]$. It follows from condition (f3) that

$$\liminf_{h \downarrow 0} h^{-1}(\Phi(u_1(t + h), u_2(t + h)) - \Phi(u_1(t), u_2(t))) \leq \omega_\nu \Phi(u_1(t), u_2(t)).$$

The desired inequality is obtained. $\qquad \square$

Proof of Theorem 2.3. Assertion (ii) follows from assertion (i) and Proposition 5.2. To verify assertion (i), let $x \in D$ and set $\tau_\infty = \tau(\psi(x))$. By $\bar{\tau}$ we denote the supremum of $\tau \in (0, \tau_\infty]$ such that there exists a solution $u \in C([0, \tau); E)$ to the mutational equation $\overset{\circ}{u}(t) \ni f(u(t))$ for $t \in [0, \tau)$ and $u(0) = x$ satisfying $\psi(u(t)) \leq m(t; \psi(x))$ for $t \in [0, \tau)$. The existence of such a solution is ensured by Proposition 5.1, and so the definition of $\bar{\tau}$ makes sense and $\bar{\tau} \in (0, \tau_\infty]$. Assume to the contrary that $\bar{\tau} < \tau_\infty$. Then there exists a unique solution u on $[0, \bar{\tau})$ to the mutational equation $\overset{\circ}{u}(t) \ni f(u(t))$ for $t \in [0, \bar{\tau})$ and $u(0) = x$ satisfying $\psi(u(t)) \leq m(t; \psi(x))$ for $t \in [0, \bar{\tau})$. Set $\nu = \sup\{m(t; \psi(x)); t \in [0, \bar{\tau}]\}$. Then we have $\nu < \infty$ and $u(t) \in D_\nu(\psi)$ for $t \in [0, \bar{\tau})$. By condition (H) there exists $r > 0$ such that $D_\nu(\psi) \subset D_r(\phi)$. Let $h \in (0, \bar{\tau})$ and $v(t) = u(t + h)$ for $t \in [0, \bar{\tau} - h)$. Since v is a solution to $\overset{\circ}{v}(t) \ni f(v(t))$ on $t \in [0, \bar{\tau} - h)$ and $v(0) = u(h)$, we apply Proposition 5.2 with $u_1 = v$ and $u_2 = u$ to get $\Phi(u(t + h), u(t)) \leq e^{\omega_\nu t} \Phi(u(h), u(0))$ for $t \in [0, \bar{\tau} - h)$. Since $\Phi(u(h), u(0)) \to 0$ as $h \downarrow 0$, we observe from condition (Φ2) that the limit $\bar{x} = \lim_{t \uparrow \bar{\tau}} u(t)$ exists in E and is in $D_\nu(\psi) \cap D_r(\phi)$. By Proposition 5.1 there exist $\sigma > 0$ and a solution w to $\overset{\circ}{w}(t) \ni f(w(t))$ for $t \in [0, \sigma)$ and $w(0) = \bar{x}$ satisfying $\psi(w(t)) \leq m(t; \psi(\bar{x}))$ for $t \in [0, \sigma)$. This means that u can be extended beyond $\bar{\tau}$, which contradicts the definition of $\bar{\tau}$. $\qquad \square$

6. Relation with Kato's quasilinear theory

This section is devoted to the study of the abstract quasilinear evolution equation described in Section 2. We use the same notations as in Section 2. The purpose is to derive the following theorem from the main theorem (Theorem 2.3).

Theorem 6.1. *Let $u_0 \in Y$ and set $\tau_0 = \tau(N_{u_0}(Su_0))$. Then there exists a unique solution $u \in C([0, \tau_0); X)$ to (QE; u_0) in the sense that $u(0) = u_0$, $u(t) \in Y$ for $t \in [0, \tau_0)$, $\|u(t)\|_Y$ is locally bounded on $[0, \tau_0)$ and $(d/dt)^+ u(t) = A(u(t))u(t)$ for $t \in [0, \tau_0)$. Moreover, the unique solution u satisfies*

$$N_{u(t)}(Su(t)) \leq m(t; N_{u_0}(Su_0)) \quad \text{for } t \in [0, \tau_0),$$

where m is the maximal solution to the initial value problem (2.5) with

(6.1) $$g(p) = (\beta_A(p) + \lambda_N(p)\lambda_B(p) + \mu_N(p)\lambda_A(p)p)p \quad \text{for } p \geq 0.$$

Proof. Let $E = X$ and $D = Y$, and consider the functional ψ from E into $[0, \infty]$ defined by $\psi(x) = N_x(Sx)$ if $x \in D$, and $\psi(x) = \infty$ otherwise. Then ψ is lower semicontinuous on E. Since $\phi(x) = \|Sx\|_X \leq N_x(Sx) = \psi(x)$ for $x \in D$, we have $D_\nu(\psi) \subset D_\nu(\phi) = Y_\nu$, which implies that condition (H) is satisfied. From Proposition 2.1 we observe that conditions (S1) and (S2) are satisfied. Define $\Phi(x, y) = N_x(x - y)$ for $(x, y) \in E \times E$. To verify condition (Φ1), let $(x, y), (\hat{x}, \hat{y}) \in D_r(\phi) \times D_r(\phi)$. By condition (N) we have

$$N_x(x - y) \leq N_{\hat{x}}(x - y)(1 + \mu_N(r)\|x - \hat{x}\|_X) \leq N_{\hat{x}}(x - y) + 2\mu_N(r)\lambda_N(r)rd(x, \hat{x}),$$

where we have used the fact that $N_{\hat{x}}(x) \leq \lambda_N(r)\|x\|_X \leq \lambda_N(r)\|x\|_Y$. Since

$$N_x(x - y) - N_{\hat{x}}(\hat{x} - \hat{y}) \leq N_{\hat{x}}(x - \hat{x} - (y - \hat{y})) + 2\mu_N(r)\lambda_N(r)rd(x, \hat{x}),$$

we have $\Phi(x, y) - \Phi(\hat{x}, \hat{y}) \leq \lambda_N(r)(1 + 2\mu_N(r)r)(d(x, \hat{x}) + d(y, \hat{y}))$. This means that condition (Φ1) with $L_r = \lambda_N(r)(1 + 2\mu_N(r)r)$ is satisfied. Since $D_\nu(\psi) \subset Y_\nu$, condition ($\Phi$2) with $m_\nu = 1$ and $M_\nu = \lambda_N(\nu)$ follows from condition (N).

We shall prove that the mapping f on D to $\Theta(X)$ defined by $f(w) = \vartheta_w$ for $w \in D$ satisfies conditions (f1) through (f3). To prove condition (f1), let $w_0, w \in D_\nu(\psi)$. Since $D_\nu(\psi) \subset D_\nu(\phi) = Y_\nu$, we see from condition (A3) that $\lim_{h \downarrow 0} h^{-1} \|T_w(h)x - T_{w_0}(h)x\|_X = \|A(w)x - A(w_0)x\|_X \leq \mu_A(\nu)\|w - w_0\|_X\|x\|_Y$ for $x \in D$; hence $D_r(f(w), f(w_0)) \leq \mu_A(\nu)r\|w - w_0\|_X$. Condition (f1) is thus satisfied. To verify (f3), let $x, y \in D_\nu(\psi)$. Then we have $x, y \in Y_\nu$. Take $r > \nu$ arbitrarily. Since $\lim_{h \downarrow 0} T_x(h)x = x$ in Y, there exists $h_0 \in (0, 1]$ such that $\|f(x)(h, x) - x\|_Y \leq r - \nu$ for $h \in (0, h_0]$. Note that $f(x)(h, x) \in D_r(\phi)$ for $h \in (0, h_0]$. By condition (N) we have

$$\Phi(f(x)(h, x), f(y)(h, y)) \leq N_x(T_x(h)x - T_y(h)y)(1 + \mu_N(r)\|T_x(h)x - x\|_X)$$
$$\leq N_x(T_x(h)(x - y)) + \lambda_N(\nu)\|T_x(h)y - T_y(h)y\|_X$$
$$+ \mu_N(r)N_x(T_x(h)x - T_y(h)y)\|T_x(h)x - x\|_X$$

for $h \in (0, h_0]$. By condition (A1) we have $N_x(T_x(h)(x - y)) \leq e^{\beta_A(\nu)h}N_x(x - y)$. It follows that

$$\limsup_{h \downarrow 0} h^{-1}(\Phi(f(x)(h, x), f(y)(h, y)) - \Phi(x, y))$$
$$\leq \beta_A(\nu)\Phi(x, y) + \lambda_N(\nu)\|A(x)y - A(y)y\|_X + \mu_N(r)\Phi(x, y)\|A(x)x\|_X.$$

From condition (A3) we see that condition (f3) with $\omega_\nu = \beta_A(\nu) + \lambda_N(\nu)\mu_A(\nu)\nu + \mu_N(\nu)\lambda_A(\nu)\nu$ is satisfied. Finally, to verify condition (f2), let $x \in D$. Take $r > 0$ arbitrarily such that $\|x\|_Y < r$, and then there exists $h_0 \in (0, 1]$ such that $f(x)(h, x) \in D_r(\phi)$ for $h \in (0, h_0]$. Let $h \in (0, h_0]$. By (2.4) we have

$$N_x(ST_x(h)x) \leq e^{\beta_A(r)h}N_x(Sx) + \int_0^h e^{\beta_A(r)(h-s)}N_x(B(x)ST_x(s)x)\,ds.$$

By condition (N) we have

$$N_{f(x)(h,x)}(Sf(x)(h,x)) \leq N_x(Sf(x)(h,x))(1 + \mu_N(r)\|f(x)(h,x) - x\|_X).$$

These inequalities together imply that

$$\limsup_{h \downarrow 0} h^{-1}(\psi(f(x)(h,x)) - \psi(x))$$
$$\leq \beta_A(r)\psi(x) + N_x(B(x)Sx) + \psi(x)\mu_N(r)\|A(x)x\|_X,$$

and the right-hand side is bounded by $(\beta_A(r) + \lambda_N(r)\lambda_B(r) + \mu_N(r)\lambda_A(r)r)\psi(x)$. Since r is arbitrarily given so that $r > \|x\|_Y$, the continuous function g defined by (6.1) satisfies condition (f2) with $x_h = f(x)(h,x)$, where we have used the fact that $\|x\|_Y \leq \psi(x)$ and functions β_A, λ_N, λ_B, μ_N and λ_A are nondecreasing. Theorem 2.3 asserts that there exist $\tau > 0$ and a unique $u \in C([0,\tau); E)$ such that $u(t) \in D$ for $t \in [0,\tau)$, the function $t \to \psi(u(t))$ is locally bounded on $[0,\tau)$ and $\lim_{h \downarrow 0} h^{-1}d(f(u(t))(h,u(t)), u(t+h)) = 0$ for $t \in [0,\tau)$. Since $D = Y$, we have $u(t) \in Y \subset D(A(u(t)))$ for $t \in [0,\tau)$ and the function $t \to \|u(t)\|_Y$ is locally bounded on $[0,\tau)$. By condition (A1) we have $\lim_{h \downarrow 0} h^{-1}(T_{u(t)}(h)u(t) - u(t)) = A(u(t))u(t)$ for $t \in [0,\tau)$. It follows that that u is right-differentiable on $[0,\tau)$ and $(d/dt)^+u(t) = A(u(t))u(t)$ for $t \in [0,\tau)$. $\qquad\square$

Remark 6.2. The main result in [5] has an advantage in that the regularity of solutions obtained is better than ours. Applying Theorem 6.1 and the linear theory developed by Kato [6], we can prove that for each $u_0 \in Y$ there exist $\tau > 0$ and a unique solution $u \in C([0,\tau); Y) \cap C^1([0,\tau); X)$ to (QE; u_0), which is due to Hughes *et. al.* [5]. Indeed, let u be a solution to (QE) on $[0,\tau)$ in the sense of Theorem 6.1 and let $\sigma \in (0,\tau)$ be arbitrary. Since u is Lipschitz continuous in X and bounded in Y on $[0,\sigma]$, we deduce from [5, Lemmas 2.1, 2.2 and 2.5] that the family $\{A(u(t)); t \in [0,\sigma]\}$ of linear operators in X generates an evolution operator $\{U^u(t,s); 0 \leq s \leq t \leq \sigma\}$ on X in the sense of [6, Theorem 1]. Since $(d/ds)^+U^u(t,s)u(s) = U^u(t,s)((d/ds)^+u(s) - A(u(s))u(s)) = 0$ for $0 \leq s \leq t$ and $t \in [0,\sigma]$, we have $u(t) = U^u(t,0)u_0$ for $t \in [0,\sigma]$, and the right-hand side is continuous in Y on $[0,\sigma]$, since $u_0 \in Y$. This proves that $u \in C([0,\tau); Y)$. Since $(d/dt)^+u(t) = A(u(t))u(t)$ for $t \in [0,\tau)$ and the right-hand side is continuous in X on $[0,\tau)$, we conclude that $u \in C^1([0,\tau); X)$ and $(d/dt)u(t) = A(u(t))u(t)$ for $t \in [0,\tau)$.

Finally, we give a sufficient condition for the global solvability in terms of β_A and λ_B. If the function β_A is taken as a negative constant and $\lim_{p \downarrow 0} \lambda_B(p) = 0$, then the maximal solution for the function g defined by (6.1) exists globally in time for sufficiently small $p > 0$, so that (QE) with small data is globally well-posed in time. For example, consider an equation of the form

$$u''(t) + \sigma(|A^{1/2}u(t)|^2)Au(t) + \gamma u'(t) = 0 \quad \text{for } t \geq 0$$

in a Hilbert space H, where A is a selfadjoint operator in H such that there exists $c_A > 0$ satisfying $\langle Au, u \rangle \geq c_A|u|^2$ for $u \in D(A)$, σ is a positive function on $[0,\infty)$ of class C^1, and $\gamma > 0$. The well-known global well-posedness for this equation with small data is deduced from Theorem 6.1. Without loss of generality, we may assume that $0 < m_\sigma \leq \sigma(r) \leq M_\sigma$ for $r \geq 0$. Let $X = D(A^{1/2}) \times H$ and $Y = D(A) \times D(A^{1/2})$. We define a family $\{N_{(w,z)}; (w,z) \in Y\}$ of norms in X by

$$N_{(w,z)}(u,v) = \{\sigma(|A^{1/2}w|^2)^{-1}(|v|^2 + |\gamma u + v|^2) + 2|A^{1/2}u|^2\}^{1/2} \quad \text{for } (u,v) \in X.$$

This is important, and then the family $\{A(w, z); (w, z) \in Y\}$ of operators in X defined by $(A(w, z))(u, v) = (v, \; -\sigma(|A^{1/2}w|^2)Au - \gamma v)$ for $(u, v) \in Y$ satisfies condition (A1) with $\beta_A(p) = -\beta_0$ for $p \geq 0$, where β_0 is a positive constant. Condition (A2) is satisfied with $S(u, v) = (A^{1/2}u, A^{1/2}v)$ for $(u, v) \in Y$ and $B = 0$. All the other conditions are verified without difficulty.

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