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BILINEAR ESTIMATES IN BESOV SPACES GENERATED BY THE DIRICHLET LAPLACIAN

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ABSTRACT. The purpose of this paper is to establish bilinear estimates in Besov spaces generated by the Dirichlet Laplacian on a domain of Euclidian spaces. These estimates are proved by using the gradient estimates for heat semigroup together with the Bony paraproduct formula and the boundedness of spectral multipliers.

1. INTRODUCTION

The bilinear estimates in Sobolev spaces or Besov spaces are of great importance to study the well-posedness for the Cauchy problem to nonlinear partial differential equations. In this paper we study the bilinear estimates of standard type in Besov spaces:

$$\|fg\|_{\dot{B}^s_{p,q}} \leq C\left(\|f\|_{\dot{B}^s_{p_1,q}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_3}}\|g\|_{\dot{B}^s_{p_4,q}}\right), \tag{1.1}$$

where $s > 0$ and p, p_1, p_2, p_3, p_4 and q satisfy

$$1 \leq p, p_1, p_2, p_3, p_4, q \leq \infty \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

We study also the inhomogeneous version of (1.1).

The basis of proving the bilinear estimates in Sobolev spaces $W^{k,p}$ $(k = 1, 2, \ldots)$ is to use the Leibniz rule and the Hölder inequality. However, when one considers the fractional order regularity, some idea would be needed. If the domain is the whole space $\mathbb{R}^n$, the Fourier transformation is one of the most powerful tools, and allows one to introduce the derivative of fractional order. It enables us to prove the bilinear estimates by using frequency decomposition called the Bony paraproduct formula (see Bony [1]) and the boundedness of Fourier multipliers. On the other hand, when the domain is different from $\mathbb{R}^n$, one cannot rely on such a kind of method. It will be revealed that the bilinear estimates hold for small regularity number in the Besov spaces generated by the Dirichlet Laplacian, of which we established several properties on open sets in $\mathbb{R}^n$ (see [7]), and that there arises a problem for large regularity essentially. The purpose of this paper is to establish the bilinear estimates in those Besov spaces.

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In the rest of this section we give a definition of Besov spaces generated by the Dirichlet Laplacian on an open set along [7]. Let Ω be an open set of $\mathbb{R}^n$, where $n \geq 1$. The Dirichlet Laplacian $\mathcal{H}$ is defined on $L^2(\Omega)$ by letting

$$\begin{cases} \mathcal{D}(\mathcal{H}) = \left\{ f \in H_0^1(\Omega) \,\middle|\, \Delta f \in L^2(\Omega) \right\}, \\ \mathcal{H}f = -\Delta f, \quad f \in \mathcal{D}(\mathcal{H}), \end{cases}$$

where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to $H^1(\Omega)$-norm. The operator $\mathcal{H}$ is a non-negative self-adjoint operator on $L^2(\Omega)$. For a Borel measurable function ϕ on $\mathbb{R}$, an operator $\phi(\mathcal{H})$ is defined by letting

$$\phi(\mathcal{H}) = \int_{-\infty}^{\infty} \phi(\lambda) \, dE_{\mathcal{H}}(\lambda)$$

with the domain

$$\mathcal{D}(\phi(\mathcal{H})) = \left\{ f \in L^2(\Omega) \,\middle|\, \int_{-\infty}^{\infty} |\phi(\lambda)|^2 d\|E_{\mathcal{H}}(\lambda) f\|^2_{L^2(\Omega)} < \infty \right\},$$

where $\{E_{\mathcal{H}}(\lambda)\}_{\lambda \in \mathbb{R}}$ is the spectral resolution of the identity for $\mathcal{H}$.

We begin by introducing the spaces of test functions on Ω and their duals. For the purpose, let us introduce the Littlewood-Paley partition of unity. Let ϕ_0 be a non-negative and smooth function on $\mathbb{R}$ such that

$$\operatorname{supp} \phi_0 \subset \{ \lambda \in \mathbb{R} \,|\, 2^{-1} \leq \lambda \leq 2 \} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \phi_0(2^{-j}\lambda) = 1 \quad \text{for } \lambda > 0, \tag{1.2}$$

and $\{\phi_j\}_{j=-\infty}^{\infty}$ is defined by letting

$$\phi_j(\lambda) := \phi_0(2^{-j}\lambda) \quad \text{for } \lambda \in \mathbb{R}. \tag{1.3}$$

Definition (Spaces of test functions and distributions on Ω).

(i) (Linear topological spaces $\mathcal{X}(\Omega)$ and $\mathcal{X}'(\Omega)$). A linear topological space $\mathcal{X}(\Omega)$ is defined by letting

$$\mathcal{X}(\Omega) := \left\{ f \in L^1(\Omega) \cap \mathcal{D}(\mathcal{H}) \,\middle|\, \mathcal{H}^M f \in L^1(\Omega) \cap \mathcal{D}(\mathcal{H}) \text{ for any } M \in \mathbb{N} \right\}$$

equipped with the family of semi-norms $\{p_M(\cdot)\}_{M=1}^{\infty}$ given by

$$p_M(f) := \|f\|_{L^1(\Omega)} + \sup_{j \in \mathbb{N}} 2^{Mj} \|\phi_j(\sqrt{\mathcal{H}}) f\|_{L^1(\Omega)}. \tag{1.4}$$

$\mathcal{X}'(\Omega)$ denotes the topological dual of $\mathcal{X}(\Omega)$.

(ii) (Linear topological spaces $\mathcal{Z}(\Omega)$ and $\mathcal{Z}'(\Omega)$). A linear topological space $\mathcal{Z}(\Omega)$ is defined by letting

$$\mathcal{Z}(\Omega) := \left\{ f \in \mathcal{X}(\Omega) \,\middle|\, \sup_{j \leq 0} 2^{M|j|} \big\|\phi_j\big(\sqrt{\mathcal{H}}\big) f\big\|_{L^1(\Omega)} < \infty \text{ for any } M \in \mathbb{N} \right\}$$

equipped with the family of semi-norms $\{q_M(\cdot)\}_{M=1}^{\infty}$ given by

$$q_M(f) := \|f\|_{L^1(\Omega)} + \sup_{j \in \mathbb{Z}} 2^{M|j|} \|\phi_j(\sqrt{\mathcal{H}}) f\|_{L^1(\Omega)}.$$

$\mathcal{Z}'(\Omega)$ denotes the topological dual of $\mathcal{Z}(\Omega)$.

In this paper we often use the notation ${}_{X'}\langle \cdot, \cdot \rangle_X$ of duality pair of a linear topological space X and its dual X'.

We proved in Lemma 4.6 from [7] that

$$\mathcal{X}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{X}'(\Omega), \tag{1.5}$$

$$\mathcal{Z}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{Z}'(\Omega) \tag{1.6}$$

for any $1 \le p \le \infty$. The inclusion relation (1.5) ((1.6) resp.) assures that

$$\int_\Omega \big| f(x)\overline{g(x)} \big| \, dx < \infty$$

for any $f \in L^p(\Omega)$, $1 \le p \le \infty$, and $g \in \mathcal{X}(\Omega)$ ($g \in \mathcal{Z}(\Omega)$ resp.). Hence we can regard functions in the Lebesgue spaces as elements in $\mathcal{X}'(\Omega)$ and $\mathcal{Z}'(\Omega)$ as follows:

Definition. For $f \in L^1(\Omega) + L^\infty(\Omega)$, we identify f as an element in $\mathcal{X}'(\Omega)$ ($\mathcal{Z}'(\Omega)$ resp.) by letting

$${}_{\mathcal{X}'(\Omega)}\langle f, g \rangle_{\mathcal{X}(\Omega)} = \int_\Omega f(x)\overline{g(x)} \, dx \quad \left({}_{\mathcal{Z}'(\Omega)}\langle f, g \rangle_{\mathcal{Z}(\Omega)} = \int_\Omega f(x)\overline{g(x)} \, dx \quad \text{resp.} \right)$$

for any $g \in \mathcal{X}(\Omega)$ ($g \in \mathcal{Z}(\Omega)$ resp.).

Next, we introduce the notion of dual operators on $\mathcal{X}'(\Omega)$ and $\mathcal{Z}'(\Omega)$.

Definition (Dual operators). Let ϕ be a real-valued Borel measurable function on $\mathbb{R}$.

(i) For a mapping $\phi(\mathcal{H}) : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$, we define $\phi(\mathcal{H}) : \mathcal{X}'(\Omega) \to \mathcal{X}'(\Omega)$ by letting

$${}_{\mathcal{X}'(\Omega)}\big\langle \phi(\mathcal{H}) f, g \big\rangle_{\mathcal{X}(\Omega)} := {}_{\mathcal{X}'(\Omega)}\big\langle f, \phi(\mathcal{H}) g \big\rangle_{\mathcal{X}(\Omega)} \tag{1.7}$$

for any $f \in \mathcal{X}'(\Omega)$ and $g \in \mathcal{X}(\Omega)$.

(ii) For a mapping $\phi(\mathcal{H}) : \mathcal{Z}(\Omega) \to \mathcal{Z}(\Omega)$, we define $\phi(\mathcal{H}) : \mathcal{Z}'(\Omega) \to \mathcal{Z}'(\Omega)$ by letting

$${}_{\mathcal{Z}'(\Omega)}\big\langle \phi(\mathcal{H}) f, g \big\rangle_{\mathcal{Z}(\Omega)} := {}_{\mathcal{Z}'(\Omega)}\big\langle f, \phi(\mathcal{H}) g \big\rangle_{\mathcal{Z}(\Omega)}$$

for any $f \in \mathcal{Z}'(\Omega)$ and $g \in \mathcal{Z}(\Omega)$.

When we consider the inhomogeneous Besov spaces, a function ψ, whose support is restricted in the neighborhood of the origin, is needed. More precisely, let $\psi \in C_0^\infty(\mathbb{R})$ be a function satisfying

$$\psi(\lambda^2) + \sum_{j=1}^\infty \phi_j(\lambda) = 1 \quad \text{for } \lambda \ge 0. \tag{1.8}$$

We are now in a position to give the definition of Besov spaces generated by $\mathcal{H}$.

Definition (Besov spaces). Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Then the Besov spaces are defined as follows:

(i) The inhomogeneous Besov spaces $B^s_{p,q}(\mathcal{H})$ are defined by letting

$$B^s_{p,q}(\mathcal{H}) := \left\{ f \in \mathcal{X}'(\Omega) \,\middle|\, \|f\|_{B^s_{p,q}(\mathcal{H})} < \infty \right\},$$

where

$$\|f\|_{B^s_{p,q}(\mathcal{H})} := \|\psi(\mathcal{H})f\|_{L^p(\Omega)} + \left\| \left\{ 2^{sj} \|\phi_j(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \right\}_{j\in\mathbb{N}} \right\|_{\ell^q(\mathbb{N})}.$$

(ii) The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H})$ are defined by letting

$$\dot{B}^s_{p,q}(\mathcal{H}) := \left\{ f \in \mathcal{Z}'(\Omega) \,\middle|\, \|f\|_{\dot{B}^s_{p,q}(\mathcal{H})} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{H})} := \left\| \left\{ 2^{sj} \|\phi_j(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})}.$$

It is proved in Theorem 2.5 from [7] that $B^s_{p,q}(\mathcal{H})$ and $\dot{B}^s_{p,q}(\mathcal{H})$ are Banach spaces, and

$$\mathcal{X}(\Omega) \hookrightarrow B^s_{p,q}(\mathcal{H}) \hookrightarrow \mathcal{X}'(\Omega),$$

$$\mathcal{Z}(\Omega) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}) \hookrightarrow \mathcal{Z}'(\Omega).$$

for any $s \in \mathbb{R}$ and $1 \le p, q \le \infty$.

We conclude this section by giving a remark on the regularity numbers such that the bilinear estimates hold. As is well known, when Ω is the whole space $\mathbb{R}^n$, one does not need to impose any restriction on the regularity number $s > 0$ of Besov spaces. On the other hand, when we consider these estimates for functions whose regularity is measured by the Dirichlet Laplacian on domains, a restriction is required on the regularity. In fact, it is possible to construct a counter-example for high regularity (see appendix A). This is because a product of functions operated by the Laplacian may not belong to the domain of the Dirichlet Laplacian generally. Hence, in general, it is impossible to get the estimates in high regularity. This can be seen from the following observation: Let Ω be a domain with smooth boundary, and let f and g be in the domain of the Dirichlet Laplacian $\mathcal{H}$. Applying the Leibniz rule to the product fg, we are confronted with the term $\nabla f \cdot \nabla g$ which is possible to be in the complement of the domain of $\mathcal{H}$, since it does not in general vanishes on the boundary. Observing the proof of Theorem 2.1 in §2, we see that the derivatives of functions must be handled even if s is not large, and the argument of estimates for derivatives requires the gradient estimates for heat semigroup in L^p. However, the range of p depends on the domain. To avoid this complexity, we assume the gradient estimate in L^∞, while that in L^p with $p \in [1, 2]$ is true without any assumption.

This paper is organized as follows. In §2 we state the main result. In §3 we prepare some useful lemmas to prove the main theorem. In §4 we prove the main theorem. In §5 we discuss the bilinear estimates in the spaces generated by the Schrödinger operators.

2. Statement of result

Let us consider a domain Ω such that the following gradient estimate

$$\|\nabla e^{-t\mathcal{H}}\|_{L^\infty(\Omega)\to L^\infty(\Omega)} \le Ct^{-\frac{1}{2}} \tag{2.1}$$

holds either for any $t \in (0,1]$ or for any $t > 0$, where $\{e^{-t\mathcal{H}}\}_{t>0}$ is the semigroup generated by $\mathcal{H}$.

We shall prove here the following:

Theorem 2.1. *Let $0 < s < 2$ and p, p_1, p_2, p_3, p_4 and q be such that*

$$1 \le p, p_1, p_2, p_3, p_4, q \le \infty \quad \textit{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then the following assertions hold:

(i) *Let Ω be a domain of $\mathbb{R}^n$ such that the gradient estimate* (2.1) *holds for any $t \in (0,1]$. Then there exists a constant $C > 0$ such that*

$$\|fg\|_{B^s_{p,q}(\mathcal{H})} \le C \Big(\|f\|_{B^s_{p_1,q}(\mathcal{H})} \|g\|_{L^{p_2}(\Omega)} + \|f\|_{L^{p_3}(\Omega)} \|g\|_{B^s_{p_4,q}(\mathcal{H})} \Big) \tag{2.2}$$

for any $f \in B^s_{p_1,q}(\mathcal{H}) \cap L^{p_3}(\Omega)$ and $g \in B^s_{p_4,q}(\mathcal{H}) \cap L^{p_2}(\Omega)$.

(ii) *Let Ω be a domain of $\mathbb{R}^n$ such that the gradient estimate* (2.1) *holds for any $t > 0$. Then there exists a constant $C > 0$ such that*

$$\|fg\|_{\dot{B}^s_{p,q}(\mathcal{H})} \le C \Big(\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})} \|g\|_{L^{p_2}(\Omega)} + \|f\|_{L^{p_3}(\Omega)} \|g\|_{\dot{B}^s_{p_4,q}(\mathcal{H})} \Big) \tag{2.3}$$

for any $f \in \dot{B}^s_{p_1,q}(\mathcal{H}) \cap L^{p_3}(\Omega)$ and $g \in \dot{B}^s_{p_4,q}(\mathcal{H}) \cap L^{p_2}(\Omega)$.

As to the range of the regularity number s in Theorem 2.1, it is not clear whether or not it is sharp. However we can find an $s \ge 2$ such that Theorem 2.1 does not hold. For more details, see appendix A.

In the rest of this section we shall give some examples of domains such that estimate (2.1) holds. We consider three cases as follows:

I. Inhomogeneous case;
II. Homogeneous case;
III. L^p-gradient estimate.

I. Inhomogeneous case. The estimate (2.1) holds for any $t \in (0,1]$ in the case when the domain Ω fulfills the following properties:

(a) Ω is a domain with uniform $C^{2,\alpha}$-boundary for some $\alpha \in (0,1)$;
(b) Ω is a bounded domain with $C^{1,1}$-boundary.

Hence the bilinear estimate (2.2) in Theorem 2.1 holds for domains of type (a) and (b). As to the case when Ω is a domain in (a), Fornaro, Metafune and Priola proved the estimate (2.1) for any $t \in (0,1]$ (see [4]). A typical example of such domains is the half space $\mathbb{R}^n_+$. When Ω is a domain in (b), the problem is reduced to the case of the half space (see, e.g., Wloka [16]), and hence, we have the estimate (2.1) for any $t \in (0,1]$.

II. Homogeneous case. The estimate (2.1) holds for any $t > 0$ in the case when Ω fulfills the following:

(a) Ω is the half space $\mathbb{R}^n_+$;
(b) Ω is a bounded domain with $C^{1,1}$-boundary.

Hence the bilinear estimate (2.3) in Theorem 2.1 holds for domains of type (a) and (b). As to (a), the estimate (2.1) for any $t > 0$ is an immediate consequence of (I-a) and the representation formula of heat kernel on $\mathbb{R}^n_+$. As to (b), we can obtain the estimate (2.1) for any $t > 0$ by combining (I-b) and the following estimate: If Ω is a bounded Lipschitz domain, then there exists a constant $C > 1$ such that

$$\|\nabla e^{-t\mathcal{H}}\|_{L^\infty(\Omega)\to L^\infty(\Omega)} \le Ce^{-\lambda_1 t} \tag{2.4}$$

for any $t > 1$, where $\lambda_1 > 0$ is the first eigenvalue of $\mathcal{H}$. Here, the estimate (2.4) is proved by combining the identity

$$\|\nabla e^{-t\mathcal{H}}\|_{L^\infty(\Omega)\to L^\infty(\Omega)} = \|\nabla e^{-t\mathcal{H}}(\cdot,\cdot)\|_{L^\infty(\Omega;L^1(\Omega))}$$

(see Lemma B.1 in appendix B) and the pointwise estimate for the kernel $e^{-t\mathcal{H}}(x,y)$ of the operator $e^{-t\mathcal{H}}$:

$$C^{-1}\phi_{\lambda_1}(x)\phi_{\lambda_1}(y)e^{-\lambda_1 t} \le e^{-t\mathcal{H}}(x,y) \le C\phi_{\lambda_1}(x)\phi_{\lambda_1}(y)e^{-\lambda_1 t}$$

for almost everywhere $x, y \in \Omega$ and any $t > 1$, where $\phi_{\lambda_1} \ge 0$ is the eigenfunction corresponding to λ_1 (see Davies [3]).

III. L^p-gradient estimate. There is a counter-example of domains in which the estimate (2.1) for any $t > 0$ does not hold. More precisely, when Ω is an exterior domain outside a compact $C^{1,1}$-obstacle, (2.1) does not hold (see (A.2) in appendix A). However, there exists a $p_0 \in [2,\infty]$ depending on Ω such that

$$\|\nabla e^{-t\mathcal{H}}\|_{L^p(\Omega)\to L^p(\Omega)} \le Ct^{-\frac{1}{2}}, \quad t > 0 \tag{2.5}$$

for any $p \in [1, p_0]$ (see (a)–(c) below). If we take p, p_j $(j = 1,2,3,4)$ as

$$1 \le p, p_1, p_2, p_3, p_4 \le p_0,$$

then we can prove estimates (2.2) and (2.3) by a trivial modification of proof of Theorem 2.1. The following are examples of domains and the possible range of p.

(a) Let Ω be an open set. Then the estimate (2.5) holds for any $p \in [1,2]$;
(b) if Ω is a bounded C^1-domain, then the estimate (2.5) holds for any $p \in [1,\infty)$;
(c) if Ω is a bounded Lipschitz domain, then the estimate (2.5) holds provided that either $1 \le p \le 3$ if $n \ge 3$, or $1 \le p \le 4$ if $n = 2$.

As to (a), see Theorem 1.2 from [8], which is stated in Proposition 3.4 in §3. The assertions (b) and (c) are immediate consequences of (a) and L^p-boundedness of the Riesz transform $\nabla\mathcal{H}^{-\frac{1}{2}}$ (see [2, 9, 12, 13, 17]).

3. Preliminaries

In this section we introduce some useful lemmas to prove Theorem 2.1. Throughout this section, we assume that Ω is an open set of $\mathbb{R}^n$. Here and below, we denote by $\mathscr{S}(\mathbb{R})$ the space of all rapidly decreasing functions on $\mathbb{R}$.

3.1. **Approximations of the identity.** The following results can be found in our previous paper [7]. The first one is the following.

Lemma 3.1 (Lemma 4.5 from [7]). (i) *For any $f \in \mathcal{X}(\Omega)$, we have*

$$f = \psi(\mathcal{H})f + \sum_{j=1}^{\infty} \phi_j(\sqrt{\mathcal{H}})f \quad \text{in } \mathcal{X}(\Omega). \tag{3.1}$$

Furthermore, for any $f \in \mathcal{X}'(\Omega)$, we have also the identity (3.1) *in $\mathcal{X}'(\Omega)$, and $\psi(\mathcal{H})f$ and $\phi_j(\sqrt{\mathcal{H}})f$ are regarded as elements in $L^\infty(\Omega)$.*

(ii) *For any $f \in \mathcal{Z}(\Omega)$, we have*

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}})f \quad \text{in } \mathcal{Z}(\Omega). \tag{3.2}$$

Furthermore, for $f \in \mathcal{Z}'(\Omega)$, we have also the identity (3.2) *in $\mathcal{Z}'(\Omega)$, and $\phi_j(\sqrt{\mathcal{H}})f$ are regarded as elements in $L^\infty(\Omega)$.*

The second one is the following.

Lemma 3.2. (i) *For any $f \in L^2(\Omega)$ and $j \in \mathbb{Z}$, we have*

$$f = \psi(2^{-2j}\mathcal{H})f + \sum_{k=j+1}^{\infty} \phi_k(\sqrt{\mathcal{H}})f \quad \text{in } L^2(\Omega) \tag{3.3}$$

and

$$f = \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}})f + \sum_{k=j+1}^{\infty} \phi_k(\sqrt{\mathcal{H}})f \quad \text{in } L^2(\Omega) \tag{3.4}$$

(ii) *Let $1 \le p < \infty$. Then for any $f \in L^p(\Omega)$, we have*

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}})f \quad \text{in } \mathcal{X}'(\Omega). \tag{3.5}$$

Proof. The assertion (i) is proved in the course of proof of Lemma 4.5 from [7]. Hence we prove the assertion (ii). Since $L^2(\Omega) \hookrightarrow \mathcal{X}'(\Omega)$, the identity (3.5) holds for any $f \in L^p(\Omega) \cap L^2(\Omega)$. Then the identity (3.5) holds for any $f \in L^p(\Omega)$ by the density argument, since $1 \le p < \infty$. The proof of Lemma 3.2 is finished. □

3.2. **Functional calculus for spectral multipliers.** This subsection is devoted to proving L^p-estimates for the operators $\psi(\mathcal{H})$ and $\phi_j(\sqrt{\mathcal{H}})$.

We recall the following two results.

Proposition 3.3 (Theorem 1.1 from [8]). *For any $\phi \in \mathscr{S}(\mathbb{R})$ and $1 \le p \le q \le \infty$ there exists a constant $C > 0$ such that*

$$\|\phi(\theta\mathcal{H})\|_{L^p(\Omega)\to L^q(\Omega)} \le C\theta^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \tag{3.6}$$

for any $\theta > 0$.

Proposition 3.4 (Theorem 1.2 from [8]). *For any $\phi \in \mathscr{S}(\mathbb{R})$ and $1 \le p \le 2$ there exists a constant $C > 0$ such that*

$$\|\nabla \phi(\theta \mathcal{H})\|_{L^p(\Omega)\to L^p(\Omega)} \le C\theta^{-\frac{1}{2}}$$

for any $\theta > 0$.

Based on Proposition 3.3, we have the following.

Lemma 3.5. *Let $1 \le p \le \infty$. Then the following assertions hold:*

(i) *For any $m \in \mathbb{N} \cup \{0\}$ there exists a constant $C > 0$ such that*

$$\left\|\mathcal{H}^m \psi(2^{-2j}\mathcal{H})\right\|_{L^p(\Omega)\to L^p(\Omega)} \le C2^{2mj} \tag{3.7}$$

for any $j \in \mathbb{Z}$.

(ii) *For any $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$\left\|\mathcal{H}^\alpha \phi_j(\sqrt{\mathcal{H}})\right\|_{L^p(\Omega)\to L^p(\Omega)} \le C2^{2\alpha j} \tag{3.8}$$

for any $j \in \mathbb{Z}$. Furthermore, for any $\alpha \ge 0$, we have

$$\left\|\mathcal{H}^\alpha \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}})\right\|_{L^p(\Omega)\to L^p(\Omega)} \le C2^{2\alpha j} \tag{3.9}$$

for any $j \in \mathbb{Z}$.

Proof. The estimate (3.7) is an immediate consequence of Proposition 3.3. In fact, noting that

$$\lambda^m \psi(\lambda) \in C_0^\infty(\mathbb{R}),$$

we conclude from (3.6) for $\theta = 2^{-2j}$ that

$$\begin{aligned}\left\|\mathcal{H}^m \psi(2^{-2j}\mathcal{H})\right\|_{L^p(\Omega)\to L^p(\Omega)} &= 2^{2mj}\left\|(2^{-2j}\mathcal{H})^m \psi(2^{-2j}\mathcal{H})\right\|_{L^p(\Omega)\to L^p(\Omega)} \\ &\le C2^{2mj}\end{aligned}$$

for any $j \in \mathbb{Z}$. In a similar way, we get (3.8), since

$$\lambda^\alpha \phi_0(\sqrt{\lambda}) \in C_0^\infty((0,\infty)).$$

It remains to prove the estimate (3.9). When $\alpha > 0$, the estimate (3.9) follows from the estimate (3.8). In fact, we estimate

$$\begin{aligned}\left\|\mathcal{H}^\alpha \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}})\right\|_{L^p(\Omega)\to L^p(\Omega)} &\le \sum_{k=-\infty}^{j} \|\mathcal{H}^\alpha \phi_k(\sqrt{\mathcal{H}})\|_{L^p(\Omega)\to L^p(\Omega)} \\ &\le C\sum_{k=-\infty}^{j} 2^{2\alpha k} \\ &\le C2^{2\alpha j}.\end{aligned}$$

We now concentrate on the case when $\alpha = 0$. We know from Lemma 3.2 that

$$f = \psi(2^{-2j}\mathcal{H})f + \sum_{k=j+1}^{\infty} \phi_k(\sqrt{\mathcal{H}})f \quad \text{in } L^2(\Omega)$$

and

$$f = \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}})f + \sum_{k=j+1}^{\infty} \phi_k(\sqrt{\mathcal{H}})f \quad \text{in } L^2(\Omega)$$

for any $j \in \mathbb{Z}$ and $f \in L^2(\Omega)$. Combining the above identities, we readily see that

$$\sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}})f = \psi(2^{-2j}\mathcal{H})f \quad \text{in } L^2(\Omega)$$

for any $j \in \mathbb{Z}$, which implies that

$$\begin{aligned}\Big\| \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}})g \Big\|_{L^p(\Omega)} &= \big\|\psi(2^{-2j}\mathcal{H})g\big\|_{L^p(\Omega)} \\ &\le C\|g\|_{L^p(\Omega)}\end{aligned}$$

for any $j \in \mathbb{Z}$ and $g \in L^p(\Omega) \cap L^2(\Omega)$. Thus, when $1 \le p < \infty$, the estimate (3.9) for $\alpha = 0$ is proved by the density argument, and the case $p = \infty$ is obtained from L^1-estimate by the duality argument. Thus the estimate (3.9) for $\alpha = 0$ is proved. The proof of Lemma 3.5 is finished. □

Based on the gradient estimate (2.1) and Proposition 3.4, we have the following estimates which play a crucial role in the proof of Theorem 2.1.

Lemma 3.6. *Let $1 \le p \le \infty$. Then the following assertions hold*:

(i) *Let Ω be an open set of $\mathbb{R}^n$ such that the estimate* (2.1) *holds for any $t \in (0,1]$. Then for any $m \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$\|\nabla \mathcal{H}^m \psi(2^{-2j}\mathcal{H})\|_{L^p(\Omega)\to L^p(\Omega)} \le C2^{(2m+1)j}, \tag{3.10}$$

$$\|\nabla \mathcal{H}^\alpha \phi_j(\sqrt{\mathcal{H}})\|_{L^p(\Omega)\to L^p(\Omega)} \le C2^{(2\alpha+1)j} \tag{3.11}$$

for any $j \in \mathbb{N}$.

(ii) *Let Ω be an open set of $\mathbb{R}^n$ such that the estimate* (2.1) *holds for any $t > 0$. Then the estimates* (3.10) *and* (3.11) *hold for any $j \in \mathbb{Z}$. Furthermore, for any $\alpha \ge 0$ there exists a constant $C > 0$ such that*

$$\Big\|\nabla \mathcal{H}^\alpha \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}})\Big\|_{L^p(\Omega)\to L^p(\Omega)} \le C2^{(2\alpha+1)j} \tag{3.12}$$

for any $j \in \mathbb{Z}$.

Proof. We prove the assertion (i). The case $p = 1$ is an immediate consequence of Proposition 3.4 for $\theta = 2^{-2j}$, since

$$\lambda^m \psi \in C_0^\infty(\mathbb{R}), \quad \lambda^\alpha \phi_0(\sqrt{\lambda}) \in C_0^\infty((0,\infty)).$$

Hence it suffices to show the case $p = \infty$. In fact, once the case $p = \infty$ is proved, Riesz-Thorin interpolation theorem allows us to conclude the estimates (3.10) and (3.11) for any $1 \le p \le \infty$.

Let $f \in L^\infty(\Omega)$. Then it follows from the estimate (2.1) for $0 < t \le 1$ that

$$\begin{aligned}\big\|\nabla \mathcal{H}^m \psi(2^{-2j}\mathcal{H})f\big\|_{L^\infty(\Omega)} &= \big\|\nabla e^{-2^{-2j}\mathcal{H}} e^{2^{-2j}\mathcal{H}}\mathcal{H}^m\psi(2^{-2j}\mathcal{H})f\big\|_{L^\infty(\Omega)} \\ &\le C2^j \big\|e^{2^{-2j}\mathcal{H}}\mathcal{H}^m\psi(2^{-2j}\mathcal{H})f\big\|_{L^\infty(\Omega)} \\ &= C2^{(2m+1)j}\big\|e^{2^{-2j}\mathcal{H}}(2^{-2j}\mathcal{H})^m\psi(2^{-2j}\mathcal{H})f\big\|_{L^\infty(\Omega)}\end{aligned} \tag{3.13}$$

for any $j \in \mathbb{N}$. Since

$$e^\lambda \lambda^m \psi(\lambda) \in C_0^\infty(\mathbb{R}),$$

it follows from the estimate (3.6) for $p = \infty$ in Proposition 3.3 that

$$\big\|e^{2^{-2j}\mathcal{H}}(2^{-2j}\mathcal{H})^m\psi(2^{-2j}\mathcal{H})f\big\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)}. \tag{3.14}$$

Thus the required estimate (3.10) for $p = \infty$ is an immediate consequence of (3.13) and (3.14). In a similar way, we get (3.11). Thus the assertion (i) is proved.

Next we prove the assertion (ii). We can prove the estimates (3.10) and (3.11) for any $j \in \mathbb{Z}$ in the same way as (i). Furthermore, the estimate (3.12) is proved by using (3.11) in the same way as the proof of (3.8) for $\alpha > 0$. Hence we may omit the details. The proof of Lemma 3.6 is finished. □

3.3. **The Leibniz rule for the Dirichlet Laplacian.** In this subsection we prove the following lemma.

Lemma 3.7. *Assume that Ω is an open set of $\mathbb{R}^n$ such that the estimate* (2.1) *holds for any $t \in (0,1]$. Let $\Phi, \Psi \in \mathscr{S}(\mathbb{R})$. Then for any $f, g \in \mathcal{X}'(\Omega)$, we have*

$$\begin{aligned}&\mathcal{H}\big(\Phi(\mathcal{H})f \cdot \Psi(\mathcal{H})g\big) \\ &= \mathcal{H}\Phi(\mathcal{H})f \cdot \Psi(\mathcal{H})g - 2\nabla\Phi(\mathcal{H})f \cdot \nabla\Psi(\mathcal{H})g + \Phi(\mathcal{H})f \cdot \mathcal{H}\Psi(\mathcal{H})g \quad \textit{in } \mathcal{X}'(\Omega).\end{aligned} \tag{3.15}$$

Proof. To begin with, we note from Lemma 3.1 that $\Phi(\mathcal{H})f$ and $\Psi(\mathcal{H})g$ are regarded as elements in $L^\infty(\Omega)$:

$$\Phi(\mathcal{H})f,\ \Psi(\mathcal{H})g \in L^\infty(\Omega). \tag{3.16}$$

Hence, Lemmas 3.5 and 3.6 for $p = \infty$ assure that all the right members of (3.15) belong to $L^\infty(\Omega)$. It suffices to show that (3.15) holds in $\mathscr{D}'(\Omega)$, where $\mathscr{D}'(\Omega)$ is the dual space of the topological space $\mathscr{D}(\Omega) = C_0^\infty(\Omega)$. In fact, if (3.15) holds in $\mathscr{D}'(\Omega)$, then (3.15) holds for almost everywhere on Ω. Hence we conclude that (3.15) holds in $\mathcal{X}'(\Omega)$.

Since

$$\mathcal{H}h = -\Delta h \quad \text{for } h \in C_0^\infty(\Omega),$$

we write, by using (3.16),

$${}_{\mathscr{D}'(\Omega)}\langle \mathcal{H}\left(\Phi(\mathcal{H})f \cdot \Psi(\mathcal{H})g\right), h\rangle_{\mathscr{D}(\Omega)} = {}_{L^\infty(\Omega)}\langle \Psi(\mathcal{H})g, \overline{\Phi(\mathcal{H})f}(-\Delta h)\rangle_{L^1(\Omega)} \tag{3.17}$$

for any $h \in \mathscr{D}(\Omega)$. Here, noting from the definition (1.7) of $\mathcal{H}$ that

$$-\Delta\Phi(\mathcal{H})f = \mathcal{H}\Phi(\mathcal{H})f,$$

we observe from the Leibniz rule that

$$\overline{\Phi(\mathcal{H})f}(-\Delta h) = -\Delta(\overline{\Phi(\mathcal{H})f}\cdot h) - (\overline{\mathcal{H}\Phi(\mathcal{H})f})h + 2\overline{\nabla\Phi(\mathcal{H})f}\cdot\nabla h \quad \text{in } \mathscr{D}'(\Omega). \tag{3.18}$$

Since all terms in (3.18) belong to $L^1(\Omega)$ by (3.16), Lemmas 3.5 and 3.6 for $p=\infty$, multiplying (3.18) by $\Psi(\mathcal{H})g$, and using (3.17), we write

$$\begin{aligned}
&{}_{\mathscr{D}'(\Omega)}\langle \mathcal{H}\left(\Phi(\mathcal{H})f\cdot\Psi(\mathcal{H})g\right), h\rangle_{\mathscr{D}(\Omega)}\\
&= {}_{L^\infty(\Omega)}\langle \Psi(\mathcal{H})g, -\Delta(\overline{\Phi(\mathcal{H})f}\cdot h)\rangle_{L^1(\Omega)} \qquad (3.19)\\
&\quad - {}_{L^\infty(\Omega)}\langle (\mathcal{H}\Phi(\mathcal{H})f)\Psi(\mathcal{H})g, h\rangle_{L^1(\Omega)} + 2{}_{L^\infty(\Omega)}\langle \Psi(\mathcal{H})g, \overline{\nabla\Phi(\mathcal{H})f}\cdot\nabla h\rangle_{L^1(\Omega)}.
\end{aligned}$$

As to the first term in the right member of (3.19), integrating by parts, we get

$${}_{L^\infty(\Omega)}\langle \Psi(\mathcal{H})g, -\Delta(\overline{\Phi(\mathcal{H})f}\cdot h)\rangle_{L^1(\Omega)} = {}_{L^\infty(\Omega)}\langle -\Delta\Psi(\mathcal{H})g, \overline{\Phi(\mathcal{H})f}\cdot h\rangle_{L^1(\Omega)}.$$

Here, we note that

$$-\Delta\Psi(\mathcal{H})g = \mathcal{H}\Psi(\mathcal{H})g \quad \text{in } \mathscr{D}'(\Omega). \tag{3.20}$$

Since $\mathcal{H}\Psi(\mathcal{H})g$ belongs to $L^\infty(\Omega)$ by (3.16) and Lemma 3.5 for $p=\infty$, the identity (3.20) holds for almost everywhere on Ω. Hence we have

$${}_{L^\infty(\Omega)}\langle -\Delta\Psi(\mathcal{H})g, \overline{\Phi(\mathcal{H})f}\cdot h\rangle_{L^1(\Omega)} = {}_{L^\infty(\Omega)}\langle \Phi(\mathcal{H})f\cdot\mathcal{H}\Psi(\mathcal{H})g, h\rangle_{L^1(\Omega)},$$

since $\overline{\Phi(\mathcal{H})f}\cdot h \in L^1(\Omega)$. Therefore, the first term is written as

$${}_{L^\infty(\Omega)}\langle \Psi(\mathcal{H})g, -\Delta(\overline{\Phi(\mathcal{H})f}\cdot h)\rangle_{L^1(\Omega)} = {}_{L^\infty(\Omega)}\langle \Phi(\mathcal{H})f\cdot\mathcal{H}\Psi(\mathcal{H})g, h\rangle_{L^1(\Omega)}.$$

In a similar way, the third term in the right member of (3.19) is written as

$$\begin{aligned}
&{}_{L^\infty(\Omega)}\langle \Psi(\mathcal{H})g, \overline{\nabla\Phi(\mathcal{H})f}\cdot\nabla h\rangle_{L^1(\Omega)}\\
&= -{}_{\mathscr{D}'(\Omega)}\langle \Delta\Phi(\mathcal{H})f\cdot\Psi(\mathcal{H})g, h\rangle_{\mathscr{D}(\Omega)} - {}_{\mathscr{D}'(\Omega)}\langle \nabla\Phi(\mathcal{H})f\cdot\nabla\Psi(\mathcal{H})g, h\rangle_{\mathscr{D}(\Omega)} \qquad (3.21)\\
&= {}_{\mathscr{D}'(\Omega)}\langle \mathcal{H}\phi(\mathcal{H})f\cdot\Psi(\mathcal{H})g, h\rangle_{\mathscr{D}(\Omega)} - {}_{\mathscr{D}'(\Omega)}\langle \nabla\Phi(\mathcal{H})f\cdot\nabla\Psi(\mathcal{H})g, h\rangle_{\mathscr{D}(\Omega)}.
\end{aligned}$$

Therefore, summarizing (3.19) and (3.21), we conclude that (3.15) holds in $\mathscr{D}'(\Omega)$. The proof of Lemma 3.7 is finished. □

3.4. **Properties of the space $\mathcal{P}(\Omega)$.** In this subsection we shall study several properties of a space $\mathcal{P}(\Omega)$ which is defined by

$$\mathcal{P}(\Omega) := \Big\{ f\in\mathcal{X}'(\Omega) \,\Big|\, {}_{\mathcal{Z}'(\Omega)}\langle f, g\rangle_{\mathcal{Z}(\Omega)} = 0 \text{ for any } g\in\mathcal{Z}(\Omega)\Big\}. \tag{3.22}$$

We recall that $\mathcal{X}'(\Omega)$ and $\mathcal{Z}'(\Omega)$ correspond to $\mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{S}_0'(\mathbb{R}^n)$ in the classical case, respectively. Here $\mathscr{S}_0'(\mathbb{R}^n)$ is the dual space of $\mathscr{S}_0(\mathbb{R}^n)$ defined by

$$\mathscr{S}_0(\mathbb{R}^n) := \Big\{ f\in\mathscr{S}(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0 \quad \text{for any } \alpha\in(\mathbb{N}\cup\{0\})^n\Big\}$$

endowed with the induced topology of $\mathscr{S}(\mathbb{R}^n)$. It is known that $\mathscr{S}_0'(\mathbb{R}^n)$ is characterized by the quotient space of $\mathscr{S}'(\mathbb{R}^n)$ modulo polynomials, i.e.,

$$\mathscr{S}_0'(\mathbb{R}^n) \cong \mathscr{S}'(\mathbb{R}^n)/\mathcal{P}, \tag{3.23}$$

where $\mathcal{P}$ is the set of all polynomials of n real variables (see, e.g., Proposition 1.1.3 from Grafakos [5]). As to the space $\mathcal{P}(\Omega)$, it is readily checked that $\mathcal{P}(\Omega)$ is a closed subspace of $\mathcal{X}'(\Omega)$, and hence, we can apply Theorem in p.127 from Schaefer [11] and Propositions 35.5 and 35.6 from Trèves [15] to obtain the isomorphism:

$$\mathcal{Z}'(\Omega) \cong \mathcal{X}'(\Omega)/\mathcal{P}(\Omega)$$

(cf. Theorem 1.1 from Sawano [10]).

We shall prove the following:

Lemma 3.8. *The space $\mathcal{P}(\Omega)$ enjoys the following*:

(i) *Let $f \in \mathcal{X}'(\Omega)$. Then the following assertions are equivalent*:
  (a) $f \in \mathcal{P}(\Omega)$;
  (b) $\phi_j(\sqrt{\mathcal{H}})f = 0$ *in* $\mathcal{X}'(\Omega)$ *for any* $j \in \mathbb{Z}$;
  (c) $\|f\|_{\dot{B}^s_{p,q}(\mathcal{H})} = 0$ *for any* $s \in \mathbb{R}$ *and* $1 \le p, q \le \infty$.

(ii) $\mathcal{P}(\Omega)$ *is a subspace of* $L^\infty(\Omega)$.

(iii) *In particular, if* Ω *is a domain such that the gradient estimate* (2.1) *holds for any* $t > 0$, *then*

$$\mathcal{P}(\Omega) = \textit{either} \quad \{0\} \quad \textit{or} \quad \{f = c \textit{ on } \Omega \,|\, c \in \mathbb{C}\}.$$

Proof. We prove the assertion (i). We readily see from the definition of $\dot{B}^s_{p,q}(\mathcal{H})$ that (c) implies (b). Conversely, we suppose that (b) holds. Since $f \in \mathcal{X}'(\Omega)$, it follows from the assertion (i) in Lemma 3.1 that

$$\phi_j(\sqrt{\mathcal{H}})f \in L^\infty(\Omega)$$

for any $j \in \mathbb{Z}$. Hence we deduce that

$$\phi_j(\sqrt{\mathcal{H}})f = 0 \quad \text{a.e. } x \in \Omega$$

for any $j \in \mathbb{Z}$, which implies that (c) holds true.

We have to prove that (a) and (b) are equivalent. Suppose that (a) holds, i.e., $f \in \mathcal{P}(\Omega)$. We note that if $g \in \mathcal{X}(\Omega)$, then

$$\phi_j(\sqrt{\mathcal{H}})g \in \mathcal{Z}(\Omega) \qquad \text{for any } j \in \mathbb{Z}. \tag{3.24}$$

In fact, fixing $j \in \mathbb{N}$, we note that

$$\phi_k(\sqrt{\mathcal{H}})\phi_j(\sqrt{\mathcal{H}})f \neq 0$$

only if $k = j-1, j, j+1$. Then, by using Proposition 3.3, we deduce that for any $M \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{k \le 0} 2^{-Mk} \|\phi_k(\sqrt{\mathcal{H}})\phi_j(\sqrt{\mathcal{H}})g\|_{L^1(\Omega)} \\
\le\, & 2^{-M(j-1)}\|\phi_{j-1}(\sqrt{\mathcal{H}})\phi_j(\sqrt{\mathcal{H}})g\|_{L^1(\Omega)} + 2^{-Mj}\|\phi_j(\sqrt{\mathcal{H}})\phi_j(\sqrt{\mathcal{H}})g\|_{L^1(\Omega)} \\
&\qquad + 2^{-M(j+1)}\|\phi_{j+1}(\sqrt{\mathcal{H}})\phi_j(\sqrt{\mathcal{H}})g\|_{L^1(\Omega)} \\
\le\, & C2^{-Mj}\|\phi_j(\sqrt{\mathcal{H}})g\|_{L^1(\Omega)} \\
\le\, & C2^{-Mj}\|g\|_{L^1(\Omega)},
\end{aligned}$$

which implies (3.24). Since $f \in \mathcal{P}(\Omega)$, it follows that

$$_{\mathcal{X}'(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}})f, g\rangle_{\mathcal{X}(\Omega)} = {}_{\mathcal{Z}'(\Omega)}\langle f, \phi_j(\sqrt{\mathcal{H}})g\rangle_{\mathcal{Z}(\Omega)} = 0$$

for any $j \in \mathbb{Z}$ and $g \in \mathcal{X}(\Omega)$, which implies (b). Conversely, let us suppose that f satisfies (b). Since $\mathcal{Z}(\Omega) \subset \mathcal{X}(\Omega)$, it follows that

$$_{\mathcal{Z}'(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}})f, g\rangle_{\mathcal{Z}(\Omega)} = {}_{\mathcal{X}'(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}})f, g\rangle_{\mathcal{X}(\Omega)} = 0 \tag{3.25}$$

for any $j \in \mathbb{Z}$ and $g \in \mathcal{Z}(\Omega)$. Here we recall the assertion (ii) in Lemma 3.1 that

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}})f \quad \text{in } \mathcal{Z}'(\Omega).$$

Then, by using this identity and (3.25), we have

$${}_{\mathcal{Z}'(\Omega)}\langle f, g\rangle_{\mathcal{Z}(\Omega)} = \sum_{j=-\infty}^{\infty} {}_{\mathcal{Z}'(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}})f, g\rangle_{\mathcal{Z}(\Omega)} = 0$$

for any $g \in \mathcal{Z}(\Omega)$, which implies (a). Thus we conclude the assertion (i).

Next we prove the assertion (ii). Let $f \in \mathcal{P}(\Omega)$. It follows from (3.1) in Lemma 3.1 that

$$f = \psi(\mathcal{H})f + \sum_{j=1}^{\infty} \phi_j(\sqrt{\mathcal{H}})f \quad \text{in } \mathcal{X}'(\Omega).$$

Applying (b) in the assertion (i) to the second term in the right member, we get

$$f = \psi(\mathcal{H})f \quad \text{in } \mathcal{X}'(\Omega). \tag{3.26}$$

Since $\psi(\mathcal{H})f \in L^\infty(\Omega)$ by the assertion (i) in Lemma 3.1, we conclude that $f \in L^\infty(\Omega)$. Therefore, the assertion (ii) is proved.

Finally we show the assertion (iii). Let $f \in \mathcal{P}(\Omega)$. Then, again by using the argument in (3.26), we see that

$$f = \psi(2^{-2k}\mathcal{H})f + \sum_{j=k}^{\infty} \phi_j(\sqrt{\mathcal{H}})f = \psi(2^{-2k}\mathcal{H})f \quad \text{in } \mathcal{X}'(\Omega) \tag{3.27}$$

for any $k \in \mathbb{Z}$. Since the gradient estimate (2.1) holds for any $t > 0$, applying (3.10) from Lemma 3.6 to the last member in (3.27), we get

$$\begin{aligned} \|\nabla f\|_{L^\infty(\Omega)} =& \|\nabla \psi(2^{-2k}\mathcal{H})f\|_{L^\infty(\Omega)} \\ \le& C2^k \|f\|_{L^\infty(\Omega)} \end{aligned}$$

for any $k \in \mathbb{Z}$, which implies that $\nabla f = 0$ in Ω. Since Ω is connected, f is a constant in Ω. Summarizing the above argument, we deduce that

$$\{0\} \subset \mathcal{P}(\Omega) \subset \{f = c \text{ on } \Omega \,|\, c \in \mathbb{C}\}.$$

Finally, we prove that if $\mathcal{P}(\Omega) \neq \{0\}$, then

$$\mathcal{P}(\Omega) = \{f = c \text{ on } \Omega \,|\, c \in \mathbb{C}\}. \tag{3.28}$$

In fact, we suppose that there exists a constant $c \neq 0$ such that $c \in \mathcal{P}(\Omega)$. Then $\alpha c \in \mathcal{P}(\Omega)$ for any $\alpha \neq 0$, since $\mathcal{P}(\Omega)$ is a linear space. Hence we must have (3.28). This proves (iii). The proof of Lemma 3.8 is finished. $\square$

3.5. **A lemma on convergence in Besov spaces.** In this subsection we shall prove a lemma in Besov spaces, which is useful in the proof of the theorem.

Lemma 3.9. *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Assume that $\{f_N\}_{N\in\mathbb{N}}$ is a bounded sequence in $\dot{B}^s_{p,q}(\mathcal{H})$, and that there exists an $f \in \mathcal{X}'(\Omega)$ such that*

$$f_N \to f \quad \text{in } \mathcal{X}'(\Omega) \text{ as } N \to \infty. \tag{3.29}$$

Then $f \in \dot{B}^s_{p,q}(\mathcal{H})$ and

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{H})} \le \liminf_{N\to\infty} \|f_N\|_{\dot{B}^s_{p,q}(\mathcal{H})}. \tag{3.30}$$

Before going to the proof, let us give a remark on the idea of proof of the lemma. When $1 < p, q < \infty$, $\dot{B}^s_{p,q}(\mathcal{H})$ are the reflexive Banach spaces for any $s \in \mathbb{R}$. This fact and the limiting properties of the weak convergence imply the inequality (3.30). Otherwise, we need the pointwise convergence of $\phi_j(\sqrt{\mathcal{H}})f_N$, which is obtained directly with a property of the kernel $\phi(\mathcal{H})(x,y)$ of the operator $\phi(\mathcal{H})$. Let us investigate the property of the kernel.

Lemma 3.10. *Let $\phi \in \mathscr{S}(\mathbb{R})$. Then*

$$\phi(\mathcal{H})(x,\cdot) \in \mathcal{X}(\Omega) \quad \text{for each } x \in \Omega. \tag{3.31}$$

Proof. Since

$$\|\phi(\mathcal{H})\|_{L^p(\Omega)\to L^\infty(\Omega)} < \infty$$

for any $1 \le p \le \infty$ by Proposition 3.3, it follows from Lemma B.1 in appendix B that

$$\|\phi(\mathcal{H})(\cdot,\cdot)\|_{L^\infty(\Omega;L^{p'}(\Omega))} = \|\phi(\mathcal{H})\|_{L^p(\Omega)\to L^\infty(\Omega)}$$

for any $1 \le p \le \infty$, where p' is the conjugate exponent of p, and we put

$$\|\phi(\mathcal{H})(\cdot,\cdot)\|_{L^\infty(\Omega;L^{p'}(\Omega))} := \sup_{x\in\Omega} \|\phi(\mathcal{H})(x,\cdot)\|_{L^{p'}(\Omega)}.$$

Hence we have

$$\phi(\mathcal{H})(x,\cdot) \in L^{p'}(\Omega) \quad \text{for each } x \in \Omega, \tag{3.32}$$

where $1 \le p' \le 2$. Let $M \in \mathbb{N}$. We denote by $K_{\mathcal{H}^M\phi(\mathcal{H})}(x,y)$ the kernel of $\mathcal{H}^M\phi(\mathcal{H})$. Then, for any $f \in \mathcal{X}(\Omega)$, we have

$$\begin{aligned}
{}_{\mathcal{X}'(\Omega)}\langle \mathcal{H}^M(\phi(\mathcal{H})(x,\cdot)), f\rangle_{\mathcal{X}(\Omega)} &= {}_{\mathcal{X}'(\Omega)}\langle \phi(\mathcal{H})(x,\cdot), \mathcal{H}^M f\rangle_{\mathcal{X}(\Omega)} \\
&= \phi(\mathcal{H})\mathcal{H}^M\overline{f}(x) \\
&= \mathcal{H}^M\phi(\mathcal{H})\overline{f}(x) \\
&= {}_{\mathcal{X}'(\Omega)}\langle K_{\mathcal{H}^M\phi(\mathcal{H})}(x,\cdot), f\rangle_{\mathcal{X}(\Omega)}
\end{aligned}$$

for any $x \in \Omega$, which implies that

$$\mathcal{H}^M(\phi(\mathcal{H})(x,\cdot))(y) = K_{\mathcal{H}^M\phi(\mathcal{H})}(x,y) \quad \text{a.e. } y \in \Omega$$

for any $x \in \Omega$. Since

$$\lambda^M\phi(\lambda) \in \mathscr{S}(\mathbb{R})$$

for any $M \in \mathbb{N}$, it follows from (3.32) that

$$K_{\mathcal{H}^M\phi(\mathcal{H})}(x,\cdot) \in L^1(\Omega) \cap L^2(\Omega)$$

for any $M \in \mathbb{N}$ and $x \in \Omega$. Hence we obtain

$$\mathcal{H}^M(\phi(\mathcal{H})(x,\cdot)) \in L^1(\Omega) \cap L^2(\Omega)$$

for any $M \in \mathbb{N}$ and $x \in \Omega$. Thus we conclude (3.31). The proof of Lemma 3.10 is finished. □

We are in a position to prove Lemma 3.9.

Proof of Lemma 3.9. First we show that

$$\phi_j(\sqrt{\mathcal{H}})f_N(x) \to \phi_j(\sqrt{\mathcal{H}})f(x) \quad \text{a.e.}\, x \in \Omega \text{ as } N \to \infty \tag{3.33}$$

for each $j \in \mathbb{Z}$. Put

$$\Phi_j = \phi_{j-1} + \phi_j + \phi_{j+1}$$

for $j \in \mathbb{Z}$. Then, noting from the assertion (i) in Lemma 3.1 that

$$\Phi_j(\sqrt{\mathcal{H}})f_N \in L^\infty(\Omega),$$

and from Lemma 3.10 that

$$\phi_j(\sqrt{\mathcal{H}})(x,\cdot) \in \mathcal{X}(\Omega) \quad \text{for each } x \in \Omega,$$

we write

$$\begin{aligned} \phi_j(\sqrt{\mathcal{H}})f_N(x) &= \phi_j(\sqrt{\mathcal{H}})\Phi_j(\sqrt{\mathcal{H}})f_N(x) \\ &= {}_{\mathcal{X}'(\Omega)}\langle \Phi_j(\sqrt{\mathcal{H}})f_N, \phi_j(\sqrt{\mathcal{H}})(x,\cdot)\rangle_{\mathcal{X}(\Omega)} \end{aligned} \tag{3.34}$$

for each $j \in \mathbb{Z}$ and $x \in \Omega$. In a similar way, we have

$$\phi_j(\sqrt{\mathcal{H}})f(x) = {}_{\mathcal{X}'(\Omega)}\langle \Phi_j(\sqrt{\mathcal{H}})f, \phi_j(\sqrt{\mathcal{H}})(x,\cdot)\rangle_{\mathcal{X}(\Omega)} \tag{3.35}$$

for each $j \in \mathbb{Z}$ and $x \in \Omega$. Since

$$\Phi_j(\sqrt{\mathcal{H}})f_N \to \Phi_j(\sqrt{\mathcal{H}})f \quad \text{in } \mathcal{X}'(\Omega) \text{ as } N \to \infty$$

for each $j \in \mathbb{Z}$ by assumption (3.29) and the continuity of $\Phi_j(\sqrt{\mathcal{H}})$ from $\mathcal{X}'(\Omega)$ into itself, we deduce that

$${}_{\mathcal{X}'(\Omega)}\langle \Phi_j(\sqrt{\mathcal{H}})f_N, \phi_j(\sqrt{\mathcal{H}})(x,\cdot)\rangle_{\mathcal{X}(\Omega)} \to {}_{\mathcal{X}'(\Omega)}\langle \Phi_j(\sqrt{\mathcal{H}})f, \phi_j(\sqrt{\mathcal{H}})(x,\cdot)\rangle_{\mathcal{X}(\Omega)} \tag{3.36}$$

for each $j \in \mathbb{Z}$ and $x \in \Omega$ as $N \to \infty$. Hence, combining (3.34)–(3.36), we get the pointwise convergence (3.33).

Let us turn to the proof of the estimate (3.30). To begin with, given $1 \le p \le \infty$, we claim that

$$\|\phi_j(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \le \liminf_{N\to\infty} \|\phi_j(\sqrt{\mathcal{H}})f_N\|_{L^p(\Omega)} \tag{3.37}$$

for each $j \in \mathbb{Z}$. When $1 \le p < \infty$, the inequality (3.37) is a consequence of (3.33) and Fatou's lemma. We have to prove the case when $p = \infty$. In this case, thanks to (3.33), the inequality (3.37) is true for $p = \infty$, since $\{\phi_j(\sqrt{\mathcal{H}})f_N\}_{N\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\Omega)$. Finally, multiplying by 2^{sj} to the both sides of (3.37), we conclude the required inequality (3.30). The proof of Lemma 3.9 is finished. □

4. Proof of Theorem 2.1

In this section we prove Theorem 2.1. In the inhomogeneous case we write

$$f = \psi(\mathcal{H})f + \sum_{k=1}^{\infty} \phi_k(\sqrt{\mathcal{H}})f \quad \text{in } \mathcal{X}'(\Omega),$$

and in the homogeneous case we write

$$f = \sum_{k=-\infty}^{\infty} \phi_k(\sqrt{\mathcal{H}})f \quad \text{in } \mathcal{Z}'(\Omega).$$

Hence it is sufficient to prove the homogeneous case (ii), since one can reduce the argument of the proof of (i) to that of (ii). Therefore, we shall concentrate on proving (ii).

Hereafter, for the sake of simplicity, we use the following notations:

$$f_j := \phi_j(\sqrt{\mathcal{H}})f, \quad S_j(f) := \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}})f, \quad j \in \mathbb{Z}.$$

We divide the proof into two cases:

$$1 \le p_2, p_3 < \infty, \text{ and } p_2 = \infty \text{ or } p_3 = \infty,$$

since the approximation by the Littlewood Paley partition of unity is available only for $p_2, p_3 < \infty$ (see (3.2)) and a constant function in $\mathcal{P}(\Omega)$ defined by (3.22) appears in the case when $p_2 = \infty$ or $p_3 = \infty$.

The case: $1 \le p_2, p_3 < \infty$**.** Let $f \in \dot{B}^s_{p_1,q}(\mathcal{H}) \cap L^{p_3}(\Omega)$ and $g \in \dot{B}^s_{p_4,q}(\mathcal{H}) \cap L^{p_2}(\Omega)$. Then we write

$$fg = \sum_{k\in\mathbb{Z}} f_k S_{k-3}(g) + \sum_{l\in\mathbb{Z}} S_{l-3}(f) g_l + \sum_{|k-l|\le 2} f_k g_l \quad \text{in } \mathcal{X}'(\Omega), \tag{4.1}$$

which is assured by the assertion (ii) in Lemma 3.2, since $p_2, p_3 < \infty$. By using Minkowski's inequality, we write

$$\|fg\|_{\dot{B}^s_{p,q}(\mathcal{H})} \le \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI},$$

where we put

$$\mathrm{I} := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \sum_{|k-j|\le 2} \left\| \phi_j(\sqrt{\mathcal{H}}) \Big(f_k S_{k-3}(g) \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$\mathrm{II} := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \sum_{|k-j|> 2} \left\| \phi_j(\sqrt{\mathcal{H}}) \Big(f_k S_{k-3}(g) \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$\mathrm{III} := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \sum_{|l-j|\le 2} \left\| \phi_j(\sqrt{\mathcal{H}}) \Big(S_{l-3}(f) g_l \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$\mathrm{IV} := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \sum_{|l-j|>2} \left\| \phi_j(\sqrt{\mathcal{H}}) \Big(S_{l-3}(f) g_l \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$\mathrm{V} := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \sum_{\substack{j-2\le k \\ \text{or} \\ j-2\le l}} \left\| \phi_j(\sqrt{\mathcal{H}}) \Big(\sum_{|k-l|\le 2} f_k g_l \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$\mathrm{VI} := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj} \sum_{\substack{j-2> k \\ \text{and} \\ j-2> l}} \left\| \phi_j(\sqrt{\mathcal{H}}) \Big(\sum_{|k-l|\le 2} f_k g_l \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}}.$$

We note that when $\Omega = \mathbb{R}^n$, the terms II, IV and VI vanishes. For example, the integrands in II satisfy

$$\phi_j(\sqrt{\mathcal{H}})\Big(f_k S_{k-3}(\sqrt{\mathcal{H}})g\Big) = \mathscr{F}^{-1}\Big[\phi_j(|\xi|)\Big\{\big(\phi_k(|\xi|)\mathscr{F}f\big) * \big(S_{k-3}(|\xi|)\mathscr{F}g\big)\Big\}\Big] = 0$$

for $|j-k|>2$, where $\mathscr{F}$ is the Fourier transform on $\mathbb{R}^n$ and

$$S_{k-3}(\sqrt{\mathcal{H}}) = \sum_{l=-\infty}^{k-3} \phi_l(\sqrt{\mathcal{H}}).$$

However, when $\Omega \neq \mathbb{R}^n$, the integrands do not vanish in general. On the other hand, the gradient estimates in Lemma 3.6 work well for getting the required estimates for the terms II, IV, VI.

Thus we estimate separately as follows:

Case A: Estimates for I, III and V   and   Case B: Estimates for II, IV and VI.

Case A: Estimates for I, III **and** V**.** These terms can be estimated in the same way as in the case when $\Omega = \mathbb{R}^n$. Since similar arguments also appear for II, IV and VI, we give the proof in a self-contained way. First we estimate the term I. Noting from the assertion (ii) in Lemma 3.5 that $f_k \in L^{p_1}(\Omega)$ and $S_{k-3}(g) \in L^{p_2}(\Omega)$ for each $k \in \mathbb{Z}$, we deduce from the estimate (3.8) in Lemma 3.5, Hölder's inequality and the estimate (3.9) for $\alpha = 0$ in Lemma 3.5 that

$$\begin{aligned} \left\| \phi_j(\sqrt{\mathcal{H}}) \Big(f_k S_{k-3}(g) \Big) \right\|_{L^p(\Omega)} &\le C \big\| f_k S_{k-3}(g) \big\|_{L^p(\Omega)} \\ &\le C \| f_k \|_{L^{p_1}(\Omega)} \| S_{k-3}(g) \|_{L^{p_2}(\Omega)} \\ &\le C \| f_k \|_{L^{p_1}(\Omega)} \| g \|_{L^{p_2}(\Omega)}, \end{aligned}$$

since $1/p = 1/p_1 + 1/p_2$. Thus we conclude from the above estimate and Minkowski's inequality that

$$\begin{aligned}
\mathrm{I} &\le C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{|k-j|\le 2}\|f_k\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&= C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{|k'|\le 2}2^{-sk'}\cdot 2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&\le C\sum_{|k'|\le 2}2^{-sk'}\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&\le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})}\|g\|_{L^{p_2}(\Omega)}.
\end{aligned}$$

As to the term III, interchanging the role of f and g in the above argument, we get

$$\mathrm{III} \le C\|f\|_{L^{p_3}(\Omega)}\|g\|_{\dot{B}^s_{p_4,q}(\mathcal{H})},$$

where $1/p = 1/p_3 + 1/p_4$.

As to the term V, we estimate the case when $j-2\le k$. Applying the estimate (3.8), and using Hölder's inequality, we estimate

$$\begin{aligned}
&\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{j-2\le k}\Big\|\phi_j(\sqrt{\mathcal{H}})\Big(\sum_{l=k-2}^{k+2}f_kg_l\Big)\Big\|_{L^p(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{j-2\le k}\|f_k\|_{L^{p_1}(\Omega)}\Big(\sum_{l=k-2}^{k+2}\|g_l\|_{L^{p_2}(\Omega)}\Big)\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{j-2\le k}\|f_k\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}.
\end{aligned}$$

Here, by applying Minkowski's inequality to the right member in the above inequality, we find that

$$\begin{aligned}
\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{j-2\le k}\|f_k\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}} &= \Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\ge -2}2^{-sk'}\cdot 2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\sum_{k'=-2}^{\infty}2^{-sk'}\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})},
\end{aligned}$$

since $s>0$. Hence, combining the above two estimates, we deduce that

$$\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{sj}\sum_{j-2\le k}\Big\|\phi_j(\sqrt{\mathcal{H}})\Big(\sum_{|k-l|\le 2}f_kg_l\Big)\Big\|_{L^p(\Omega)}\Big)^q\Big\}^{\frac{1}{q}} \le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})}\|g\|_{L^{p_2}(\Omega)}.$$

In a similar way, we can proceed the above argument in the case when $j-2 \le l$; thus we conclude that

$$\mathrm{V} \le C\big(\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})}\|g\|_{L^{p_2}(\Omega)} + \|f\|_{L^{p_3}(\Omega)}\|g\|_{\dot{B}^s_{p_4,q}(\mathcal{H})}\big).$$

Case B: Estimates for II, IV **and** VI. First let us estimate the term II. When $k-j>2$, we deduce from the same argument as in I that

$$\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\sum_{k-j>2}\Big\|\phi_j(\sqrt{\mathcal{H}})\Big(f_k S_{k-3}(g)\Big)\Big\|_{L^p(\Omega)}\right)^q\right\}^{\frac{1}{q}} \le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})}\|g\|_{L^{p_2}(\Omega)}.$$

Hence all we have to do is to prove the case when $k-j<-2$, i.e.,

$$\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\sum_{k-j<-2}\Big\|\phi_j(\sqrt{\mathcal{H}})\Big(f_k S_{k-3}(g)\Big)\Big\|_{L^p(\Omega)}\right)^q\right\}^{\frac{1}{q}} \le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})}\|g\|_{L^{p_2}(\Omega)}. \quad (4.2)$$

In fact, noting from Lemma 3.1 that

$$f_k, S_{k-3}(g) \in L^\infty(\Omega),$$

and from (1.6) that

$$L^\infty(\Omega) \hookrightarrow \mathcal{X}'(\Omega),$$

we have

$$f_k S_{k-3}(g) \in \mathcal{X}'(\Omega).$$

Then we write

$$\phi_j(\sqrt{\mathcal{H}})\big(f_k S_{k-3}(g)\big) = \mathcal{H}^{-1}\phi_j(\sqrt{\mathcal{H}})\mathcal{H}\big(f_k S_{k-3}(g)\big) \quad \text{in } \mathcal{X}'(\Omega). \quad (4.3)$$

Here it should be noted that the operator $\mathcal{H}^{-1}$ in (4.3) is well-defined, since

$$\mathcal{H}^{-1}\phi_j(\sqrt{\mathcal{H}})h \in \mathcal{X}'(\Omega)$$

for any $h \in \mathcal{X}'(\Omega)$. Hence, applying the Leibniz rule in Lemma 3.7 to the identities (4.3), we have:

$$\begin{aligned}&\phi_j(\sqrt{\mathcal{H}})\big(f_k S_{k-3}(g)\big) &(4.4)\\ =&\mathcal{H}^{-1}\phi_j(\sqrt{\mathcal{H}})\Big\{(\mathcal{H}f_k)S_{k-3}(g) - 2\nabla f_k\cdot\nabla S_{k-3}(g) + f_k\big(\mathcal{H}S_{k-3}(g)\big)\Big\} \quad \text{in } \mathcal{X}'(\Omega).\end{aligned}$$

Thanks to estimates (3.8) and (3.9) from Lemma 3.5, the first term in the right member in (4.4) is estimated as

$$\begin{aligned}\Big\|\mathcal{H}^{-1}\phi_j(\sqrt{\mathcal{H}})\Big\{(\mathcal{H}f_k)S_{k-3}(g)\Big\}\Big\|_{L^p(\Omega)} &\le C2^{-2j}\big\|(\mathcal{H}f_k)S_{k-3}(g)\big\|_{L^p(\Omega)}\\ &\le C2^{-2j}\|\mathcal{H}f_k\|_{L^{p_1}(\Omega)}\|S_{k-3}(g)\|_{L^{p_2}(\Omega)}\\ &\le C2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)}.\end{aligned}$$

In a similar way, we estimate the third term as

$$\Big\|\mathcal{H}^{-1}\phi_j(\sqrt{\mathcal{H}})\Big\{f_k\mathcal{H}S_{k-3}(g)\Big\}\Big\|_{L^p(\Omega)} \le C2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)}.$$

As to the second, thanks to (3.11) and (3.12) from Lemma 3.6, we estimate

$$\begin{aligned}
\Big\|\mathcal{H}^{-1}\phi_j(\sqrt{\mathcal{H}})\Big\{\nabla f_k\cdot\nabla S_{k-3}(g)\Big\}\Big\|_{L^p(\Omega)} &\le C2^{-2j}\big\|\nabla f_k\cdot\nabla S_{k-3}(g)\big\|_{L^p(\Omega)}\\
&\le C2^{-2j}\|\nabla f_k\|_{L^{p_1}(\Omega)}\|\nabla S_{k-3}(g)\|_{L^{p_2}(\Omega)}\\
&\le C2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)}.
\end{aligned}$$

Hence, combining the identity (4.4) with the above three estimates, we get

$$\Big\|\phi_j(\sqrt{\mathcal{H}})\Big(f_kS_{k-3}(g)\Big)\Big\|_{L^p(\Omega)} \le C2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)}$$

for any $j,k\in\mathbb{Z}$. Therefore, we conclude from this estimate that

$$\begin{aligned}
&\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\sum_{k-j<-2}\Big\|\phi_j(\sqrt{\mathcal{H}})\Big(f_kS_{k-3}(g)\Big)\Big\|_{L^p(\Omega)}\right)^q\right\}^{\frac{1}{q}}\\
&\le C\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\sum_{k-j<-2}2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\right)^q\right\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&= C\left\{\sum_{j\in\mathbb{Z}}\left(\sum_{k'<-2}2^{(2-s)k'}\cdot 2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\right)^q\right\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&\le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})}\|g\|_{L^{p_2}(\Omega)},
\end{aligned}$$

since $s<2$, which proves (4.2). Thus we conclude that

$$\mathrm{II}\le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})}\|g\|_{L^{p_2}(\Omega)}.$$

As to the term IV, interchanging the role of f and g in the above argument, we get

$$\mathrm{IV}\le C\|f\|_{L^{p_3}(\Omega)}\|g\|_{\dot{B}^s_{p_4,q}(\mathcal{H})}.$$

As to the term VI, we estimate in a similar way to II;

$$\begin{aligned}
\mathrm{VI} &\le \left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\sum_{j-2>k}\Big\|\phi_j(\sqrt{\mathcal{H}})\Big(\sum_{|k-l|\le 2}f_kg_l\Big)\Big\|_{L^p(\Omega)}\right)^q\right\}^{\frac{1}{q}}\\
&\le C\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\sum_{j-2>k}2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\right)^q\right\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&= C\left\{\sum_{j\in\mathbb{Z}}\left(\sum_{k'<-2}2^{(2-s)k'}\cdot 2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\right)^q\right\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&\le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H})}\|g\|_{L^{p_2}(\Omega)},
\end{aligned}$$

since $s<2$.

Summarizing cases A and B, we arrive at the required estimate (2.3). The proof of the case when $1\le p_2,p_3<\infty$ is finished.

It remains to prove the case when $p_2 = \infty$ or $p_3 = \infty$.

The case: $p_2 = \infty$ or $p_3 = \infty$. We may prove only the case when $p_2 = p_3 = \infty$, since the other cases are proved in a similar way. In this case, we note that

$$p_1 = p_4 = p.$$

Let $f, g \in \dot{B}^s_{p,q}(\mathcal{H}) \cap L^\infty(\Omega)$. Then it follows from Lemma 3.5 that

$$\Big\| \sum_{j=k}^{\infty} f_j \Big\|_{L^\infty(\Omega)} \le C \|f\|_{L^\infty(\Omega)} \tag{4.5}$$

for any $k \in \mathbb{Z}$. Hence there exist a subsequence

$$\Big\{ \sum_{j=k_l}^{\infty} f_j \Big\}_{l \in \mathbb{N}}$$

and a function $F \in L^\infty(\Omega)$ such that

$$\sum_{j=k_l}^{\infty} f_j \rightharpoonup F \quad \text{weakly* in } L^\infty(\Omega) \tag{4.6}$$

as $l \to \infty$, which also yields the convergence in $\mathcal{X}'(\Omega)$ and $\mathcal{Z}'(\Omega)$ by the embedding

$$L^\infty(\Omega) \hookrightarrow \mathcal{X}'(\Omega) \hookrightarrow \mathcal{Z}'(\Omega).$$

On the other hand, it follows from Lemma 3.1 that

$$\sum_{j=k_l}^{\infty} f_j \to f \quad \text{in } \mathcal{Z}'(\Omega)$$

as $l \to \infty$. Hence we see that $F = f$ in $\mathcal{Z}'(\Omega)$, which implies that

$$P_f := f - F \in \mathcal{P}(\Omega),$$

since $f \in L^\infty(\Omega)$. Therefore we conclude from (4.6) that

$$\sum_{j=k_l}^{\infty} f_j \rightharpoonup f - P_f \quad \text{weakly* in } L^\infty(\Omega) \tag{4.7}$$

as $l \to \infty$. In a similar way, there exist a subsequence

$$\Big\{ \sum_{j=k_{l'}}^{\infty} g_j \Big\}_{l' \in \mathbb{N}}$$

and $P_g \in \mathcal{P}(\Omega)$ such that

$$\sum_{j=k_{l'}}^{\infty} g_j \rightharpoonup g - P_g \quad \text{weakly* in } L^\infty(\Omega) \tag{4.8}$$

as $l' \to \infty$. Hence, by (4.7) and (4.8), there exists a subsequence $\{l'(l)\}_{l=1}^\infty$ of $\{l'\}_{l'=1}^\infty$ such that

$$\Big(\sum_{j=k_l}^{\infty} f_j \Big) \Big(\sum_{j=k_{l'(l)}}^{\infty} g_j \Big) \rightharpoonup (f - P_f)(g - P_g) \quad \text{weakly* in } L^\infty(\Omega)$$

as $l \to \infty$. Hence we have

$$\Big(\sum_{j=k_l}^{\infty} f_j\Big)\Big(\sum_{j=k_{l'(l)}}^{\infty} g_j\Big) \to (f-P_f)(g-P_g) \quad \text{in } \mathcal{X}'(\Omega) \tag{4.9}$$

as $l \to \infty$, since $L^\infty(\Omega) \hookrightarrow \mathcal{X}'(\Omega)$. Now, the estimate of $\dot{B}^s_{p,q}$-norm of the left member in (4.9) is obtained by the argument as in the previous case $1 \le p_2, p_3 < \infty$. Hence, there exists a constant $C > 0$ such that

$$\Big\|\Big(\sum_{j=k_l}^{\infty} f_j\Big)\Big(\sum_{j=k_{l'(l)}}^{\infty} g_j\Big)\Big\|_{\dot{B}^s_{p,q}(\mathcal{H})} \le C\Big(\|f\|_{\dot{B}^s_{p,q}(\mathcal{H})}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{\dot{B}^s_{p,q}(\mathcal{H})}\Big) \tag{4.10}$$

for any $l \in \mathbb{N}$. Here, we note that P_f and P_g are constants by the assertion (iii) from Lemma 3.8. As a consequence of (4.9) and (4.10), we conclude from Lemma 3.9 that

$$\begin{aligned}\|fg\|_{\dot{B}^s_{p,q}(\mathcal{H})} \le\ & \liminf_{l\to\infty}\Big\|\Big(\sum_{j=k_l}^{\infty} f_j\Big)\Big(\sum_{j=k_{l'(l)}}^{\infty} g_j\Big)\Big\|_{\dot{B}^s_{p,q}(\mathcal{H})} \\ &+ \|fP_g\|_{\dot{B}^s_{p,q}(\mathcal{H})} + \|P_f g\|_{\dot{B}^s_{p,q}(\mathcal{H})} + \|P_f P_g\|_{\dot{B}^s_{p,q}(\mathcal{H})} \\ \le\ & C\Big(\|f\|_{\dot{B}^s_{p,q}(\mathcal{H})}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{\dot{B}^s_{p,q}(\mathcal{H})}\Big) \\ &+ \|f\|_{\dot{B}^s_{p,q}(\mathcal{H})}|P_g| + |P_f|\|g\|_{\dot{B}^s_{p,q}(\mathcal{H})} + \|P_f P_g\|_{\dot{B}^s_{p,q}(\mathcal{H})}.\end{aligned}$$

Here, combining part (c) in (i) and the assertion (iii) from Lemma 3.8, we deduce that

$$\|P_f P_g\|_{\dot{B}^s_{p,q}(\mathcal{H})} = 0.$$

Hence, all we have to do is to prove that

$$|P_f| \le C\|f\|_{L^\infty(\Omega)}, \tag{4.11}$$

$$|P_g| \le C\|g\|_{L^\infty(\Omega)}. \tag{4.12}$$

Noting (4.7), we estimate, by using (4.5),

$$|P_f| \le \|f\|_{L^\infty(\Omega)} + \liminf_{l\to\infty}\Big\|\sum_{j=k_l}^{\infty} f_j\Big\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)}.$$

This proves (4.11). In a similar way, we get (4.12). The proof of Theorem 2.1 is finished.

5. The case of Schrödinger operators

In this section we shall derive the bilinear estimates in Besov spaces generated by the Schrödinger operator $\mathcal{H} + V$, which is obtained as a corollary of Theorem 2.1 and the isomorphism of Besov spaces generated by the Dirichlet Laplacian and Schrödinger operators (see Proposition 5.1 below).

To begin with, let us give definitions of the Schrödinger operator and function spaces generated by the Schrödinger operator along [7]. Let Ω be an open set in

$\mathbb{R}^n$, where $n \geq 1$. We denote by $\mathcal{H}_V$ the self-adjoint realization of $-\Delta + V$ with the domain

$$\mathcal{D}(\mathcal{H}_V) = \{ f \in H_0^1(\Omega) \,|\, \sqrt{V_+} f,\, \mathcal{H}_V f \in L^2(\Omega) \},$$

where $V = V(x)$ is a real-valued measurable function on Ω such that

$$V = V_+ - V_-, \quad V_\pm \geq 0, \quad V_+ \in L^1_{\mathrm{loc}}(\Omega) \text{ and } V_- \in K_n(\Omega). \tag{5.1}$$

Here, $V_- \in K_n(\Omega)$ if and only if

$$\begin{cases} \displaystyle\lim_{r\to 0} \sup_{x\in\Omega} \int_{\Omega\cap\{|x-y|<r\}} \frac{|V_-(y)|}{|x-y|^{n-2}}\, dy = 0 & \text{for } n \geq 3, \\ \displaystyle\lim_{r\to 0} \sup_{x\in\Omega} \int_{\Omega\cap\{|x-y|<r\}} \log(|x-y|^{-1})|V_-(y)|\, dy = 0 & \text{for } n = 2, \\ \displaystyle\sup_{x\in\Omega} \int_{\Omega\cap\{|x-y|<1\}} |V_-(y)|\, dy < \infty & \text{for } n = 1. \end{cases}$$

Then we define a linear topological space $\mathcal{X}_V(\Omega)$, its dual space $\mathcal{X}_V'(\Omega)$ and inhomogeneous Besov spaces $B^s_{p,q}(\mathcal{H}_V)$ in a similar way to definitions in §2. Furthermore, if we assume the additional condition that

$$\begin{cases} \displaystyle\sup_{x\in\Omega} \int_\Omega \frac{V_-(y)}{|x-y|^{n-2}}\, dy < \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}-1\right)} & \text{if } n \geq 3, \\ V_- = 0 & \text{if } n = 1, 2, \end{cases} \tag{5.2}$$

then we also define a linear topological space $\mathcal{Z}_V(\Omega)$, its dual space $\mathcal{Z}_V'(\Omega)$ and homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_V)$ in a similar way to definitions in §2.

By refining how to handle the low spectrum part in the proof of Proposition 3.5 in [7], we can consider potentials in a wider class for the inhomogeneous Besov spaces. We have the following.

Proposition 5.1. *Let Ω be an open set of $\mathbb{R}^n$, and let $1 \leq p, q \leq \infty$ and s be such that*

$$\begin{cases} -\min\left\{2, n\left(1 - \dfrac{1}{p}\right)\right\} < s < \min\left\{\dfrac{n}{p}, 2\right\} & \text{if } n \geq 3, \\ -2 + \dfrac{2}{p} < s < \dfrac{2}{p} & \text{if } n = 1, 2. \end{cases} \tag{5.3}$$

Then the following assertions hold:

(i) *Suppose that the potential V satisfies the assumption* (5.1) *and*

$$\begin{cases} V \in L^{\frac{n}{2},\infty}(\Omega) + L^\infty(\Omega) & \text{if } n \geq 3, \\ V \in K_n(\Omega) & \text{if } n = 1, 2, \end{cases}$$

where $L^{\frac{n}{2},\infty}(\Omega)$ is the Lorentz space. Then

$$B^s_{p,q}(\mathcal{H}_V) \cong B^s_{p,q}(\mathcal{H}). \tag{5.4}$$

(ii) *Let* $n \geq 2$. *Suppose that the potential* V *satisfies the assumption* (5.2) *and*

$$\begin{cases} V \in L^{\frac{n}{2},\infty}(\Omega) & \text{if } n \geq 3, \\ V \in L^1(\Omega) & \text{if } n = 2. \end{cases}$$

Then

$$\dot{B}^s_{p,q}(\mathcal{H}_V) \cong \dot{B}^s_{p,q}(\mathcal{H}). \tag{5.5}$$

Proof. The assertion (ii) is proved in Proposition 3.5 from [7]. We prove the assertion (i). For this purpose, we prove that for $p = 1$ if $n = 1, 2$ and for $1 \leq p < n/2$ if $n \geq 3$, there exists a constant $C > 0$ such that

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \leq C2^{-2(j-k)}\|f\|_{L^p(\Omega)}, \tag{5.6}$$

$$\|\phi_k(\sqrt{\mathcal{H}})\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \leq C2^{-2(k-j)}\|f\|_{L^p(\Omega)} \tag{5.7}$$

for any $j, k \in \mathbb{N}$ and $f \in L^p(\Omega)$, where we put

$$\Phi_j = \phi_{j-1} + \phi_j + \phi_{j+1}.$$

Once (5.6) and (5.7) are established, the required isomorphism (5.4) is proved in the completely same way as in the proof of Proposition 3.5 from [7].

We divide the proof into two cases: $n \geq 3$ and $n = 1, 2$.

The case $n \geq 3$. We write

$$V = V_1 + V_2, \quad V_1 \in L^{\frac{n}{2},\infty}(\Omega), \quad V_2 \in L^\infty(\Omega).$$

Let $1 \leq p < n/2$ and $f \in L^p(\Omega) \cap L^2(\Omega)$. By the estimate (3.8) in Lemma 3.5, we have

$$\begin{aligned} &\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \\ =& \|\phi_j(\sqrt{\mathcal{H}_V})\mathcal{H}_V^{-1}\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \\ \leq& C2^{-2j}\Big\{\|\mathcal{H}\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} + \|V_1\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} + \|V_2\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)}\Big\} \end{aligned}$$

for any $j, k \in \mathbb{N}$. As to the first term, we estimate, by using (3.8) from Lemma 3.5,

$$\|\mathcal{H}\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \leq C2^{2k}\|f\|_{L^p(\Omega)}$$

for any $k \in \mathbb{N}$. As to the third term, we see from Proposition 3.3 that

$$\begin{aligned} \|V_2\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} &\leq 2^{2k}\|V_2\|_{L^\infty(\Omega)}\|\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \\ &\leq C2^{2k}\|V_2\|_{L^\infty(\Omega)}\|f\|_{L^p(\Omega)} \end{aligned}$$

for any $k \in \mathbb{N}$. As to the second term, we use the following estimate: For any $1 \leq p < p_0 < \infty$ and $1 \leq q \leq \infty$, there exists a constant $C > 0$ such that

$$\|\phi_k(\sqrt{\mathcal{H}_V})f\|_{L^{p_0,q}(\Omega)} \leq C2^{n(\frac{1}{p}-\frac{1}{p_0})k}\|f\|_{L^p(\Omega)} \tag{5.8}$$

for any $k \in \mathbb{Z}$ and $f \in L^{p_0}(\Omega)$ (see Lemma 9.1 in [7]). Thanks to (5.8), we estimate

$$\begin{aligned} \|V_1\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} &\leq \|V_1\|_{L^{\frac{n}{2},\infty}(\Omega)}\|\Phi_k(\sqrt{\mathcal{H}})f\|_{L^{p_0,p}(\Omega)} \\ &\leq C2^{2k}\|V_1\|_{L^{\frac{n}{2},\infty}(\Omega)}\|f\|_{L^p(\Omega)} \end{aligned}$$

for any $k \in \mathbb{N}$, where p_0 is a real number with $1/p = 2/n + 1/p_0$. Hence, combining the estimates obtained now, we get

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}})f\|_{L^p(\Omega)} \le C2^{-2j}2^{2k}\|f\|_{L^p(\Omega)}$$

for any $j, k \in \mathbb{N}$. Therefore (5.6) is obtained by the density argument. In a similar way, we get (5.7).

The case $n = 1, 2$. Since $V \in K_n(\Omega)$, the infimum of the spectrum $\sigma(\mathcal{H}_V)$ of $\mathcal{H}_V$ is finite. First let us check that for sufficiently large $M > -\inf \sigma(\mathcal{H}_V)$ there exists a constant $C > 0$ such that

$$\|V(\mathcal{H}_V + M)^{-1}\|_{L^1(\Omega)\to L^1(\Omega)} \le C. \tag{5.9}$$

To prove (5.9), we need the following pointwise estimate for the kernel $e^{-t\mathcal{H}_V}(x, y)$ of the operator $e^{-t\mathcal{H}_V}$, which is established in Proposition 3.1 from [8]:

There exist two constants $C > 0$ and $\omega \ge -\inf \sigma(\mathcal{H}_V)$ such that

$$0 \le e^{-t\mathcal{H}_V}(x, y) \le Ce^{\omega t}e^{2t\Delta}(x, y) \quad \text{a.e.}\, x, y \in \Omega \tag{5.10}$$

for any $t > 0$, where $e^{t\Delta}(x, y)$ is the kernel of free heat semigroup $e^{t\Delta}$ on $L^2(\mathbb{R}^n)$. More precisely, we have

$$e^{t\Delta}(x, y) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}}.$$

Now, let $f \in L^1(\Omega) \cap L^2(\Omega)$. Taking M so that $M > \omega$, and using the following formula:

$$(\mathcal{H}_V + M)^{-1} = \int_0^\infty e^{-Mt}e^{-t\mathcal{H}_V}\, dt,$$

we see from (5.10) that

$$\begin{aligned} \left| \left[(\mathcal{H}_V + M)^{-1}f\right](x)\right| &\le \int_0^\infty e^{-Mt}\left|(e^{-t\mathcal{H}_V}f)(x)\right| dt \\ &\le C\int_0^\infty e^{-(M-\omega)t}(e^{2t\Delta}|\tilde{f}|)(x)\, dt \\ &= C\left[(-2\Delta + M - \omega)^{-1}|\tilde{f}|\right](x) \end{aligned} \tag{5.11}$$

for almost everywhere $x \in \Omega$, where $\tilde{f}$ is the zero extension of f to $\mathbb{R}^n$. Let $\tilde{V}$ be the zero extension of V to $\mathbb{R}^n$. Since $\tilde{V} \in K_n(\mathbb{R}^n)$, we deduce from Proposition A.2.3 in Simon [14] that

$$\left\|\tilde{V}(-2\Delta + M - \omega)^{-1}|\tilde{f}|\right\|_{L^1(\mathbb{R}^n)} \le C\|\tilde{f}\|_{L^1(\mathbb{R}^n)} = C\|f\|_{L^1(\Omega)} \tag{5.12}$$

Therefore, combining (5.11) and (5.12), we obtain

$$\|V(\mathcal{H}_V + M)^{-1}f\|_{L^1(\Omega)} \le C\|f\|_{L^1(\Omega)}.$$

Hence, (5.9) is proved by the density argument.

Let us turn to the proof of (5.6). We estimate, by using Proposition 3.3,

$$
\begin{aligned}
&\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}})f\|_{L^1(\Omega)}\\
=&\|\phi_j(\sqrt{\mathcal{H}_V})\mathcal{H}_V^{-1}\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}})f\|_{L^1(\Omega)}\\
\le& C2^{-2j}\Big\{\|\mathcal{H}\Phi_k(\sqrt{\mathcal{H}})f\|_{L^1(\Omega)}+\|V\Phi_k(\sqrt{\mathcal{H}})f\|_{L^1(\Omega)}\Big\}
\end{aligned}
\tag{5.13}
$$

for any $j \in \mathbb{N}$. Here, thanks to (3.8) from Lemma 3.5, the first term in the right member of (5.13) is dominated by $2^{2k}\|f\|_{L^1(\Omega)}$. As to the second, we estimate, by using (5.9),

$$
\begin{aligned}
\|V\Phi_k(\sqrt{\mathcal{H}})f\|_{L^1(\Omega)} &\le C\|V(\mathcal{H}+M)^{-1}(\mathcal{H}+M)\Phi_k(\sqrt{\mathcal{H}})f\|_{L^1(\Omega)}\\
&\le C\|(\mathcal{H}+M)\Phi_k(\sqrt{\mathcal{H}})f\|_{L^1(\Omega)}\\
&\le C2^{2k}\|f\|_{L^1(\Omega)}
\end{aligned}
$$

for any $k \in \mathbb{N}$. Combining these estimates obtained now, we get (5.6). In a similar way, we get (5.7). The proof of Proposition 5.1 is finished. □

Combining Theorem 2.1 with Proposition 5.1, we have the following:

Corollary 5.2. *Let p, p_1, p_2, p_3, p_4 and q be such that*

$$
1 \le p, p_1, p_2, p_3, p_4, q \le \infty \quad \text{and} \quad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4},
$$

and let s be such that

$$
0 < s < \min\left\{\frac{n}{p_1}, \frac{n}{p_4}, 2\right\} \quad \text{if } n \ge 3; \quad 0 < s < \min\left\{\frac{2}{p_1}, \frac{2}{p_4}\right\} \quad \text{if } n = 1, 2.
$$

Then, under the same assumption on V in Proposition 5.1, the assertions (i) *and* (ii) *in Theorem 2.1 hold for $B^s_{p,q}(\mathcal{H}_V)$ and $\dot{B}^s_{p,q}(\mathcal{H}_V)$, respectively.*

Appendix A. (High regularity case)

In this appendix we check that the bilinear estimates do not necessarily hold for some $s \ge 2$. Let us consider the bilinear estimate (2.3) in the case when

$$
p = 1, \quad p_1 = p_2 = p_3 = p_4 = q = 2 \quad \text{and} \quad f = g,
$$

namely,

$$
\|f^2\|_{\dot{B}^s_{1,2}(\mathcal{H})} \le C\|f\|_{\dot{B}^s_{2,2}(\mathcal{H})}\|f\|_{L^2(\Omega)} \tag{A.1}
$$

for any $f \in \dot{B}^s_{2,2}(\mathcal{H}) \cap L^2(\Omega)$. We note that the estimate (A.1) is already proved for any $s < 2$ on an arbitrary open set (see (III-a) in §2). We shall show that the estimate (A.1) does not hold for some $s \ge 2$.

Let $n \ge 3$ and Ω be an exterior domain in $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is compact and its boundary is of $C^{1,1}$. Then it is known that

$$
\|\nabla e^{-t\mathcal{H}}\|_{L^1(\Omega)\to L^\infty(\Omega)} \gtrsim t^{-\frac{n}{2}} \tag{A.2}
$$

for sufficiently large $t > 0$ (see Ishige and Kabeya [6], and also Zhang [18]). However we can claim the following:

Claim A.1. Let $\varepsilon > 0$. If the estimate (A.1) holds for any $s \in [2, n+2+\varepsilon]$, then there exists a constant $C > 0$ such that

$$\|\nabla e^{-t\mathcal{H}}\|_{L^1(\Omega)\to L^\infty(\Omega)} \le C t^{-\frac{n}{2}-\frac{1}{2}+\frac{\varepsilon}{4}} \tag{A.3}$$

for sufficiently large $t > 0$.

The estimate (A.3) contradicts (A.2) if we choose ε sufficiently small. Thus, if Claim A.1 is proved, then we conclude that when Ω is the exterior domain whose boundary is compact and of $C^{1,1}$, the bilinear estimate (2.3) does not hold for some $s \ge 2$. In the rest of this section, we prove Claim A.1.

Let $f \in L^1(\Omega)$. By the Leibniz rule, we have

$$\mathcal{H}(e^{-t\mathcal{H}}f)^2 = 2(\mathcal{H}e^{-t\mathcal{H}}f)(e^{-t\mathcal{H}}f) - 2|\nabla e^{-t\mathcal{H}}f|^2 \quad \text{in } \mathscr{D}'(\Omega),$$

and

$$\begin{aligned}\|\nabla e^{-t\mathcal{H}}f\|_{L^\infty(\Omega)}^2 &\le \|\mathcal{H}(e^{-t\mathcal{H}}f)^2\|_{L^\infty(\Omega)} + \|(\mathcal{H}e^{-t\mathcal{H}}f)(e^{-t\mathcal{H}}f)\|_{L^\infty(\Omega)} \\ &=: \mathrm{I} + \mathrm{II}.\end{aligned} \tag{A.4}$$

We readily see from Proposition 3.3 that

$$\begin{aligned}\mathrm{II} &\le \|\mathcal{H}e^{-t\mathcal{H}}f\|_{L^\infty(\Omega)}\|e^{-t\mathcal{H}}f\|_{L^\infty(\Omega)} \\ &\le C t^{-\frac{n}{2}-1}\|f\|_{L^1(\Omega)} \cdot t^{-\frac{n}{2}}\|f\|_{L^1(\Omega)} \\ &= C t^{-n-1}\|f\|_{L^1(\Omega)}^2.\end{aligned} \tag{A.5}$$

To estimate for I, we recall that

$$\phi_j = \Phi_j \phi_j, \tag{A.6}$$

where

$$\Phi_j = \phi_{j-1} + \phi_j + \phi_{j+1}.$$

Then, by using identities (A.6) and Lemma 3.5, we find that

$$\begin{aligned}\mathrm{I} &\le \sum_{j\in\mathbb{Z}} \|\phi_j(\sqrt{\mathcal{H}})\mathcal{H}(e^{-t\mathcal{H}}f)^2\|_{L^\infty(\Omega)} \\ &= \sum_{j\in\mathbb{Z}} \|\mathcal{H}\Phi_j(\sqrt{\mathcal{H}})\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^\infty(\Omega)} \\ &\le C\sum_{j\in\mathbb{Z}} 2^{2j}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^\infty(\Omega)}.\end{aligned}$$

Here, by using (A.6) and Proposition 3.3, we estimate

$$\begin{aligned}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^\infty(\Omega)} &= \|\Phi_j(\sqrt{\mathcal{H}})\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^\infty(\Omega)} \\ &\le C2^{nj}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^1(\Omega)}.\end{aligned}$$

Hence, combining these estimates obtained now, we get

$$\begin{aligned}\mathrm{I} &\le C\sum_{j\in\mathbb{Z}} 2^{(n+2)j}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^1(\Omega)} \\ &=: C(\mathrm{I}_1 + \mathrm{I}_2),\end{aligned}$$

where

$$\mathrm{I}_1 = \sum_{j\le 0} 2^{(n+2)j}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^1(\Omega)},$$

$$\mathrm{I}_2 = \sum_{j\ge 1} 2^{(n+2)j}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^1(\Omega)}.$$

Here, writing

$$\mathrm{I}_1 = \sum_{j\le 0} 2^{\varepsilon j}\cdot 2^{-\varepsilon j}\cdot 2^{(n+2)j}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^1(\Omega)},$$

$$\mathrm{I}_2 = \sum_{j\ge 1} 2^{-\varepsilon j}\cdot 2^{\varepsilon j}\cdot 2^{(n+2)j}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^1(\Omega)}$$

for any $\varepsilon > 0$, we estimate

$$\begin{aligned}\mathrm{I}_1 &\le \Big\{\sum_{j\le 0} 2^{2\varepsilon j}\Big\}^{\frac12}\Big\{\sum_{j\le 0}\big(2^{(n+2-\varepsilon)j}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^1(\Omega)}\big)^2\Big\}^{\frac12}\\ &\le C\|(e^{-t\mathcal{H}}f)^2\|_{\dot B^{n+2-\varepsilon}_{1,2}(\mathcal{H})},\\ \mathrm{I}_2 &\le \Big\{\sum_{j\ge 1} 2^{-2\varepsilon j}\Big\}^{\frac12}\Big\{\sum_{j\ge 1}\big(2^{(n+2+\varepsilon)j}\|\phi_j(\sqrt{\mathcal{H}})(e^{-t\mathcal{H}}f)^2\|_{L^1(\Omega)}\big)^2\Big\}^{\frac12}\\ &\le C\|(e^{-t\mathcal{H}}f)^2\|_{\dot B^{n+2+\varepsilon}_{1,2}(\mathcal{H})},\end{aligned}$$

respectively, which imply that

$$\mathrm{I} \le C\Big\{\|(e^{-t\mathcal{H}}f)^2\|_{\dot B^{n+2-\varepsilon}_{1,2}(\mathcal{H})} + \|(e^{-t\mathcal{H}}f)^2\|_{\dot B^{n+2+\varepsilon}_{1,2}(\mathcal{H})}\Big\} \tag{A.7}$$

for any $\varepsilon > 0$. Now, since $f \in L^1(\Omega)$, it follows from L^1-L^2-estimate for heat semigroup $e^{-t\mathcal{H}}$ that

$$e^{-t\mathcal{H}}f \in \dot B^s_{2,2}(\mathcal{H}) \cap L^2(\Omega) \quad \text{for any } s \in [0, n+2+\varepsilon] \text{ and } t > 0.$$

Hence, applying the assumption that (A.1) holds for any $s \in [2, n+2+\varepsilon]$, we deduce that

$$\|(e^{-t\mathcal{H}}f)^2\|_{\dot B^{n+2-\varepsilon}_{1,2}(\mathcal{H})} \le C\|e^{-t\mathcal{H}}f\|_{\dot B^{n+2-\varepsilon}_{2,2}(\mathcal{H})}\|e^{-t\mathcal{H}}f\|_{L^2(\Omega)}. \tag{A.8}$$

Since

$$\|g\|_{\dot B^s_{2,2}(\mathcal{H})} \simeq \|\mathcal{H}^{\frac{s}{2}}g\|_{L^2(\Omega)}, \quad g \in \dot B^s_{2,2}(\mathcal{H})$$

for any $s \in \mathbb{R}$, the first factor in the right member of (A.8) is estimated as

$$\begin{aligned}\|e^{-t\mathcal{H}}f\|_{\dot B^{n+2-\varepsilon}_{2,2}(\mathcal{H})} &\le C\|\mathcal{H}^{\frac{n}{2}+1-\frac{\varepsilon}{2}}e^{-t\mathcal{H}}f\|_{L^2(\Omega)}\\ &\le Ct^{-\frac{n}{2}-1+\frac{\varepsilon}{2}}\|e^{-\frac{t}{2}\mathcal{H}}f\|_{L^2(\Omega)}\\ &\le Ct^{-\frac{3n}{4}-1+\frac{\varepsilon}{2}}\|f\|_{L^1(\Omega)},\end{aligned}$$

where we used Proposition 3.3 in the second step, and L^1-L^2-estimate for heat semigroup $e^{-\frac{t}{2}\mathcal{H}}$ in the last step. Again, by L^1-L^2-estimate for heat semigroup $e^{-\frac{t}{2}\mathcal{H}}$, we have

$$\|e^{-t\mathcal{H}}f\|_{L^2(\Omega)} \le Ct^{-\frac{n}{4}}\|f\|_{L^1(\Omega)}.$$

Hence, combining all the estimates obtained now, we get

$$\|(e^{-t\mathcal{H}}f)^2\|_{\dot{B}^{\frac{n}{2}+2-\varepsilon}_{1,2}(\mathcal{H})} \le Ct^{-n-1+\frac{\varepsilon}{2}}\|f\|^2_{L^1(\Omega)}. \tag{A.9}$$

In a similar way, we have

$$\|(e^{-t\mathcal{H}}f)^2\|_{\dot{B}^{\frac{n}{2}+2+\varepsilon}_{1,2}(\mathcal{H})} \le Ct^{-n-1-\frac{\varepsilon}{2}}\|f\|^2_{L^1(\Omega)}. \tag{A.10}$$

Therefore, combining the estimates (A.7), (A.9) and (A.10), we conclude that

$$\mathrm{I} \le C\left(t^{-n-1+\frac{\varepsilon}{2}} + t^{-n-1-\frac{\varepsilon}{2}}\right)\|f\|^2_{L^1(\Omega)}. \tag{A.11}$$

Thus, combining (A.4), (A.5) and (A.11), we arrive at (A.3). Claim A.1 is proved.

Appendix B

In this appendix we prove the following.

Lemma B.1. *Let $1 \le p \le \infty$ and T be a bounded linear operator from $L^p(\Omega)$ to $L^\infty(\Omega)$, and $T(x,y)$ the kernel of T. Then*

$$\|T\|_{L^p(\Omega)\to L^\infty(\Omega)} = \|T(\cdot,\cdot)\|_{L^\infty(\Omega;L^{p'}(\Omega))}, \tag{B.1}$$

where p' is the conjugate exponent of p.

Proof. It follows from Hölder's inequality that

$$\|T\|_{L^p(\Omega)\to L^\infty(\Omega)} \le \|T(\cdot,\cdot)\|_{L^\infty(\Omega;L^{p'}(\Omega))} \tag{B.2}$$

for any $1 \le p \le \infty$. In fact, letting $f \in L^p(\Omega)$, we have

$$\begin{aligned}|Tf(x)| &= \Big|\int_\Omega T(x,y)f(y)\,dy\Big| \\ &\le \|T(x,\cdot)\|_{L^{p'}(\Omega)}\|f\|_{L^p(\Omega)}\end{aligned}$$

for any $x \in \Omega$. Hence we obtain

$$\|Tf\|_{L^\infty(\Omega)} \le \|T(\cdot,\cdot)\|_{L^\infty(\Omega;L^{p'}(\Omega))}\|f\|_{L^p(\Omega)},$$

which implies (B.2). Therefore it suffices to prove the converse:

$$\|T(\cdot,\cdot)\|_{L^\infty(\Omega;L^{p'}(\Omega))} \le \|T\|_{L^p(\Omega)\to L^\infty(\Omega)} \tag{B.3}$$

for any $1 \le p \le \infty$. When $1 \le p < \infty$, we estimate

$$\begin{aligned}\|T(x,\cdot)\|_{L^{p'}(\Omega)} &= \sup_{f\in L^p(\Omega),\, \|f\|_{L^p(\Omega)}=1}\Big|\int_\Omega T(x,y)f(y)\,dy\Big| \\ &= \sup_{f\in L^p(\Omega),\, \|f\|_{L^p(\Omega)}=1}\big|Tf(x)\big| \\ &\le \sup_{f\in L^p(\Omega),\, \|f\|_{L^p(\Omega)}=1}\|T\|_{L^p(\Omega)\to L^\infty(\Omega)}\|f\|_{L^p(\Omega)} \\ &\le \|T\|_{L^p(\Omega)\to L^\infty(\Omega)}\end{aligned}$$

for any $x \in \Omega$. This proves (B.3) for $1 \le p < \infty$. When $p = \infty$, fixing $x_0 \in \Omega$, we estimate

$$\begin{aligned}
\|T(x_0,\cdot)\|_{L^1(\Omega)} &= \int_\Omega |T(x_0,y)|\,dy \\
&= \int_\Omega T(x_0,y)e^{-i\arg\{T(x_0,y)\}}\,dy \\
&\le \sup_{x\in\Omega}\Big|\int_\Omega T(x,y)e^{-i\arg\{T(x_0,y)\}}\,dy\Big| \\
&= \sup_{x\in\Omega}\big|Te^{-i\arg\{T(x_0,\cdot)\}}(x)\big| \\
&\le \|T\|_{L^\infty(\Omega)\to L^\infty(\Omega)}\big\|e^{-i\arg\{T(x_0,\cdot)\}}\big\|_{L^\infty(\Omega)} \\
&= \|T\|_{L^\infty(\Omega)\to L^\infty(\Omega)}.
\end{aligned}$$

Thus (B.1) is proved for $p = \infty$. □

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