Boundedness of spectral multipliers for Schrödinger operators and its applications to Besov spaces

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Chapter 1

Introduction

The theory of function spaces and partial differential equations on Euclidian spaces or Lie groups has been developed on the basis of Fourier analysis. In particular, the Fourier multiplier defined via the Fourier transform is one of the powerful tools, and enables one to introduce the derivative of fractional order and solution operators of the Cauchy problem for partial differential equations. However, when one considers the Fourier multipliers on measure spaces not having the invariance properties of measures, these operators cannot be well-defined in general. For example, it is difficult to define the Fourier transform on non-smooth domains or domains with unbounded boundary. To overcome this difficulty, we arrived at the idea of the spectral multiplier which is a generalization of the Fourier multiplier. This thesis is concerned with boundedness of spectral multipliers for Schrödinger operators on open sets of Euclidian spaces, and its application to the theory of Besov spaces. This framework is the most general in the setting of Euclidian spaces. The motivation of the study of this thesis comes from obtaining several estimates for solutions to the initial-boundary value problem of partial differential equations on unbounded domains. On account of defining the Besov spaces on open sets, we can discuss the bilinear estimates on these spaces.

Since the 1970s, many authors have investigated the spectral multipliers for the Laplace operators acting on Lie groups of polynomial growth and for the Laplace-Beltrami operators on compact manifolds. In 1990 Hebisch proved boundedness of spectral multipliers for Schrödinger operators with positive potential on Euclidian spaces (see [33]). There are also several results on the Schrödinger operators with more general potentials. For example, Jensen and Nakamura dealt with potentials admitting negative part of Kato class on Euclidian spaces (see [44, 45], and also D'Ancona and Pierfelice [18] and Duong, Ouhabaz and Sikora [20]). Since the 1990s, the above results have been applied to the theory of function spaces (see [1, 9, 18, 28, 44, 88]), and there are a lot of literatures on Besov spaces. As is well known, the Besov spaces were introduced by Besov in around 1960 (see [2, 3]). These spaces play an important role in studying approximation and regularity of functions, and have various characterizations (see, e.g., Triebel [81, 82, 84]). Among

other things, the characterization by Peetre via the Fourier multipliers has many applications to partial differential equations on Euclidian spaces or Lie groups (see [62–64]). Recently, many authors have investigated Besov spaces on domains via the spectral approach instead of Fourier multipliers (see [6, 38, 49]). However, to the best of our knowledge, it is necessary to impose some smoothness assumptions on the domains in order to define the inhomogeneous and homogeneous Besov spaces with full range of indices.

The purpose in chapter 3 is to prove L^p-boundedness of spectral multipliers for Schrödinger operators with potentials of Kato class $K_d(\Omega)$ on an open set Ω. The Coulomb potential is a typical example of potentials of this class which is defined in section 2.2 of chapter 2. The advantage of introducing $K_d(\Omega)$ is twofold; we need not impose any assumption on decay and smoothness of potentials of $K_d(\Omega)$. Self-adjointness of Schrödinger operators is discussed in section 2.3. We need Gaussian upper bounds on Ω, which are proved in section 2.4. The results on spectral multipliers are described in section 3.1 of chapter 3. As a by-product, the result on gradient estimates for spectral multipliers are obtained.

The purpose in chapter 4 is to define the Besov spaces generated by the Schrödinger operators on open sets *without any geometrical and smoothness assumption on the boundary*, based on the spectral theory by referring to the idea of Peetre. In chapter 4 we give the definitions of Besov spaces and prove the fundamental properties such as completeness, duality, lifting properties, embedding relations and equivalence relations between the perturbed Besov spaces and the free ones. In the formulation we will face on the problem how to determine topological vector spaces over open sets corresponding to the Schwartz space and the Lizorkin test function space on $\mathbb{R}^d$. In section 4.1 we introduce new test function spaces on open sets and show their properties similar to the Schwartz space and the Lizorkin test function space. This is a main novelty in this thesis.

In chapter 5 we discuss bilinear estimates in Besov spaces. These estimates are also called the fractional Leibniz rule. The bilinear estimates in Sobolev spaces or Besov spaces are of great importance to study the well-posedness for the Cauchy problem to nonlinear partial differential equations such as the KdV equations and Navier-Stokes equations (see [13, 31, 32, 48]). These estimates for the Dirichlet Laplacian or more general operators are important to study the initial-boundary value problem of nonlinear partial differential equations. The purpose in this chapter is to prove the bilinear estimates in Besov spaces generated by the Dirichlet Laplacian on domains. More precisely, we reveal that these estimates hold for some small regularity number in Besov spaces, and as to the large regularity we present a counter-example. The gradient estimates for heat equation play an important role.

In chapter 6 we derive the gradient estimates for heat equation with the Dirichlet boundary condition in an exterior domain. These estimates are not only of interest itself, but also have some applications. As is mentioned above, these es-

timates are related to the bilinear estimates in Besov spaces generated by the Dirichlet Laplacian. It is well known that the gradient estimates hold for solutions to heat equation on $\mathbb{R}^d$, and that their time decay rate is $t^{-1/2}$. It is also true in half spaces $\mathbb{R}^d_+$ and bounded domains. However, it is not true in general in exterior domains. More precisely, the time decay rate is not necessarily the rate $t^{-1/2}$ in this case (see [37, 38, 51]). The purpose in this chapter is to reveal the sharp time decay rates in exterior domains.

Finally we consider the case of the Laplace operator with the Neumann boundary condition on a Lipschitz domain. In particular, we are interested in the case of bounded domains, since the situation is different from the case of Dirichlet boundary condition: Zero is not an eigenvalue of the Dirichlet Laplacian, but that of the Neumann Laplacian. In chapter 7 we state the results on spectral multipliers and Besov spaces for the Neumann Laplacian.

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Chapter 2

Preliminaries

In this chapter we shall give some notations, and state assumptions on potentials. Furthermore, we prove self-adjointness of Schrödinger operators, and finally Gaussian upper estimates for heat semigroups. These results will play an important role in the later chapters.

2.1 Notations

Let E be a measurable set of $\mathbb{R}^d$ with $d \geq 1$. For $0 < p \leq \infty$, we denote by $L^p(E)$ the Lebesgue space, i.e., $f \in L^p(E)$ if and only if $\|f\|_{L^p(E)} < \infty$, where

$$\|f\|_{L^p(E)} := \begin{cases} \left(\displaystyle\int_E |f(x)|^p \, dx \right)^{\frac{1}{p}} & \text{if } 0 < p < \infty, \\ \operatorname*{ess.\,sup}_{x \in E} |f(x)| & \text{if } p = \infty. \end{cases}$$

Let Ω be an open set of $\mathbb{R}^d$ with $d \geq 1$, and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $1 \leq p \leq \infty$ and $m \in \mathbb{N}_0$, we denote by $W^{m,p}(\Omega)$ the Sobolev spaces over Ω, i.e., $f \in W^{m,p}(\Omega)$ if and only if

$$\partial_x^\alpha f \in L^p(\Omega) \quad \text{for any multi-index } \alpha = (\alpha_1, \cdots, \alpha_d) \text{ with } |\alpha| \leq m,$$

where $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. Here the norm of $W^{m,p}(\Omega)$ is given by

$$\|f\|_{W^{m,p}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq m} \|\partial_x^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \displaystyle\sum_{|\alpha| \leq m} \|\partial_x^\alpha f\|_{L^\infty(\Omega)} & \text{if } p = \infty. \end{cases}$$

The space $C_0^\infty(\Omega)$ is the set of all C^∞-functions on Ω having compact supports in Ω. Then we denote by $W_0^{m,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$. In particular case $p = 2$, we write

$$H^m(\Omega) := W^{m,2}(\Omega) \quad \text{and} \quad H_0^m(\Omega) := W_0^{m,2}(\Omega).$$

We denote by $L^1_{\mathrm{loc}}(\Omega)$ the space of locally integrable functions on Ω and by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz space, i.e., the space of all rapidly decreasing functions on $\mathbb{R}^d$.

The convolution of measurable functions f and g on $\mathbb{R}^d$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y)\,dy, \quad \text{a.e. } x \in \mathbb{R}^d.$$

We use the notation $\mathscr{B}(X,Y)$ for the space of all bounded linear operators from a Banach space X to another one Y with operator norm $\|\cdot\|_{\mathscr{B}(X,Y)}$. When $X = Y$, we write $\mathscr{B}(X) = \mathscr{B}(X,X)$. We denote by $\mathcal{D}(T)$ the domain of an operator T, and by $\sigma(T)$ the spectrum of T.

We use the notation ${}_{X'}\langle\cdot,\cdot\rangle_X$ for the duality pair of a topological vector space X and its dual X'. We say that a sequence $\{f_N\}_{N=1}^{\infty}$ in X' converges to $f \in X'$ if

$${}_{X'}\langle f_N, \varphi\rangle_X \to {}_{X'}\langle f, \varphi\rangle_X \quad \text{as } N \to \infty \quad \text{for any } \varphi \in X.$$

2.2 Assumptions on potentials

Throughout this thesis we assume that the potential $V = V(x)$ is a real-valued measurable function on an open set Ω of $\mathbb{R}^d$ whose negative part belongs to the Kato class. More precisely, we impose the following assumption on V:

Assumption A. *V is a real-valued measurable function on Ω, and is decomposed into $V = V_+ - V_-$ such that $V_\pm \geq 0$, $V_+ \in L^1_{\mathrm{loc}}(\Omega)$ and $V_- \in K_d(\Omega)$, where $K_d(\Omega)$ is the Kato class of potentials.*

Here, let us give the definition of $K_d(\Omega)$ as follows:

Definition. We say that V_- belongs to the class $K_d(\Omega)$ if

$$\begin{cases} \limsup_{r\to 0} \sup_{x\in\Omega} \displaystyle\int_{\Omega\cap\{|x-y|<r\}} \frac{V_-(y)}{|x-y|^{d-2}}\,dy = 0 & \text{for } d \geq 3, \\ \limsup_{r\to 0} \sup_{x\in\Omega} \displaystyle\int_{\Omega\cap\{|x-y|<r\}} \log(|x-y|^{-1})V_-(y)\,dy = 0 & \text{for } d = 2, \\ \sup_{x\in\Omega} \displaystyle\int_{\Omega\cap\{|x-y|<1\}} V_-(y)\,dy < \infty & \text{for } d = 1 \end{cases}$$

(see Kato [47] and Schechter [73]).

It is noted that the potentials of Kato class assure the self-adjointness and the lower bound of the Schrödinger operator (see section 2.3).

Under assumption A, we can obtain uniform L^p-estimates in high frequency part of the spectral multipliers which are useful in the study of inhomogeneous Besov spaces (see part (i) in Theorem 3.1 below). To study homogeneous Besov spaces we need to discuss uniform L^p-estimates in low frequency part, and hence, we need to impose a smallness assumption on the negative part of V as follows:

Assumption B. *The negative part V_- of V satisfies*

$$\begin{cases} \sup_{x\in\Omega} \int_\Omega \dfrac{V_-(y)}{|x-y|^{d-2}}\, dy < \gamma_d & \text{if } d \geq 3, \\ V_- = 0 & \text{if } d = 1, 2, \end{cases}$$

where γ_d is the absolutely constant such that

$$\gamma_d := \frac{\pi^{\frac{d}{2}}}{\Gamma(d/2-1)}$$

for $d \geq 3$ with the Gamma function $\Gamma(\cdot)$.

Throughout this thesis we use the following notation:

$$\|V_-\|_{K_d(\Omega)} := \sup_{x\in\Omega} \int_\Omega \frac{V_-(y)}{|x-y|^{d-2}}\, dy$$

for $d \geq 3$.

2.3 Self-adjointness of Schrödinger operators

In this section we show self-adjointness of Schrödinger operators with the Dirichlet boundary condition by using the theory of quadratic forms. We make the assumption as follows:

Assumption C. *The negative part V_- of V satisfies*

$$\begin{cases} \sup_{x\in\Omega} \int_\Omega \dfrac{V_-(y)}{|x-y|^{d-2}}\, dy < 4\gamma_d & \text{if } d \geq 3, \\ V_- = 0 & \text{if } d = 1, 2. \end{cases}$$

We note here that assumption C is weaker than assumption B.

The purpose in this section is to prove the following:

Proposition 2.1. *Suppose that the potential V satisfies assumption* A. *Let q be a quadratic form defined by*

$$q(f,g) = \int_\Omega \nabla f(x) \cdot \overline{\nabla g(x)}\, dx + \int_\Omega V(x) f(x) \overline{g(x)}\, dx, \quad f, g \in \mathcal{Q}(q),$$

where

$$\mathcal{Q}(q) = \big\{ f \in H_0^1(\Omega) : \sqrt{V_+} f \in L^2(\Omega) \big\}.$$

Then the following assertions hold:

(i) *There exists a unique semi-bounded self-adjoint operator $\mathcal{H}_V$ on $L^2(\Omega)$ such that*

$$\begin{cases} \mathcal{D}(\mathcal{H}_V) = \big\{ f \in \mathcal{Q}(q) : \exists\, h_f \in L^2(\Omega) \text{ such that} \\ \qquad\qquad\qquad\qquad q(f,g) = (h_f, g)_{L^2(\Omega)} \text{ for any } g \in \mathcal{Q}(q) \big\}, \\ \mathcal{H}_V f = h_f, \quad f \in \mathcal{D}(\mathcal{H}_V), \end{cases} \tag{2.1}$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ stands for the inner product of $L^2(\Omega)$.

(ii) *If V_- further satisfies assumption* C, *then $\mathcal{H}_V$ is non-negative on $L^2(\Omega)$, and zero is not an eigenvalue of $\mathcal{H}_V$.*

We note that $\mathcal{D}(\mathcal{H}_V)$ can be simply written as

$$\mathcal{D}(\mathcal{H}_V) = \big\{ f \in \mathcal{Q}(q) : \mathcal{H}_V f \in L^2(\Omega) \big\}.$$

We recall a notion of quadratic forms on a Hilbert space (see p. 276 in Reed and Simon [65]).

Definition. Let $\mathscr{H}$ be a Hilbert space with norm $\|\cdot\|$. A quadratic form $\tilde{q}$ is a map

$$\tilde{q} : \mathcal{Q}(\tilde{q}) \times \mathcal{Q}(\tilde{q}) \to \mathbb{C},$$

where $\mathcal{Q}(\tilde{q})$ is a dense linear subset of $\mathscr{H}$ called the form domain of $\tilde{q}$, such that $\tilde{q}(\cdot, g)$ is linear and $\tilde{q}(f,\cdot)$ is conjugate linear for $f, g \in \mathcal{Q}(\tilde{q})$. A quadratic form $\tilde{q}$ is called semi-bounded if there exists a real number M such that

$$\tilde{q}(f,f) \geq -M\|f\|^2$$

for any $f \in \mathcal{Q}(\tilde{q})$, and in particular, $\tilde{q}$ is called non-negative if

$$\tilde{q}(f,f) \geq 0$$

for any $f \in \mathcal{Q}(\tilde{q})$. We say that a semi-bounded quadratic form $\tilde{q}$ is closed if $\mathcal{Q}(\tilde{q})$ is complete with respect to the norm

$$\|f\|_{+1} := \sqrt{\tilde{q}(f,f) + (M+1)\|f\|^2}. \tag{2.2}$$

The proof of Proposition 2.1 is done by using the following two lemmas.

Lemma 2.2. *Let $\mathscr{H}$ be a Hilbert space with the inner product $(\cdot,\cdot)$, and let*

$$\tilde{q} : \mathcal{Q}(\tilde{q}) \times \mathcal{Q}(\tilde{q}) \to \mathbb{C}$$

be a densely defined semi-bounded closed quadratic form. Then there exists a semi-bounded self-adjoint operator T on $\mathscr{H}$ uniquely such that

$$\begin{cases} \mathcal{D}(T) = \big\{ f \in \mathcal{Q}(\tilde{q}) : \exists\, h_f \in \mathscr{H} \text{ such that } \tilde{q}(f,g) = (h_f, g) \text{ for any } g \in \mathcal{Q}(\tilde{q}) \big\}, \\ Tf = h_f, \quad u \in \mathcal{D}(T). \end{cases}$$

For the proof of Lemma 2.2, see Theorem VIII.15 in [65] (see also subsection 1.2.3 in Ouhabaz [60] and Theorem 5.37 in Weidmann [86]).

The following lemma states that the negative part V_- of the potential is relatively form-bounded with respect to the Dirichlet Laplacian.

Lemma 2.3. *Suppose that V_- belongs to $K_d(\Omega)$. Then the following assertions hold*:

(i) *For any $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that*

$$\int_\Omega V_-(x)|f(x)|^2\,dx \le \varepsilon \|\nabla f\|^2_{L^2(\Omega)} + b_\varepsilon \|f\|^2_{L^2(\Omega)} \tag{2.3}$$

for any $f \in H^1_0(\Omega)$.

(ii) *Let $d \ge 3$. Assume further that V_- satisfies $\|V_-\|_{K_d(\Omega)} < \infty$. Then*

$$\int_\Omega V_-(x)|f(x)|^2\,dx \le \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d} \|\nabla f\|^2_{L^2(\Omega)} \tag{2.4}$$

for any $f \in H^1_0(\Omega)$.

Proof. The proof is done by reducing the problem to the whole space case, and by the similar argument of Lemma 3.1 from D'Ancona and Pierfelice [18] who treated mainly three dimensional case.

First we show the assertion (i). Let $f \in C^\infty_0(\Omega)$, and let $\tilde{f}$ and $\tilde{V}_-$ be the zero extensions of f and V_- to $\mathbb{R}^d$, respectively. We prove that for any $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^d} \tilde{V}_-(x)|\tilde{f}(x)|^2\,dx \le \varepsilon \|\nabla \tilde{f}\|^2_{L^2(\mathbb{R}^d)} + b_\varepsilon \|\tilde{f}\|^2_{L^2(\mathbb{R}^d)}. \tag{2.5}$$

The inequality (2.5) is equivalent to

$$\begin{aligned} \int_{\mathbb{R}^d} \tilde{V}_-(x)|\tilde{f}(x)|^2\,dx &\le \varepsilon(-\Delta \tilde{f}, \tilde{f})_{L^2(\mathbb{R}^d)} + b_\varepsilon \|\tilde{f}\|^2_{L^2(\mathbb{R}^d)} \\ &= \varepsilon \big\| (-\Delta + b_\varepsilon \varepsilon^{-1})^{\frac{1}{2}} \tilde{f} \big\|^2_{L^2(\mathbb{R}^d)}, \end{aligned}$$

where we note that $-\Delta$ is the self-adjoint operator with domain $H^2(\mathbb{R}^d)$. Put

$$g := (-\Delta + b_\varepsilon \varepsilon^{-1})^{\frac{1}{2}} \tilde{f}.$$

Then the inequality (2.5) takes the form

$$\big\| \tilde{V}_-^{\frac{1}{2}} (-\Delta + b_\varepsilon \varepsilon^{-1})^{-\frac{1}{2}} g \big\|^2_{L^2(\mathbb{R}^d)} \le \varepsilon \|g\|^2_{L^2(\mathbb{R}^d)}.$$

This estimate can be obtained if we show that

$$\|TT^*\|_{\mathscr{B}(L^2(\mathbb{R}^d))} \le \varepsilon, \tag{2.6}$$

where we set

$$T := \tilde{V}_-^{\frac{1}{2}} (-\Delta + b_\varepsilon \varepsilon^{-1})^{-\frac{1}{2}}.$$

Thus, our goal is to show that for any $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that the estimate (2.6) holds.

Let $\varepsilon > 0$ be fixed arbitrarily, and let $b > 0$. Let $G_0(x-y; M)$ be the kernel of $(-\Delta + M)^{-1}$ for $M \ge 0$. By the definition of G_0 and the Schwarz inequality, we estimate

$$\begin{aligned}
&\|TT^* g\|^2_{L^2(\mathbb{R}^d)} = \big\| \tilde{V}_-^{\frac{1}{2}} (-\Delta + b\varepsilon^{-1})^{-1} \tilde{V}_-^{\frac{1}{2}} g \big\|^2_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} \tilde{V}_-(x) \left| \int_{\mathbb{R}^d} G_0(x-y; b\varepsilon^{-1}) \tilde{V}_-^{\frac{1}{2}}(y) g(y)\, dy \right|^2 dx \\
&\le \int_{\mathbb{R}^d} \tilde{V}_-(x) \bigg(\int_{\mathbb{R}^d} G_0(x-y; b\varepsilon^{-1}) \tilde{V}_-(y)\, dy \bigg) \bigg(\int_{\mathbb{R}^d} G_0(x-y; b\varepsilon^{-1}) |g(y)|^2\, dy \bigg) dx \\
&\le \big\| (-\Delta + b\varepsilon^{-1})^{-1} \tilde{V}_- \big\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \tilde{V}_-(x) \bigg(\int_{\mathbb{R}^d} G_0(x-y; b\varepsilon^{-1}) |g(y)|^2\, dy \bigg) dx.
\end{aligned}$$

Applying Fubini-Tonelli theorem to the integral on the right, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} \tilde{V}_-(x) \bigg(\int_{\mathbb{R}^d} G_0(x-y; b\varepsilon^{-1}) |g(y)|^2\, dy \bigg) dx \\
&= \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} G_0(x-y; b\varepsilon^{-1}) \tilde{V}_-(x)\, dx \bigg) |g(y)|^2\, dy \\
&\le \big\| (-\Delta + b\varepsilon^{-1})^{-1} \tilde{V}_- \big\|_{L^\infty(\mathbb{R}^d)} \|g\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Combining the above two estimates, we obtain

$$\|TT^* g\|^2_{L^2(\mathbb{R}^d)} \le \big\| (-\Delta + b\varepsilon^{-1})^{-1} \tilde{V}_- \big\|^2_{L^\infty(\mathbb{R}^d)} \|g\|^2_{L^2(\mathbb{R}^d)}.$$

Using the fact that $V \in K_d(\mathbb{R}^d)$ is equivalent to

$$\lim_{M \to \infty} \big\| (-\Delta + M)^{-1} |V| \big\|_{L^\infty(\mathbb{R}^d)} = 0$$

(see Proposition A.2.3 in [76]), we see that there exists a constant $b_\varepsilon > 0$ such that

$$\left\|(-\Delta + b_\varepsilon \varepsilon^{-1})^{-1}\tilde{V}_-\right\|_{L^\infty(\mathbb{R}^d)} \le \varepsilon, \tag{2.7}$$

since $\tilde{V}_- \in K_d(\mathbb{R}^d)$, which implies (2.6). Hence (2.5) is proved.

Now the required inequality (2.3) follows from (2.5). In fact, by using (2.5), we estimate

$$\begin{aligned}\int_\Omega V_-(x)|f(x)|^2\,dx &= \int_{\mathbb{R}^d} \tilde{V}_-(x)|\tilde{f}(x)|^2\,dx \\ &\le \varepsilon\|\nabla\tilde{f}\|^2_{L^2(\mathbb{R}^d)} + b_\varepsilon\|\tilde{f}\|^2_{L^2(\mathbb{R}^d)} \\ &= \varepsilon\|\nabla f\|^2_{L^2(\Omega)} + b_\varepsilon\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

As a consequence, the inequality (2.3) is proved by density argument.

Next we show the assertion (ii). The proof of (2.4) is almost identical to that of (2.3) by regarding b_ε as 0. The only difference is the estimate (2.7). We use the following pointwise estimate:

$$0 < G_0(x;0) \le \frac{1}{4\gamma_d|x|^{d-2}}, \quad x \ne 0$$

for $d \ge 3$. Instead of (2.7), we can apply the following estimate:

$$\begin{aligned}\|(-\Delta)^{-1}\tilde{V}_-\|_{L^\infty(\mathbb{R}^d)} &= \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} G_0(x-y;0)\tilde{V}_-(y)\,dy \\ &\le \frac{1}{4\gamma_d}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\tilde{V}_-(y)}{|x-y|^{d-2}}\,dy \\ &= \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d},\end{aligned}$$

whence the argument in the proof of (2.3) works well in this case, and we get (2.4). The proof of Lemma 2.3 is complete. □

We are now in a position to prove Proposition 2.1.

Proof of Proposition 2.1. It is clear that q is densely defined on $L^2(\Omega)$. Moreover, q is semi-bounded. In fact, it follows from the inequality (2.3) for $\varepsilon = 1$ that

$$q(f,f) \ge \|\nabla f\|^2_{L^2(\Omega)} - \int_\Omega V_-(x)|f(x)|^2\,dx \ge -b_1\|f\|^2_{L^2(\Omega)} \tag{2.8}$$

for any $f \in \mathcal{Q}(q)$. Hence, if we show that q is closed, then Lemma 2.2 ensures the unique existence of the semi-bounded self-adjoint operator $\mathcal{H}_V$ on $L^2(\Omega)$ satisfying (2.1).

We show that q is closed. Put

$$q_1(f,g) = \int_\Omega \nabla f(x) \cdot \overline{\nabla g(x)}\, dx - \int_\Omega V_-(x) f(x) \overline{g(x)}\, dx, \quad f,g \in \mathcal{Q}_1(q) := H_0^1(\Omega),$$

$$q_2(f,g) = \int_\Omega V_+(x) f(x) \overline{g(x)}\, dx, \quad f,g \in \mathcal{Q}_2(q) := \big\{ f \in L^2(\Omega) : \sqrt{V_+} f \in L^2(\Omega) \big\}.$$

Then we have

$$q(f,g) = q_1(f,g) + q_2(f,g), \quad f,g \in \mathcal{Q}_1(q) \cap \mathcal{Q}_2(q).$$

Since the sum of two closed quadratic forms is also closed, it suffices to show that q_1 and q_2 are closed. First we show that q_1 is closed. All we have to do is to show that the norm $\|\cdot\|_{+1}$ is equivalent to that of $H_0^1(\Omega)$, where $\|\cdot\|_{+1}$ is defined in (2.2), i.e.,

$$\|f\|_{+1} = \sqrt{q_1(f,f) + (b_1+1)\|f\|^2_{L^2(\Omega)}}.$$

Since $V_- \geq 0$, we see that

$$\|f\|^2_{+1} \leq \|\nabla f\|^2_{L^2(\Omega)} + (b_1+1)\|f\|^2_{L^2(\Omega)} \leq (b_1+1)\|f\|^2_{H^1(\Omega)}$$

for any $f \in H_0^1(\Omega)$, and by using the inequality (2.3), we have

$$\begin{aligned} \|f\|^2_{+1} &= \|\nabla f\|^2_{L^2(\Omega)} - \int_\Omega V_-(x)|f(x)|^2\, dx + (b_1+1)\|f\|^2_{L^2(\Omega)} \\ &\geq (1-\varepsilon)\|\nabla f\|^2_{L^2(\Omega)} + (b_1 - b_\varepsilon + 1)\|f\|^2_{L^2(\Omega)} \end{aligned}$$

for any $f \in H_0^1(\Omega)$, where we choose $\varepsilon \in (0,1)$ and b_ε such that $b_1 < b_\varepsilon < b_1 + 1$. The above two inequalities imply that $\|\cdot\|_{+1}$ is equivalent to $\|\cdot\|_{H^1(\Omega)}$. Hence q_1 is closed.

Next we show that q_2 is closed. Put $q_2(f) = q_2(f,f)$ for simplicity. Assume that

$$f \in L^2(\Omega), \quad f_n \in \mathcal{Q}(q_2), \quad q_2(f_n - f_m) \to 0, \quad \|f_n - f\|_{L^2(\Omega)} \to 0 \quad \text{as } n,m \to \infty,$$

and we prove that

$$f \in \mathcal{Q}(q_2) \quad \text{and} \quad q_2(f_n - f) \to 0 \quad \text{as } n \to \infty. \tag{2.9}$$

Since $\{\sqrt{V_+} f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\Omega)$, there exists $g \in L^2(\Omega)$ such that

$$\sqrt{V_+} f_n \to g \quad \text{in } L^2(\Omega).$$

Hence the sequence $\{\sqrt{V_+} f_n\}_{n=1}^\infty$ converges to g almost everywhere along a subsequence denoted by the same, namely,

$$\sqrt{V_+} f_n(x) \to g(x) \quad \text{a.e. } x \in \Omega \text{ as } n \to \infty.$$

On the other hand, since any convergent sequence in $L^2(\Omega)$ contains a subsequence which converges almost everywhere in Ω, it follows that

$$\sqrt{V_+}f_n(x) \to \sqrt{V_+}f(x) \quad \text{a.e. } x \in \Omega \text{ as } n \to \infty.$$

Summarizing three convergences obtained now, we get

$$\sqrt{V_+}f = g \in L^2(\Omega).$$

This proves (2.9). Thus q is closed.

Next, we prove the assertion (ii). We estimate by using the inequality (2.4) from Lemma 2.3 and assumption C on V_-,

$$\begin{aligned}(\mathcal{H}_V f, f)_{L^2(\Omega)} &\geq \|\nabla f\|^2_{L^2(\Omega)} - \int_\Omega V_-(x)|f(x)|^2\,dx \\ &\geq \left(1 - \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d}\right)\|\nabla f\|^2_{L^2(\Omega)} \\ &\geq 0\end{aligned}$$

for any $f \in \mathcal{D}(\mathcal{H}_V)$. Hence $\mathcal{H}_V$ is non-negative on $L^2(\Omega)$.

Finally, we prove that zero is not an eigenvalue of $\mathcal{H}_V$, namely, f satisfies

$$f \in \mathcal{D}(\mathcal{H}_V) \quad \text{and} \quad \mathcal{H}_V f = 0 \quad \text{in } L^2(\Omega), \tag{2.10}$$

then $f = 0$. We consider the case $d \geq 3$. It follows from the assertion (ii) in Lemma 2.3 and assumption (2.10) that

$$0 = (\mathcal{H}_V f, f)_{L^2(\Omega)} \geq \left(1 - \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d}\right)\|\nabla f\|^2_{L^2(\Omega)},$$

which implies that $f = 0$, since $u \in \mathcal{D}(\mathcal{H}_V) \subset H^1_0(\Omega)$. The case $d = 1, 2$ is similar, since $V_- = 0$. The proof of Proposition 2.1 is complete. □

2.4 Gaussian upper estimates for heat semigroup

In this section we shall prove L^p-L^q-estimates for semigroup $\{e^{-t\mathcal{H}_V}\}_{t>0}$ generated by $\mathcal{H}_V$ and pointwise estimates for the kernel of $e^{-t\mathcal{H}_V}$. We denote by $e^{-tL}(x, y)$ the kernel of semigroup $\{e^{-tL}\}_{t>0}$ generated by an operator L.

When $\Omega = \mathbb{R}^d$ and $V = 0$, i.e., $\mathcal{H}_V = -\Delta$ on $L^2(\mathbb{R}^d)$, it is well known that the kernel $e^{t\Delta}(x, y)$ is written as

$$e^{t\Delta}(x, y) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}} \tag{2.11}$$

for any $t > 0$ and $x, y \in \mathbb{R}^d$. This representation is fundamental in the study of the Cauchy problem to heat equations on $\mathbb{R}^d$, since various properties on solutions of heat equations are derived from (2.11). Our goal in this section is to prove some estimates for the kernel $e^{-t\mathcal{H}_V}(x, y)$.

The main result in this section is the following:

Proposition 2.4. *Let $1 \le p \le q \le \infty$. Suppose that the potential V satisfies assumption* A. *Then $e^{-t\mathcal{H}_V}$ is extended to a bounded linear operator from $L^p(\Omega)$ to $L^q(\Omega)$ for each $t > 0$. Furthermore, the following assertions hold*:

(i) *There exist two constants $\omega \ge -\inf \sigma(\mathcal{H}_V)$ and $C_1 > 0$ such that*

$$\|e^{-t\mathcal{H}_V} f\|_{L^q(\Omega)} \le C_1 t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} e^{\omega t} \|f\|_{L^p(\Omega)} \tag{2.12}$$

for any $t > 0$ and $f \in L^p(\Omega)$.

(ii) *There exist two constants $\omega \ge -\inf \sigma(\mathcal{H}_V)$ and $C_2 > 0$ such that the kernel $e^{-t\mathcal{H}_V}(x, y)$ fulfills with the following estimate*:

$$0 \le e^{-t\mathcal{H}_V}(x, y) \le C_2 t^{-\frac{d}{2}} e^{\omega t} e^{-\frac{|x-y|^2}{8t}} \quad a.e.\, x, y \in \Omega \tag{2.13}$$

for any $t > 0$.

(iii) *Assume further that V_- satisfies*

$$\begin{cases} \|V_-\|_{K_d(\Omega)} < 2\gamma_d & \text{if } d \ge 3, \\ V_- = 0 & \text{if } d = 1, 2. \end{cases} \tag{2.14}$$

Then

$$\|e^{-t\mathcal{H}_V} f\|_{L^q(\Omega)} \le \begin{cases} \dfrac{(2\pi t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}{\left(1 - \|V_-\|_{K_d(\Omega)}/2\gamma_d\right)^2} \|f\|_{L^p(\Omega)} & \text{if } d \ge 3, \\ (4\pi t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)} & \text{if } d = 1, 2 \end{cases} \tag{2.15}$$

for any $t > 0$ and $f \in L^p(\Omega)$.

(iv) *If V_- further satisfies assumption* B, *then*

$$0 \le e^{-t\mathcal{H}_V}(x, y) \le \begin{cases} \dfrac{(2\pi t)^{-\frac{d}{2}}}{1 - \|V_-\|_{K_d(\Omega)}/\gamma_d} e^{-\frac{|x-y|^2}{8t}} & \text{if } d \ge 3, \\ (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}} & \text{if } d = 1, 2, \end{cases} \quad a.e.\, x, y \in \Omega \tag{2.16}$$

for any $t > 0$.

We denote by $\tilde{\mathcal{H}}_{\tilde{V}}$ and $\tilde{\mathcal{H}}_{\tilde{V}_-}$ the self-adjoint realizations of $-\Delta+\tilde{V}$ and $-\Delta-\tilde{V}_-$ on $L^2(\mathbb{R}^d)$, respectively, where $\tilde{V}$ and $\tilde{V}_-$ are the zero extensions of $V$and V_- to $\mathbb{R}^d$, respectively. Then, under assumption A, we have

$$\mathcal{D}(\tilde{\mathcal{H}}_{\tilde{V}}) = \Big\{ f \in H^1(\mathbb{R}^d) : \sqrt{\tilde{V}_+} f \in L^2(\mathbb{R}^d),\ \tilde{\mathcal{H}}_{\tilde{V}} f \in L^2(\mathbb{R}^d) \Big\},$$

$$\mathcal{D}(\tilde{\mathcal{H}}_{\tilde{V}_-}) = \big\{ f \in H^1(\mathbb{R}^d) : \tilde{\mathcal{H}}_{\tilde{V}_-} f \in L^2(\mathbb{R}^d) \big\}.$$

The following lemma is crucial in the proof of Proposition 2.4.

Lemma 2.5. *Suppose that the potential V satisfies assumption* A. *Let $\tilde{V}$ and $\tilde{V}_-$ be the zero extensions of V and V_- to $\mathbb{R}^d$, respectively. Then for any non-negative function $f \in L^2(\Omega)$, the following estimates hold*:

$$\big(e^{-t\mathcal{H}_V} f\big)(x) \geq 0 \quad a.e.\, x \in \Omega, \tag{2.17}$$

$$\big(e^{-t\mathcal{H}_V} f\big)(x) \leq \big(e^{-t\tilde{\mathcal{H}}_{\tilde{V}}} \tilde{f}\,\big)(x) \quad a.e.\, x \in \Omega, \tag{2.18}$$

$$\big(e^{-t\tilde{\mathcal{H}}_{\tilde{V}}} \tilde{f}\,\big)(x) \leq \big(e^{-t\tilde{\mathcal{H}}_{\tilde{V}_-}} \tilde{f}\,\big)(x) \quad a.e.\, x \in \Omega \tag{2.19}$$

for any $t > 0$, where $\tilde{f}$ is the zero extension of f to $\mathbb{R}^d$.

The proof of Lemma 2.5 is rather long, and will be postponed.

Proof of Proposition 2.4. The assertion (i) is an immediate consequence of the assertion (ii) and Young's inequality. Hence we concentrate on proving the assertion (ii). We adopt a sequence $\{j_\varepsilon(x)\}_{\varepsilon>0}$ of functions on $\mathbb{R}^d$ defined by letting

$$j_\varepsilon(x) := \frac{1}{\varepsilon^d}\, j\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d, \tag{2.20}$$

where

$$j(x) = \begin{cases} A_d\, e^{-\frac{1}{1-|x|^2}} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

with

$$A_d := \left(\int_{|x|<1} e^{-\frac{1}{1-|x|^2}}\, dx \right)^{-1}.$$

As is well known, the sequence $\{j_\varepsilon(x)\}_{\varepsilon>0}$ enjoys the following property:

$$j_\varepsilon(\cdot - y) \to \delta_y \quad \text{in } \mathscr{S}'(\mathbb{R}^d) \text{ as } \varepsilon \to 0, \tag{2.21}$$

where δ_y is the Dirac delta function at $y \in \Omega$ and $\mathscr{S}'(\mathbb{R}^d)$ is the topological dual of $\mathscr{S}(\mathbb{R}^d)$. Let $y \in \Omega$ be fixed, and let $\tilde{K}(t,x,y)$ be the kernel of $e^{-t\tilde{\mathcal{H}}_{\tilde{V}}}$. Taking

$\varepsilon > 0$ sufficiently small so that $\operatorname{supp} j_\varepsilon(\cdot - y) \Subset \Omega$, and applying (2.17) and (2.18) from Lemma 2.5 to both f and $\tilde{f}$ replaced by $j_\varepsilon(\cdot - y)$, we get

$$0 \le \int_\Omega e^{-t\mathcal{H}_V}(x,z) j_\varepsilon(z-y)dz \le \int_{\mathbb{R}^d} e^{-t\tilde{\mathcal{H}}_{\tilde{V}}}(x,z) j_\varepsilon(z-y)dz \quad \text{a.e. } x \in \Omega.$$

Noting (2.21) and taking the limit of the previous inequality as $\varepsilon \to 0$, we get

$$0 \le e^{-t\mathcal{H}_V}(x,y) \le e^{-t\tilde{\mathcal{H}}_{\tilde{V}}}(x,y) \quad \text{a.e. } x, y \in \Omega$$

for any $t > 0$. Finally, by using the pointwise estimates:

$$e^{-t\tilde{\mathcal{H}}_{\tilde{V}}}(x,y) \le Ct^{-\frac{d}{2}} e^{\omega t} e^{-\frac{|x-y|^2}{8t}} \quad \text{a.e. } x, y \in \Omega \tag{2.22}$$

for any $t > 0$ (see Proposition B.6.7 in [76]), we obtain the estimate (2.13), as desired. Thus the assertion (ii) is proved.

Finally, we prove the estimates (2.15) in (iii) and (2.16) in (iv). We recall Proposition 5.1 in [18] that if $d \ge 3$, then

$$\left\| e^{-t\tilde{\mathcal{H}}_{\tilde{V}}} |\tilde{f}| \right\|_{L^q(\mathbb{R}^d)} \le \frac{(2\pi t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}{\left(1 - \|\tilde{V}_-\|_{K_d(\mathbb{R}^d)}/2\gamma_d\right)^2} \|\tilde{f}\|_{L^p(\mathbb{R}^d)}$$

for any $t > 0$, and

$$e^{-t\tilde{\mathcal{H}}_{\tilde{V}}}(x,y) \le \frac{(2\pi t)^{-\frac{d}{2}}}{1 - \|\tilde{V}_-\|_{K_d(\mathbb{R}^d)}/\gamma_d} e^{-\frac{|x-y|^2}{8t}} \left(= \frac{(2\pi t)^{-\frac{d}{2}}}{1 - \|V_-\|_{K_d(\Omega)}/\gamma_d} e^{-\frac{|x-y|^2}{8t}} \right)$$

for a.e. $x, y \in \Omega$ and any $t > 0$. When $d = 1, 2$, we have

$$\left\| e^{-t\tilde{\mathcal{H}}_{\tilde{V}}} |\tilde{f}| \right\|_{L^q(\mathbb{R}^d)} \le (4\pi t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|\tilde{f}\|_{L^p(\mathbb{R}^d)},$$

$$e^{-t\tilde{\mathcal{H}}_{\tilde{V}}}(x,y) \le (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}} \quad \text{a.e. } x, y \in \Omega$$

for any $t > 0$. Then, applying the above estimates to the argument of the derivations of (2.12) and (2.13), we conclude (2.15) and (2.16). The proof of Proposition 2.4 is finished. □

In the rest of this section we shall prove Lemma 2.5. For this purpose, we need two lemmas. The first one is concerned with the existence and uniqueness of solutions for evolution equations in abstract setting.

Lemma 2.6. *Let $\mathscr{H}$ be a Hilbert space with norm $\|\cdot\|$. Assume that A is a non-negative self-adjoint operator on $\mathscr{H}$. Let $\{T(t)\}_{t\ge 0}$ be the semigroup generated by A, and let $f \in \mathscr{H}$ and $u(t) = T(t)f$. Then u is a unique solution of the following problem:*

$$\begin{cases} u \in C([0,\infty); \mathscr{H}) \cap C((0,\infty); \mathcal{D}(A)) \cap C^1((0,\infty); \mathscr{H}), \\ u'(t) + Au(t) = 0, \quad t > 0, \\ u(0) = f, \end{cases}$$

where $\mathcal{D}(A)$ means the Banach space with graph norm $\|\cdot\| + \|A\cdot\|$.

For the proof of Lemma 2.6, see, e.g., Theorem 3.2.1 in [10].

Remark. It is known that for any non-negative self-adjoint operator on a Hilbert space, its domain is a Banach space with respect to the graph norm of its operator (see Corollary 2.2.9 in Cazenave and Haraux [10]).

The second one is about the differentiability properties for composite functions of Lipschitz continuous functions and $W^{1,p}$-functions.

Lemma 2.7. *Consider the positive and negative parts of a real-valued function $u \in W^{1,p}(\Omega)$ for $1 \le p \le \infty$:*

$$u^+ = \chi_{\{u>0\}} u \quad \text{and} \quad u^- = -\chi_{\{u<0\}} u.$$

Then $u^\pm \in W^{1,p}(\Omega)$ and

$$\partial_{x_j} u^+ = \chi_{\{u>0\}} \partial_{x_j} u, \quad \partial_{x_j} u^- = -\chi_{\{u<0\}} \partial_{x_j} u$$

for $j = 1, 2, \ldots, d$, where $\partial_{x_j} = \partial/\partial x_j$. Furthermore, if $u \in W_0^{1,p}(\Omega)$ for $1 \le p < \infty$, then

$$u^\pm \in W_0^{1,p}(\Omega). \tag{2.23}$$

Proof. Since the first part of the lemma is well known, we omit the proof. For the proof, see Lemma 7.6 in Gilbarg and Trudinger [29]. Hence we prove only the latter part.

Since $u \in W_0^{1,p}(\Omega)$ with $1 \le p < \infty$, there exists a sequence $\{\phi_n\}_n$ in $C_0^\infty(\Omega)$ such that

$$\phi_n \to u \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to \infty. \tag{2.24}$$

Let us take a non-negative function $\psi \in C^\infty(\mathbb{R})$ as

$$\psi(x) \begin{cases} = -x & \text{if } x \le -1, \\ \le -x & \text{if } -1 < x < 0, \\ = 0 & \text{if } x \ge 0, \end{cases}$$

and put

$$\psi_n(x) := \frac{1}{n} \psi(nx), \quad n \in \mathbb{N}. \tag{2.25}$$

Then there exists a constant $C_0 > 0$ such that

$$|\psi_n'(x)| \le C_0, \quad n \in \mathbb{N}. \tag{2.26}$$

Let us consider two kinds of composite functions $\psi_n \circ \phi_n$ and $\psi_n \circ u$. We show that

$$\psi_n \circ \phi_n - \psi_n \circ u \to 0 \quad \text{in } W^{1,p}(\Omega), \tag{2.27}$$

$$\psi_n \circ u - u^- \to 0 \quad \text{in } W^{1,p}(\Omega) \tag{2.28}$$

as $n \to \infty$. In fact, noting (2.26), we deduce from the mean value theorem that

$$\begin{aligned} \|\psi_n \circ \phi_n - \psi_n \circ u\|_{L^p(\Omega)} &= \left\| \int_0^1 \psi_n'\big(\theta\phi_n + (1-\theta)u\big)(\phi_n - u)\, d\theta \right\|_{L^p(\Omega)} \\ &\le C_0 \|\phi_n - u\|_{L^p(\Omega)}. \end{aligned} \tag{2.29}$$

As to the derivatives of $\psi_n \circ \phi_n - \psi_n \circ u$, we write

$$\begin{aligned} &\|\partial_{x_j}(\psi_n \circ \phi_n - \psi_n \circ u)\|_{L^p(\Omega)} \\ &= \|\psi_n'(\phi_n)\partial_{x_j}\phi_n - \psi_n'(u)\partial_{x_j}u\|_{L^p(\Omega)} \\ &\le \|\psi_n'(\phi_n)(\partial_{x_j}\phi_n - \partial_{x_j}u)\|_{L^p(\Omega)} + \big\|[\psi_n'(\phi_n) - \psi_n'(u)]\partial_{x_j}u\big\|_{L^p(\Omega)} \\ &\le C_0\|\partial_{x_j}\phi_n - \partial_{x_j}u\|_{L^p(\Omega)} + \big\|[\psi_n'(\phi_n) - \psi_n'(u)]\partial_{x_j}u\big\|_{L^p(\Omega)}, \end{aligned} \tag{2.30}$$

where we used again (2.26) in the last step. Noting the pointwise convergence and uniform boundedness with respect to n:

$$[\psi_n'(\phi_n)(x) - \psi_n'(u)(x)]\partial_{x_j}u(x) \to 0 \quad \text{a.e. } x \in \Omega \text{ as } n \to \infty,$$

$$\big|[\psi_n'(\phi_n)(x) - \psi_n'(u)(x)]\partial_{x_j}u(x)\big| \le 2C_0|\partial_{x_j}u(x)| \in L^p(\Omega),$$

we can apply Lebesgue's dominated convergence theorem to obtain

$$\big\|[\psi_n'(\phi_n) - \psi_n'(u)]\partial_{x_j}u\big\|_{L^p(\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{2.31}$$

Hence, summarizing (2.24) and (2.29)–(2.31), we obtain (2.27).

As to the latter convergence (2.28), since

$$|(\psi_n \circ u)(x) - u^-(x)| \le 2|u(x)| \in L^p(\Omega),$$

$$|\partial_{x_j}(\psi_n \circ u)(x) - \partial_{x_j}u^-(x)| \le (C_0+1)|\partial_{x_j}u(x)| \in L^p(\Omega),$$

and since

$$(\psi_n \circ u)(x) - u^-(x) \to 0, \quad \text{a.e. } x \in \Omega,$$

$$\partial_{x_j}(\psi_n \circ u)(x) - \partial_{x_j}u^-(x) = [\psi_n'(u) - \chi_{\{u<0\}}]\partial_{x_j}u(x) \to 0, \quad \text{a.e. } x \in \Omega$$

as $n \to \infty$, Lebesgue's dominated convergence theorem allows us to conclude (2.28).

It follows from (2.27) and (2.28) that

$$\psi_n \circ \phi_n - u^- \to 0 \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to \infty.$$

Since $\{\psi_n \circ \phi_n\}$ is a sequence in $C_0^\infty(\Omega)$, we conclude (2.23) from the above convergence. The proof of Lemma 2.7 is finished. □

Proof of Lemma 2.5. We start by proving (2.17). Let M be a real number satisfying

$$M > -\inf \sigma(\mathcal{H}_V).$$

Then $\mathcal{H}_V + M$ is the non-negative self-adjoint operator on $L^2(\Omega)$ with domain

$$\mathcal{D}(\mathcal{H}_V + M) = \big\{u \in H_0^1(\Omega) : \sqrt{V_+}u \in L^2(\Omega),\ \mathcal{H}_V u \in L^2(\Omega)\big\}.$$

Put

$$u(t) = e^{-t(\mathcal{H}_V+M)}f, \quad t \geq 0$$

for a non-negative function $f \in L^2(\Omega)$. Lemma 2.6 implies that $u(t)$ satisfies

$$\begin{cases} u \in C([0,\infty); L^2(\Omega)) \cap C([0,\infty); \mathcal{D}(\mathcal{H}_V + M)) \cap C^1((0,\infty); L^2(\Omega)), \\ \partial_t u(t) + (\mathcal{H}_V + M)u(t) = 0, \quad t > 0, \\ u(0) = f. \end{cases}$$

If we show that

$$\|u^-(t)\|^2_{L^2(\Omega)} \text{ is monotonically decreasing with respect to } t \geq 0, \tag{2.32}$$

then we obtain

$$u^-(t,x) = 0 \quad \text{a.e. } x \in \Omega$$

for each $t > 0$, since

$$u^-(0,x) = f^-(x) = 0 \quad \text{a.e. } x \in \Omega.$$

This means that

$$u(t,x) \geq 0 \qquad \text{a.e. } x \in \Omega$$

for each $t > 0$; thus we conclude (2.17). Now the assertion (2.32) is an immediate consequence of the following:

$$\frac{d}{dt}\int_\Omega (u^-)^2\,dx \leq 0. \tag{2.33}$$

Hence we pay attention to prove (2.33). Here and below, the time variable t may be omitted, since no confusion arises.

By the definition of u^+, we have

$$\partial_t u^+(t,x) = 0 \quad \text{ for } x \in \{u < 0\} \text{ and each } t > 0.$$

We compute

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u^-)^2\,dx &= 2\int_\Omega u^-\partial_t u^-\,dx = 2\int_{\{u<0\}} u^-\partial_t(u^+ - u)\,dx \\ &= -2\int_{\{u<0\}} u^-\partial_t u\,dx = 2\int_\Omega [(\mathcal{H}_V + M)u]\,u^-\,dx \end{aligned} \tag{2.34}$$

where we use the equation

$$\partial_t u + (\mathcal{H}_V + M)u = 0$$

in the last step. Since $u^- \in H_0^1(\Omega)$ and $\sqrt{V_+}u^- \in L^2(\Omega)$ by Lemma 2.7 and $\sqrt{V_+}u \in L^2(\Omega)$, we have, by going back to (2.1) in the definition of $\mathcal{H}_V$,

$$\int_\Omega [(\mathcal{H}_V + M)u]\, u^-\, dx = \int_\Omega \nabla u \cdot \nabla u^-\, dx + \int_\Omega V u u^-\, dx + \int_\Omega M u u^-\, dx. \quad (2.35)$$

Here we see from Lemma 2.7 that

$$\nabla u^- = -\chi_{\{u<0\}} \nabla u,$$

and hence, the first term on the right of (2.35) is written as

$$\int_\Omega \nabla u \cdot \nabla u^-\, dx = -\int_\Omega |\nabla u^-|^2\, dx.$$

As to the second, by the estimate (2.3) for $\varepsilon = 1$ from Lemma 2.3, we have

$$\int_\Omega V u u^-\, dx \le \int_\Omega V_- |u^-|^2\, dx \le \|u^-\|_{L^2(\Omega)}^2 + b_1 \|\nabla u^-\|_{L^2(\Omega)}^2;$$

thus, by choosing M as

$$M > b_1 (\ge -\inf \sigma(\mathcal{H}_V)), \quad (2.36)$$

we find that

$$\int_\Omega [(\mathcal{H}_V + M)u]\, u^-\, dx \le (b_1 - M)\|\nabla u^-\|_{L^2(\Omega)}^2 \le 0.$$

Hence, combining this inequality and (2.34), we conclude (2.33).

Next, we prove (2.18). Let us define two functions $v^{(1)}(t)$ and $v^{(2)}(t)$ as follows:

$$v^{(1)}(t) := e^{-t(\tilde{\mathcal{H}}_{\tilde{V}} + M)} \tilde{f} \quad \text{and} \quad v^{(2)}(t) := e^{-t(\mathcal{H}_V + M)} f$$

for $t \ge 0$. Then it follows from Lemma 2.6 that $v^{(1)}$ and $v^{(2)}$ satisfy

$$\begin{cases} v^{(1)} \in C([0,\infty); L^2(\mathbb{R}^d)) \cap C((0,\infty); \mathcal{D}(\tilde{\mathcal{H}}_{\tilde{V}} + M)) \cap C^1((0,\infty); L^2(\mathbb{R}^d)), \\ \partial_t v^{(1)}(t) + (\tilde{\mathcal{H}}_{\tilde{V}} + M) v^{(1)}(t) = 0, \quad t > 0, \\ v^{(1)}(0) = \tilde{f} \end{cases} \quad (2.37)$$

and

$$\begin{cases} v^{(2)} \in C([0,\infty); L^2(\Omega)) \cap C((0,\infty); \mathcal{D}(\mathcal{H}_V + M)) \cap C^1((0,\infty); L^2(\Omega)), \\ \partial_t v^{(2)}(t) + (\mathcal{H}_V + M) v^{(2)}(t) = 0, \quad t > 0, \\ v^{(2)}(0) = f \end{cases} \quad (2.38)$$

for each $t > 0$, respectively. We define a new function v as

$$v(t) := v^{(1)}(t)|_\Omega - v^{(2)}(t)$$

for $t \geq 0$, where $v^{(1)}(t)|_\Omega$ is the restriction of $v^{(1)}(t)$ to Ω. Let us consider the negative part of v:

$$v^- = -\chi_{\{v<0\}} v.$$

Then, thanks to (2.37) and (2.38), we have

$$v^- \in C([0,\infty); L^2(\Omega)) \cap C^1((0,\infty); L^2(\Omega)).$$

Moreover, we have $v^- \in H^1(\Omega)$ by using Lemma 2.7, since $v \in H^1(\Omega)$, and we immediately have $\sqrt{V_+}v^- \in L^2(\Omega)$, since $\sqrt{V_+}v \in L^2(\Omega)$. Once we prove that

$$v^- \in H_0^1(\Omega), \tag{2.39}$$

we obtain

$$\frac{d}{dt}\int_\Omega (v^-)^2\,dx \leq 0 \tag{2.40}$$

by the previous argument. In fact, in a similar way to (2.34), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (v^-)^2\,dx &= -2\int_{\{v<0\}} v^-\partial_t v^{(1)}\,dx + 2\int_{\{v<0\}} v^-\partial_t v^{(2)}\,dx \\ &= 2\int_{\mathbb{R}^d} \big\{(\tilde{\mathcal{H}}_{\tilde{V}} + M)v^{(1)}\big\}\tilde{v}^-\,dx - 2\int_\Omega \big\{(\mathcal{H}_V + M)v^{(2)}\big\}v^-\,dx,\end{aligned}$$

where $\tilde{v}^-$ is the zero extension of v^- to $\mathbb{R}^d$. Since $v^- \in H_0^1(\Omega)$ and $\sqrt{V_+}v^- \in L^2(\Omega)$ by (2.39), we have, by the definitions of $\tilde{\mathcal{H}}_{\tilde{V}}$ and $\mathcal{H}_V$,

$$\begin{aligned}&\int_{\mathbb{R}^d} \big[(\tilde{\mathcal{H}}_{\tilde{V}} + M)v^{(1)}\big]\tilde{v}^-\,dx - \int_\Omega \big[(\mathcal{H}_V + M)v^{(2)}\big]v^-\,dx \\ &= \int_\Omega \nabla v\cdot\nabla v^-\,dx + \int_\Omega V v v^-\,dx + \int_\Omega M v v^-\,dx \\ &\leq (b_1 - M)\|v^-\|_{L^2(\Omega)}^2 \\ &\leq 0,\end{aligned}$$

since M is chosen as in (2.36). Hence we obtain (2.40), which implies the required inequality (2.18).

We have to prove (2.39). The proof is similar to that of Lemma 2.7. Since $v^{(2)}(t) \in H_0^1(\Omega)$ for each $t > 0$ by (2.38), there exists a sequence $\{\phi_n(t)\}$ in $C_0^\infty(\Omega)$ such that

$$\phi_n(t) \to v^{(2)}(t) \quad \text{in } H^1(\Omega) \text{ as } n \to \infty$$

for each $t > 0$. Put

$$v_n(t) := v^{(1)}(t)|_\Omega - \phi_n(t), \quad n \in \mathbb{N}.$$

Let $\{\psi_n\}$ be the sequence as in (2.25). As in the proof of Lemma 2.7, we can show that

$$\psi_n \circ v_n^- \to v^- \quad \text{in } H^1(\Omega) \text{ as } n \to \infty.$$

Since v_n^- have compact supports in $\operatorname{supp}\phi_n$ by $v^{(1)} \geq 0$ on Ω, it follows that the functions $\psi_n \circ v_n^-$ also have compact supports in Ω. Let $(\psi_n \circ v_n^-)\tilde{}$ be the zero extension of $\psi_n \circ v_n^-$ to $\mathbb{R}^d$, and $j_\varepsilon(x)$ be the functions defined in (2.20). Taking ε along a sequence $\{\varepsilon_n\}$ such that

$$\varepsilon_n \searrow 0 \quad \text{and} \quad \operatorname{supp} j_{\varepsilon_n} * (\psi_n \circ v_n^-)\tilde{} \Subset \Omega \quad \text{for any } n \in \mathbb{N},$$

we have

$$j_{\varepsilon_n} * (\psi_n \circ v_n^-)\tilde{}\,|_\Omega \in C_0^\infty(\Omega) \quad \text{for any } n \in \mathbb{N}.$$

Since

$$j_{\varepsilon_n} * (\psi_n \circ v_n^-)\tilde{}\,|_\Omega \to v^- \quad \text{in } H^1(\Omega) \text{ as } n \to \infty,$$

we conclude (2.39).

Finally, as to the inequality (2.19), letting $f \in L^2(\Omega)$ be non-negative, we put

$$w^{(1)}(t) := e^{-t(\tilde{\mathcal{H}}_{\tilde{V}_-}+M)}\tilde{f}, \quad w^{(2)}(t) := e^{-t(\tilde{\mathcal{H}}_{\tilde{V}}+M)}\tilde{f}, \quad w(t) := w^{(1)}(t) - w^{(2)}(t)$$

for $t \geq 0$. Noting that $w^{(1)}(t) \in \mathcal{D}(\tilde{\mathcal{H}}_{\tilde{V}_-})$ and $w^{(2)}(t) \in \mathcal{D}(\tilde{\mathcal{H}}_{\tilde{V}})$, it suffices to show that

$$\frac{d}{dt}\int_\Omega (w^-)^2\,dx \leq 0. \tag{2.41}$$

We prove (2.41). In a similar way to (2.39), we have $w^- \in H_0^1(\Omega)$. Hence we estimate

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (w^-)^2\,dx &= -2\int_\Omega (\partial_t w)w^-\,dx \\
&= 2\int_\Omega \big[(\tilde{\mathcal{H}}_{\tilde{V}_-}+M)w^{(1)}\big]w^-\,dx - 2\int_\Omega \big[(\tilde{\mathcal{H}}_{\tilde{V}}+M)w^{(2)}\big]w^-\,dx \\
&= -2\int_\Omega (|\nabla w^-|^2 - \tilde{V}_-|w^-|^2 + M|w^-|^2)\,dx - 2\int_\Omega (\tilde{V}_+ w^{(2)})w^-\,dx \\
&\leq -2\int_\Omega (\tilde{V}_+ w^{(2)})w^-\,dx,
\end{aligned}$$

where we used the inequality (2.3) in the last step. Since $w^{(2)}(t) \geq 0$ by (2.17) and (2.18), we conclude the required inequality (2.41), which proves the inequality (2.19). The proof of Lemma 2.5 is complete. □

Chapter 3

Boundedness of spectral multipliers for Schrödinger operators

In this chapter the functional calculus of spectral multipliers for Schrödinger operators is developed, which was discussed in Iwabuchi, Matsuyama and Taniguchi [43].

3.1 Spectral multipliers

We consider the self-adjoint realization $\mathcal{H}_V$ of Schrödinger operator $-\Delta + V(x)$ whose existence is discussed in section 2.3. Let $\{E_{\mathcal{H}_V}(\lambda)\}_{\lambda\in\mathbb{R}}$ be the spectral resolution of the identity for $\mathcal{H}_V$. Here the resolution $\{E_{\mathcal{H}_V}(\lambda)\}_{\lambda\in\mathbb{R}}$ is uniquely determined for $\mathcal{H}_V$ by the spectral theorem. Then for any Borel measurable function ϕ on $\mathbb{R}$, an operator $\phi(\mathcal{H}_V)$ is defined by letting

$$\phi(\mathcal{H}_V) = \int_{-\infty}^{\infty} \phi(\lambda)\, dE_{\mathcal{H}_V}(\lambda)$$

with domain

$$\mathcal{D}(\phi(\mathcal{H}_V)) = \left\{ f \in L^2(\Omega) : \int_{-\infty}^{\infty} |\phi(\lambda)|^2\, d(E_{\mathcal{H}_V}(\lambda)f, f)_{L^2(\Omega)} < \infty \right\}.$$

The operator $\phi(\mathcal{H}_V)$ is called the spectral multiplier for $\mathcal{H}_V$.

The purpose in this chapter is to study functional calculus of spectral multipliers $\phi(\mathcal{H}_V)$. More precisely, we prove uniform L^p-L^q-estimates and gradient estimates for $\phi(\theta\mathcal{H}_V)$ with respect to a parameter $\theta > 0$. The motivation comes from the point of view of harmonic analysis and partial differential equations. For instance, the spectral multiplier is a generalization of Fourier multiplier in the

following sense: When $\Omega = \mathbb{R}^d$ and $V = 0$, i.e., $\mathcal{H}_V = -\Delta$ on $L^2(\mathbb{R}^d)$, the spectral multiplier coincides with the Fourier multiplier, i.e.,

$$\phi(-\Delta) = \mathscr{F}^{-1}\big[\phi(|\cdot|^2)\mathscr{F}\big],$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and inverse Fourier transform on $\mathbb{R}^d$. These estimates play a fundamental role in studying functions spaces such as Sobolev spaces, Hardy space, BMO spaces, Besov spaces and Triebel-Lizorkin spaces generated by the Schrödinger operators (see [1,9,18,21,28,38,44,49,53,88]). The theory of spectral multipliers is also related to the study of convergence of the Riesz means or convergence of eigenfunction expansion of self-adjoint operators (see, e.g., Chapter IX in Stein [77]).

We shall prove the following:

Theorem 3.1. *Let $\phi \in \mathscr{S}(\mathbb{R})$ and $1 \le p \le q \le \infty$. Suppose that the potential V satisfies assumption* A. *Then $\phi(\mathcal{H}_V)$ is extended to a bounded linear operator from $L^p(\Omega)$ to $L^q(\Omega)$. Furthermore, the following assertions hold*:

(i) *There exists a constant $C > 0$ such that*

$$\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \tag{3.1}$$

for any $0 < \theta \le 1$.

(ii) *If V_- further satisfies assumption* B, *then the estimate* (3.1) *holds for any $\theta > 0$.*

Theorem 3.2. *Let $\phi \in \mathscr{S}(\mathbb{R})$ and $1 \le p \le q \le 2$. Suppose that the potential V satisfies assumption* A. *Then $\phi(\mathcal{H}_V)$ is extended to a bounded linear operator from $L^p(\Omega)$ to $W^{1,q}(\Omega)$. Furthermore, the following assertions hold*:

(i) *There exists a constant $C > 0$ such that*

$$\|\nabla\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \tag{3.2}$$

for any $0 < \theta \le 1$.

(ii) *If V_- further satisfies assumption* B, *then the estimate* (3.2) *holds for any $\theta > 0$.*

Remark. The potential like

$$V(x) \simeq -c|x|^{-2} \quad \text{as } |x| \to \infty, \quad c > 0$$

is very interesting. However, it is excluded from assumption A on V. The reason is that the uniform boundedness in Theorem 3.1 would not be generally obtained, since

$$\lim_{t\to\infty} \|e^{-t\mathcal{H}_V}\|_{\mathscr{B}(L^p(\Omega))} = \infty$$

for some $p \neq 2$ which was proved in [34, 35].

Remark. We can weaken the assumption that $\phi \in \mathscr{S}(\mathbb{R})$ in Theorems 3.1 and 3.2. In fact, we have

$$\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \leq C_0 \left\|(1+|\cdot|^2)^{\frac{\beta+m}{2}}\phi\right\|_{H^m(\mathbb{R})},$$

where m is an integer with $m > (d+1)/2$ and β is a real number with $\beta > d/4 + (d/2)(1/p - 1/q)$. For the details, see section 3.7.

One of the main ingredients of this chapter is to reveal that we are able to deal with a potential whose negative part is of Kato class on open sets. The advantage of the present argument is to provide a unified treatment of the proof of Theorems 3.1 and 3.2. For this purpose, we introduce scaled amalgam spaces on Ω and apply the resolvent estimates in the amalgam spaces and some commutator estimates. This idea comes from Jensen and Nakamura [44, 45].

This chapter is organized as follow. Section 3.2 is devoted to proving the uniform estimates in scaled amalgam spaces for the resolvent of $\mathcal{H}_V$. In section 3.3 some commutator estimates are derived. In section 3.4 we prove estimates for the spectral multipliers in amalgam spaces. Based on these estimates, the proofs of Theorem 3.1 and Theorem 3.2 are given in sections 3.5 and 3.6, respectively.

3.2 Resolvent estimates in amalgam spaces

In this section we shall prove boundedness of the resolvent of $\theta\mathcal{H}_V$ in scaled amalgam spaces. The result in this section plays an important role in the proof of Theorem 3.1.

Following Fournier and Stewart [23], let us give the definition of scaled amalgam spaces on Ω as follows.

Definition. Let $1 \leq p, q \leq \infty$ and $\theta > 0$. The space $l^p(L^q)_\theta$ is defined by letting

$$l^p(L^q)_\theta = l^p(L^q)_\theta(\Omega) := \left\{ f \in L^q_{\mathrm{loc}}(\overline{\Omega}) : \sum_{n\in\mathbb{Z}^d} \|f\|^p_{L^q(C_\theta(n))} < \infty \right\}$$

with norm

$$\|f\|_{l^p(L^q)_\theta} = \begin{cases} \left(\sum_{n\in\mathbb{Z}^d} \|f\|^p_{L^q(C_\theta(n))} \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \sup_{n\in\mathbb{Z}^d} \|f\|_{L^q(C_\theta(n))} & \text{for } p = \infty, \end{cases}$$

where $C_\theta(n)$ is the intersection of Ω and the cube centered at $\theta^{1/2}n$ ($n \in \mathbb{Z}^d$) with side length $\theta^{1/2}$:

$$C_\theta(n) = \left\{ x = (x_1, x_2, \cdots, x_d) \in \Omega : \max_{j=1,\cdots,d} |x_j - \theta^{\frac{1}{2}} n_j| \le \frac{\theta^{\frac{1}{2}}}{2} \right\}.$$

Here we adopt the Euclidean norm for $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$:

$$|n| = \sqrt{n_1^2 + n_2^2 + \cdots + n_d^2}.$$

Let us give a few remarks on the properties of amalgam spaces. The spaces $l^p(L^q)_\theta$ are complete with respect to the norm $\|\cdot\|_{l^p(L^q)_\theta}$. Furthermore, these spaces enjoy the following embedding:

$$l^p(L^q)_\theta \hookrightarrow L^p(\Omega) \cap L^q(\Omega) \tag{3.3}$$

for any $\theta > 0$, provided $1 \le p \le q \le \infty$. In fact, noting that

$$|C_\theta(n)| \le \theta^{\frac{d}{2}},$$

we estimate, by using Hölder's inequality,

$$\begin{aligned} \|f\|_{L^p(\Omega)} &= \left(\sum_{n \in \mathbb{Z}^d} \|f\|^p_{L^p(C_\theta(n))} \right)^{\frac{1}{p}} \\ &\le \left(\sum_{n \in \mathbb{Z}^d} |C_\theta(n)|^{1-\frac{p}{q}} \|f\|^p_{L^q(C_\theta(n))} \right)^{\frac{1}{p}} \\ &\le \theta^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{l^p(L^q)_\theta} \end{aligned} \tag{3.4}$$

for any $\theta > 0$ and $1 \le p \le q \le \infty$, which implies that

$$l^p(L^q)_\theta \hookrightarrow L^p(\Omega). \tag{3.5}$$

On the other hand, since $l^p \hookrightarrow l^q$ for $1 \le p \le q \le \infty$, and since $l^q(L^q)_\theta = L^q(\Omega)$ for $\theta > 0$ and $1 \le q \le \infty$, we deduce that

$$l^p(L^q)_\theta \hookrightarrow l^q(L^q)_\theta = L^q(\Omega). \tag{3.6}$$

Thus, (3.5) and (3.6) imply (3.3).

We have the Young inequality for scaled amalgam spaces:

$$\big\|f * g\big\|_{l^p(L^q)_\theta(\mathbb{R}^d)} \le 3^d \|f\|_{l^{p_1}(L^{q_1})_\theta(\mathbb{R}^d)} \|g\|_{l^{p_2}(L^{q_2})_\theta(\mathbb{R}^d)} \tag{3.7}$$

for any $f \in l^{p_1}(L^{q_1})_\theta(\mathbb{R}^d)$ and $g \in l^{p_2}(L^{q_2})_\theta(\mathbb{R}^d)$, provided $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - 1 \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - 1.$$

For the proof of (3.7), see [23], and also [45].

The goal in this section is to prove the following:

Proposition 3.3. *Let $1 \le p \le q \le \infty$, and β be such that*

$$\beta > \frac{d}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{3.8}$$

Suppose that the potential V satisfies assumption A. *Let $z \in \mathbb{C}$ with*

$$\mathrm{Re}(z) < \min\{-\omega, 0\}, \tag{3.9}$$

where ω is the constant as in Proposition 2.4. Then $(\mathcal{H}_V - z)^{-\beta}$ is extended to a bounded linear operator from $L^p(\Omega)$ to $l^p(L^q)_\theta$ with $\theta = 1$. Furthermore, the following assertions hold:

(i) *There exists a constant C depending on d, p, q, β and z such that*

$$\|(\theta\mathcal{H}_V - z)^{-\beta}\|_{\mathscr{B}(L^p(\Omega), L^q(\Omega))} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}, \tag{3.10}$$

$$\|(\theta\mathcal{H}_V - z)^{-\beta}\|_{\mathscr{B}(L^p(\Omega), l^p(L^q)_\theta)} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \tag{3.11}$$

for any $0 < \theta \le 1$.

(ii) *Assume further that V_- satisfies* (2.14). *Let $z \in \mathbb{C}$ be such that*

$$\mathrm{Re}(z) < 0.$$

Then the estimate (3.10) *holds for any $\theta > 0$. Moreover, If V_- further satisfies assumption* B, *then the estimate* (3.11) *holds for any $\theta > 0$.*

Proof. First we prove (3.10). Let $0 < \theta \le 1$. We use the following formula:

$$(\mathcal{H}_V - z)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{zt} e^{-t\mathcal{H}_V}\, dt \tag{3.12}$$

for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) < \inf \sigma(\mathcal{H}_V)$ and $\beta > 0$. Thanks to (3.12) and L^p-L^q-estimates (2.12) for $e^{-t\theta\mathcal{H}_V}$ in Proposition 2.4, we estimate

$$\begin{aligned}
&\|(\theta\mathcal{H}_V - z)^{-\beta} f\|_{L^q(\Omega)} \\
&\le \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{\mathrm{Re}(z)t} \|e^{-t\theta\mathcal{H}_V} f\|_{L^q(\Omega)}\, dt \\
&\le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \bigg(\int_0^\infty t^{\beta-1} e^{[\mathrm{Re}(z) - \min\{-\omega,0\}]t}\, t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\, dt \bigg) \|f\|_{L^p(\Omega)}
\end{aligned}$$

for any $f \in L^p(\Omega)$, provided $1 \le p \le q \le \infty$, where C is independent of θ. Here, let us take z as in (3.9). Then the integral on the right is absolutely convergent, since β satisfies the inequality (3.8). This proves (3.10).

Let us turn to the proof of (3.11). If we prove that there exists a constant $C > 0$ such that

$$\|e^{-t\theta\mathcal{H}_V} f\|_{l^p(L^q)_\theta} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\big\{t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} + 1\big\}e^{-\min\{-\omega,0\}t}\|f\|_{L^p(\Omega)} \tag{3.13}$$

for any $t > 0$ and $f \in L^p(\Omega)$ provided $1 \le p \le q \le \infty$, then the estimate (3.11) is obtained by combining (3.12) and (3.13). In fact, by using (3.12), we estimate

$$\begin{aligned}
&\|(\theta\mathcal{H}_V - z)^{-\beta} f\|_{l^p(L^q)_\theta} \\
&\le \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{\mathrm{Re}(z)t} \|e^{-t\theta\mathcal{H}_V} f\|_{l^p(L^q)_\theta}\, dt \\
&\le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \left(\int_0^\infty t^{\beta-1} e^{[\mathrm{Re}(z)-\min\{-\omega,0\}]t} \big\{t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} + 1\big\}\, dt \right) \|f\|_{L^p(\Omega)}.
\end{aligned}$$

Here the integral on the right is absolutely convergent, since z satisfies (3.9) and β satisfies (3.8). This proves (3.11). Therefore, all we have to do is to prove the estimate (3.13).

To this end, we recall the estimate (2.13) from Proposition 2.4. We define the right member of (2.13) as $K_0(t, x-y)$, i.e.,

$$K_0(t,x) = C_2 t^{-\frac{d}{2}} e^{\omega t} e^{-\frac{|x|^2}{8t}}, \quad t > 0, \quad x \in \mathbb{R}^d.$$

Now, letting $1 \le r \le \infty$, we prove that

$$\|K_0(\theta t, \cdot)\|_{l^1(L^r)_\theta} \le C\theta^{-\frac{d}{2}(1-\frac{1}{r})}\big\{t^{-\frac{d}{2}(1-\frac{1}{r})} + 1\big\}e^{-\min\{-\omega,0\}t} \tag{3.14}$$

for any $t > 0$, where $C > 0$ is independent of θ. We estimate $L^r(C_\theta(n))$-norms of $K_0(\theta t, \cdot)$ for the case $n = 0$ and $n \ne 0$, separately.

The case $n = 0$: When $1 \le r < \infty$, we estimate

$$\begin{aligned}
\|K_0(\theta t, \cdot)\|_{L^r(C_\theta(0))} &\le C_2(\theta t)^{-\frac{d}{2}} e^{-\min\{-\omega,0\}\theta t} \left(\int_{\mathbb{R}^d} e^{-\frac{r|x|^2}{8\theta t}}\, dx \right)^{\frac{1}{r}} \\
&\le C_2(\theta t)^{-\frac{d}{2}(1-\frac{1}{r})} e^{-\min\{-\omega,0\}t} \left(\int_{\mathbb{R}^d} e^{-\frac{r|x|^2}{8}}\, dx \right)^{\frac{1}{r}} \qquad (3.15) \\
&= \frac{(8\pi)^{\frac{d}{2r}} C_2}{r^{\frac{d}{2r}}} (\theta t)^{-\frac{d}{2}(1-\frac{1}{r})} e^{-\min\{-\omega,0\}t}.
\end{aligned}$$

When $r = \infty$, we estimate

$$\begin{aligned}
\|K_0(\theta t, \cdot)\|_{L^\infty(C_\theta(0))} &= C_2(\theta t)^{-\frac{d}{2}} e^{-\min\{-\omega,0\}\theta t} \left(\sup_{x \in C_\theta(0)} e^{-\frac{|x|^2}{8\theta t}} \right) \qquad (3.16) \\
&\le C_2(\theta t)^{-\frac{d}{2}} e^{-\min\{-\omega,0\}t}.
\end{aligned}$$

The case $n \neq 0$: We estimate

$$\begin{aligned}\sum_{n\neq 0} \|K_0(\theta t,\cdot)\|_{L^r(C_\theta(n))} &\le \sum_{n\neq 0} \|K_0(\theta t,\cdot)\|_{L^\infty(C_\theta(n))}|C_\theta(n)|^{\frac{1}{r}} \\ &= C_2(\theta t)^{-\frac{d}{2}} e^{-\min\{-\omega,0\}\theta t} \sum_{n\neq 0}\left(\sup_{x\in C_\theta(n)} e^{-\frac{|x|^2}{8\theta t}}\right)\cdot |C_\theta(n)|^{\frac{1}{r}} \\ &\le C_2(\theta t)^{-\frac{d}{2}} e^{-\min\{-\omega,0\}t}\left(\sum_{n\neq 0} e^{-\frac{|n|^2}{32t}}\right)(\theta^{\frac{d}{2}})^{\frac{1}{r}},\end{aligned}$$

where we used in the last step

$$\frac{|\theta^{\frac{1}{2}}n|}{2} \le |x| \left(\le 2|\theta^{\frac{1}{2}}n|\right), \quad x \in C_\theta(n).$$

Here, by an explicit calculation, we see that

$$\sum_{n\neq 0} e^{-\frac{|n|^2}{32t}} = 2^d\left(\sum_{j=1}^{\infty} e^{-\frac{j^2}{32t}}\right)^d \le 2^d\left(\int_0^\infty e^{-\frac{\sigma^2}{32t}}\,d\sigma\right)^d = (8\sqrt{2})^d \pi^{\frac{d}{2}} t^{\frac{d}{2}}.$$

Summarizing the estimates obtained now, we conclude that

$$\begin{aligned}\sum_{n\neq 0} \|K_0(\theta t,\cdot)\|_{L^r(C_\theta(n))} &\le C_2(\theta t)^{-\frac{d}{2}} e^{-\min\{-\omega,0\}t}\cdot (8\sqrt{2})^d \pi^{\frac{d}{2}} t^{\frac{d}{2}} \cdot (\theta^{\frac{d}{2}})^{\frac{1}{r}} \\ &= (8\sqrt{2})^d \pi^{\frac{d}{2}} C_2 \theta^{-\frac{d}{2}(1-\frac{1}{r})} e^{-\min\{-\omega,0\}t}\end{aligned} \tag{3.17}$$

for any $r \in [1,\infty]$.

Combining the estimates (3.15), (3.16) and (3.17), we get (3.14), as desired.

We are now in a position to prove the key estimate (3.13). Let $f \in L^p(\Omega)$ and $\tilde{f}$ be a zero extension of f to $\mathbb{R}^d$. Thanks to the estimate (2.13) from Proposition 2.4, i.e.,

$$0 \le e^{-t\mathcal{H}_V}(x,y) \le K_0(t,x-y) \quad \text{a.e. } x,y \in \Omega$$

for any $t > 0$, we estimate

$$\begin{aligned}\|e^{-t\theta\mathcal{H}_V} f\|_{l^p(L^q)_\theta} &\le \left\| \int_\Omega K(\theta t,\cdot,y)|f(y)|\,dy\right\|_{l^p(L^q)_\theta} \\ &\le \left\| \int_{\mathbb{R}^d} K_0(\theta t,\cdot - y)|\tilde{f}(y)|\,dy\right\|_{l^p(L^q)_\theta(\mathbb{R}^d)}.\end{aligned}$$

Applying the Young inequality (3.7) to the right member, and using the inequality (3.14), we deduce that

$$\begin{aligned}\|e^{-t\theta\mathcal{H}_V} f\|_{l^p(L^q)_\theta} &\le 3^d \|K_0(\theta t,\cdot)\|_{l^1(L^r)_\theta(\mathbb{R}^d)} \|\tilde{f}\|_{l^p(L^p)_\theta(\mathbb{R}^d)} \\ &\le C\theta^{-\frac{d}{2}(1-\frac{1}{r})}\left\{t^{-\frac{d}{2}(1-\frac{1}{r})}+1\right\} e^{-\min\{-\omega,0\}t}\|\tilde{f}\|_{L^p(\mathbb{R}^d)} \\ &= C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\left\{t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}+1\right\} e^{-\min\{-\omega,0\}t}\|f\|_{L^p(\Omega)},\end{aligned}$$

provided that p, q, r satisfy $1 \le p, q, r \le \infty$ and $1/p + 1/r - 1 = 1/q$. This proves (3.13).

Finally, the proof of the assertion (ii) is done by the same argument as in (i), if we apply (2.15) and (2.16) to the identity (3.12). So we may omit the details. The proof of Proposition 3.3 is finished. □

3.3 Commutator estimates

In this section we shall prepare commutator estimates. These estimates will be also an important tool in the proof of Theorem 3.1. To begin with, we introduce operators $\mathrm{Ad}^k(L)$ for some operator L as follows:

Definition. Let X and Y be topological vector spaces, and let A and B be continuous linear operators from X and Y into themselves, respectively. For a continuous linear operator L from X into Y, the operators $\mathrm{Ad}^k(L)$ from X into Y, $k = 0, 1, \ldots$, are successively defined by

$$\mathrm{Ad}^0(L) = L, \quad \mathrm{Ad}^k(L) = \mathrm{Ad}^{k-1}(BL - LA), \quad k \ge 1.$$

It is known that the following recursive formula holds: There exists a set of constants $\{C(n,m) : n \ge 0,\, 0 \le m \le n\}$ such that

$$B^n L = \sum_{m=0}^{n} C(n,m) \mathrm{Ad}^m(L) A^{n-m} \tag{3.18}$$

(see Lemma 3.1 in [45]).

The result in this section is concerned with L^2-boundedness for $\mathrm{Ad}^k(e^{-itR_{V,\theta}})$, where $R_{V,\theta}$ is the resolvent operator defined by letting

$$R_{V,\theta} := (\theta \mathcal{H}_V + M)^{-1}, \quad \theta > 0$$

for a fixed constant M with $M > \max\{-\inf \sigma(\mathcal{H}_V), 0\}$. Hereafter we put

$$X = \mathscr{D}(\Omega), \quad Y = \mathscr{D}'(\Omega),$$

where we denote by $\mathscr{D}(\Omega)$ the topological vector space consisting of smooth functions on Ω with compact support, and by $\mathscr{D}'(\Omega)$ its dual space, and we take A and B as

$$A = B = x_j - \theta^{\frac{1}{2}} n_j \quad \text{for some } j \in \{1, \cdots, d\}. \tag{3.19}$$

Then we shall prove here the following.

Proposition 3.4. *Suppose that the potential V satisfies assumption* A. *Let A and B be the operators as in* (3.19). *Then for any non-negative integer k, the following assertions hold*:

(i) *There exists a constant $C > 0$ depending on d, k and M such that*

$$\|\mathrm{Ad}^k(e^{-itR_{V,\theta}})\|_{\mathscr{B}(L^2(\Omega))} \le C\theta^{\frac{k}{2}}(1+|t|)^k \tag{3.20}$$

for any $t \in \mathbb{R}$ and $0 < \theta \le 1$.

(ii) *If V_- further satisfies assumption C given in section 2.3, then the estimate* (3.20) *holds for any $t \in \mathbb{R}$ and $\theta > 0$.*

We prepare some lemmas in order to prove Proposition 3.4. First, we show L^2-boundedness of $R_{V,\theta}$ and $\nabla R_{V,\theta}$.

Lemma 3.5. *Suppose that the potential V satisfies assumption* A. *Then the following assertions hold*:

(i) *There exists a constant $C > 0$ such that*

$$\|R_{V,\theta}\|_{\mathscr{B}(L^2(\Omega))} \le \frac{1}{M + \min\{\inf\sigma(\mathcal{H}_V), 0\}}, \tag{3.21}$$

$$\|\nabla R_{V,\theta}\|_{\mathscr{B}(L^2(\Omega))} \le C\theta^{-\frac{1}{2}} \tag{3.22}$$

for any $0 < \theta \le 1$.

(ii) *If V_- further satisfies $\|V_-\|_{K_d(\Omega)} < \infty$, then*

$$\|R_{V,\theta}\|_{\mathscr{B}(L^2(\Omega))} \le M^{-1}, \tag{3.23}$$

$$\|\nabla R_{V,\theta}\|_{\mathscr{B}(L^2(\Omega))} \le M^{-\frac{1}{2}}\left(1 - \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d}\right)^{-\frac{1}{2}}\theta^{-\frac{1}{2}} \tag{3.24}$$

for any $\theta > 0$.

Proof. First we prove the assertion (i). Since $\mathcal{H}_V$ is the self-adjoint operator on $L^2(\Omega)$, we obtain (3.21), (3.22), (3.23) and (3.24) by the spectral resolution. In fact, we have

$$\begin{aligned}\|R_{V,\theta}f\|^2_{L^2(\Omega)} &= \int_{\inf\sigma(\mathcal{H}_V)}^{\infty} \frac{1}{(\theta\lambda + M)^2}\, d\|E_{\mathcal{H}_V}(\lambda)f\|^2_{L^2(\Omega)} \\ &\le \begin{cases} \dfrac{1}{M^2}\|f\|^2_{L^2(\Omega)} & \text{if } \inf\sigma(\mathcal{H}_V) \ge 0, \\ \dfrac{1}{[M + \inf\sigma(\mathcal{H}_V)]^2}\|f\|^2_{L^2(\Omega)} & \text{if } \inf\sigma(\mathcal{H}_V) < 0 \end{cases}\end{aligned}$$

for any $f \in L^2(\Omega)$, since $0 < \theta \le 1$. This proves (3.21).

Next we consider the estimate for $\nabla R_{V,\theta} f$. Since $R_{V,\theta} f \in \mathcal{D}(\mathcal{H}_V)$ for any $f \in L^2(\Omega)$, we estimate

$$\begin{aligned}
\|\nabla R_{V,\theta} f\|_{L^2(\Omega)}^2 &= \int_\Omega \Big(\nabla R_{V,\theta} f \cdot \nabla R_{V,\theta} f + V|R_{V,\theta} f|^2 - V|R_{V,\theta} f|^2 \Big)\, dx \\
&= (\mathcal{H}_V R_{V,\theta} f, R_{V,\theta} f)_{L^2(\Omega)} + \int_\Omega (V_- - V_+)|R_{V,\theta} f|^2\, dx \\
&\le (\mathcal{H}_V R_{V,\theta} f, R_{V,\theta} f)_{L^2(\Omega)} + \int_\Omega V_- |R_{V,\theta} f|^2\, dx \\
&=: I + II.
\end{aligned}$$

Then we estimate the first term I as

$$\begin{aligned}
I &= \int_{\inf \sigma(\mathcal{H}_V)}^\infty \frac{\lambda}{(\theta\lambda + M)^2}\, d\|E_{\mathcal{H}_V}(\lambda) f\|_{L^2(\Omega)}^2 \\
&\le \int_{\max\{\inf \sigma(\mathcal{H}_V), 0\}}^\infty \theta^{-1} \cdot \frac{\theta\lambda}{\theta\lambda + M} \cdot \frac{1}{\theta\lambda + M}\, d\|E_{\mathcal{H}_V}(\lambda) f\|_{L^2(\Omega)}^2 \\
&\le M^{-1}\theta^{-1} \int_{\inf \sigma(\mathcal{H}_V)}^\infty d\|E_{\mathcal{H}_V}(\lambda) f\|_{L^2(\Omega)}^2 \\
&= M^{-1}\theta^{-1} \|f\|_{L^2(\Omega)}^2.
\end{aligned}$$

As to the second term II, by using the inequality (2.3) for $\varepsilon \in (0,1)$ from Lemma 2.3 and estimate (3.21), we have

$$\begin{aligned}
II &\le \varepsilon \|\nabla R_{V,\theta} f\|_{L^2(\Omega)}^2 + b_\varepsilon \|R_{V,\theta} f\|_{L^2(\Omega)}^2 \\
&\le \varepsilon \|\nabla R_{V,\theta} f\|_{L^2(\Omega)}^2 + C b_\varepsilon \theta^{-1} \|f\|_{L^2(\Omega)}^2,
\end{aligned} \tag{3.25}$$

since $0 < \theta \le 1$. Combining the above three estimates, we conclude the estimate (3.22).

We now turn to the proof of (ii). In this case we have $\inf \sigma(\mathcal{H}_V) \ge 0$. It is sufficient to prove only the estimate (3.24) for $\nabla R_{V,\theta} f$, since the proof of (3.23) is similar to (3.21). If V_- satisfies $\|V_-\|_{K_d(\Omega)} < \infty$, then we have, by using the inequality (2.4) from Lemma 2.3,

$$II \le \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d} \|\nabla R_{V,\theta} f\|_{L^2(\Omega)}^2.$$

Using this estimate instead of (3.25), the estimate (3.24) is proved for any $\theta > 0$ in the same way as (3.22). The proof of Lemma 3.5 is complete. □

Next, we shall introduce two formulas on the operator Ad.

Lemma 3.6. *The sequence* $\{\mathrm{Ad}^k(R_{V,\theta})\}_{k=0}^\infty$ *of operators satisfies the following recursive formula*:

$$\mathrm{Ad}^0(R_{V,\theta}) = R_{V,\theta}, \quad \mathrm{Ad}^1(R_{V,\theta}) = -2\theta R_{V,\theta} \partial_{x_j} R_{V,\theta}, \tag{3.26}$$

$$\mathrm{Ad}^k(R_{V,\theta}) = \theta\big\{ -2k\mathrm{Ad}^{k-1}(R_{V,\theta})\partial_{x_j}R_{V,\theta} + k(k-1)\mathrm{Ad}^{k-2}(R_{V,\theta})R_{V,\theta}\big\} \tag{3.27}$$

for $k \geq 2$.

Lemma 3.7. *For all* $k \geq 0$*, the following formulas hold*:

$$\begin{aligned} &\mathrm{Ad}^{k+1}(e^{-itR_{V,\theta}}) \\ &= -i\int_0^t \sum_{k_1+k_2+k_3=k} \Gamma(k_1,k_2,k_3)\mathrm{Ad}^{k_1}(e^{-isR_{V,\theta}})\mathrm{Ad}^{k_2+1}(R_{V,\theta})\mathrm{Ad}^{k_3}(e^{-i(t-s)R_{V,\theta}})\,ds \end{aligned} \tag{3.28}$$

for each $t \in \mathbb{R}$*, where the constants* $\Gamma(k_1,k_2,k_3)$ $(k_1,k_2,k_3 \geq 0)$ *are trinomial coefficients*:

$$\Gamma(k_1,k_2,k_3) = \frac{k!}{k_1!k_2!k_3!}.$$

Proof of Lemma 3.6. When $k=0$, the first equation in (3.26) is trivial. Hence it is sufficient to prove the case when $k \geq 1$. For the sake of simplicity, we perform a formal argument without considering the domain of operators. The rigorous argument is given in the final part.

Let us introduce the generalized binomial coefficients $\Gamma(k,m)$ as follows:

$$\Gamma(k,m) = \begin{cases} \dfrac{k!}{(k-m)!m!}, & k \geq m \geq 0, \\ 0, & k < m \text{ or } k < 0. \end{cases}$$

Once the following recursive formula is established:

$$\mathrm{Ad}^k(R_{V,\theta}) = -\sum_{m=0}^{k-1}\Gamma(k,m)\mathrm{Ad}^m(R_{V,\theta})\mathrm{Ad}^{k-m}(\theta\mathcal{H}_V)R_{V,\theta}, \quad k \geq 1, \tag{3.29}$$

the identities (3.26) and (3.27) are an immediate consequence of (3.29), since

$$\mathrm{Ad}^1(\theta\mathcal{H}_V) = 2\theta\partial_{x_j}, \quad \mathrm{Ad}^2(\theta\mathcal{H}_V) = -2\theta, \quad \mathrm{Ad}^k(\theta\mathcal{H}_V) = 0, \quad k \geq 3.$$

Hence, all we have to do is to prove (3.29). We proceed the argument by induction. For $k=1$, it can be readily checked that

$$\begin{aligned} \mathrm{Ad}^1(R_{V,\theta}) &= x_jR_{V,\theta} - R_{V,\theta}x_j \\ &= R_{V,\theta}(\theta\mathcal{H}_V + M)x_jR_{V,\theta} - R_{V,\theta}x_j(\theta\mathcal{H}_V + M)R_{V,\theta} \\ &= R_{V,\theta}\big(\theta\mathcal{H}_Vx_j - x_j\cdot\theta\mathcal{H}_V\big)R_{V,\theta} \\ &= -R_{V,\theta}\mathrm{Ad}^1(\theta\mathcal{H}_V)R_{V,\theta} \\ &= -\Gamma(1,0)\mathrm{Ad}^0(R_{V,\theta})\mathrm{Ad}^1(\theta\mathcal{H}_V)R_{V,\theta}. \end{aligned}$$

Hence (3.29) is true for $k = 1$. For $l \geq 2$, let us suppose that (3.29) holds for $k = 1, \ldots, l$. Writing

$$\mathrm{Ad}^{l+1}(R_{V,\theta}) = x_j \mathrm{Ad}^l(R_{V,\theta}) - \mathrm{Ad}^l(R_{V,\theta}) x_j, \tag{3.30}$$

we see that the first term becomes

$$\begin{aligned}
&x_j \mathrm{Ad}^l(R_{V,\theta}) \\
&= x_j \Big\{ -\sum_{m=0}^{l-1} \Gamma(l,m) \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l-m}(\theta \mathcal{H}_V) \Big\} R_{V,\theta} \\
&= -\sum_{m=0}^{l-1} \Gamma(l,m) \big\{ \mathrm{Ad}^{m+1}(R_{V,\theta}) \mathrm{Ad}^{l-m}(\theta \mathcal{H}_V) + \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l-m+1}(\theta \mathcal{H}_V) \big\} R_{V,\theta} \\
&\quad - \sum_{m=0}^{l-1} \Gamma(l,m) \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l-m}(\theta \mathcal{H}_V) x_j R_{V,\theta} \\
&=: I_1 + I_2.
\end{aligned}$$

Here I_1 is written as

$$\begin{aligned}
I_1 &= -\sum_{m=1}^{l} \Gamma(l,m-1) \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l-m+1}(\theta \mathcal{H}_V) R_{V,\theta} \\
&\quad - \sum_{m=0}^{l-1} \Gamma(l,m) \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l-m+1}(\theta \mathcal{H}_V) R_{V,\theta} \\
&= -\sum_{m=0}^{l} \Gamma(l,m-1) \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l+1-m}(\theta \mathcal{H}_V) R_{V,\theta} \\
&\quad - \sum_{m=0}^{l} \Gamma(l,m) \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l-m+1}(\theta \mathcal{H}_V) R_{V,\theta} + \mathrm{Ad}^l(R_{V,\theta}) \mathrm{Ad}^1(\theta \mathcal{H}_V) R_{V,\theta} \\
&= -\sum_{m=0}^{l} \Gamma(l+1,m) \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l+1-m}(\theta \mathcal{H}_V) R_{V,\theta} + \mathrm{Ad}^l(R_{V,\theta}) \mathrm{Ad}^1(\theta \mathcal{H}_V) R_{V,\theta},
\end{aligned}$$

where we used in the last step

$$\Gamma(l,m-1) + \Gamma(l,m) = \Gamma(l+1,m).$$

As to I_2, we write as

$$\begin{aligned}
I_2 &= -\Big\{ \sum_{m=0}^{l-1} \Gamma(l,m) \mathrm{Ad}^m(R_{V,\theta}) \mathrm{Ad}^{l-m}(\theta \mathcal{H}_V) R_{V,\theta} \Big\} (\theta \mathcal{H}_V + M) x_j R_{V,\theta} \\
&= \mathrm{Ad}^l(R_{V,\theta}) (\theta \mathcal{H}_V + M) x_j R_{V,\theta}.
\end{aligned}$$

Hence, summarizing the previous equations, we get

$$x_j \mathrm{Ad}^l(R_{V,\theta}) = -\sum_{m=0}^{l} \Gamma(l+1,m)\mathrm{Ad}^m(R_{V,\theta})\mathrm{Ad}^{l+1-m}(\theta\mathcal{H}_V) + \mathrm{Ad}^l(R_{V,\theta})\big\{\mathrm{Ad}^1(\theta\mathcal{H}_V) + (\theta\mathcal{H}_V + M)x_j\big\}R_{V,\theta}.$$

Therefore, going back to (3.30), and noting

$$\mathrm{Ad}^1(\theta\mathcal{H}_V) + (\theta\mathcal{H}_V + M)x_j = x_j(\theta\mathcal{H}_V + M),$$

we conclude that

$$\begin{aligned}\mathrm{Ad}^{l+1}(R_{V,\theta}) &= -\sum_{m=0}^{l} \Gamma(l+1,m)\mathrm{Ad}^m(R_{V,\theta})\mathrm{Ad}^{l+1-m}(\theta\mathcal{H}_V) \\ &\quad + \mathrm{Ad}^l(R_{V,\theta})\big\{\mathrm{Ad}^1(\theta\mathcal{H}_V) + (\theta\mathcal{H}_V + M)x_j\big\}R_{V,\theta} - \mathrm{Ad}^l(R_{V,\theta})x_j \\ &= -\sum_{m=0}^{l} \Gamma(l+1,m)\mathrm{Ad}^m(R_{V,\theta})\mathrm{Ad}^{l+1-m}(\theta\mathcal{H}_V) \\ &\quad + \mathrm{Ad}^l(R_{V,\theta})x_j(\theta\mathcal{H}_V + M)R_{V,\theta} - \mathrm{Ad}^l(R_{V,\theta})x_j \\ &= -\sum_{m=0}^{l} \Gamma(l+1,m)\mathrm{Ad}^m(R_{V,\theta})\mathrm{Ad}^{l+1-m}(\theta\mathcal{H}_V).\end{aligned}$$

Hence (3.29) is true for $k = l+1$.

The above proof is formal in the sense that the domain of operators is not taken into account in the argument. In fact, even for $f \in C_0^\infty(\Omega)$, each $x_j R_{V,\theta} f$ does not necessarily belong to the domain of $\mathcal{H}_V$, since we only know the fact that

$$R_{V,\theta} f \in \mathcal{D}(\mathcal{H}_V) = \big\{u \in H_0^1(\Omega) : \sqrt{V_+}u \in L^2(\Omega),\ \mathcal{H}_V u \in L^2(\Omega)\big\}.$$

Therefore, we should perform the argument by using a duality pair ${}_{\mathscr{D}'(\Omega)}\langle\cdot,\cdot\rangle_{\mathscr{D}(\Omega)}$ of $\mathscr{D}'(\Omega)$ and $\mathscr{D}(\Omega)$ in a rigorous way. We may prove the lemma only for $k = 1$. For, as to the case $k > 1$, the argument is done in a similar manner. Now we write

$$\begin{aligned}{}_{\mathscr{D}'(\Omega)}\langle \mathrm{Ad}^1(R_{V,\theta})f, g\rangle_{\mathscr{D}(\Omega)} &= (R_{V,\theta}f, x_j g)_{L^2(\Omega)} - (x_j f, R_{V,\theta}g)_{L^2(\Omega)} \\ &=: I - II\end{aligned}$$

for $f, g \in C_0^\infty(\Omega)$. Since $R_{V,\theta}f, R_{V,\theta}g \in H_0^1(\Omega)$, there exist two sequences $\{f_n\}_n$, $\{g_m\}_m$ in $C_0^\infty(\Omega)$ such that

$$f_n \to R_{V,\theta}f \quad \text{and} \quad g_m \to R_{V,\theta}g \quad \text{in } H^1(\Omega) \text{ as } n, m \to \infty.$$

Hence, since $x_j f_n, x_j g_m \in C_0^\infty(\Omega)$, we see that

$$\begin{aligned}
I &= \lim_{n\to\infty} (f_n, x_j g)_{L^2(\Omega)} \\
&= \lim_{n\to\infty} (x_j f_n, (\theta\mathcal{H}_V + M)R_{V,\theta} g)_{L^2(\Omega)} \\
&= \lim_{n\to\infty} \big\{\theta(\nabla(x_j f_n), \nabla R_{V,\theta} g)_{L^2(\Omega)} + ((\theta V + M)x_j f_n, R_{V,\theta} g)_{L^2(\Omega)}\big\} \\
&= \lim_{n,m\to\infty} \big\{\theta(\nabla(x_j f_n), \nabla g_m)_{L^2(\Omega)} + ((\theta V + M)x_j f_n, g_m)_{L^2(\Omega)}\big\} \\
&= \lim_{n,m\to\infty} \big\{\theta(f_n, \partial_{x_j} g_m)_{L^2(\Omega)} + \theta(x_j \nabla f_n, \nabla g_m)_{L^2(\Omega)} + ((\theta V + M)x_j f_n, g_m)_{L^2(\Omega)}\big\},
\end{aligned}$$

and in a similar way,

$$II = \lim_{n,m\to\infty} \big\{\theta(\partial_{x_j} f_n, g_m)_{L^2(\Omega)} + \theta(x_j \nabla f_n, \nabla g_m)_{L^2(\Omega)} + ((\theta V + M)x_j f_n, g_m)_{L^2(\Omega)}\big\}.$$

Then, combining the above equations, we deduce that

$$\begin{aligned}
{}_{\mathscr{D}'(\Omega)}\langle \mathrm{Ad}^1(R_{V,\theta}) f, g\rangle_{\mathscr{D}(\Omega)} &= \lim_{n,m\to\infty} \theta\big\{(f_n, \partial_{x_j} g_m)_{L^2(\Omega)} - (\partial_{x_j} f_n, g_m)_{L^2(\Omega)}\big\} \\
&= \lim_{n,m\to\infty} \theta(-2\partial_{x_j} f_n, g_m)_{L^2(\Omega)} \\
&= (-2\theta\partial_{x_j} R_{V,\theta} f, R_{V,\theta} g)_{L^2(\Omega)} \\
&= (-2\theta R_{V,\theta}\partial_{x_j} R_{V,\theta} f, g)_{L^2(\Omega)}
\end{aligned}$$

for any $f, g \in C_0^\infty(\Omega)$. Thus (3.26) is valid in a distributional sense. In a similar way, (3.27) can be also shown in a distributional sense. The proof of Lemma 3.6 is finished. □

Proof of Lemma 3.7. It is sufficient to prove the lemma without taking account of the domain of operators as in the proof of Lemma 3.6. We consider the case $k = 0$:

$$\mathrm{Ad}^1(e^{-itR_{V,\theta}}) = -i \int_0^t e^{-isR_{V,\theta}} \mathrm{Ad}^1(R_{V,\theta}) e^{-i(t-s)R_{V,\theta}}\, ds \tag{3.31}$$

for $t \geq 0$. We write

$$\begin{aligned}
\mathrm{Ad}^1(e^{-itR_{V,\theta}}) &= x_j e^{-itR_{V,\theta}} - e^{-itR_{V,\theta}} x_j \\
&= -\int_0^t \frac{d}{ds}\big(e^{-isR_{V,\theta}} x_j e^{-i(t-s)R_{V,\theta}}\big)\, ds \\
&= -i \int_0^t e^{-isR_{V,\theta}} (x_j R_{V,\theta} - R_{V,\theta} x_j) e^{-i(t-s)R_{V,\theta}}\, ds \\
&= -i \int_0^t e^{-isR_{V,\theta}} \mathrm{Ad}^1(R_{V,\theta}) e^{-i(t-s)R_{V,\theta}}\, ds.
\end{aligned}$$

This proves (3.31). The proof of (3.28) is performed by induction argument. So we may omit the details. The proof of Lemma 3.7 is complete. □

We are now in a position to prove Proposition 3.4.

Proof of Proposition 3.4. By Lemma 3.6, we have

$$\mathrm{Ad}^0(R_{V,\theta}) = R_{V,\theta}, \quad \mathrm{Ad}^1(R_{V,\theta}) = -2\theta R_{V,\theta}\partial_{x_j} R_{V,\theta}, \tag{3.32}$$

$$\mathrm{Ad}^k(R_{V,\theta}) = \theta\big\{ -2k\mathrm{Ad}^{k-1}(R_{V,\theta})\partial_{x_j} R_{V,\theta} + k(k-1)\mathrm{Ad}^{k-2}(R_{V,\theta})R_{V,\theta}\big\} \tag{3.33}$$

for $k \geq 2$.

First we prove the assertion (i). Let $0 < \theta \leq 1$. Since $R_{V,\theta}$ and $\partial_{x_j} R_{V,\theta}$ are bounded on $L^2(\Omega)$ by (3.21) and (3.22) from Lemma 3.5, operators $\mathrm{Ad}^k(R_{V,\theta})$ are also bounded on $L^2(\Omega)$ for each $k \geq 0$. Before going to prove the estimates (3.20), we prepare the following estimates for $\mathrm{Ad}^k(R_{V,\theta})$: For $k \geq 0$, there exists a constant $C_k > 0$ such that

$$\|\mathrm{Ad}^k(R_{V,\theta})\|_{\mathscr{B}(L^2(\Omega))} \leq C_k \theta^{\frac{k}{2}} \tag{3.34}$$

for any $0 < \theta \leq 1$. We prove (3.34) by induction. For $k = 0, 1$, it follows from the identity (3.32) and estimates (3.21) and (3.22) in Lemma 3.5 that

$$\|\mathrm{Ad}^0(R_{V,\theta})\|_{\mathscr{B}(L^2(\Omega))} = \|R_{V,\theta}\|_{\mathscr{B}(L^2(\Omega))} \leq C_0,$$

$$\|\mathrm{Ad}^1(R_{V,\theta})\|_{\mathscr{B}(L^2(\Omega))} = 2\theta\|R_{V,\theta}\partial_{x_j} R_{V,\theta}\|_{\mathscr{B}(L^2(\Omega))} \leq C_1\theta^{\frac{1}{2}}.$$

Let us suppose that (3.34) is true for $k \in \{0, 1, \ldots, l\}$. Combining identities (3.33) and estimates (3.21) and (3.24) from Lemma 3.5, we get (3.34) for $k = l + 1$:

$$\begin{aligned}
&\|\mathrm{Ad}^{l+1}(R_{V,\theta})\|_{\mathscr{B}(L^2(\Omega))} \\
&= \big\|\theta\big\{ -2(l+1)Ad^l(R_{V,\theta})\partial_{x_j} R_{V,\theta} + l(l+1)\mathrm{Ad}^{l-1}(R_{V,\theta})R_{V,\theta}\big\}\big\|_{\mathscr{B}(L^2(\Omega))} \\
&\leq 2l(l+1)\theta\big\{\|\mathrm{Ad}^l(R_{V,\theta})\|_{\mathscr{B}(L^2(\Omega))}\|\partial_{x_j} R_{V,\theta}\|_{\mathscr{B}(L^2(\Omega))} \\
&\qquad\qquad + \|\mathrm{Ad}^{l-1}(R_{V,\theta})\|_{\mathscr{B}(L^2(\Omega))}\|R_{V,\theta}\|_{\mathscr{B}(L^2(\Omega))}\big\} \\
&\leq C_{l+1}\theta\big\{\theta^{\frac{l}{2}} \cdot \theta^{-\frac{1}{2}} + \theta^{\frac{l-1}{2}}\big\} \\
&\leq C_{l+1}\theta^{\frac{l+1}{2}}.
\end{aligned}$$

Thus (3.34) is true for any $k \geq 0$.

We prove (3.20) also by induction. Clearly, (3.20) is true for $k = 0$. As to the case $k = 1$, by using the estimate (3.34) and the formula (3.28) with $k = 0$ in Lemma 3.7:

$$\mathrm{Ad}^1(e^{-itR_{V,\theta}}) = -i\int_0^t e^{-isR_{V,\theta}}\mathrm{Ad}^1(R_{V,\theta})e^{-i(t-s)R_{\theta,V}}\,ds$$

for each $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\|\mathrm{Ad}^1(e^{-itR_{V,\theta}})\|_{\mathscr{B}(L^2(\Omega))} \\
&\leq \int_0^{|t|} \|e^{-isR_{V,\theta}}\|_{\mathscr{B}(L^2(\Omega))}\|\mathrm{Ad}^1(R_{V,\theta})\|_{\mathscr{B}(L^2(\Omega))}\|e^{-i(t-s)R_{V,\theta}}\|_{\mathscr{B}(L^2(\Omega))}\,ds \\
&\leq C_1 \int_0^{|t|} \theta^{\frac{1}{2}}\,ds \leq C_1\theta^{\frac{1}{2}}(1+|t|)
\end{aligned}$$

for any $t \in \mathbb{R}$. Hence, (3.20) is true for $k = 1$. Let us suppose that (3.20) holds for $k \in \{0, 1, \ldots, l\}$. Then, by using the estimate (3.34) and the formula (3.28) in Lemma 3.7, we estimate

$$\begin{aligned}
&\left\|\mathrm{Ad}^{l+1}(e^{-itR_{V,\theta}})\right\|_{\mathscr{B}(L^2(\Omega))} \\
&\leq C_{l+1}\int_0^t \sum_{l_1+l_2+l_3=l} \left\|\mathrm{Ad}^{l_1}(e^{-isR_{V,\theta}})\right\|_{\mathscr{B}(L^2(\Omega))} \\
&\qquad\qquad \times \left\|\mathrm{Ad}^{l_2+1}(R_{V,\theta})\right\|_{\mathscr{B}(L^2(\Omega))}\left\|\mathrm{Ad}^{l_3}(e^{-i(t-s)R_{V,\theta}})\right\|_{\mathscr{B}(L^2(\Omega))}\,ds \\
&\leq C_{l+1}\int_0^t \sum_{l_1+l_2+l_3=l} \theta^{\frac{l_1}{2}}(1+|s|)^{l_1}\cdot\theta^{\frac{l_2+1}{2}}\cdot\theta^{\frac{l_3}{2}}(1+|t-s|)^{l_3}\,ds \\
&\leq C_{l+1}\theta^{\frac{l+1}{2}}(1+|t|)^{l+1}
\end{aligned}$$

for any $t \in \mathbb{R}$. Hence (3.20) is true for $k = l+1$. Thus (3.20) holds for any $k \geq 0$.

The assertion (ii) is proved in the same way as assertion (i) by using the estimate (3.24) from Lemma 3.5 instead of (3.22). The proof of Proposition 3.4 is complete. □

3.4 Estimates for spectral multipliers in amalgam spaces

In this section we prove the following. The following lemma is a result on estimates for spectral multipliers $\phi(\theta\mathcal{H}_V)$ in scaled amalgam spaces.

Lemma 3.8. *Let $\phi \in \mathscr{S}(\mathbb{R})$. Suppose that the potential V satisfies assumption* A. *Then the following assertions hold*:

(i) *There exists a constant $C > 0$ such that*

$$\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(l^1(L^2)_\theta)} \leq C \tag{3.35}$$

for any $0 < \theta \leq 1$.

(ii) *If V_- further satisfies assumption* B, *then the estimate* (3.35) *holds for any $\theta > 0$.*

For this purpose, let us introduce a family $\mathscr{A}_\alpha$ of operators, which is useful to prove the lemma.

Definition. Let $\alpha > 0$ and $\theta > 0$. We say that $L \in \mathscr{A}_\alpha (= \mathscr{A}_{\alpha,\theta})$ if $L \in \mathscr{B}(L^2(\Omega))$ and

$$|||L|||_\alpha := \sup_{n\in\mathbb{Z}^d} \big\| |\cdot - \theta^{\frac{1}{2}} n|^\alpha L\chi_{C_\theta(n)} \big\|_{\mathscr{B}(L^2(\Omega))} < \infty. \tag{3.36}$$

First we prepare two lemmas.

Lemma 3.9. *Let $\theta > 0$, and let $L \in \mathscr{A}_\alpha$ for some $\alpha > d/2$. Then there exists a constant $C > 0$ depending only on α and d such that*

$$\|Lf\|_{l^1(L^2)_\theta} \le C\Big(\|L\|_{\mathscr{B}(L^2(\Omega))} + \theta^{-\frac{d}{4}} |||L|||_\alpha^{\frac{d}{2\alpha}} \|L\|_{\mathscr{B}(L^2(\Omega))}^{1-\frac{d}{2\alpha}} \Big) \|f\|_{l^1(L^2)_\theta} \tag{3.37}$$

for any $f \in l^1(L^2)_\theta$.

Proof. If we prove that

$$\begin{aligned} &\sum_{m\in\mathbb{Z}^d} \|\chi_{C_\theta(m)} L\chi_{C_\theta(n)} f\|_{L^2(\Omega)} \\ &\le C\Big(\|L\|_{\mathscr{B}(L^2(\Omega))} + \theta^{-\frac{d}{4}} |||L|||_\alpha^{\frac{d}{2\alpha}} \|L\|_{\mathscr{B}(L^2(\Omega))}^{1-\frac{d}{2\alpha}} \Big) \|\chi_{C_\theta(n)} f\|_{L^2(\Omega)} \end{aligned}$$

for any $\theta > 0$ and $n \in \mathbb{Z}^d$, then, summing up (5.23) with respect to $n \in \mathbb{Z}^d$, we conclude the required estimate (3.37):

$$\begin{aligned} \|Lf\|_{l^1(L^2)_\theta} &\le \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} \|\chi_{C_\theta(m)} L\chi_{C_\theta(n)} f\|_{L^2(\Omega)} \\ &\le C\Big(\|L\|_{\mathscr{B}(L^2(\Omega))} + \theta^{-\frac{d}{4}} |||L|||_\alpha^{\frac{d}{2\alpha}} \|L\|_{\mathscr{B}(L^2(\Omega))}^{1-\frac{d}{2\alpha}} \Big) \|f\|_{l^1(L^2)_\theta} \end{aligned}$$

for any $\theta > 0$ and $f \in l^1(L^2)_\theta$. Hence we have only to prove the estimate (5.23).

Let $n \in \mathbb{Z}^d$ be fixed. For any $\omega > 0$, we write

$$\begin{aligned} &\sum_{m\in\mathbb{Z}^d} \|\chi_{C_\theta(m)} L\chi_{C_\theta(n)} f\|_{L^2(\Omega)} \\ &= \sum_{|m-n|>\omega} |\theta^{\frac{1}{2}} m - \theta^{\frac{1}{2}} n|^{-\alpha} |\theta^{\frac{1}{2}} m - \theta^{\frac{1}{2}} n|^\alpha \|\chi_{C_\theta(m)} L\chi_{C_\theta(n)} f\|_{L^2(\Omega)} \\ &\qquad + \sum_{|m-n|\le\omega} \|\chi_{C_\theta(m)} L\chi_{C_\theta(n)} f\|_{L^2(\Omega)} \\ &=: I(n) + II(n). \end{aligned}$$

By using the Schwarz inequality we estimate $I(n)$ as

$$\begin{aligned} I(n) \le \theta^{-\frac{\alpha}{2}} &\Bigg(\sum_{|m-n|>\omega} |m-n|^{-2\alpha} \Bigg)^{\frac{1}{2}} \\ &\times \Bigg(\sum_{|m-n|>\omega} |\theta^{\frac{1}{2}} m - \theta^{\frac{1}{2}} n|^{2\alpha} \|\chi_{C_\theta(m)} L\chi_{C_\theta(n)} f\|_{L^2(\Omega)}^2 \Bigg)^{\frac{1}{2}}. \end{aligned} \tag{3.38}$$

The first factor of (3.38) is estimated as

$$\begin{aligned}\sum_{|m-n|>\omega}|m-n|^{-2\alpha}&=\sum_{|m|>\omega}|m|^{-2\alpha}\le C\prod_{j=1}^{d}\sum_{|m_j|>\frac{\omega}{\sqrt{d}}}(1+|m_j|)^{-\frac{2\alpha}{d}}\\&\le C\prod_{j=1}^{d}\int_{\{\sigma>\frac{\omega}{\sqrt{d}}\}}\sigma^{-\frac{2\alpha}{d}}\,d\sigma\le C\prod_{j=1}^{d}\omega^{-\frac{2\alpha}{d}+1}=C\omega^{-2\alpha+d},\end{aligned}\tag{3.39}$$

since $\alpha>d/2$. As to the second factor of (3.38), noting that

$$\frac{|\theta^{\frac12}m-\theta^{\frac12}n|}{2}\le|x-\theta^{\frac12}n|$$

for any $x\in C_\theta(m)$, we estimate

$$\begin{aligned}&\sum_{|m-n|>\omega}|\theta^{\frac12}m-\theta^{\frac12}n|^{2\alpha}\|\chi_{C_\theta(m)}L\chi_{C_\theta(n)}f\|^2_{L^2(\Omega)}\\&\qquad=\sum_{|m-n|>\omega}|\theta^{\frac12}m-\theta^{\frac12}n|^{2\alpha}\int_{C_\theta(m)}|L\chi_{C_\theta(n)}f|^2\,dx\\&\qquad\le 2^{2\alpha}\sum_{|m-n|>\omega}\int_{C_\theta(m)}\left||x-\theta^{\frac12}n|^{\alpha}L\chi_{C_\theta(n)}f\right|^2dx.\end{aligned}$$

Moreover, by the definition (3.36) of $|||L|||_\alpha$, we estimate

$$\begin{aligned}\sum_{|m-n|>\omega}\int_{C_\theta(m)}\left||x-\theta^{\frac12}n|^{\alpha}L\chi_{C_\theta(n)}f\right|^2dx&\le\left\||\cdot-\theta^{\frac12}n|^{\alpha}L\chi_{C_\theta(n)}f\right\|^2_{L^2(\Omega)}\\&\le|||L|||^2_\alpha\|\chi_{C_\theta(n)}f\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence, summarizing the above two estimates, we deduce that

$$\sum_{|m-n|>\omega}|\theta^{\frac12}m-\theta^{\frac12}n|^{2\alpha}\|\chi_{C_\theta(m)}L\chi_{C_\theta(n)}f\|^2_{L^2(\Omega)}\le 2^{2\alpha}|||L|||^2_\alpha\|\chi_{C_\theta(n)}f\|^2_{L^2(\Omega)}.\tag{3.40}$$

Thus we find from (3.38)–(3.40) that

$$I(n)\le C(d,\alpha)\theta^{-\frac{\alpha}{2}}\omega^{-(\alpha-\frac{d}{2})}|||L|||_\alpha\|\chi_{C_\theta(n)}f\|_{L^2(\Omega)}.\tag{3.41}$$

Let us turn to the estimation of $II(n)$. It is readily to see that

$$\begin{aligned}II(n)&\le\Big(\sum_{|m-n|\le\omega}1\Big)^{\frac12}\Big(\sum_{|m-n|\le\omega}\|\chi_{C_\theta(m)}L\chi_{C_\theta(n)}f\|^2_{L^2(\Omega)}\Big)^{\frac12}\\&\le(1+\omega^{\frac{d}{2}})\Big(\sum_{|m-n|\le\omega}\|\chi_{C_\theta(m)}L\chi_{C_\theta(n)}f\|^2_{L^2(\Omega)}\Big)^{\frac12}.\end{aligned}$$

Since

$$\Big(\sum_{|m-n|\le\omega}\|\chi_{C_\theta(m)}L\chi_{C_\theta(n)}f\|_{L^2(\Omega)}^2\Big)^{\frac12}\le\|L\|_{\mathscr{B}(L^2(\Omega))}\|\chi_{C_\theta(n)}f\|_{L^2(\Omega)},$$

we deduce that

$$II(n)\le(1+\omega^{\frac{d}{2}})\|L\|_{\mathscr{B}(L^2(\Omega))}\|\chi_{C_\theta(n)}f\|_{L^2(\Omega)}. \tag{3.42}$$

Combining the estimates (3.41) and (3.42), we get

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^d}\|\chi_{C_\theta(m)}L\chi_{C_\theta(n)}f\|_{L^2(\Omega)}\\&\le C(d,\alpha)\Big\{\theta^{-\frac{\alpha}{2}}\omega^{-(\alpha-\frac{d}{2})}|||L|||_\alpha+(1+\omega^{\frac{d}{2}})\|L\|_{\mathscr{B}(L^2(\Omega))}\Big\}\|\chi_{C_\theta(n)}f\|_{L^2(\Omega)}.\end{aligned}$$

Finally, taking

$$\omega=\left(\frac{|||L|||_\alpha}{\|L\|_{\mathscr{B}(L^2(\Omega))}}\right)^{\frac1\alpha}\cdot\theta^{-\frac12},$$

we obtain the required estimate (5.23). The proof of Lemma 3.9 is complete. □

Lemma 3.10. *Let $\phi\in\mathscr{S}(\mathbb{R})$. Suppose that the potential V satisfies assumption* A. *Then for any $\alpha>0$ the following assertions hold*:

(i) *The operator $\phi(\theta\mathcal{H}_V)$ belongs to $\mathscr{A}_\alpha$ for any $0<\theta\le1$. Furthermore, there exist a constant $C>0$ such that*

$$|||\phi(\theta\mathcal{H}_V)|||_\alpha\le C\theta^{\frac{\alpha}{2}} \tag{3.43}$$

for any $0<\theta\le1$.

(ii) *If V_- further satisfies assumption* C *given in section 2.3, then the same conclusion as in the assertion* (i) *holds for any $\theta>0$.*

Proof. To begin with, we prove the assertion (i). Let $0<\theta\le1$ and M be a real number such that

$$M>\max\{-\inf\sigma(\mathcal{H}_V),0\}. \tag{3.44}$$

We may assume that $\operatorname{supp}\phi\subset[-M,\infty)$ without loss of generality. Let us choose $\psi\in C_0^\infty(\mathbb{R})$ such that

$$\psi(\mu)=\chi(\mu)\phi(\mu^{-1}-M),$$

where χ is a smooth function on $\mathbb{R}$ such that

$$\chi(\mu)=\begin{cases}1 & \text{for } 0\le\mu\le\dfrac{1}{M+\inf\sigma(\mathcal{H}_V)}+1,\\[2ex] 0 & \text{for } \mu\le-1 \text{ and } \mu\ge\dfrac{1}{M+\inf\sigma(\mathcal{H}_V)}+2.\end{cases} \tag{3.45}$$

When we consider the operator $\theta\mathcal{H}_V$ for $0 < \theta \leq 1$, it is possible to take, independently of θ, the real number M satisfying (3.44). Then we write

$$\psi(R_{V,\theta}) = \psi\big((\theta\mathcal{H}_V + M)^{-1}\big) = \phi(\theta\mathcal{H}_V).$$

In order to prove the estimate (3.43), it suffices to show that

$$|||\psi(R_{V,\theta})|||_\alpha \leq C\theta^{\frac{\alpha}{2}} \int_{-\infty}^{\infty} (1+|t|)^\alpha |\hat{\psi}(t)|\, dt, \tag{3.46}$$

where $\hat{\psi}$ is the Fourier transform of ψ on $\mathbb{R}$ and the integral on the right is absolutely convergent, since $\hat{\psi} \in \mathscr{S}(\mathbb{R})$. The proof is based on the formula:

$$\psi(R_{V,\theta}) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-itR_{V,\theta}} \hat{\psi}(t)\, dt. \tag{3.47}$$

Applying the formula (3.47), we obtain

$$\begin{aligned}
|||\psi(R_{V,\theta})|||_\alpha &= \sup_{n\in\mathbb{Z}^d} \big\| |\cdot - \theta^{\frac{1}{2}} n|^\alpha \psi(R_{V,\theta}) \chi_{C_\theta(n)} \big\|_{\mathscr{B}(L^2(\Omega))} \\
&\leq (2\pi)^{-\frac{1}{2}} \sup_{n\in\mathbb{Z}^d} \int_{-\infty}^{\infty} \big\| |\cdot - \theta^{\frac{1}{2}} n|^\alpha e^{-itR_{V,\theta}} \chi_{C_\theta(n)} \big\|_{\mathscr{B}(L^2(\Omega))} |\hat{\psi}(t)|\, dt.
\end{aligned}$$

Let N be a positive integer. Thanks to the formula (3.18) with $A = B = x_j - \theta^{1/2} n_j$ and $L = e^{-itR_{V,\theta}}$, we find from the assertion (i) in Proposition 3.4 that

$$\begin{aligned}
&\big\| |\cdot - \theta^{\frac{1}{2}} n|^N e^{-itR_{V,\theta}} \chi_{C_\theta(n)} \big\|_{\mathscr{B}(L^2(\Omega))} \\
&\leq \sum_{k=0}^{N} C(N,k) \|\mathrm{Ad}^k(e^{-itR_{V,\theta}})\|_{\mathscr{B}(L^2(\Omega))} \big\| |\cdot - \theta^{\frac{1}{2}} n|^{N-k} \chi_{C_\theta(n)} \big\|_{\mathscr{B}(L^2(\Omega))} \\
&\leq \sum_{k=0}^{N} C(N,k) \theta^{\frac{k}{2}} (1+|t|)^k \theta^{\frac{N-k}{2}} \\
&\leq C\theta^{\frac{N}{2}} (1+|t|)^N.
\end{aligned}$$

Now, it follows from Calderón-Lions interpolation theorem (see Theorem IX.20 in Reed and Simon [66]) that

$$\big\| |\cdot - \theta^{\frac{1}{2}} n|^\alpha e^{-itR_{V,\theta}} \chi_{C_\theta(n)} \big\|_{\mathscr{B}(L^2(\Omega))} \leq C\theta^{\frac{\alpha}{2}} (1+|t|)^\alpha$$

for any $\alpha > 0$ and $t \in \mathbb{R}$. Thus we conclude (3.46), which proves (3.43).

As to the assertion (ii), noting that $\inf \sigma(\mathcal{H}_V) \geq 0$, we can prove the estimate (3.43) for any $\theta > 0$ in the same way as assertion (i) by using the assertion (ii) in Proposition 3.4 instead of assertion (i) in Proposition 3.4. The proof of Lemma 3.10 is finished. □

Proof of Lemma 3.8. We prove only the assertion (i), since the proof of (ii) is the same as that of (i). Let $0 < \theta \le 1$. By Lemma 3.10, the operator $\phi(\theta\mathcal{H}_V)$ belongs to $\mathscr{A}_\alpha$ for any $\alpha > 0$. Choosing $\alpha > d/2$, and applying Lemma 3.9 to $\phi(\theta\mathcal{H}_V)$, we have

$$\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(l^1(L^2)_\theta)} \le C\Big(\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^2(\Omega))} + \theta^{-\frac{d}{4}}\|\!|\phi(\theta\mathcal{H}_V)|\!\|_\alpha^{\frac{d}{2\alpha}}\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^2(\Omega))}^{1-\frac{d}{2\alpha}}\Big).$$

Hence, combining the above estimate with (3.43) in Lemma 3.10, we conclude (3.35). Thus the proof of Lemma 3.8 is finished. □

3.5 L^p-L^q-estimates for spectral multipliers

In this section we prove Theorem 3.1; uniform L^p-L^q-estimates for $\phi(\theta\mathcal{H}_V)$ with respect to a parameter θ.

For this purpose, we prove the following uniform L^p-estimates.

Theorem 3.11. *Let $\phi \in \mathscr{S}(\mathbb{R})$ and $1 \le p \le \infty$. Suppose that the potential V satisfies assumption* A. *Then $\phi(\mathcal{H}_V)$ is extended to a bounded linear operator on $L^p(\Omega)$. Furthermore, the following assertions hold*:

(i) *There exists a constant $C > 0$ such that*

$$\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega))} \le C \tag{3.48}$$

for any $0 < \theta \le 1$.

(ii) *If V_- further satisfies assumption* B*, then the estimate* (3.48) *holds for any $\theta > 0$.*

Proof. First we prove the assertion (i). Let $0 < \theta \le 1$. It suffices to show L^1-estimate for $\phi(\theta\mathcal{H}_V)$. In fact, if L^1-estimate is proved, then L^∞-estimate is also obtained by duality argument, and hence, the Riesz-Thorin interpolation theorem allows us to conclude L^p-estimates (3.48) for $1 \le p \le \infty$.

Let us proceed the proof of L^1-estimate. Let $f \in L^1(\Omega) \cap L^2(\Omega)$. By (3.4) for $p = 1$ and $q = 2$, we estimate

$$\|\phi(\theta\mathcal{H}_V)f\|_{L^1(\Omega)} \le \theta^{\frac{d}{4}}\|\phi(\theta\mathcal{H}_V)f\|_{l^1(L^2)_\theta}. \tag{3.49}$$

Here, given a real number $\beta > d/4$, we choose $\tilde{\phi} \in \mathscr{S}(\mathbb{R})$ as

$$\tilde{\phi}(\lambda) = (\lambda + M)^\beta \phi(\lambda) \quad \text{for } \lambda \in \sigma(\mathcal{H}_V), \tag{3.50}$$

where M is a real number such that $M > \max\{\omega, 0\}$, where ω is the constant in Proposition 2.4. Then, by Lemma 3.8 and Proposition 3.3, we obtain

$$\begin{aligned}\|\phi(\theta\mathcal{H}_V)f\|_{l^1(L^2)_\theta} &= \|\phi(\theta\mathcal{H}_V)(\theta\mathcal{H}_V + M)^\beta(\theta\mathcal{H}_V + M)^{-\beta}f\|_{l^1(L^2)_\theta} \\ &= \|\tilde{\phi}(\theta\mathcal{H}_V)(\theta\mathcal{H}_V + M)^{-\beta}f\|_{l^1(L^2)_\theta} \\ &\le C\|(\theta\mathcal{H}_V + M)^{-\beta}f\|_{l^1(L^2)_\theta} \\ &\le C\theta^{-\frac{d}{4}}\|f\|_{L^1(\Omega)}.\end{aligned}$$

Therefore, combining (3.49) with the above estimate, we conclude that

$$\|\phi(\theta\mathcal{H}_V)f\|_{L^1(\Omega)} \le C\|f\|_{L^1(\Omega)}$$

for any $0 < \theta \le 1$ and $f \in L^1(\Omega)$, where the constant C is independent of θ.

The assertion (ii) is proved in the same way as assertion (i) by using assertions (ii) in Proposition 3.3 and Lemma 3.10 instead of assertions (i) in Proposition 3.3 and Lemma 3.10, respectively. The proof of Theorem 3.11 is complete. □

Proof of Theorem 3.1. We prove only the assertion (i), since the assertion (ii) is proved in the same way as assertion (i). Let $0 < \theta \le 1$ and M be a real number such that $M > \max\{\omega, 0\}$, where ω is the constant in Proposition 2.4. Given a positive real number β satisfying $\beta > (d/2)(1/p - 1/q)$, we choose $\tilde{\phi} \in \mathscr{S}(\mathbb{R})$ as

$$\tilde{\phi}(\lambda) = (\lambda + M)^\beta\phi(\lambda) \quad \text{for } \lambda \in \sigma(\mathcal{H}_V).$$

By using Proposition 3.3 and Theorem 3.11, we estimate

$$\begin{aligned}\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} &= \|\phi(\theta\mathcal{H}_V)(\theta\mathcal{H}_V + M)^\beta(\theta\mathcal{H}_V + M)^{-\beta}\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \\ &\le \|\tilde{\phi}(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^q(\Omega))}\|(\theta\mathcal{H}_V + M)^{-\beta}\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \\ &\le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\end{aligned}$$

for any $1 \le p \le q \le \infty$. The proof of Theorem 3.1 is complete. □

3.6 Gradient estimates for spectral multipliers

In this section we prove Theorem 3.2; L^p-L^q-estimates for $\nabla\phi(\theta\mathcal{H}_V)$.

For this purpose, we prove the following lemma.

Lemma 3.12. *Let $\phi \in \mathscr{S}(\mathbb{R})$. Suppose that the potential V satisfies assumption* A. *Then the following assertions hold*:

(i) *There exists a constant $C > 0$ such that*

$$\|\nabla\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(l^1(L^2)_\theta)} \le C\theta^{-\frac{1}{2}} \tag{3.51}$$

for any $0 < \theta \le 1$.

(ii) *If V_- further satisfies assumption* B*, then the estimate* (3.51) *holds for any* $\theta > 0$.

Lemma 3.12 follows from the following lemma in the same way as the proof of Lemma 3.8.

Lemma 3.13. *Let $\phi \in \mathscr{S}(\mathbb{R})$. Suppose that the potential V satisfies assumption* A*. Then for any $\alpha > 0$, the following assertions hold*:

(i) *The operator $\nabla\phi(\theta\mathcal{H}_V)$ belongs to $\mathscr{A}_\alpha$ for any $0 < \theta \leq 1$. Furthermore, there exist a constant $C > 0$ such that*

$$\|\nabla\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^2(\Omega))} \leq C\theta^{-\frac{1}{2}}, \tag{3.52}$$

$$\|\!|\!|\nabla\phi(\theta\mathcal{H}_V)|\!|\!|_\alpha \leq C\theta^{\frac{\alpha-1}{2}} \tag{3.53}$$

for any $0 < \theta \leq 1$.

(ii) *If V_- further satisfies assumption* C *given in section 2.3, then the same conclusion as in the assertion* (i) *holds for any $\theta > 0$.*

Proof. First we prove the assertion (i). Let $0 < \theta \leq 1$. We prove the estimate (3.52). Since $\phi(\theta\mathcal{H}_V)f \in \mathcal{D}(\mathcal{H}_V)$ for any $f \in L^2(\Omega)$, we estimate

$$\begin{aligned}
\|\nabla\phi(\theta\mathcal{H}_V)f\|^2_{L^2(\Omega)} &= (\mathcal{H}_V\phi(\theta\mathcal{H}_V)f, \phi(\theta\mathcal{H}_V)f)_{L^2(\Omega)} - \int_\Omega V|\phi(\theta\mathcal{H}_V)f|^2\,dx \\
&\leq (\mathcal{H}_V\phi(\theta\mathcal{H}_V)f, \phi(\theta\mathcal{H}_V)f)_{L^2(\Omega)} + \int_\Omega V_-|\phi(\theta\mathcal{H}_V)f|^2\,dx \\
&=: I + II.
\end{aligned}$$

Then, applying Theorem 3.1 to $\mathcal{H}_V\phi(\theta\mathcal{H}_V)f$ and $\phi(\theta\mathcal{H}_V)f$, we estimate I as

$$I \leq \|\mathcal{H}_V\phi(\theta\mathcal{H}_V)f\|_{L^2(\Omega)}\|\phi(\theta\mathcal{H}_V)f\|_{L^2(\Omega)} \leq C\theta^{-1}\|f\|^2_{L^2(\Omega)}.$$

As to the second term II, by using the inequality (2.3) from Lemma 2.3, we have

$$\begin{aligned}
II &\leq \varepsilon\|\nabla\phi(\theta\mathcal{H}_V)f\|^2_{L^2(\Omega)} + b_\varepsilon\|\phi(\theta\mathcal{H}_V)f\|^2_{L^2(\Omega)} \\
&\leq \varepsilon\|\nabla\phi(\theta\mathcal{H}_V)f\|^2_{L^2(\Omega)} + Cb_\varepsilon\theta^{-1}\|f\|^2_{L^2(\Omega)}
\end{aligned}$$

for any $\varepsilon > 0$. Here we choose ε as $0 < \varepsilon < 1$. Then, combining the above three estimates, we conclude the estimate (3.52).

Next we prove the estimate (3.53). Let M be a real number such that

$$M > \max\{-\inf\sigma(\mathcal{H}_V), 0\}.$$

We may assume that $\operatorname{supp}\phi \subset [-M,\infty)$ without loss of generality. Let us choose $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\psi(\mu) = \chi(\mu)\mu^{-1}\phi(\mu^{-1} - M), \tag{3.54}$$

where χ is a smooth function on $\mathbb{R}$ satisfying (3.45). Then we write

$$\nabla\phi(\theta\mathcal{H}_V) = \nabla R_{V,\theta}\psi(R_{V,\theta}).$$

Hence we have only to show that there exists a constant $C > 0$ such that

$$|||\nabla R_{V,\theta}\psi(R_{V,\theta})|||_\alpha \le C\theta^{\frac{\alpha-1}{2}} \tag{3.55}$$

for any $0 < \theta \le 1$.

It suffices to show the estimate (3.55) for positive integers α by using Calderón-Lions interpolation theorem. We prove (3.55) only for $\alpha = 1$, since the cases $\alpha \ge 2$ are proved by the induction with Lemmas 3.6 and 3.7. Let $j \in \{1, 2, \dots, d\}$ be fixed. By the formula (3.47), we have

$$\begin{aligned}
&|||\partial_{x_j} R_{V,\theta}\psi(R_{V,\theta})|||_1 \\
&= \sup_{n\in\mathbb{Z}^d} \left\| |\cdot - \theta^{\frac{1}{2}}n|\, \partial_{x_j} R_{V,\theta}\psi(R_{V,\theta})\chi_{C_\theta(n)} \right\|_{\mathscr{B}(L^2(\Omega))} \\
&\le \sup_{n\in\mathbb{Z}^d} (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left\| |\cdot - \theta^{\frac{1}{2}}n|\, \partial_{x_j} R_{V,\theta} e^{-itR_\theta}\chi_{C_\theta(n)} \right\|_{\mathscr{B}(L^2(\Omega))} |\hat{\psi}(t)|\, dt.
\end{aligned} \tag{3.56}$$

If we show that

$$\left\| |\cdot - \theta^{\frac{1}{2}}n|\, \partial_{x_j} R_{V,\theta} e^{-itR_\theta}\chi_{C_\theta(n)} \right\|_{\mathscr{B}(L^2(\Omega))} \le C(1 + |t|) \tag{3.57}$$

for any $t \in \mathbb{R}$ and $n \in \mathbb{Z}^d$, then we conclude from (3.56) that

$$|||\partial_{x_j} R_{V,\theta}\psi(R_{V,\theta})|||_1 \le C(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (1 + |t|)|\hat{\psi}(t)|\, dt \big(= C\theta^{\frac{1-1}{2}} \big)$$

for $j = 1, 2, \dots, d$, which is the estimate (3.55) for $\alpha = 1$. Hence we pay attention to prove (3.57). We write

$$\begin{aligned}
&(x_k - \theta^{\frac{1}{2}}n_k)\partial_{x_j} R_{V,\theta} e^{-itR_{V,\theta}} \\
&= \partial_{x_j}\left[(x_k - \theta^{\frac{1}{2}}n_k) R_{V,\theta} e^{-itR_{V,\theta}} \right] - \delta_{jk} R_{V,\theta} e^{-itR_{V,\theta}} \\
&= \partial_{x_j} R_{V,\theta}(x_k - \theta^{\frac{1}{2}}n_k) e^{-itR_{V,\theta}} + \partial_{x_j}\mathrm{Ad}^1(R_{V,\theta}) e^{-itR_{V,\theta}} - \delta_{jk} R_{V,\theta} e^{-itR_{V,\theta}} \\
&= \partial_{x_j} R_{V,\theta} e^{-itR_{V,\theta}}(x_k - \theta^{\frac{1}{2}}n_k) + \partial_{x_j} R_{V,\theta}\mathrm{Ad}^1(e^{-itR_{V,\theta}}) \\
&\qquad + \partial_{x_j}\mathrm{Ad}^1(R_{V,\theta}) e^{-itR_{V,\theta}} - \delta_{jk} R_{V,\theta} e^{-itR_{V,\theta}}
\end{aligned}$$

for $k = 1, 2, \dots, d$, where we choose A and B in the operators $\mathrm{Ad}^1(R_{V,\theta})$ and $\mathrm{Ad}^1(e^{-itR_{V,\theta}})$ as $A = B = x_k - \theta^{1/2} n_k$ and δ_{jk} is Kronecker's delta. Then we estimate

$$\begin{aligned}
&\big\|(x_k - \theta^{\frac{1}{2}} n_k)\partial_{x_j} R_{V,\theta} e^{-itR_\theta} \chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} \\
&\le \big\|\partial_{x_j} R_{V,\theta} e^{-itR_{V,\theta}} (x_k - \theta^{\frac{1}{2}} n_k)\chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} \\
&\quad + \big\|\partial_{x_j} R_{V,\theta} \mathrm{Ad}^1(e^{-itR_{V,\theta}})\chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} \\
&\quad + \big\|\partial_{x_j} \mathrm{Ad}^1(R_{V,\theta}) e^{-itR_{V,\theta}} \chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} + \big\|\delta_{jk} R_{V,\theta} e^{-itR_{V,\theta}} \chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} \\
&=: I + II + III + IV
\end{aligned}$$

for $k = 1, 2, \dots, d$. Noting that there exists a constant $C > 0$ such that

$$\big\|(x_k - \theta^{\frac{1}{2}} n_k)\chi_{C_\theta(n)} f\big\|_{L^2(\Omega)} \le C\theta^{\frac{1}{2}} \|f\|_{L^2(\Omega)} \tag{3.58}$$

for any $\theta > 0$ and $n \in \mathbb{Z}^d$, we use the estimate (3.22) from Lemma 3.5 to deduce that

$$I \le \big\|\partial_{x_j} R_{V,\theta}\big\|_{\mathscr{B}(L^2(\Omega))} \|e^{-itR_{V,\theta}}\|_{\mathscr{B}(L^2(\Omega))} \|(x_k - \theta^{\frac{1}{2}} n_k)\chi_{C_\theta(n)}\|_{\mathscr{B}(L^2(\Omega))} \le C.$$

As to the second term II, by using (3.20) for $k = 1$ from Proposition 3.4 and (3.22) from Lemma 3.5, we estimate

$$\begin{aligned}
II &\le \big\|\partial_{x_j} R_{V,\theta}\big\|_{\mathscr{B}(L^2(\Omega))} \big\|\mathrm{Ad}^1(e^{-itR_{V,\theta}})\big\|_{\mathscr{B}(L^2(\Omega))} \|\chi_{C_\theta(n)}\|_{\mathscr{B}(L^2(\Omega))} \\
&\le C\theta^{-\frac{1}{2}} \cdot \theta^{\frac{1}{2}}(1 + |t|) = C(1 + |t|).
\end{aligned}$$

As to the third term III, we use the formula (3.26) from Lemma 3.6:

$$\mathrm{Ad}^1(R_{V,\theta}) = -2\theta R_{V,\theta} \partial_{x_k} R_{V,\theta}.$$

Then, by using (3.22) from Lemma 3.5, we estimate

$$\begin{aligned}
III &= 2\theta \big\|\partial_{x_j} R_{V,\theta} \partial_{x_k} R_{V,\theta} e^{-itR_{V,\theta}} \chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} \\
&\le 2\theta \big\|\partial_{x_j} R_{V,\theta}\big\|_{\mathscr{B}(L^2(\Omega))} \big\|\partial_{x_k} R_{V,\theta}\big\|_{\mathscr{B}(L^2(\Omega))} \big\|e^{-itR_{V,\theta}} \chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} \\
&\le C\theta \cdot \theta^{-\frac{1}{2}} \cdot \theta^{-\frac{1}{2}} = C.
\end{aligned}$$

As to the fourth term IV, we readily see that

$$IV \le \big\|R_{V,\theta}\big\|_{\mathscr{B}(L^2(\Omega))} \big\|e^{-itR_{V,\theta}}\big\|_{\mathscr{B}(L^2(\Omega))} \|\chi_{C_\theta(n)}\|_{\mathscr{B}(L^2(\Omega))} \le C.$$

Combining all the above estimates, we arrive at

$$\big\|(x_k - \theta^{\frac{1}{2}} n_k)\partial_{x_j} R_{V,\theta} e^{-itR_\theta} \chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} \le C(1 + |t|)$$

for $k = 1, 2, \dots, d$, which imply the estimate (3.57). The assertion (ii) is proved in the similar way to assertion (i). In fact, we have only to use the inequality (2.4) instead of (2.3), and assertions (ii) instead of assertions (i) from Lemmas 3.5 and 3.10. Thus the proof of Lemma 3.13 is finished. □

Proof of Theorem 3.2. We prove only the assertion (i), since the assertion (ii) is proved in the same way as assertion (i). Let $0 < \theta \leq 1$. It suffices to show L^1-estimate for $\nabla\phi(\theta\mathcal{H}_V)$. In fact, L^2-estimate has been already proved in (3.52). If L^1-estimate are proved, then L^p-estimates are obtained by the Riesz-Thorin interpolation theorem. Hence we conclude the required L^p-L^q-estimates (3.2) in a similar way to the proof of Theorem 3.1.

Let us turn to the proof of the L^1-estimate. By (3.4) for $p = 1$ and $q = 2$, we estimate

$$\|\nabla\phi(\theta\mathcal{H}_V)f\|_{L^1(\Omega)} \leq \theta^{\frac{d}{4}}\|\nabla\phi(\theta\mathcal{H}_V)f\|_{l^1(L^2)_\theta}.$$

Let $M > \max\{\omega, 0\}$ and $\beta > d/4$, where ω is the constant in Proposition 2.4. Here we choose $\tilde{\phi} \in \mathscr{S}(\mathbb{R})$ as

$$\tilde{\phi}(\lambda) = (\lambda + M)^\beta\phi(\lambda) \quad \text{for } \lambda \in \sigma(\mathcal{H}_V).$$

By Lemma 3.12 and Proposition 3.3, we estimate

$$\begin{aligned}\|\nabla\phi(\theta\mathcal{H}_V)f\|_{l^1(L^2)_\theta} &= \|\nabla\tilde{\phi}(\theta\mathcal{H}_V)(\theta\mathcal{H}_V + M)^{-\beta}f\|_{l^1(L^2)_\theta} \\ &\leq C\theta^{-\frac{1}{2}} \cdot \theta^{-\frac{d}{4}}\|f\|_{L^1(\Omega)}.\end{aligned}$$

Thus, combining the estimates obtained now, we conclude the required L^1-estimate

$$\|\nabla\phi(\theta\mathcal{H}_V)f\|_{L^1(\Omega)} \leq C\theta^{-\frac{1}{2}}\|f\|_{L^1(\Omega)}$$

for any $0 < \theta \leq 1$ and $f \in L^1(\Omega)$. The proof of Theorem 3.2 is complete. □

3.7 A remark on smoothness of symbols

In this section we show to weaken the assumption that $\phi \in \mathscr{S}(\mathbb{R})$ in Theorems 3.1 and 3.2.

We can improve the assumption that $\phi \in \mathscr{S}(\mathbb{R})$ in Theorem 3.1, and even Theorem 3.2. Namely, the function ϕ in Theorem 3.1 can be taken from the weighted Sobolev spaces. In fact, let m be an integer with $m > (d+1)/2$, and β a real number with $\beta > d/4 + (d/2)(1/p - 1/q)$. If the measurable potential V satisfies assumption A, then there exists a constant $C_0 > 0$, independent of ϕ, such that

$$\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \leq C_0 \left\|(1 + |\cdot|^2)^{\frac{\beta+m}{2}}\phi\right\|_{H^m(\mathbb{R})} \tag{3.59}$$

for any $0 < \theta \leq 1$. Needless to say, once the estimate (3.59) is established for $0 < \theta \leq 1$, after some trivial changes, if we further suppose assumption B on V, then the estimate (3.59) holds for any $\theta > 0$. We prove this estimate only for $0 < \theta \leq 1$. To begin with, we show (3.59) for $p = q$. Let us define $\tilde{\phi}$ as in (3.50):

$$\tilde{\phi}(\lambda) = (\lambda + M)^\beta\phi(\lambda) \quad \text{for } \lambda \in \sigma(\mathcal{H}_V),$$

where $\beta > d/4$. We note that

$$\|\tilde{\phi}(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^2(\Omega))} \le \|\tilde{\phi}\|_{L^\infty(\mathbb{R})}$$

for any $0 < \theta \le 1$. Indeed, we have:

$$\begin{aligned}\|\tilde{\phi}(\theta\mathcal{H}_V)f\|^2_{L^2(\Omega)} &= \int_{\inf\sigma(\mathcal{H}_V)}^{\infty} |\tilde{\phi}(\theta\lambda)|^2\, d\|E_{\mathcal{H}_V}(\lambda)f\|^2_{L^2(\Omega)} \\ &\le \|\tilde{\phi}\|^2_{L^\infty(\mathbb{R})}\|f\|^2_{L^2(\Omega)}\end{aligned}$$

for any $0 < \theta \le 1$. Then, following the proof of Theorem 3.11, we estimate

$$\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega))} \le C\Big(\|\tilde{\phi}\|_{L^\infty(\mathbb{R})} + \theta^{-\frac{d}{4}}|||\tilde{\phi}(\theta\mathcal{H}_V)|||_\alpha^{\frac{d}{2\alpha}}\|\tilde{\phi}\|^{1-\frac{d}{2\alpha}}_{L^\infty(\mathbb{R})}\Big) \tag{3.60}$$

for any $\alpha > d/2$ and $0 < \theta \le 1$. To estimate the quantity $|||\tilde{\phi}(\theta\mathcal{H}_V)|||_\alpha$, let us choose ψ such that

$$\psi(\mu) = \chi(\mu)\tilde{\phi}(\mu^{-1} - M), \tag{3.61}$$

where χ is a smooth function on $\mathbb{R}$ satisfying (3.45). Then we write

$$\tilde{\phi}(\theta\mathcal{H}_V) = \psi(R_{V,\theta}).$$

From the estimate (3.46) in the proof of Theorem 3.11, we get, by using Schwarz' inequality and Plancherel's identity,

$$\begin{aligned}|||\tilde{\phi}(\theta\mathcal{H}_V)|||_\alpha &\le C\theta^{\frac{\alpha}{2}}\int_{-\infty}^{\infty}(1+|t|)^\alpha|\hat{\psi}(t)|\,dt, \\ &\le C\theta^{\frac{\alpha}{2}}\big\|(1+|\cdot|)^{\alpha-m}\big\|_{L^2(\mathbb{R})}\big\|(1+|\cdot|)^m\hat{\psi}\big\|_{L^2(\mathbb{R})} \\ &= C\theta^{\frac{\alpha}{2}}\|\psi\|_{H^m(\mathbb{R})},\end{aligned}$$

provided that the integer m satisfies

$$m > \alpha + \frac{1}{2} > \frac{d+1}{2}.$$

Hence, noting from the definition (3.61) of ψ that

$$\begin{aligned}&\|\psi\|^2_{H^m(\mathbb{R})} \\ &= \sum_{k=0}^m\int_{\mathbb{R}}\left|\frac{d^k}{d\mu^k}\Big\{\chi(\mu)\tilde{\phi}(\mu^{-1}-M)\Big\}\right|^2 d\mu \\ &\le C\left\{\int_{\inf\sigma(\mathcal{H}_V)}^{\infty}|\tilde{\phi}(\lambda)|^2(\lambda+M)^{-2}\,d\lambda + \sum_{k=1}^m\int_{\inf\sigma(\mathcal{H}_V)}^{\infty}\left|\frac{d^k}{d\lambda^k}\Big\{(\lambda+M)^k\tilde{\phi}(\lambda)\Big\}\right|^2 d\lambda\right\} \\ &\le C\left\|(1+|\cdot|^2)^{\frac{m}{2}}\tilde{\phi}\right\|^2_{H^m(\mathbb{R})} \\ &\le C\left\|(1+|\cdot|^2)^{\frac{\beta+m}{2}}\phi\right\|^2_{H^m(\mathbb{R})},\end{aligned}$$

we obtain

$$\|\!|\!|\tilde{\phi}(\theta\mathcal{H}_V)|\!|\!|_\alpha \le C\theta^{\frac{\alpha}{2}} \left\|(1+|\cdot|^2)^{\frac{\beta+m}{2}}\phi\right\|_{H^m(\mathbb{R})}. \tag{3.62}$$

Furthermore, by using Sobolev's inequality, we have

$$\|\tilde{\phi}\|_{L^\infty(\mathbb{R})} = \left\|(\cdot+M)^\beta\phi\right\|_{L^\infty(\mathbb{R})} \le C\left\|(1+|\cdot|)^\beta\phi\right\|_{H^1(\mathbb{R})}. \tag{3.63}$$

Therefore, applying (3.62) and (3.63) to (3.60), we conclude that

$$\|\phi(\theta\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega))} \le C_0 \left\|(1+|\cdot|^2)^{\frac{\beta+m}{2}}\phi\right\|_{H^m(\mathbb{R})},$$

which implies (3.59) for $p = q$.

In the case when $p < q$, we can also prove (3.59) by the same way as above, if ϕ in the above argument is replaced by $(\lambda+M)^{\beta'}\phi$ for $\beta' > (d/2)(1/p - 1/q)$.

Chapter 4

Besov spaces on open sets

In 1959–61 Besov introduced the Besov spaces in his papers [2, 3]. Besov spaces play an important role in studying approximation and regularity of functions, and have many applications to partial differential equations. There are a lot of literatures on characterization of Besov spaces, and we refer to the books of Triebel [81, 82, 84] and Sawano [71] for the details. We are concerned with Besov spaces characterized by differential operators via the spectral approach (see [1, 6, 9, 18, 28, 44, 49] and references therein).

If Ω is the half space $\mathbb{R}^d_+$, a bounded domain or an exterior domain in $\mathbb{R}^d$ with smooth boundary, then the theory of Besov spaces is well established by extending functions on Ω to $\mathbb{R}^d$ or restricting functions on $\mathbb{R}^d$ to Ω (see [59, 68, 69, 80–85]). In this thesis we adopt a direct way, namely, we shall define Besov spaces on an open set Ω as subspaces of the collection of distributions on Ω via explicit norms. In the formulation we will face on the problem how to determine test function spaces over Ω corresponding to the Schwartz space and the Lizorkin test function space on $\mathbb{R}^d$. Recently, Bui, Duong and Yan introduced some test function spaces to define the Besov spaces $\dot{B}^s_{p,q}$ on an arbitrary open set, where s, p and q satisfy $|s| < 1$ and $1 \leq p, q \leq \infty$ (see [6]). They also proved the equivalence relation among the Besov spaces generated by the Laplacian and some operators, including the Schrödinger operators, on the whole space $\mathbb{R}^d$ with some additional conditions such as Hölder continuity for the kernel of semigroup generated by them. As to the results on the Besov spaces generated by the elliptic operators on manifolds or Hermite operators, we refer to [1, 6–9, 19, 49] and the references therein.

To the best of our knowledge, it is necessary to impose some smoothness assumptions on the boundary $\partial\Omega$ in order to define the Besov spaces $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ with all indices s, p, q satisfying $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. In this chapter we shall define the Besov spaces $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ generated by the Schrödinger operator $-\Delta + V$ with the Dirichlet boundary condition for all indices s, p, q *without any geometrical and smoothness assumption on the boundary* $\partial\Omega$, and shall prove the fundamental properties such as completeness, duality and embedding relations, etc. Furthermore, regarding the Besov spaces generated by the Dirichlet

Laplacian as the standard one, and adopting the potential V that belongs to the Lorentz space $L^{\frac{d}{2},\infty}(\Omega)$, we shall establish the equivalence relation between the Besov spaces generated by the Dirichlet Laplacian and the Schrödinger operators. The motivation of the study of such properties and equivalence relations comes from their applications to partial differential equations, and one can consult the papers of Jensen and Nakamura [44, 45], Georgiev and Visciglia [28], D'Ancona and Pierfelice [18] and Iwabuchi [40].

The arguments in this chapter are based on ones in Iwabuchi, Matsuyama, and Taniguchi [41]. Due to the spectral decomposition of $\mathcal{H}_V$, we can define the Sobolev spaces $H^s(\mathcal{H}_V)$ by letting

$$H^s(\mathcal{H}_V) = \left\{ f \in L^2(\Omega) : (I + \mathcal{H}_V)^{\frac{s}{2}} f \in L^2(\Omega) \right\} \quad \text{for } s \geq 0. \tag{4.1}$$

Then, the regularity and boundary value of functions in $H^s(\mathcal{H}_V)$ are determined by $\mathcal{H}_V$. In particular case $\Omega = \mathbb{R}^d$ and $V = 0$, i.e., $\mathcal{H}_V = -\Delta$ on $L^2(\mathbb{R}^d)$, $H^s(-\Delta)$ coincide with the standard Sobolev spaces defined via the Fourier transform (also called the Bessel-potential spaces). We shall apply the above characterization of $H^s(\mathcal{H}_V)$ to those of the inhomogeneous and homogeneous Besov spaces.

Let us recall the definition of test function spaces on $\mathbb{R}^d$ and the classical Besov spaces, i.e., spaces when $\Omega = \mathbb{R}^d$ and $V = 0$. It is well known that the inhomogeneous Besov spaces and homogeneous ones are characterized as subspaces of $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}'_0(\mathbb{R}^d)$ by the Littlewood-Paley dyadic decomposition of the spectrum of $\sqrt{-\Delta}$, namely, $B^s_{p,q}(\mathbb{R}^d)$ and $\dot{B}^s_{p,q}(\mathbb{R}^d)$ consist of all $f \in \mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}'_0(\mathbb{R}^d)$ such that

$$\begin{aligned} \|f\|_{B^s_{p,q}(\mathbb{R}^d)} = {} & \left\| \mathscr{F}^{-1} \psi(|\xi|) \mathscr{F} f \right\|_{L^p(\mathbb{R}^d)} \\ & + \left\| \left\{ 2^{sj} \| \mathscr{F}^{-1} \phi_j(|\xi|) \mathscr{F} f \|_{L^p(\mathbb{R}^d)} \right\}_{j \in \mathbb{N}} \right\|_{l^q(\mathbb{N})} < \infty, \end{aligned} \tag{4.2}$$

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left\| \left\{ 2^{sj} \| \mathscr{F}^{-1} \phi_j(|\xi|) \mathscr{F} f \|_{L^p(\mathbb{R}^d)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} < \infty, \tag{4.3}$$

respectively, where $\{\psi\} \cup \{\phi_j\}_j$ is the Littlewood-Paley dyadic decomposition (see (4.12)–(4.14) below). Here $\mathscr{S}'(\mathbb{R}^d)$ is the space of the tempered distributions on $\mathbb{R}^d$, which is the topological dual of the Schwartz space $\mathscr{S}(\mathbb{R}^d)$. The space $\mathscr{S}(\mathbb{R}^d)$ consists of rapidly decreasing functions equipped with the family of semi-norms

$$\sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{\frac{M}{2}} \sum_{|\alpha| \leq M} |\partial_x^\alpha f(x)|, \quad M = 1, 2, \cdots. \tag{4.4}$$

$\mathscr{S}'_0(\mathbb{R}^d)$ is the topological dual of the Lizorkin test function space $\mathscr{S}_0(\mathbb{R}^d)$, which is the subspace of $\mathscr{S}(\mathbb{R}^d)$ defined by

$$\mathscr{S}_0(\mathbb{R}^d) := \left\{ f \in \mathscr{S}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x^\alpha f(x)\, dx = 0 \text{ for any } \alpha \in \mathbb{N}_0^d \right\} \tag{4.5}$$

endowed with the induced topology of $\mathscr{S}(\mathbb{R}^d)$. It is known that $\mathscr{S}_0'(\mathbb{R}^d)$ is characterized by the quotient space of $\mathscr{S}'(\mathbb{R}^d)$ modulo polynomials, i.e.,

$$\mathscr{S}_0'(\mathbb{R}^d) \cong \mathscr{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d), \tag{4.6}$$

where $\mathcal{P}(\mathbb{R}^d)$ is the space of all polynomials of d real variables (see, e.g., Proposition 1.1.3 in Grafakos [30]).

When $\Omega \neq \mathbb{R}^d$, the following question naturally arises in the formulation:

Question. *What are spaces on Ω corresponding to $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}_0(\mathbb{R}^d)$?*

An answer will be given in section 4.1; we shall introduce the spaces $\mathcal{X}_V(\Omega)$ and $\mathcal{Z}_V(\Omega)$ corresponding to $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}_0(\mathbb{R}^d)$, respectively. There we will encounter with two problems in the formulations:

(a) To handle the neighborhood of zero spectrum in the definition of the homogeneous Besov spaces;

(b) To develop the dyadic resolution of identity operators on our spaces $\mathcal{X}_V'(\Omega)$ and $\mathcal{Z}_V'(\Omega)$; dyadic resolution lifted from $L^2(\Omega)$.

Let us explain the problem (a). Looking at the definition (4.5) of $\mathscr{S}_0(\mathbb{R}^d)$, one understands that the low frequency part of f is treated by

$$\int_{\mathbb{R}^d} x^\alpha f(x)dx = 0 \quad \text{for any } \alpha \in \mathbb{N}_0^d. \tag{4.7}$$

However, when $\Omega \neq \mathbb{R}^d$ it seems difficult to get an idea corresponding to (4.7). To overcome this difficulty, instead of (4.7), we propose

$$\sup_{j\leq 0} 2^{M|j|}\|\phi_j(\sqrt{\mathcal{H}_V})f\|_{L^1(\Omega)} < \infty, \quad M = 1, 2, \cdots \tag{4.8}$$

in semi-norms $q_{V,M}(\cdot)$ of a test function space $\mathcal{Z}_V(\Omega)$ (see (4.16) and Proposition 4.9 below). This is probably a main novelty in our work. The condition (4.8) seems one of important ingredients to introduce test function spaces not only for Besov spaces but also for other spaces of homogeneous type. We note that the problem of zero spectrum does not appear in the inhomogeneous Besov spaces, and hence, our spaces $\mathcal{X}_V(\Omega)$ and $\mathcal{X}_V'(\Omega)$ may be analogous to the test function spaces and their duals introduced by Kerkyacharian and Petrushev [49] (see also Ruzhansky and Tokmagambetov [67] who treat $H^s(\mathcal{H}_V)$ on a bounded open set, and the operator $\mathcal{H}_V$ does not have to be self-adjoint).

We turn to explain the problem (b). For the sake of simplicity, let us consider the case when $V = 0$. As is well known, the identity operator is resolved by the dyadic decomposition of the spectrum for $\mathcal{H}_0$ in $L^2(\Omega)$, namely,

$$I = \psi(\mathcal{H}_0) + \sum_{j=1}^{\infty} \phi_j(\sqrt{\mathcal{H}_0}), \tag{4.9}$$

which is assured by the spectral theorem. Initially, the resolution (4.9) holds in $L^2(\Omega)$, and then, it is lifted to the space $\mathcal{X}_0'(\Omega)$. This argument is accomplished in Lemma 4.5 below. When one considers $\mathcal{Z}_0'(\Omega)$, (4.9) is replaced by

$$I = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_0}). \tag{4.10}$$

Thanks to these resolutions (4.9) and (4.10), the well known methods in the classical Besov spaces on $\mathbb{R}^d$ work well also in the present case. The starting point of this argument is to extend the spectral restriction operators $\phi_j(\sqrt{\mathcal{H}_0})$ on $L^2(\Omega)$ to those on $L^1(\Omega)$. There, the uniform boundedness on $L^1(\Omega)$ of $\{\phi_j(\sqrt{\mathcal{H}_0})\}_{j=-\infty}^{\infty}$, i.e.,

$$\sup_{j\in\mathbb{Z}} \|\phi_j(\sqrt{\mathcal{H}_0})\|_{\mathscr{B}(L^1(\Omega))} < \infty \tag{4.11}$$

plays a crucial role in proving (4.9) in $\mathcal{X}_0'(\Omega)$ and (4.10) in $\mathcal{Z}_0'(\Omega)$, respectively. This uniform boundedness follows from Theorem 3.1 in chapter 2 (see Lemma 4.1 below). Furthermore, (4.11) guarantees the independence of the choice of $\{\psi\} \cup \{\phi_j\}_j$, when we define spaces $\mathcal{X}_0(\Omega)$, $\mathcal{X}_0'(\Omega)$, $\mathcal{Z}_0(\Omega)$, $\mathcal{Z}_0'(\Omega)$ and Besov spaces defined in subsections 4.1.1 and 4.2.1. We should also note that our argument can be applied not only to the Schrödinger operator $\mathcal{H}_V$ but also to more general self-adjoint operators $\mathcal{L}$ such that the Gaussian upper bounds for $e^{-t\mathcal{L}}$ hold.

This chapter is organized as follow. In section 4.1 we define test function and distribution spaces on Ω and prove their fundamental properties. In section 4.2 we give definitions of Besov spaces generated by $\mathcal{H}_V$ and prove their fundamental properties.

4.1 Test functions and distribution spaces

In this section we give definitions and show fundamental properties of test function and distribution spaces on Ω, which provide the basis for the study of our Besov spaces.

4.1.1 Definitions and notations

To define and investigate test functions and distribution spaces, let us introduce the Littlewood-Paley dyadic decomposition. Let ϕ_0 be a non-negative and smooth function on $\mathbb{R}$ such that

$$\operatorname{supp}\phi_0 \subset \{\,\lambda \in \mathbb{R} : 2^{-1} \le \lambda \le 2\,\} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \phi_0(2^{-j}\lambda) = 1 \quad \text{for } \lambda > 0, \tag{4.12}$$

and $\{\phi_j\}_{j=-\infty}^{\infty}$ is defined by letting

$$\phi_j(\lambda) := \phi_0(2^{-j}\lambda) \quad \text{for } \lambda \in \mathbb{R}. \tag{4.13}$$

Let $\psi \in C_0^\infty(\mathbb{R})$ be a non-negative function on $\mathbb{R}$ such that

$$\psi(\lambda) = 1 \quad \text{for } \lambda \in [-\lambda_0, 0] \quad \text{and} \quad \psi(\lambda^2) + \sum_{j=1}^{\infty} \phi_j(\lambda) = 1 \quad \text{for } \lambda \geq 0, \tag{4.14}$$

where λ_0 is a positive constant such that $\lambda_0 > -\inf \sigma(\mathcal{H}_V)$.

Definition (Spaces of test functions and distributions on Ω). Suppose that the potential V satisfies assumption A.

(i) (Linear topological spaces $\mathcal{X}_V(\Omega)$ and $\mathcal{X}'_V(\Omega)$). A linear topological space $\mathcal{X}_V(\Omega)$ is defined by letting

$$\mathcal{X}_V(\Omega) := \left\{ f \in L^1(\Omega) \cap \mathcal{D}(\mathcal{H}_V) : \mathcal{H}_V^M f \in L^1(\Omega) \cap \mathcal{D}(\mathcal{H}_V) \text{ for any } M \in \mathbb{N} \right\} \tag{4.15}$$

equipped with the family of semi-norms $\{p_{V,M}(\cdot)\}_{M=1}^\infty$ given by

$$p_{V,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j \in \mathbb{N}} 2^{Mj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)}.$$

$\mathcal{X}'_V(\Omega)$ denotes the topological dual of $\mathcal{X}_V(\Omega)$.

(ii) (Linear topological spaces $\mathcal{Z}_V(\Omega)$ and $\mathcal{Z}'_V(\Omega)$). Suppose that V_- satisfies assumption B. A linear topological space $\mathcal{Z}_V(\Omega)$ is defined by letting

$$\mathcal{Z}_V(\Omega) := \left\{ f \in \mathcal{X}_V(\Omega) : \sup_{j \leq 0} 2^{M|j|} \left\| \phi_j\big(\sqrt{\mathcal{H}_V}\big) f \right\|_{L^1(\Omega)} < \infty \text{ for any } M \in \mathbb{N} \right\}$$

equipped with the family of semi-norms $\{q_{V,M}(\cdot)\}_{M=1}^\infty$ given by

$$q_{V,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j \in \mathbb{Z}} 2^{M|j|} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)}. \tag{4.16}$$

$\mathcal{Z}'_V(\Omega)$ denotes the topological dual of $\mathcal{Z}_V(\Omega)$.

Let us recall the notation ${}_{X'}\langle \cdot, \cdot \rangle_X$ for the duality pair of a topological vector space X and its dual X'. It is proved in Proposition 4.2 below that $\mathcal{X}_V(\Omega)$ and $\mathcal{Z}_V(\Omega)$ are Fréchet spaces, and in Proposition 4.6 below that

$$\mathcal{X}_V(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{X}'_V(\Omega), \tag{4.17}$$

$$\mathcal{Z}_V(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{Z}'_V(\Omega) \tag{4.18}$$

for any $1 \leq p \leq \infty$. The embedding relation (4.17) ((4.18) resp.) assures that

$$\int_\Omega \left| f(x) \overline{g(x)} \right| dx < \infty$$

for any $f \in L^p(\Omega)$, $1 \leq p \leq \infty$, and $g \in \mathcal{X}_V(\Omega)$ ($g \in \mathcal{Z}_V(\Omega)$ resp.). Hence we can regard functions in the Lebesgue spaces as elements in $\mathcal{X}'_V(\Omega)$ and $\mathcal{Z}'_V(\Omega)$ as follows:

Definition. For $f \in L^1(\Omega)+L^\infty(\Omega)$, we identify f as an element in $\mathcal{X}_V'(\Omega)$ ($\mathcal{Z}_V'(\Omega)$ resp.) by letting

$$ {}_{\mathcal{X}_V'(\Omega)}\langle f, g\rangle_{\mathcal{X}_V(\Omega)} = \int_\Omega f(x)\overline{g(x)}\,dx \quad \left({}_{\mathcal{Z}_V'(\Omega)}\langle f, g\rangle_{\mathcal{Z}_V(\Omega)} = \int_\Omega f(x)\overline{g(x)}\,dx \quad \text{resp.} \right) $$

for any $g \in \mathcal{X}_V(\Omega)$ ($g \in \mathcal{Z}_V(\Omega)$ resp.).

For a mapping $\phi(\mathcal{H}_V)$ on $\mathcal{X}_V(\Omega)$ ($\mathcal{Z}_V(\Omega)$ resp.), we define the dual operator of $\phi(\mathcal{H}_V)$ on $\mathcal{X}_V'(\Omega)$ ($\mathcal{Z}_V'(\Omega)$ resp.) induced naturally from that on $L^2(\Omega)$.

Definition. Let ϕ be a real-valued Borel measurable function on $\mathbb{R}$.

(i) For a mapping $\phi(\mathcal{H}_V) : \mathcal{X}_V(\Omega) \to \mathcal{X}_V(\Omega)$, we define $\phi(\mathcal{H}_V) : \mathcal{X}_V'(\Omega) \to \mathcal{X}_V'(\Omega)$ by letting

$$ {}_{\mathcal{X}_V'}\langle \phi(\mathcal{H}_V)f, g\rangle_{\mathcal{X}_V} := {}_{\mathcal{X}_V'}\langle f, \phi(\mathcal{H}_V)g\rangle_{\mathcal{X}_V} \tag{4.19} $$

for any $g \in \mathcal{X}_V(\Omega)$.

(ii) For a mapping $\phi(\mathcal{H}_V) : \mathcal{Z}_V(\Omega) \to \mathcal{Z}_V(\Omega)$, we define $\phi(\mathcal{H}_V) : \mathcal{Z}_V'(\Omega) \to \mathcal{Z}_V'(\Omega)$ by letting

$$ {}_{\mathcal{Z}_V'}\langle \phi(\mathcal{H}_V)f, g\rangle_{\mathcal{Z}_V} := {}_{\mathcal{Z}_V'}\langle f, \phi(\mathcal{H}_V)g\rangle_{\mathcal{Z}_V} \tag{4.20} $$

for any $g \in \mathcal{Z}_V(\Omega)$.

Let us give a few remarks on $\mathcal{X}_0(\Omega)$ and $\mathcal{Z}_0(\Omega)$ as follows:

- When $\Omega = \mathbb{R}^d$ and $V = 0$, the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ is contained in $\mathcal{X}_0(\mathbb{R}^d)$, and the inclusion for tempered distributions are just opposite. Namely, it can be readily checked that

$$ \begin{aligned} &\mathscr{S}(\mathbb{R}^d) \hookrightarrow \mathcal{X}_0(\mathbb{R}^d) \hookrightarrow \mathcal{X}_0'(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d), \\ &\mathscr{S}_0(\mathbb{R}^d) \hookrightarrow \mathcal{Z}_0(\mathbb{R}^d) \hookrightarrow \mathcal{Z}_0'(\mathbb{R}^d) \hookrightarrow \mathscr{S}_0'(\mathbb{R}^d), \\ &C_0^\infty(\mathbb{R}^d) \subset \mathcal{X}_0(\mathbb{R}^d), \quad C_0^\infty(\mathbb{R}^d) \not\subset \mathcal{Z}_0(\mathbb{R}^d). \end{aligned} $$

- When $\Omega = \mathbb{R}^d$ and $V = 0$, the restriction of low frequency in the definition (4.16) of $q_{0,M}(f)$ is natural, since one can show that any element $f \in \mathscr{S}(\mathbb{R}^d)$ belongs to $\mathscr{S}_0(\mathbb{R}^d)$ if and only if $q_{0,M}(f) < \infty$ for $M = 1, 2, \ldots$ (see Proposition 4.9 below).

- When $\Omega \neq \mathbb{R}^d$ and $\partial\Omega$ is smooth, any $f \in \mathcal{X}_0(\Omega)$ or $\mathcal{Z}_0(\Omega)$ satisfies

$$ f \equiv 0 \quad \text{on } \partial\Omega, $$

since $f \in H_0^1(\Omega)$. Hence, the condition $p_{0,M}(f) < \infty$ not only determines smoothness and integrability of f but also assures the Dirichlet boundary condition. Also, such an f contacts with $\partial\Omega$ of order infinity in the following way:

$$ \mathcal{H}_0^M f \equiv 0 \quad \text{on } \partial\Omega, \quad M = 0, 1, 2, \cdots. $$

The same assertion holds for $\mathcal{X}_V(\Omega)$, $\mathcal{Z}_V(\Omega)$ and $\mathcal{H}_V$.

- In order to simplify the argument, instead of the polynomial weights appearing in semi-norms (4.4) of $\mathscr{S}(\mathbb{R}^d)$, we adopted the integrability condition on f.

4.1.2 Properties of test functions and distribution spaces

In this section we prove the fundamental properties of $\mathcal{X}_V(\Omega)$, $\mathcal{Z}_V(\Omega)$ and their dual spaces. For this purpose, we prepare a lemma on L^p-estimates for operators $\psi(\mathcal{H}_V)$ and $\phi_j(\sqrt{\mathcal{H}_V})$ for $1 \le p \le \infty$.

Based on Theorem 3.1, we have the following.

Lemma 4.1. *Let $1 \le r \le p \le \infty$. Suppose that the potential V satisfies assumption* A. *Then we have the following assertions*:

(i) *For any $\phi \in C_0^\infty(\mathbb{R})$ and $m \in \mathbb{N}_0$ there exists a constant $C > 0$ such that*

$$\|\mathcal{H}_V^m \phi(\mathcal{H}_V) f\|_{L^p(\Omega)} \le C \|f\|_{L^r(\Omega)} \tag{4.21}$$

for any $f \in L^r(\Omega)$.

(ii) *For any $\phi \in C_0^\infty((0,\infty))$ and $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$\|\mathcal{H}_V^\alpha \phi(2^{-j}\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \le C 2^{d(\frac{1}{r}-\frac{1}{p})j+2\alpha j} \|f\|_{L^r(\Omega)} \tag{4.22}$$

for any $j \in \mathbb{N}$ and $f \in L^r(\Omega)$.

(iii) *Suppose that V satisfies assumption* B. *Then for any $\phi \in C_0^\infty((0,\infty))$ and $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$\|\mathcal{H}_V^\alpha \phi(2^{-j}\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \le C 2^{d(\frac{1}{r}-\frac{1}{p})j+2\alpha j} \|f\|_{L^r(\Omega)} \tag{4.23}$$

for any $j \in \mathbb{Z}$ and $f \in L^r(\Omega)$.

Proof. Let $m \in \mathbb{N}_0$. Since

$$\lambda^m \phi(\lambda) \in C_0^\infty(\mathbb{R}),$$

the estimate (4.21) is an immediate consequence of (i) in Theorem 3.1. As to the estimate (4.22), we deduce from (i) in Theorem 3.1 that

$$\begin{aligned}\|\mathcal{H}_V^\alpha \phi(2^{-j}\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} &= 2^{2\alpha j} \|(2^{-2j}\mathcal{H}_V)^\alpha \phi(2^{-j}\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \\ &\le C 2^{d(\frac{1}{r}-\frac{1}{p})j+2\alpha j} \|f\|_{L^r(\Omega)}\end{aligned}$$

for any $\alpha \in \mathbb{R}$ and $j \in \mathbb{N}$, since

$$\lambda^{2\alpha} \phi(\lambda) \in C_0^\infty((0,\infty)).$$

Hence (4.22) is proved. The estimate (4.23) is also proved in the analogous way to the above argument by applying (ii) in Theorem 3.1 instead of (i) in Theorem 3.1. The proof of Lemma 4.1 is finished. □

Let us discuss the fundamental properties of $\mathcal{X}_V(\Omega)$, $\mathcal{Z}_V(\Omega)$ and their dual spaces. The first result is the following.

Proposition 4.2. *Suppose that the potential V satisfies assumption* A. *Then $\mathcal{X}_V(\Omega)$ is complete. Furthermore, if V satisfies assumption* B*, then $\mathcal{Z}_V(\Omega)$ is complete.*

Proof. We first show the completeness of $\mathcal{X}_V(\Omega)$. We start by proving that for any $M \in \mathbb{N}_0$ there exists a constant $C > 0$ such that

$$\|\mathcal{H}_V^M f\|_{L^1 \cap L^2} \leq C p_{V,M'}(f) \tag{4.24}$$

provided that $f \in L^1(\Omega) \cap L^2(\Omega)$ with $p_{V,M'}(f) < \infty$, where we put

$$M' := \begin{cases} 2M + \dfrac{d}{2} + 1 & \text{if } d \text{ is even,} \\ 2M + \dfrac{d+1}{2} + 1 & \text{if } d \text{ is odd.} \end{cases}$$

Put

$$\Phi_1 := \psi + \phi_1 + \phi_2 \quad \text{and} \quad \Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \text{ for } j = 2, 3, \cdots .$$

It follows from the dyadic resolution in $L^2(\Omega)$, identities $\phi_j = \Phi_j \phi_j$ and estimate (4.22) for $\alpha = M$ in Lemma 4.1 that

$$\begin{aligned} \|\mathcal{H}_V^M f\|_{L^1 \cap L^2} &\leq \|\mathcal{H}_V^M \psi(\mathcal{H}_V) f\|_{L^1 \cap L^2} + \sum_{j=1}^{\infty} \|\mathcal{H}_V^M \Phi_j(\sqrt{\mathcal{H}_V}) \phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1 \cap L^2} \\ &\leq C\|f\|_{L^1(\Omega)} + C \sum_{j=1}^{\infty} 2^{(2M + \frac{d}{2})j} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} \\ &\leq C p_{V,0}(f) + C p_{V,M'}(f) \sum_{j=1}^{\infty} 2^{-j} \\ &\leq C p_{V,M'}(f), \end{aligned}$$

which proves (4.24). We turn to prove the completeness of $\mathcal{X}_V(\Omega)$. Let $\{f_N\}_{N=1}^{\infty}$ be a Cauchy sequence in $\mathcal{X}_V(\Omega)$. Then, for any $M \in \mathbb{N}$, there exists $C_M > 0$ such that

$$p_{V,M}(f_N) \leq C_M \quad \text{for all } N \in \mathbb{N}. \tag{4.25}$$

Since $\{f_N\}$ is a Cauchy sequence in $L^1(\Omega)$, there exists a function $f \in L^1(\Omega)$ such that

$$f_N \to f \quad \text{ in } L^1(\Omega) \text{ as } N \to \infty.$$

Combining this convergence with the boundedness of $2^{Mj}\phi_j(\sqrt{\mathcal{H}_V})$ from $L^1(\Omega)$ to itself, which is assured by (4.22) for $\alpha = 0$ and (4.25), we have

$$2^{Mj}\|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1} = \lim_{N \to \infty} 2^{Mj}\|\phi_j(\sqrt{\mathcal{H}_V}) f_N\|_{L^1},$$

and hence,

$$p_{V,M}(f) \le C_M \tag{4.26}$$

for any $M \in \mathbb{N}$. Here, noting that the Sobolev spaces $H^s(\mathcal{H}_V)$ defined in (4.1) are complete and that $\{f_N\}_N$ is also a Cauchy sequence in $H^{2M}(\mathcal{H}_V)$ for $M \in \mathbb{N}_0$ by the estimate (4.24), we deduce from the inequalities (4.24) and (4.26) that

$$\mathcal{H}_V^M f \in L^1(\Omega) \cap L^2(\Omega) \quad \text{for any } M \in \mathbb{N}_0.$$

Hence we obtain $f \in \mathcal{X}_V(\Omega)$. We next show the convergence of f_N to f in $\mathcal{X}_V(\Omega)$. For each M, let us take a subsequence $\{f_{N(k)}\}_{k=1}^{\infty}$ such that

$$p_{V,M}(f_{N(k)} - f_{N(k-1)}) \le 2^{-k},$$

where we put $f_{N(0)} = 0$. Hence we have

$$\sum_{k=1}^{\infty} p_{V,M}(f_{N(k)} - f_{N(k-1)}) < \infty. \tag{4.27}$$

Since $\{f_{N(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^1(\Omega)$, f is written by

$$f = \lim_{L\to\infty} f_{N(L)} = \lim_{L\to\infty} \sum_{k=1}^{L} (f_{N(k)} - f_{N(k-1)}) \quad \text{in } L^1(\Omega). \tag{4.28}$$

Then (4.27) and (4.28) yield the convergence of $p_{V,M}(f_{N(L)} - f)$ to zero as $L \to \infty$, and hence,

$$p_{V,M}(f_N - f) \to 0 \quad \text{as } N \to \infty \quad \text{for any } M \in \mathbb{N}.$$

Therefore $\mathcal{X}_V(\Omega)$ is complete.

We next show the completeness of $\mathcal{Z}_V(\Omega)$. Let $\{f_N\}_{N=1}^{\infty}$ be a Cauchy sequence in $\mathcal{Z}_V(\Omega)$. Since $\mathcal{Z}_V(\Omega)$ is a subspace of $\mathcal{X}_V(\Omega)$ and $\mathcal{X}_V(\Omega)$ is complete, $\{f_N\}_{N=1}^{\infty}$ is also a Cauchy sequence in $\mathcal{X}_V(\Omega)$ and there exists an element $f \in \mathcal{X}_V(\Omega)$ such that f_N converges to f in $\mathcal{X}_V(\Omega)$ as $N \to \infty$. In order to prove $f \in \mathcal{Z}_V(\Omega)$, we show that

$$\sup_{j\le 0} 2^{M|j|} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} < \infty \quad \text{for any } M \in \mathbb{N}. \tag{4.29}$$

Since f_N converges to f in $L^1(\Omega)$ as $N \to \infty$ and $\phi_j(\sqrt{\mathcal{H}_V})$ is bounded on $L^1(\Omega)$ for each $j \in \mathbb{Z}$ by (4.23) for $\alpha = 0$, it follows that

$$\lim_{N\to\infty} \|\phi_j(\sqrt{\mathcal{H}_V}) f_N\|_{L^1(\Omega)} = \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} \quad \text{for any } j \in \mathbb{Z}.$$

Since $\{f_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathcal{Z}_V(\Omega)$, $\{q_{V,M}(f_N)\}_{N=1}^{\infty}$ is a bounded sequence for each M and there exists a constant $C_M > 0$ depending only on M such that

$$2^{M|j|} \|\phi_j(\sqrt{-\Delta}) f_N\|_{L^1(\Omega)} \le C_M \quad \text{for all } j \le 0 \text{ and } N = 1, 2, \cdots.$$

By taking the limit as $N \to \infty$ in the above inequality, we conclude that f satisfies (4.29), and hence, $f \in \mathcal{Z}_V(\Omega)$. Finally, the convergence of f_N to f in $\mathcal{Z}_V(\Omega)$ follows from the analogous argument to (4.27) and (4.28):

$$\sum_{k=1}^{\infty} q_{V,M}(f_{N(k)} - f_{N(k-1)}) < \infty,$$

$$f = \lim_{L\to\infty} \sum_{k=1}^{L} (f_{N(k)} - f_{N(k-1)}) \quad \text{in } L^1(\Omega),$$

where $f_{N(0)} = 0$, which imply that

$$q_{V,M}(f_N - f) \to 0 \quad \text{as } N \to \infty \quad \text{for any } M \in \mathbb{N}.$$

Thus we conclude that $\mathcal{Z}_V(\Omega)$ is complete. The proof of Proposition 4.2 is now finished. □

Proposition 4.3. *Suppose that the potential V satisfies assumption* A. *Then the following assertions hold*:

(i) *For any $f \in \mathcal{X}_V'(\Omega)$, there exist a number $M_0 \in \mathbb{N}$ and a constant $C_f > 0$ such that*

$$|{}_{\mathcal{X}_V'}\langle f, g\rangle_{\mathcal{X}_V}| \le C_f\, p_{V,M_0}(g) \quad \textit{for any } g \in \mathcal{X}_V(\Omega).$$

(ii) *Furthermore, if V satisfies assumption* B, *then for any $f \in \mathcal{Z}_V'(\Omega)$, there exist a number $M_1 \in \mathbb{N}$ and a constant $C_f > 0$ such that*

$$|{}_{\mathcal{Z}_V'}\langle f, g\rangle_{\mathcal{Z}_V}| \le C_f q_{V,M_1}(g) \quad \textit{for any } g \in \mathcal{Z}_V(\Omega).$$

Proof. Suppose that (i) is not true. Then, for any $m \in \mathbb{N}$, there exists $g_m \in \mathcal{X}_V(\Omega)$ such that

$$|{}_{\mathcal{X}_V'}\langle f, g_m\rangle_{\mathcal{X}_V}| > m p_{V,m}(g_m). \tag{4.30}$$

Put

$$\widetilde{g}_m := \frac{g_m}{m p_{V,m}(g_m)}.$$

Noting that $p_{V,k}(\widetilde{g}_m)$ is monotonically increasing in $k \in \{1, 2, \dots, m\}$, we have

$$p_{V,k}(\widetilde{g}_m) \le p_{V,m}(\widetilde{g}_m) = \frac{1}{m} \quad \text{for } k = 1, 2, \cdots, m.$$

Hence it follows that for any fixed $k \in \mathbb{N}$

$$p_{V,k}(\widetilde{g}_m) \to 0 \quad \text{as } m \to \infty;$$

thus we find that

$$\widetilde{g}_m \to 0 \quad \text{in } \mathcal{X}_V(\Omega) \text{ as } m \to \infty.$$

The above convergence yields that

$$|{}_{\mathcal{X}'_V}\langle f, \widetilde{g}_m\rangle_{\mathcal{X}_V}| \to 0 \quad \text{as } m \to \infty. \tag{4.31}$$

However, the assumption (4.30) implies that

$$|{}_{\mathcal{X}'_V}\langle f, \widetilde{g}_m\rangle_{\mathcal{X}_V}| > 1 \quad \text{for all } m \in \mathbb{N};$$

therefore this inequality contradicts (4.31). Thus the assertion (i) holds. The assertion (ii) follows analogously. This ends the proof of Proposition 4.3. □

Proposition 4.4. *Suppose that the potential V satisfies assumption* A*. Then the following assertions hold*:

(i) *For any $\phi \in C_0^\infty(\mathbb{R})$, $\phi(\mathcal{H}_V)$ maps continuously from $\mathcal{X}_V(\Omega)$ into itself, and maps continuously from $\mathcal{X}'_V(\Omega)$ into itself.*

(ii) *Furthermore, if V satisfies assumption* B*, then for any $\phi \in C_0^\infty((0,\infty))$, $\phi(\mathcal{H}_V)$ maps continuously from $\mathcal{Z}_V(\Omega)$ into itself, and maps continuously from $\mathcal{Z}'_V(\Omega)$ into itself.*

Proof. First we prove the assertion (i). Let $f \in \mathcal{X}_V(\Omega)$. It follows from (4.21) in Lemma 4.1 that

$$\mathcal{H}_V^m \phi(\mathcal{H}_V) f \in \mathcal{D}(\mathcal{H}_V), \quad p_{V,M}(\phi(\mathcal{H}_V)f) \le C p_{V,M}(f) \tag{4.32}$$

for any $m \in \mathbb{N}_0$ and $M \in \mathbb{N}$. This proves that $\phi(\mathcal{H}_V)$ is continuous from $\mathcal{X}_V(\Omega)$ into itself. The continuity of $\phi(\mathcal{H}_V)$ from $\mathcal{X}'_V(\Omega)$ into itself follows from the definition (4.19).

As to the assertion (ii), since V satisfies assumption A, $\phi(\mathcal{H}_V)$ enjoys the assertion (i), and hence, we conclude that

$$\phi(\mathcal{H}_V) f \in \mathcal{X}_V(\Omega) \quad \text{for any } f \in \mathcal{Z}_V(\Omega).$$

We show that

$$q_{V,M}(\phi(\mathcal{H}_V)f) \le C q_{V,M}(f) \tag{4.33}$$

for any $M \in \mathbb{N}$. Indeed, recalling the definition (4.16) of $q_{V,M}(f)$ and noting that

$$q_{V,M}(\phi(\mathcal{H}_V)f) \le p_{V,M}(\phi(\mathcal{H}_V)f) + \sup_{j\le 0} 2^{M|j|} \|\phi_j(\sqrt{\mathcal{H}_V})\phi(\mathcal{H}_V)f\|_{L^1(\Omega)},$$

we apply (4.32) to the first term to obtain

$$p_{V,M}(\phi(\mathcal{H}_V)f) \le C p_{V,M}(f) \le C q_{V,M}(f).$$

For the second term in $q_{V,M}(\phi(\mathcal{H}_V)f)$, again applying (4.21) for $m = 0$, we estimate

$$\sup_{j\le 0} 2^{M|j|} \|\phi_j(\sqrt{\mathcal{H}_V})\phi(\mathcal{H}_V)f\|_{L^1(\Omega)} \le C \sup_{j\le 0} 2^{M|j|} \|\phi_j(\sqrt{\mathcal{H}_V})f\|_{L^1(\Omega)} \le C q_{V,M}(f)$$

for any $M \in \mathbb{N}$. Therefore, the above two estimates imply (4.33), which concludes the continuity of $\phi(\mathcal{H}_V)$ from $\mathcal{Z}_V(\Omega)$ into itself. Finally, the continuity of $\phi(\mathcal{H}_V)$ from $\mathcal{Z}'_V(\Omega)$ into itself follows from the definition (4.20). The proof of Lemma 4.4 is finished. □

The approximation of identity is established by the following lemma.

Proposition 4.5. *Suppose that the potential V satisfies assumption* A. *Then the following assertions hold*:

(i) *For any $f \in \mathcal{X}_V(\Omega)$, we have*

$$f = \psi(\mathcal{H}_V)f + \sum_{j=1}^{\infty} \phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } \mathcal{X}_V(\Omega). \tag{4.34}$$

Furthermore, for any $f \in \mathcal{X}'_V(\Omega)$, we have also the identity (4.34) *in $\mathcal{X}'_V(\Omega)$, and $\psi(\mathcal{H}_V)f$ and $\phi_j(\sqrt{\mathcal{H}_V})f$ are regarded as elements in $L^\infty(\Omega)$.*

(ii) *Furthermore, if V satisfies assumption* B, *then for any $f \in \mathcal{Z}_V(\Omega)$, we have*

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } \mathcal{Z}_V(\Omega). \tag{4.35}$$

Furthermore, for $f \in \mathcal{Z}'_V(\Omega)$, we have also the identity (4.35) *in $\mathcal{Z}'_V(\Omega)$, and $\phi_j(\sqrt{\mathcal{H}_V})f$ are regarded as elements in $L^\infty(\Omega)$.*

Proof. First we prove the assertion (i). Let $f \in \mathcal{X}_V(\Omega)$. Then we have $f \in L^2(\Omega)$, and f is written as

$$f = \psi(\mathcal{H}_V)f + \sum_{j=1}^{\infty} \phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } L^2(\Omega). \tag{4.36}$$

It is sufficient to verify that the series in the right member is absolutely convergent in $\mathcal{X}_V(\Omega)$. Let $M \in \mathbb{N}$ be arbitrarily fixed. Applying (4.22) for $\alpha = 0, 1$ from Lemma 4.1, we have

$$p_{V,M}\big(\psi(\mathcal{H}_V)f\big) \le C p_{V,M}(f),$$

$$p_{V,M}\big(\phi_j(\sqrt{\mathcal{H}_V})f\big) \le C2^{-2j} p_{V,M}\big(\mathcal{H}_V \phi_j(\sqrt{\mathcal{H}_V})f\big) \le C2^{-2j} p_{V,M+2}(f),$$

which imply that

$$\sum_{j=1}^{\infty} p_{V,M}\big(\phi_j(\sqrt{\mathcal{H}_V})f\big) \le C p_{V,M+2}(f) \sum_{j=1}^{\infty} 2^{-2j} < \infty. \tag{4.37}$$

Hence (4.34) holds for $f \in \mathcal{X}_V(\Omega)$. As to the expansion (4.34) for $f \in \mathcal{X}_V'(\Omega)$, applying the identity (4.34) for $g \in \mathcal{X}_V(\Omega)$, we have formally the following identity:

$$\begin{aligned} {}_{\mathcal{X}_V'}\langle f, g\rangle_{\mathcal{X}_V} &= {}_{\mathcal{X}_V'}\langle f, \psi(\mathcal{H}_V)g\rangle_{\mathcal{X}_V} + \sum_{j=1}^{\infty} {}_{\mathcal{X}_V'}\langle f, \phi_j(\sqrt{\mathcal{H}_V})g\rangle_{\mathcal{X}_V} \\ &= {}_{\mathcal{X}_V'}\langle \psi(\mathcal{H}_V)f, g\rangle_{\mathcal{X}_V} + \sum_{j=1}^{\infty} {}_{\mathcal{X}_V'}\langle \phi_j(\sqrt{\mathcal{H}_V})f, g\rangle_{\mathcal{X}_V}, \end{aligned} \tag{4.38}$$

where the second equality is valid due to the definition (4.19). We must prove the absolute convergence of the series in (4.38). By Lemma 4.3 (i), there exist $M_0 \in \mathbb{N}$ and $C > 0$ such that

$$|{}_{\mathcal{X}_V'}\langle \phi_j(\sqrt{\mathcal{H}_V})f, g\rangle_{\mathcal{X}_V}| = |{}_{\mathcal{X}_V'}\langle f, \phi_j(\sqrt{\mathcal{H}_V})g\rangle_{\mathcal{X}_V}| \le C_f p_{V,M_0}\big(\phi_j(\sqrt{\mathcal{H}_V})g\big).$$

Then, the above estimate and (4.37) yield the absolute convergence of the series in (4.38).

For the proof of $\psi(\mathcal{H}_V)f \in L^\infty(\Omega)$, we begin by proving that

$$|{}_{\mathcal{X}_V'}\langle \psi(\mathcal{H}_V)f, g\rangle_{\mathcal{X}_V}| \le C\|g\|_{L^1(\Omega)} \quad \text{for all } g \in \mathcal{X}_V(\Omega). \tag{4.39}$$

By the definition (4.19), Lemma 4.3 (i) and (4.21) for $m = 0$, there exist $M_0 \in \mathbb{N}$ and $C_f, C_{f,\psi} > 0$ such that

$$\begin{aligned} |{}_{\mathcal{X}_V'}\langle \psi(\mathcal{H}_V)f, g\rangle_{\mathcal{X}_V}| &= |{}_{\mathcal{X}_V'}\langle f, \psi(\mathcal{H}_V)g\rangle_{\mathcal{X}_V}| \\ &\le C_f p_{V,M_0}(\psi(\mathcal{H}_V)g) \\ &\le C_{f,\psi}\|g\|_{L^1(\Omega)}, \end{aligned}$$

which proves (4.39). Thanks to (4.39), the Hahn-Banach theorem allows us to deduce that the mapping

$${}_{\mathcal{X}_V'}\langle \psi(\mathcal{H}_V)f, \cdot\rangle_{\mathcal{X}_V} : \mathcal{X}_V(\Omega) \to \mathbb{C}$$

is extended as a mapping from $L^1(\Omega)$ to $\mathbb{C}$. Since $L^1(\Omega)^* = L^\infty(\Omega)$, there exists a function $F \in L^\infty(\Omega)$ such that

$${}_{\mathcal{X}_V'}\langle \psi(\mathcal{H}_V)f, g\rangle_{\mathcal{X}_V} = \int_\Omega F(x)\overline{g(x)}\,dx \quad \text{for all } g \in \mathcal{X}_V(\Omega).$$

Then we conclude that $\psi(\mathcal{H}_V)f \in L^\infty(\Omega)$. In a similar way, it is possible to prove that $\phi_j(\sqrt{\mathcal{H}_V})f \in L^\infty(\Omega)$. The proof of (i) is now complete.

As to the assertion (ii), noting that any $f \in \mathcal{Z}_V(\Omega)$ is in $L^2(\Omega)$, we first prove that

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } L^2(\Omega) \tag{4.40}$$

for any $f \in L^2(\Omega)$. Put

$$g_L := \int_{-\infty}^{\infty} \Big(1 - \sum_{j=L}^{\infty} \phi_j(\sqrt{\lambda})\Big) dE_{\mathcal{H}_V}(\lambda) f. \tag{4.41}$$

It is readily checked that $\{g_L\}_L$ is a Cauchy sequence in $L^2(\Omega)$, so we put

$$g := \lim_{L \to -\infty} g_L \quad \text{in } L^2(\Omega).$$

Noting that $\mathcal{H}_V$ is non-negative on $L^2(\Omega)$ and that the support of $1 - \sum_{j=L}^{\infty} \phi_j(\sqrt{\lambda})$ is contained in the interval $(-\infty, 2^{2L}]$, we find that

$$\begin{aligned} \|\mathcal{H}_V g_L\|^2_{L^2(\Omega)} &= \int_{-\infty}^{2^{2L}} \Big|\lambda\Big(1 - \sum_{j=L}^{\infty} \phi_j(\sqrt{\lambda})\Big)\Big|^2 d\|E_{\mathcal{H}_V}(\lambda) f\|^2_{L^2(\Omega)} \\ &\leq C 2^{4L} \|f\|^2_{L^2(\Omega)} \to 0 \quad \text{as } L \to -\infty. \end{aligned}$$

Hence we deduce that

$$g \in \mathcal{D}(\mathcal{H}_V) \quad \text{and} \quad \mathcal{H}_V g = 0 \quad \text{in } L^2(\Omega)$$

by the fact that $g_L \in \mathcal{D}(\mathcal{H}_V)$, the definition of g, and the closeness of $\mathcal{H}_V$ on $L^2(\Omega)$. Since zero is not an eigenvalue of $\mathcal{H}_V$ by the assertion (ii) in Proposition 2.1, we conclude that $g = 0$, which proves (4.40) for any $f \in L^2(\Omega)$.

Now, as in the previous argument, it is sufficient to show that the series in the right member of (4.40) is absolutely convergent in $\mathcal{Z}_V(\Omega)$. For the series (4.40) with $j \geq 1$, the absolute convergence is obtained by the same argument as (4.37). For the case $j \leq 0$, it follows from (4.23) for $\alpha = \pm 1$ that

$$q_{V,M}\big(\phi_j(\sqrt{\mathcal{H}_V}) f\big) \leq C 2^{2j} q_{V,M}\big(\mathcal{H}_V^{-1} \phi_j(\sqrt{\mathcal{H}_V}) f\big) \leq C 2^{2j} q_{V,M+2}(f),$$

which imply that

$$\sum_{j=-\infty}^{0} q_{V,M}\big(\phi_j(\sqrt{\mathcal{H}_V}) f\big) \leq C q_{V,M+2}(f) \sum_{j=-\infty}^{0} 2^{2j} < \infty$$

for all $M \in \mathbb{N}$. Therefore, (4.35) is verified for $f \in \mathcal{Z}_V(\Omega)$.

Finally, as to the identity (4.35) for $f \in \mathcal{Z}'_V(\Omega)$, we proceed the analogous argument to that with replacing the assertion (i) for $p_{V,M}$ and Lemma 4.3 (i) by $q_{V,M}$ and Lemma 4.3 (ii), respectively. The proof of $\phi_j(\sqrt{\mathcal{H}_V}) f \in L^\infty(\Omega)$ also follows from the analogous argument to that of the assertion (i) as above. So we may omit the details. The proof of Lemma 4.5 is complete. □

The spaces $\mathcal{X}_V(\Omega)$ and $\mathcal{Z}_V(\Omega)$ are subspaces of $L^p(\Omega)$. More precisely, we have:

Proposition 4.6. *Suppose that the potential V satisfies assumption* A. *Then*

$$\mathcal{X}_V(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{X}'_V(\Omega) \tag{4.42}$$

for any $1 \le p \le \infty$. Furthermore, if V further satisfies assumption B, *we have*

$$\mathcal{Z}_V(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{Z}'_V(\Omega) \tag{4.43}$$

for any $1 \le p \le \infty$.

Proof. Let $f \in \mathcal{X}_V(\Omega)$. Then it follows from the definition of semi-norms $p_{V,M}(\cdot)$ that

$$\|f\|_{L^1(\Omega)} \le p_{V,0}(f).$$

Put

$$\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \quad \text{for } j \in \mathbb{Z}.$$

As to the L^∞-norm, we deduce from the identities (4.34), $\phi_j = \Phi_j \phi_j$ and the estimate (4.22) for $\alpha = 0$ that

$$\begin{aligned}
\|f\|_{L^\infty(\Omega)} &\le \|\psi(\mathcal{H}_V) f\|_{L^\infty(\Omega)} + \sum_{j=1}^{\infty} \|\Phi_j(\sqrt{\mathcal{H}_V})\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^\infty(\Omega)} \\
&\le C\|f\|_{L^1(\Omega)} + C\sum_{j=1}^{\infty} 2^{-j} \cdot 2^j 2^{dj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} \\
&\le C p_{V,0}(f) + C\sum_{j=1}^{\infty} 2^{-j} \sup_{k\in\mathbb{N}} 2^{(d+1)k} \|\phi_k(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} \\
&\le C p_{V,d+1}(f).
\end{aligned}$$

Summarizing L^1 and L^∞-estimates for $f \in \mathcal{X}_V(\Omega)$, we conclude that

$$\mathcal{X}_V(\Omega) \hookrightarrow L^1(\Omega) \cap L^\infty(\Omega) \hookrightarrow L^p(\Omega) \quad \text{for any } 1 \le p \le \infty,$$

and hence, we have

$$L^p(\Omega) \hookrightarrow \mathcal{X}'_V(\Omega) \quad \text{for any } 1 \le p \le \infty$$

by duality argument. Thus we get the embedding (4.42).

The embedding (4.43) follows from (4.42) and embeddings

$$\mathcal{Z}_V(\Omega) \hookrightarrow \mathcal{X}_V(\Omega) \quad \text{and} \quad \mathcal{X}'_V(\Omega) \hookrightarrow \mathcal{Z}'_V(\Omega).$$

The proof of Proposition 4.6 is complete. □

Next we shall characterize the space $\mathcal{Z}'_V(\Omega)$ by the quotient space of $\mathcal{X}'_V(\Omega)$. Let us define a space $\mathcal{P}_V(\Omega)$ by

$$\mathcal{P}_V(\Omega) := \big\{ f \in \mathcal{X}'_V(\Omega) : {}_{\mathcal{Z}'_V(\Omega)}\langle J(f), g\rangle_{\mathcal{Z}_V(\Omega)} = 0 \text{ for any } g \in \mathcal{Z}_V(\Omega) \big\}, \tag{4.44}$$

where $J(f)$ is the restriction of f on the subspace $\mathcal{Z}_V(\Omega)$ of $\mathcal{X}_V(\Omega)$. It is readily checked that $\mathcal{P}_V(\Omega)$ is a closed subspace of $\mathcal{X}'_V(\Omega)$, and hence, the quotient space $\mathcal{X}'_V(\Omega)/\mathcal{P}_V(\Omega)$ is a linear topological space endowed with the quotient topology.

Then we have the following:

Proposition 4.7. *Suppose that the potential V satisfies assumptions* A *and* B. *Then $\mathcal{Z}'_V(\Omega)$ is isomorphic to $\mathcal{X}'_V(\Omega)/\mathcal{P}_V(\Omega)$:*

$$\mathcal{Z}'_V(\Omega) \cong \mathcal{X}'_V(\Omega)/\mathcal{P}_V(\Omega).$$

Proposition 4.7 corresponds to the isomorphism (4.6). The proof is done by using Theorem in p.126 from Schaefer [72] and Propositions 35.5 and 35.6 from Tréves [79] (see also Theorem 1.1 in Sawano [70]).

The space $\mathcal{P}_V(\Omega)$ enjoys the following:

Proposition 4.8. *Suppose that the potential V satisfies assumptions* A *and* B.

(i) *Let $f \in \mathcal{X}'_V(\Omega)$. Then the following assertions are equivalent*:

  (a) $f \in \mathcal{P}_V(\Omega)$;
  (b) $\phi_j(\sqrt{\mathcal{H}_V})f = 0$ *in* $\mathcal{X}'_V(\Omega)$ *for any* $j \in \mathbb{Z}$;
  (c) $\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} = 0$ *for any* $s \in \mathbb{R}$ *and* $1 \le p, q \le \infty$.

(ii) $\mathcal{P}_V(\Omega)$ *is a subspace of* $L^\infty(\Omega)$.

Proof. We prove the assertion (i). It is readily seen from the definition of $\dot{B}^s_{p,q}(\mathcal{H}_V)$ that (c) implies (b), since

$$\phi_j(\sqrt{\mathcal{H}_V})f = 0 \quad \text{in } L^p(\Omega)$$

for any $j \in \mathbb{Z}$, and since $L^p(\Omega) \hookrightarrow \mathcal{X}'_V(\Omega)$. Conversely, we suppose that (b) holds. Since $f \in \mathcal{X}'_V(\Omega)$, it follows from part (i) of Proposition 4.5 that

$$\phi_j(\sqrt{\mathcal{H}_V})f \in L^\infty(\Omega)$$

for any $j \in \mathbb{Z}$. Hence, thanks to fundamental lemma of the calculus of variations, we deduce that

$$\phi_j(\sqrt{\mathcal{H}_V})f(x) = 0 \quad \text{a.e. } x \in \Omega$$

for any $j \in \mathbb{Z}$, which implies that (c) holds true.

We have to prove that (a) and (b) are equivalent. Suppose that (a) holds, i.e., $f \in \mathcal{P}_V(\Omega)$. We note that if $g \in \mathcal{X}_V(\Omega)$, then

$$\phi_j(\sqrt{\mathcal{H}_V})g \in \mathcal{Z}_V(\Omega) \quad \text{for any } j \in \mathbb{Z}. \tag{4.45}$$

In fact, fixing $j \in \mathbb{Z}$, we have

$$\phi_k(\sqrt{\mathcal{H}})\phi_j(\sqrt{\mathcal{H}})g \neq 0$$

only if $k = j-1, j, j+1$. Then, we deduce from Lemma 4.1 that

$$\begin{aligned}\sup_{k\leq 0} 2^{-Mk}\|\phi_k(\sqrt{\mathcal{H}_V})\phi_j(\sqrt{\mathcal{H}_V})g\|_{L^1(\Omega)} &\leq C \max_{k=j-1,j,j+1} 2^{-Mk}\|\phi_j(\sqrt{\mathcal{H}_V})g\|_{L^1(\Omega)} \\ &\leq C2^{-Mj}\|\phi_j(\sqrt{\mathcal{H}_V})g\|_{L^1(\Omega)} \\ &\leq C2^{-Mj}\|g\|_{L^1(\Omega)} \\ &< \infty\end{aligned}$$

for any $M \in \mathbb{N}$, which implies (4.45). Since $f \in \mathcal{P}_V(\Omega)$, thanks to (4.45), it follows that

$${}_{\mathcal{X}'_V(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}_V})f, g\rangle_{\mathcal{X}_V(\Omega)} = {}_{\mathcal{Z}'_V(\Omega)}\langle f, \phi_j(\sqrt{\mathcal{H}_V})g\rangle_{\mathcal{Z}_V(\Omega)} = 0$$

for any $j \in \mathbb{Z}$ and $g \in \mathcal{X}_V(\Omega)$, which implies (b). Conversely, let us suppose that (b) holds. Since $\mathcal{Z}_V(\Omega) \subset \mathcal{X}_V(\Omega)$, it follows that

$${}_{\mathcal{Z}'_V(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}_V})f, g\rangle_{\mathcal{Z}_V(\Omega)} = {}_{\mathcal{X}'_V(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}_V})f, g\rangle_{\mathcal{X}_V(\Omega)} = 0 \tag{4.46}$$

for any $j \in \mathbb{Z}$ and $g \in \mathcal{Z}_V(\Omega)$. Here, we recall part (ii) of Proposition 4.5 that

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } \mathcal{Z}'_V(\Omega).$$

Then, by using this identity and (4.46), we have

$${}_{\mathcal{Z}'_V(\Omega)}\langle f, g\rangle_{\mathcal{Z}_V(\Omega)} = \sum_{j=-\infty}^{\infty} {}_{\mathcal{Z}'_V(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}_V})f, g\rangle_{\mathcal{Z}_V(\Omega)} = 0$$

for any $g \in \mathcal{Z}_V(\Omega)$, which implies that $f \in \mathcal{P}_V(\Omega)$. Hence (a) holds true. Thus we conclude the assertion (i).

Next we prove the assertion (ii). Let $f \in \mathcal{P}_V(\Omega)$. It follows from (4.34) in Proposition 4.5 that

$$f = \psi(\mathcal{H}_V)f + \sum_{j=1}^{\infty} \phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } \mathcal{X}'_V(\Omega).$$

Applying (b) in the assertion (i) to the second term in the right member, we get

$$f = \psi(\mathcal{H}_V)f \quad \text{in } \mathcal{X}'_V(\Omega).$$

Since $\psi(\mathcal{H}_V)f \in L^\infty(\Omega)$ by the assertion (i) in Proposition 4.5, we conclude that $f \in L^\infty(\Omega)$. Therefore the assertion (ii) is proved. The proof of Proposition 4.8 is now finished. □

When $\Omega = \mathbb{R}^d$ and $V = 0$, we observe that $\mathcal{Z}_0(\mathbb{R}^d)$ corresponds to $\mathscr{S}_0(\mathbb{R}^d)$. In fact, the following proposition assures this fact.

Proposition 4.9. *Let $f \in \mathscr{S}(\mathbb{R}^d)$. Then the following assertions are equivalent*:

(i) $\sup_{j\le 0} 2^{M|j|}\|\phi_j(\sqrt{-\Delta})f\|_{L^1(\mathbb{R}^d)} < \infty$ *for any* $M \in \mathbb{N}$;

(ii) $\displaystyle\int_{\mathbb{R}^d} x^\alpha f(x)\,dx = 0$ *for any* $\alpha \in \mathbb{N}_0^d$.

Now, let $f \in \mathscr{S}(\mathbb{R}^d)(\subset \mathcal{X}_0(\mathbb{R}^d))$. Then we conclude from Proposition 4.9 that

$$f \in \mathcal{Z}_0(\mathbb{R}^d) \text{ if and only if } f \in \mathscr{S}_0(\mathbb{R}^d).$$

We have to prove the proposition.

Proof of Proposition 4.9. Let $f \in \mathscr{S}(\mathbb{R}^d)$. We divide the proof into two steps.

First step. We prove that the assertion (ii) is equivalent to the following:

$$\sup_{j\le 0} 2^{M|j|}\|\phi_j(|\cdot|)\mathscr{F}f\|_{L^\infty(\mathbb{R}^d)} < \infty \quad \text{for any } M \in \mathbb{N}. \tag{4.47}$$

Indeed, the assertion (ii) implies that

$$\partial_\xi^\alpha(\mathscr{F}f)(0) = \int_{\mathbb{R}^d} x^\alpha f(x)\,dx = 0 \quad \text{for any } \alpha \in \mathbb{N}_0^d. \tag{4.48}$$

Hence, it follows that

$$|(\mathscr{F}f)(\xi)| \le C|\xi|^M, \quad |\xi| \le 2 \tag{4.49}$$

for any $M \in \mathbb{N}$. Here, since

$$\operatorname{supp}\phi_j \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\},$$

it follows that

$$\phi_j(|\xi|)|\xi|^M \le C2^{Mj} \quad \text{on } \operatorname{supp}\phi_j$$

for any $j \le 0$ and $M \in \mathbb{N}$. Therefore, we deduce from (4.49) that

$$|\phi_j(|\xi|)(\mathscr{F}f)(\xi)| \le C2^{Mj}, \quad \xi \in \mathbb{R}^d$$

for any $j \le 0$ and $M \in \mathbb{N}$, which implies (4.47). Conversely, we suppose (4.47). Then

$$|\phi_j(|\xi|)(\mathscr{F}f)(\xi)| \le C2^{Mj} \le C|\xi|^M \quad \text{on } \operatorname{supp}\phi_j$$

for any $j \le 0$ and $M \in \mathbb{N}$, which implies (4.49) for any $M \in \mathbb{N}$. Since $\mathscr{F}f$ is $C^\infty(\mathbb{R}^d)$, we conclude from (4.49) that (4.48) holds. Hence, the assertion (ii) is

true. Thus the equivalence between (ii) and (4.47) is proved.

Second step. Thanks to the result in the first step, it is sufficient to show that the assertion (i) is equivalent to (4.47). Suppose that (i) holds. Then, by using L^1-L^∞-boundedness of the Fourier transform $\mathscr{F}$, we find that

$$\|\phi_j(|\cdot|)\mathscr{F}f\|_{L^\infty(\mathbb{R}^d)} = \big\|\mathscr{F}[\phi_j(\sqrt{-\Delta})f]\big\|_{L^\infty(\mathbb{R}^d)} \le (2\pi)^{-\frac{d}{2}}\|\phi_j(\sqrt{-\Delta})f\|_{L^1(\mathbb{R}^d)}$$

for any $j \le 0$. Hence, multiplying the both sides by $2^{M|j|}$ and taking the supremum with respect to $j \le 0$, we get (4.47).

Conversely, we suppose that (4.47) holds. We estimate

$$\begin{aligned}\|\phi_j(\sqrt{-\Delta})f\|_{L^1(\mathbb{R}^d)} &= \big\|\mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}f]\big\|_{L^1(\mathbb{R}^d)}\\ &\le \big\|\mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}f]\big\|^{\frac12}_{L^\infty(\mathbb{R}^d)}\big\|\mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}f]\big\|^{\frac12}_{L^{\frac12}(\mathbb{R}^d)}\end{aligned}$$

Since $\operatorname{supp}\phi_j \subset \{|\xi| \le 2\}$ for $j \le 0$, by using L^1-L^∞-boundedness of $\mathscr{F}^{-1}$, the first factor in the right member is estimated as

$$\begin{aligned}\big\|\mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}f]\big\|_{L^\infty(\mathbb{R}^d)} &\le (2\pi)^{-\frac{d}{2}}\|\phi_j(|\cdot|)\mathscr{F}f\|_{L^1(\mathbb{R}^d)}\\ &\le (2\pi)^{-\frac{d}{2}}\cdot 2^d v_d\|\phi_j(|\cdot|)\mathscr{F}f\|_{L^\infty(\mathbb{R}^d)}\end{aligned}$$

for any $j \le 0$, where v_d is the volume of the unit ball in $\mathbb{R}^d$. Therefore, by using (4.47), we deduce that

$$\sup_{j\le 0} 2^{M|j|}\big\|\mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}f]\big\|^{\frac12}_{L^\infty(\mathbb{R}^d)} \le (2\pi)^{-\frac{d}{4}}\cdot 2^{\frac{d}{2}}v_d^{\frac12}\sup_{j\le 0} 2^{M|j|}\|\phi_j(|\cdot|)\mathscr{F}f\|^{\frac12}_{L^\infty(\mathbb{R}^d)} < \infty.$$

As to the second factor, applying Theorem 1.5.2 from Triebel [81], and noting that

$$\phi_j = \phi_j(\psi + \phi_1) \quad \text{for } j \le 0,$$

we find that there exists a constant $C > 0$, independent of j, such that

$$\begin{aligned}\big\|\mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}f]\big\|_{L^{\frac12}(\mathbb{R}^d)} &= \big\|\mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}\mathscr{F}^{-1}(\psi+\phi_1)(|\cdot|)\mathscr{F}f]\big\|_{L^{\frac12}(\mathbb{R}^d)}\\ &\le C\big\|\mathscr{F}^{-1}(\psi+\phi_1)(|\cdot|)\mathscr{F}f]\big\|_{L^{\frac12}(\mathbb{R}^d)}\end{aligned}$$

for any $j \le 0$. Since $f \in \mathscr{S}(\mathbb{R}^d)$, it follows that

$$\sup_{j\le 0}\big\|\mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}f]\big\|_{L^{\frac12}(\mathbb{R}^d)} \le C\big\|\mathscr{F}^{-1}(\psi+\phi_1)(|\cdot|)\mathscr{F}f]\big\|_{L^{\frac12}(\mathbb{R}^d)} < \infty.$$

Thus, combining the estimates obtained now, we conclude that

$$\begin{aligned}&\sup_{j\le 0} 2^{M|j|}\|\phi_j(\sqrt{-\Delta})f\|_{L^1(\mathbb{R}^d)}\\ &\le C\sup_{j\le 0} 2^{M|j|}\|\phi_j(|\cdot|)\mathscr{F}f\|^{\frac12}_{L^\infty(\mathbb{R}^d)}\big\|\mathscr{F}^{-1}(\psi+\phi_1)(|\cdot|)\mathscr{F}f]\big\|^{\frac12}_{L^{\frac12}(\mathbb{R}^d)} < \infty,\end{aligned}$$

which implies the assertion (i). The proof of Proposition 4.9 is finished. □

4.2 Besov spaces generated by Schrödinger operators

4.2.1 Definition of Besov spaces

In this section we give definition of Besov spaces generated by $\mathcal{H}_V$. Throughout this section we always impose assumption A on the potential V. We recall the Littilewood-Paley dyadic decomposition $\{\psi\} \cup \{\phi_j\}_j$ defined by (4.12), (4.13) and (4.14).

Let us define Besov spaces generated by $\mathcal{H}_V$.

Definition (Besov spaces). Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$.

(i) The inhomogeneous Besov spaces $B^s_{p,q}(\mathcal{H}_V)$ are defined by letting

$$B^s_{p,q}(\mathcal{H}_V) := \big\{ f \in \mathcal{X}'_V(\Omega) : \|f\|_{B^s_{p,q}(\mathcal{H}_V)} < \infty \big\},$$

where

$$\|f\|_{B^s_{p,q}(\mathcal{H}_V)} := \|\psi(\mathcal{H}_V) f\|_{L^p(\Omega)} + \Big\| \big\{ 2^{sj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \big\}_{j \in \mathbb{N}} \Big\|_{l^q(\mathbb{N})}. \quad (4.50)$$

(ii) Suppose further that V satisfies assumption B. Then the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_V)$ are defined by letting

$$\dot{B}^s_{p,q}(\mathcal{H}_V) := \big\{ f \in \mathcal{Z}'_V(\Omega) : \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} < \infty \big\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} := \Big\| \big\{ 2^{sj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \big\}_{j \in \mathbb{Z}} \Big\|_{l^q(\mathbb{Z})}. \quad (4.51)$$

When $\Omega = \mathbb{R}^d$ and $V = 0$, i.e., $\mathcal{H}_V = -\Delta$ on $L^2(\mathbb{R}^d)$, the norms (4.50) and (4.51) are equivalent to the classical ones (4.2) and (4.3), respectively, since spectral multiplies $\psi(-\Delta)$ and $\phi_j(\sqrt{-\Delta})$ coincide with the Fourier multipliers:

$$\psi(-\Delta) = \mathscr{F}^{-1}\big[\psi(|\cdot|^2)\mathscr{F}\big] \quad \text{and} \quad \phi_j(\sqrt{-\Delta}) = \mathscr{F}^{-1}[\phi_j(|\cdot|)\mathscr{F}].$$

Furthermore, $B^s_{p,q}(-\Delta)$ are isomorphic to the classical Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ defined as subspaces of $\mathscr{S}'(\mathbb{R}^d)$, since $\mathscr{S}(\mathbb{R}^d)$ is a subspace of $\mathcal{X}_0(\mathbb{R}^d)$ and dense in $B^s_{p,q}(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $1 \le p, q < \infty$. Similarly, $\dot{B}^s_{p,q}(-\Delta)$ are isomorphic to $\dot{B}^s_{p,q}(\mathbb{R}^d)$ defined as subspaces of $\mathscr{S}'_0(\mathbb{R}^d)$. We also mention an abstract theory to characterize the Besov spaces by means of the real interpolation between two spaces, which are $L^2(\Omega)$ and the domain of the operator $\mathcal{D}(A_V)$ for instance (see papers by Lions [54, 55] and also Mayeli [58]). The real interpolation also works for our Besov spaces (see [40]).

4.2.2 Completeness, duality, lifting properties and embedding relations

In this subsection we prove the fundamental properties such as completeness, duality, lifting properties and embedding relations.

Theorem 4.10. *Assume that the potential V satisfies assumption* A. *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Then the following assertions hold*:

(i) (*Inhomogeneous Besov spaces*)

(a) $B^s_{p,q}(\mathcal{H}_V)$ *is independent of the choice of* $\{\psi\} \cup \{\phi_j\}_{j\in\mathbb{N}}$ *satisfying* (4.12), (4.13) *and* (4.14), *and enjoys the following*:

$$\mathcal{X}_V(\Omega) \hookrightarrow B^s_{p,q}(\mathcal{H}_V) \hookrightarrow \mathcal{X}'_V(\Omega). \tag{4.52}$$

(b) $B^s_{p,q}(\mathcal{H}_V)$ *is a Banach space.*

(ii) (*Homogeneous Besov spaces*) *Suppose further that V satisfies assumption* B.

(a) $\dot{B}^s_{p,q}(\mathcal{H}_V)$ *is independent of the choice of* $\{\phi_j\}_{j\in\mathbb{Z}}$ *satisfying* (4.12) *and* (4.13), *and enjoys the following*:

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}_V) \hookrightarrow \mathcal{Z}'_V(\Omega). \tag{4.53}$$

(b) $\dot{B}^s_{p,q}(\mathcal{H}_V)$ *is a Banach space.*

The following result states the fundamental properties of the Besov spaces such as duality, lifting properties, and embedding relations.

Theorem 4.11. *Suppose that V satisfies the same assumptions as in Theorem 4.10. Let $s, s_0 \in \mathbb{R}$ and $1 \le p, q, q_0, r \le \infty$. Then the following assertions hold*:

(i) *If $1 \le p, q < \infty$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, then the dual spaces of $B^s_{p,q}(\mathcal{H}_V)$ and $\dot{B}^s_{p,q}(\mathcal{H}_V)$ are $B^{-s}_{p',q'}(\mathcal{H}_V)$ and $\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)$, respectively.*

(ii) (a) *The inhomogeneous Besov spaces enjoy the following properties*:

$$(I + \mathcal{H}_V)^{\frac{s_0}{2}} f \in B^{s-s_0}_{p,q}(\mathcal{H}_V) \quad \text{for any } f \in B^s_{p,q}(\mathcal{H}_V);$$

$$B^{s+\varepsilon}_{p,q}(\mathcal{H}_V) \hookrightarrow B^s_{p,q_0}(\mathcal{H}_V) \quad \text{for any } \varepsilon > 0;$$

$$B^{s+d(\frac{1}{r}-\frac{1}{p})}_{r,q}(\mathcal{H}_V) \hookrightarrow B^s_{p,q_0}(\mathcal{H}_V) \quad \text{if } 1 \le r \le p \le \infty \text{ and } q \le q_0.$$

(b) *The homogeneous Besov spaces enjoy the following properties*:

$$\mathcal{H}_V^{\frac{s_0}{2}} f \in \dot{B}^{s-s_0}_{p,q}(\mathcal{H}_V) \quad \text{for any } f \in \dot{B}^s_{p,q}(\mathcal{H}_V);$$

$$\dot{B}^{s+d(\frac{1}{r}-\frac{1}{p})}_{r,q}(\mathcal{H}_V) \hookrightarrow \dot{B}^s_{p,q_0}(\mathcal{H}_V) \quad \text{if } 1 \le r \le p \le \infty \text{ and } q \le q_0.$$

(iii) *We have*

$$L^p(\Omega) \hookrightarrow B^0_{p,2}(\mathcal{H}_V), \dot{B}^0_{p,2}(\mathcal{H}_V) \quad \text{if } 1 < p \le 2;$$
$$B^0_{p,2}(\mathcal{H}_V), \dot{B}^0_{p,2}(\mathcal{H}_V) \hookrightarrow L^p(\Omega) \quad \text{if } 2 \le p < \infty.$$

Proof of Theorem 4.10. We divide the proof into three parts: Independence of choice of ψ and $\{\phi_j\}$; embedding relations (4.52) and (4.53); completeness of $B^s_{p,q}(\mathcal{H}_V)$ and $\dot{B}^s_{p,q}(\mathcal{H}_V)$.

Proof of independence of the choice of ψ and $\{\phi_j\}$. The proof of the independence in (i-a) and (ii-a) is similar to that of Triebel [81]. As to (i-a), let us take $\psi = \psi^{(k)}$, $\phi_j = \phi_j^{(k)}$ $(k = 1, 2)$ satisfying (4.12), (4.13) and (4.14). Since $\psi^{(1)}$ and $\phi_j^{(1)}$ satisfy

$$\begin{cases} \psi^{(1)} = \psi^{(1)}\big(\psi^{(2)} + \phi_1^{(2)}\big), \quad \phi_1^{(1)} = \phi_1^{(1)}\big(\psi^{(2)} + \phi_1^{(2)} + \phi_2^{(2)}\big), \\ \phi_j^{(1)} = \phi_j^{(1)}\big(\phi_{j-1}^{(2)} + \phi_j^{(2)} + \phi_{j+1}^{(2)}\big) \quad \text{for } j = 2, 3, \cdots, \end{cases} \tag{4.54}$$

it follows from (4.21) and (4.22) in Lemma 4.1 that

$$\begin{aligned} &\|\psi^{(1)}(\mathcal{H}_V)f\|_{L^p(\Omega)} + \|\phi_1^{(1)}(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \\ &\le C\Big\{\|\psi^{(2)}(\mathcal{H}_V)f\|_{L^p(\Omega)} + \sum_{k=1}^{2} \|\phi_k^{(2)}(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}\Big\}, \end{aligned}$$

$$\|\phi_j^{(1)}(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \le C \sum_{k=-1}^{1} \|\phi_{j+k}^{(2)}(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \quad \text{for } j = 2, 3, \cdots,$$

which imply that

$$\begin{aligned} &\|\psi^{(1)}(\mathcal{H}_V)f\|_{L^p(\Omega)} + \big\|\{2^{sj}\|\phi_j^{(1)}(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}\}_{j\in\mathbb{N}}\big\|_{l^q(\mathbb{N})} \\ &\le C\Big\{\|\psi^{(2)}(\mathcal{H}_V)f\|_{L^p(\Omega)} + \big\|\{2^{sj}\|\phi_j^{(2)}(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}\}_{j\in\mathbb{N}}\big\|_{l^q(\mathbb{N})}\Big\}. \end{aligned}$$

This proves the independence in (i-1) for the inhomogeneous Besov spaces.

As to (ii-a), we use the identity (4.54) for all $j \in \mathbb{Z}$ and apply (4.23) for $\alpha = 0$ in Lemma 4.1 to get

$$\big\|\{2^{sj}\|\phi_j^{(1)}(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}\}_{j\in\mathbb{Z}}\big\|_{l^q(\mathbb{Z})} \le C\Big\{\big\|\{2^{sj}\|\phi_j^{(2)}(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}\}_{j\in\mathbb{Z}}\big\|_{l^q(\mathbb{Z})}\Big\}.$$

This ends the proof of the required independence of the choice of ψ and $\{\phi_j\}$.

Proof of embedding relations (4.52) and (4.53). Let p' and q' be such that $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. First we prove the embedding (4.52), namely,

$$\mathcal{X}_V(\Omega) \hookrightarrow B^s_{p,q}(\mathcal{H}_V) \hookrightarrow \mathcal{X}'_V(\Omega).$$

Take Ψ and Φ_j such that

$$\Psi := \psi + \phi_1, \quad \Phi_1 := \psi + \phi_1 + \phi_2, \quad \Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \text{ for } j = 2, 3, \cdots .$$

Let $M \in \mathbb{N}$ be such that $M > s + d(1 - 1/p)$. Then, for any $f \in \mathcal{X}_V(\Omega)$, we deduce from the identities $\phi_j = \Phi_j \phi_j$ and the estimate (4.22) for $\alpha = 0$ in Lemma 4.1 that

$$\begin{aligned}
\|f\|_{B^s_{p,q}(\mathcal{H}_V)} &= \|\psi(\mathcal{H}_V) f\|_{L^p(\Omega)} + \Big\{ \sum_{j=1}^{\infty} \Big(2^{sj} \|\Phi_j(\sqrt{\mathcal{H}_V}) \phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \Big)^q \Big\}^{\frac{1}{q}} \\
&\le C \|f\|_{L^1(\Omega)} + C \Big\{ \sum_{j=1}^{\infty} \Big(2^{sj} 2^{d(1-\frac{1}{p})j} 2^{-Mj} \cdot 2^{Mj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} \Big)^q \Big\}^{\frac{1}{q}} \\
&\le C p_{V,M}(f) + C \Big\{ \sum_{j=1}^{\infty} \Big(2^{sj} 2^{d(1-\frac{1}{p})j} 2^{-Mj} \Big)^q \Big\}^{\frac{1}{q}} p_{V,M}(f) \\
&\le C p_{V,M}(f)
\end{aligned}$$

for any $f \in \mathcal{X}_V(\Omega)$. Thus we obtain the first embedding

$$\mathcal{X}_V(\Omega) \hookrightarrow B^s_{p,q}(\mathcal{H}_V). \tag{4.55}$$

To prove the second embedding

$$B^s_{p,q}(\mathcal{H}_V) \hookrightarrow \mathcal{X}'_V(\Omega), \tag{4.56}$$

we take $M' \in \mathbb{N}$ such that $M' > -s + d(1 - 1/p')$. Applying (i) in Proposition 4.5, the identities $\psi = \Psi\psi$, $\phi_j = \Phi_j \phi_j$, Hölder's inequality and the embedding (4.55) for s, p, q replaced by $-s, p', q'$, i.e.,

$$\mathcal{X}_V(\Omega) \hookrightarrow B^{-s}_{p',q'}(\mathcal{H}_V),$$

we have

$$\begin{aligned}
&|{}_{\mathcal{X}'_V}\langle f, g\rangle_{\mathcal{X}_V}| \\
&= \Big| {}_{\mathcal{X}'_V}\big\langle \psi(\mathcal{H}_V) f, \Psi(\mathcal{H}_V) g \big\rangle_{\mathcal{X}_V} + \sum_{j=1}^{\infty} {}_{\mathcal{X}'_V}\big\langle \phi_j(\sqrt{\mathcal{H}_V}) f, \Phi_j(\sqrt{\mathcal{H}_V}) g \big\rangle_{\mathcal{X}_V} \Big| \\
&\le \|\psi(\mathcal{H}_V) f\|_{L^p(\Omega)} \|\Psi(\mathcal{H}_V) g\|_{L^{p'}(\Omega)} \\
&\qquad + \big\| \big\{ 2^{sj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \big\}_{j \in \mathbb{N}} \big\|_{l^q(\mathbb{N})} \big\| \big\{ 2^{-sj} \|\Phi_j(\sqrt{\mathcal{H}_V}) g\|_{L^{p'}(\Omega)} \big\}_{j \in \mathbb{N}} \big\|_{l^{q'}(\mathbb{N})} \\
&\le C \|f\|_{B^s_{p,q}(\mathcal{H}_V)} \|g\|_{B^{-s}_{p',q'}(\mathcal{H}_V)} \\
&\le C \|f\|_{B^s_{p,q}(\mathcal{H}_V)} p_{M'}(g)
\end{aligned}$$

for any $f \in B^s_{p,q}(\mathcal{H}_V)$ and $g \in \mathcal{X}_V(\Omega)$. Therefore, (4.56) is proved, and as a result, we get the embedding (4.52).

Next we show the embedding (4.53), namely,

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}_V) \hookrightarrow \mathcal{Z}'_V(\Omega).$$

Put

$$\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \quad \text{for all } j \in \mathbb{Z}.$$

Let $L \in \mathbb{N}$ be such that $L > |s| + d(1 - 1/p)$. For any $f \in \mathcal{Z}(\Omega)$, we deduce from the identity $\phi_j = \Phi_j \phi_j$ and the estimate (4.23) for $\alpha = 0$ that

$$\begin{aligned}
\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} &= \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \|\Phi_j(\sqrt{\mathcal{H}_V}) \phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \Big)^q \Big\}^{\frac{1}{q}} \\
&\le C \Big\{ \Big(\sum_{j=-\infty}^{0} + \sum_{j=1}^{\infty} \Big) \Big(2^{sj} 2^{d(1-\frac{1}{p})j} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} \Big)^q \Big\}^{\frac{1}{q}} \\
&\le C \Big(\sup_{j \le 0} 2^{-Lj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} \Big) \Big\{ \sum_{j=-\infty}^{0} \Big(2^{sj} 2^{d(1-\frac{1}{p})j} 2^{Lj} \Big)^q \Big\}^{\frac{1}{q}} \\
&\quad + C \Big(\sup_{j \ge 1} 2^{Lj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^1(\Omega)} \Big) \Big\{ \sum_{j=1}^{\infty} \Big(2^{sj} 2^{d(1-\frac{1}{p})j} 2^{-Lj} \Big)^q \Big\}^{\frac{1}{q}} \\
&\le C q_{V,L}(f),
\end{aligned}$$

which implies that

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}_V). \tag{4.57}$$

To prove the second embedding

$$\dot{B}^s_{p,q}(\mathcal{H}_V) \hookrightarrow \mathcal{Z}'_V(\Omega),$$

we take $L' \in \mathbb{N}$ such that $L' > |s| + d(1 - 1/p')$. For any $f \in \dot{B}^s_{p,q}(\mathcal{H}_V)$ and $g \in \mathcal{Z}_V(\Omega)$, using the identities $\phi_j = \Phi_j \phi_j$, Hölder's inequality and the embedding (4.57) for s, p, q replaced by $-s, p', q'$, i.e.,

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^{-s}_{p',q'}(\mathcal{H}_V),$$

we estimate

$$\begin{aligned}
&|{}_{\mathcal{Z}'_V}\langle f, g\rangle_{\mathcal{Z}_V}| \\
&= \Big| \sum_{j=-\infty}^{\infty} {}_{\mathcal{Z}'_V}\big\langle \phi_j(\sqrt{\mathcal{H}_V}) f, \Phi_j(\sqrt{\mathcal{H}_V}) g \big\rangle_{\mathcal{Z}_V} \Big| \\
&\le \Big\| \big\{ 2^{sj} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \big\}_{j \in \mathbb{Z}} \Big\|_{l^q(\mathbb{Z})} \Big\| \big\{ 2^{-sj} \|\Phi_j(\sqrt{\mathcal{H}_V}) g\|_{L^{p'}(\Omega)} \big\}_{j \in \mathbb{Z}} \Big\|_{l^{q'}(\mathbb{Z})} \\
&\le C \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} \|g\|_{\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)} \\
&\le C \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} q_{L'}(g).
\end{aligned}$$

Thus we conclude (4.52).

Proof of the completeness of $B^s_{p,q}(\mathcal{H}_V)$ and $\dot{B}^s_{p,q}(\mathcal{H}_V)$. We have only to prove the completeness of the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_V)$, since the inhomogeneous case is similar. The proof is done by the analogous argument to that by Triebel [81]. Indeed, let $\{f_N\}_{N=1}^{\infty}$ be a Cauchy sequence in $\dot{B}^s_{p,q}(\mathcal{H}_V)$. We may assume that

$$\|f_{N+1} - f_N\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} \le 2^{-N} \tag{4.58}$$

without loss of generality. Then $\{f_N\}_{N=1}^{\infty}$ is also a Cauchy sequence in $\mathcal{Z}'_V(\Omega)$ by the embedding relation (4.52), and hence, there exists an element $f \in \mathcal{Z}'_V(\Omega)$ with the property that

$$f_N \to f \quad \text{in } \mathcal{Z}'_V(\Omega) \text{ as } N \to \infty,$$

since $\mathcal{Z}'_V(\Omega)$ is complete. This together with the boundedness of $\phi_j(\sqrt{\mathcal{H}_V})$ on $\mathcal{Z}'_V(\Omega)$ imply that

$$\phi_j(\sqrt{\mathcal{H}_V})f_N \to \phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } \mathcal{Z}'_V(\Omega) \text{ as } N \to \infty, \tag{4.59}$$

and we have $\phi_j(\sqrt{\mathcal{H}_V})f \in L^\infty(\Omega)$ by Lemma 4.5 (ii). Furthermore, fixing $j \in \mathbb{Z}$, we see that $\{\phi_j(\sqrt{\mathcal{H}_V})f_N\}_{N=1}^{\infty}$ is also a Cauchy sequence in $L^p(\Omega)$, and there exists $F_j \in L^p(\Omega)$ such that

$$\phi_j(\sqrt{\mathcal{H}_V})f_N \to F_j \quad \text{in } L^p(\Omega) \text{ as } N \to \infty,$$

which implies that

$$F_j(x) = \phi_j(\sqrt{\mathcal{H}_V})f(x) \quad \text{almost every } x \in \Omega,$$

and the convergence (4.59) also holds in the topology of $L^p(\Omega)$.

It remains to show that $f \in \dot{B}^s_{p,q}(\mathcal{H}_V)$ and f_N tends to f in $\dot{B}^s_{p,q}(\mathcal{H}_V)$ for the above $f \in \mathcal{Z}'_V(\Omega)$. Since $\left\{\{2^{sj}\|\phi_j(\sqrt{\mathcal{H}_V})f_N\|_{L^p(\Omega)}\}_{j\in\mathbb{Z}}\right\}_{N=1}^{\infty}$ is a Cauchy sequence in $l^q(\mathbb{Z})$ and

$$2^{sj}\|\phi_j(\sqrt{\mathcal{H}_V})f_N\|_{L^p(\Omega)} \to 2^{sj}\|\phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \quad \text{as } N \to \infty,$$

we get

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} < \infty,$$

and hence,

$$f \in \dot{B}^s_{p,q}(\mathcal{H}_V).$$

For the convergence of f_N to f, writing

$$f = \sum_{k=1}^{\infty}(f_k - f_{k-1}) = \lim_{N\to\infty} f_N \quad \text{in } \mathcal{Z}'_V(\Omega),$$

where $f_0 = 0$, we conclude from (4.58) that the above series converges absolutely in the topology of $\dot{B}^s_{p,q}(\mathcal{H}_V)$. Thus the completeness of $\dot{B}^s_{p,q}(\mathcal{H}_V)$ is proved. The proof of Theorem 4.10 is now finished. □

Next we prove Theorem 4.11, namely, results on duality, embedding relations and lifting properties.

Proof of the assertion (i) *in Theorem 4.11.* We treat only the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_V)$, since the inhomogeneous case follows analogously. We prove that

$$\dot{B}^s_{p,q}(\mathcal{H}_V)^* = \dot{B}^{-s}_{p',q'}(\mathcal{H}_V) \tag{4.60}$$

for any $s \in \mathbb{R}$ and $1 \le p, q < \infty$. Let us first show that

$$\dot{B}^{-s}_{p',q'}(\mathcal{H}_V) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}_V)^*. \tag{4.61}$$

Put

$$\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \quad \text{for } j \in \mathbb{Z}.$$

For any $f \in \dot{B}^{-s}_{p',q'}(\mathcal{H}_V)$, we define an operator T_f as

$$T_f g := \sum_{j=-\infty}^{\infty} \int_\Omega \Big(\phi_j(\sqrt{\mathcal{H}_V})f\Big)\, \overline{\Phi_j(\sqrt{\mathcal{H}_V})g}\, dx \quad \text{for } g \in \dot{B}^s_{p,q}(\mathcal{H}_V).$$

Then

$$\begin{aligned} |T_f g| &\le \big\|\{2^{-sj}\|\phi_j(\sqrt{\mathcal{H}_V})f\|_{L^{p'}(\Omega)}\}_{j\in\mathbb{Z}}\big\|_{l^{q'}(\mathbb{Z})} \big\|\{2^{sj}\|\Phi_j(\sqrt{\mathcal{H}_V})g\|_{L^p(\Omega)}\}_{j\in\mathbb{Z}}\big\|_{l^q(\mathbb{Z})} \\ &\le C\|f\|_{\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)}\|g\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)}, \end{aligned}$$

which implies that the operator norm $\|T_f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)^*}$ is bounded by $C\|f\|_{\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)}$. This proves the embedding (4.61).

We prove the converse embedding:

$$\dot{B}^s_{p,q}(\mathcal{H}_V)^* \hookrightarrow \dot{B}^{-s}_{p',q'}(\mathcal{H}_V). \tag{4.62}$$

Let $F \in \dot{B}^s_{p,q}(\mathcal{H}_V)^*$. We define an operator

$$T : l^q(\mathbb{Z}\,; L^p(\Omega)) \to \mathbb{C}$$

as follows. For $G = \{G_j\}_{j\in\mathbb{Z}} \in l^q(\mathbb{Z}\,; L^p(\Omega))$, we put

$$T(G) := F\Big(\sum_{j=-\infty}^{\infty} 2^{-sj}\phi_j(\sqrt{\mathcal{H}_V})G_j\Big).$$

Here we estimate

$$
\begin{aligned}
&\Big\| \sum_{j=-\infty}^{\infty} 2^{-sj}\phi_j(\sqrt{\mathcal{H}_V})G_j\Big\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)}\\
&=\Big\{ \sum_{k=-\infty}^{\infty} \Big(2^{sk}\Big\|\phi_k(\sqrt{\mathcal{H}_V}) \sum_{j=k-1}^{k+1} 2^{-sj}\phi_j(\sqrt{\mathcal{H}_V})G_j\Big\|_{L^p(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&=\Big\{ \sum_{k=-\infty}^{\infty} \Big(2^{sk}\Big\|\phi_k(\sqrt{\mathcal{H}_V}) \sum_{r=-1}^{1} 2^{-s(k+r)}\phi_{k+r}(\sqrt{\mathcal{H}_V})G_{k+r}\Big\|_{L^p(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\sum_{r=-1}^{1} 2^{-sr}\Big\{ \sum_{k=-\infty}^{\infty} \|G_k\|^q_{L^p(\Omega)}\Big\}^{\frac{1}{q}}\\
&\le C\|G\|_{l^q(\mathbb{Z}\,;L^p(\Omega))},
\end{aligned}
$$

where we used the estimate (4.23) for $\alpha = 0$ in Lemma 4.1. Hence we deduce that

$$
\begin{aligned}
|T(G)| \le&\|F\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)^*}\Big\| \sum_{j=-\infty}^{\infty} 2^{-sj}\phi_j(\sqrt{\mathcal{H}_V})G_j\Big\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)}\\
\le&C\|F\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)^*}\|G\|_{l^q(\mathbb{Z}\,;L^p(\Omega))}.
\end{aligned}
$$

Since $(l^q(\mathbb{Z}\,;L^p(\Omega)))^* = l^{q'}(\mathbb{Z}\,;L^{p'}(\Omega))$, there exists $\{F_j\}_{j\in\mathbb{Z}} \in l^{q'}(\mathbb{Z}\,;L^{p'}(\Omega))$ such that

$$
T(G) = \sum_{j=-\infty}^{\infty}\int_\Omega F_j(x)\overline{G_j(x)}\,dx, \quad \|\{F_j\}_{j\in\mathbb{Z}}\|_{l^{q'}(\mathbb{Z}\,;L^{p'}(\Omega))} \le C\|F\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)^*}. \tag{4.63}
$$

Then for any $g \in \dot{B}^s_{p,q}(\mathcal{H}_V)$, let us take $G = \{G_j\}_{j\in\mathbb{Z}}$ as

$$
G_j = 2^{sj}\Phi_j(\sqrt{\mathcal{H}_V})g.
$$

It follows from $g \in \mathcal{Z}'_V(\Omega)$, (ii) in Lemma 4.5 and the identities $\phi_j = \phi_j\Phi_j$ that

$$
\begin{aligned}
F(g) &= F\Big(\sum_{j=-\infty}^{\infty} 2^{-sj}\phi_j(\sqrt{\mathcal{H}_V})\big(2^{sj}\Phi_j(\sqrt{\mathcal{H}_V})g\big)\Big)\\
&= T(G)\\
&= \sum_{j=-\infty}^{\infty}\int_\Omega F_j(x)\overline{G_j(x)}\,dx\\
&= \sum_{j=-\infty}^{\infty}\int_\Omega F_j(x)\overline{2^{sj}\Phi_j(\sqrt{\mathcal{H}_V})g}\,dx\\
&= \sum_{j=-\infty}^{\infty}\int_\Omega \Big(2^{sj}\Phi_j(\sqrt{\mathcal{H}_V})F_j(x)\Big)\overline{g}\,dx.
\end{aligned}
$$

Taking f as

$$f = \sum_{j=-\infty}^{\infty} 2^{sj}\Phi_j(\sqrt{\mathcal{H}_V})F_j,$$

we deduce from (4.63) that

$$\|f\|_{\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)} \le C\|\{F_j\}_{j\in\mathbb{Z}}\|_{l^{q'}(\mathbb{Z};L^{p'}(\Omega))} \le C\|F\|_{\dot{B}^{s}_{p,q}(\mathcal{H}_V)^*},$$

which implies that $f \in \dot{B}^{-s}_{p',q'}(\mathcal{H}_V)$. Hence F is regarded as an element in $\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)$, and we get the inclusion (4.62); thus we conclude the isomorphism (4.60). This ends the proof of the assertion (i) in Theorem 4.11. □

Proof of the assertion (ii) *in Theorem 4.11.* The embedding relations are immediate consequences of Lemma 4.1 and of the embedding in the sequence spaces. The main point is to prove the lifting properties.

First we prove the homogeneous case, namely,

$$\mathcal{H}_V^{\frac{s_0}{2}} f \in \dot{B}^{s-s_0}_{p,q}(\mathcal{H}_V) \quad \text{for any } f \in \dot{B}^{s}_{p,q}(\mathcal{H}_V).$$

To begin with, we show that

$$\mathcal{H}_V^{\frac{s_0}{2}} \text{ is a continuous operator from } \mathcal{Z}'_V(\Omega) \text{ to itself.} \tag{4.64}$$

By the definition (4.20), it is sufficient to verify that $\mathcal{H}_V^{s_0/2}$ is the continuous operator from $\mathcal{Z}_V(\Omega)$ to itself. Let us take $M_0 \in \mathbb{N}$ such that $M_0 > |s_0|$. It follows from (4.23) for $\alpha = s_0/2$ and (4.35) that

$$q_{V,M}\big(\mathcal{H}_V^{\frac{s_0}{2}} g\big) \le C q_{V,M+M_0}(g)$$

for any $g \in \mathcal{Z}_V(\Omega)$, which implies that $\mathcal{H}_V^{s_0/2} g \in \mathcal{Z}_V(\Omega)$. This proves (4.64). Hence, all we have to do is to prove that $f \in \dot{B}^{s}_{p,q}(\mathcal{H}_V)$ satisfies

$$\|\mathcal{H}_V^{\frac{s_0}{2}} f\|_{\dot{B}^{s-s_0}_{p,q}(\mathcal{H}_V)} \le C\|f\|_{\dot{B}^{s}_{p,q}(\mathcal{H}_V)}. \tag{4.65}$$

In fact, let

$$\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}.$$

We note that $\Phi_j(\lambda)\lambda^{s_0} \in C_0^\infty((0,\infty))$. Writing

$$\Phi_j(\lambda)\lambda^{s_0} = 2^{s_0 j} \cdot \Phi_j(\lambda) \cdot (2^{-s_0 j}\lambda^{s_0}),$$

we get

$$\begin{aligned}\|\phi_j(\sqrt{\mathcal{H}_V})\mathcal{H}_V^{\frac{s_0}{2}} f\|_{L^p(\Omega)} &= 2^{s_0 j}\big\|\big\{\Phi_j(\sqrt{\mathcal{H}_V})2^{-s_0 j}\mathcal{H}_V^{\frac{s_0}{2}}\big\}\phi_j(\sqrt{\mathcal{H}_V})f\big\|_{L^p(\Omega)} \\ &\le C2^{s_0 j}\|\phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}.\end{aligned}$$

Hence, multiplying $2^{(s-s_0)j}$ to the above inequality and taking the $l^q(\mathbb{Z})$-norm, we obtain the required inequality (4.65).

As to inhomogeneous case, we have to consider the operators

$$(\lambda_0^2+1+\mathcal{H}_V)^{\frac{s_0}{2}}\phi_j(\sqrt{\mathcal{H}_V}).$$

The only different point from the homogeneous case is to show the following estimates:

$$\big\|(\lambda_0^2+1+\mathcal{H}_V)^{\frac{s_0}{2}}\phi_j(\sqrt{\mathcal{H}_V})f\big\|_{L^p(\Omega)} \le C2^{s_0 j}\big\|\phi_j(\sqrt{\mathcal{H}_V})f\big\|_{L^p(\Omega)} \tag{4.66}$$

for any $j\in\mathbb{N}$. We write

$$\begin{aligned}(\lambda_0^2+1+\mathcal{H}_V)^{\frac{s_0}{2}} =& \Big[2^{s_0 j}\big\{2^{-2j}(\lambda_0^2+1)+2^{-2j}\mathcal{H}_V\big\}^{\frac{s_0}{2}} \\ &\qquad - 2^{s_0 j}\big(2^{-2j}\mathcal{H}_V\big)^{\frac{s_0}{2}}\Big] + 2^{s_0 j}\big(2^{-2j}\mathcal{H}_V\big)^{\frac{s_0}{2}} \\ =:& \, T_1+T_2.\end{aligned}$$

As to $T_2\phi_j(\sqrt{\mathcal{H}_V})f$, it follows from (4.22) for $\alpha=s_0/2$ in Lemma 4.1 that

$$\|T_2\phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \le C2^{s_0 j}\|\phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}.$$

Writing

$$\begin{aligned}T_1 =& \, 2^{s_0 j}\int_0^{2^{-2j}(\lambda_0^2+1)} \partial_\eta(\eta+2^{-2j}\mathcal{H}_V)^{\frac{s_0}{2}}\,d\eta \\ =& \, 2^{s_0 j}\int_0^{2^{-2j}(\lambda_0^2+1)} \frac{s_0}{2}(\eta+2^{-2j}\mathcal{H}_V)^{\frac{s_0}{2}-1}\,d\eta,\end{aligned}$$

we estimate $T_1\phi_j(\sqrt{\mathcal{H}_V})f$ as

$$\big\|T_1\phi_j(\sqrt{\mathcal{H}_V})f\big\|_{L^p(\Omega)} \le C2^{s_0 j}\int_0^{2^{-2j}(\lambda_0^2+1)} \big\|(\eta+2^{-2j}\mathcal{H}_V)^{\frac{s_0}{2}-1}\phi_j(\sqrt{\mathcal{H}_V})f\big\|_{L^p(\Omega)}\,d\eta.$$

When $p=2$, we use the spectral theorem on the Hilbert space $L^2(\Omega)$ to obtain

$$\begin{aligned}&\big\|(\eta+2^{-2j}\mathcal{H}_V)^{\frac{s_0}{2}-1}\phi_j(\sqrt{\mathcal{H}_V})f\big\|^2_{L^2(\Omega)} \\ =&\int_{2^{2(j-1)}}^{2^{2(j+1)}} (\eta+2^{-2j}\lambda)^{s_0-2}\,d\big\|E_{\mathcal{H}_V}(\lambda)\phi_j(\sqrt{\mathcal{H}_V})f\big\|^2_{L^2(\Omega)} \\ \le& C\int_{2^{2(j-1)}}^{2^{2(j+1)}} (2^{-2j}\lambda)^{s_0-2}\,d\big\|E_{\mathcal{H}_V}(\lambda)\phi_j(\sqrt{\mathcal{H}_V})f\big\|^2_{L^2(\Omega)} \\ \le& C\big\|\phi_j(\sqrt{\mathcal{H}_V})f\big\|^2_{L^2(\Omega)},\end{aligned}$$

since $j \in \mathbb{N}$ and $0 \le \eta \le 2^{-2j}(\lambda_0^2+1)$. When $p \neq 2$, we have to obtain the following estimate:

$$\big\|(\eta + 2^{-2j}\mathcal{H}_V)^{\frac{s_0}{2}-1}\phi_j(\sqrt{\mathcal{H}_V})f\big\|_{L^p(\Omega)} \le C\big\|\phi_j(\sqrt{\mathcal{H}_V})f\big\|_{L^p(\Omega)}. \tag{4.67}$$

Since η is small compared with the spectrum of $2^{-2j}\mathcal{H}_V\phi_j(\sqrt{\mathcal{H}_V})$, η is able to be neglected. Hence, the proof of estimate (4.67) is done by the argument of chapter 3. So, we may omit the details. Summarizing the estimates obtained now, we conclude the estimate (4.66). The proof of the assertion (ii) in Theorem 4.11 is finished. □

Next we prove the assertion (iii) in Theorem 4.11. For this purpose, we prepare the following two lemmas

Lemma 4.12 (The Khintchine inequality [30, 50])**.** *Let $\{r_j(t)\}_{j=1}^{\infty}$ be a sequence of Rademacher functions, i.e.,*

$$r_j(t) := \sum_{k=1}^{2^j}(-1)^{k-1}\chi_{[(k-1)2^{-j},k2^{-j})}(t) \quad \text{for } t \in [0,1],$$

where χ_I denotes the characteristic function on the interval I. Then for any p with $1 < p < \infty$, there exists a constant $C > 0$ such that

$$C^{-1}\|a\|_{l^2(\mathbb{N})} \le \Big\|\sum_{j=1}^{\infty} a_j r_j\Big\|_{L^p(0,1)} \le C\|a\|_{l^2(\mathbb{N})} \tag{4.68}$$

for all $a = \{a_j\}_{j\in\mathbb{Z}} \in l^2(\mathbb{N})$.

We also utilize the Hörmander type spectral multiplier theorem from Theorem 3.1 in Duong, Ouhabaz and Sikora [20]. This idea of applying such a kind of theorem can be found in several papers (see, e.g., Ivanovici and Planchon [38]). The following proposition states the spectral multiplier theorem under the assumption stronger than Theorem 3.1 in [20], which is sufficient for our operators.

Proposition 4.13. *Suppose that $\mathcal{L}$ is a non-negative self-adjoint operator on $L^2(\Omega)$ such that the kernel $e^{-t\mathcal{L}}(x,y)$ of semigroup generated by $\mathcal{L}$ enjoys the Gaussian upper bound*

$$|e^{-t\mathcal{L}}(x,y)| \le Ct^{-\frac{d}{2}}\exp\Big(-\frac{|x-y|^2}{Ct}\Big) \tag{4.69}$$

for any $t > 0$ and almost every $x, y \in \Omega$. For $s > n/2$ and $\eta \in C_0^{\infty}((0,\infty))$, let F be a bounded Borel function on $\mathbb{R}$ such that

$$\sup_{\theta>0}\|\eta F(\theta\cdot)\|_{W^{s,\infty}(\mathbb{R})} = \sup_{\theta>0}\big\|\big(1-\partial_\lambda^2\big)^{\frac{s}{2}}\big(\eta F(\theta\cdot)\big)\big\|_{L^\infty(\mathbb{R})} < \infty. \tag{4.70}$$

Then the operator $F(\mathcal{L})$ is bounded on $L^p(\Omega)$ for any $1 < p < \infty$.

Proof of the assertion (iii) *in Theorem 4.11.* First we consider the homogeneous case. We prove the embedding

$$L^p(\Omega) \hookrightarrow \dot{B}^0_{p,2}(\mathcal{H}_V) \quad \text{for } 1 < p \le 2. \tag{4.71}$$

Let $\{r_j(t)\}$ be the sequence of Rademacher functions. If we show that there exists a constant $C > 0$ such that

$$\Big\| \sum_{j=1}^{N} r_j(t)\phi_j(\sqrt{\mathcal{H}_V})f \Big\|_{L^p(\Omega)} + \Big\| \sum_{j=-N}^{-1} r_{-j}(t)\phi_j(\sqrt{\mathcal{H}_V})f \Big\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)} \tag{4.72}$$

for any $t \in [0,1]$ and $N \in \mathbb{N}$, then (4.71) is verified. Indeed, by using (4.68) and (4.72), we estimate

$$\begin{aligned}
\Big\| \Big(\sum_{j=1}^{N} \big|\phi_j(\sqrt{\mathcal{H}_V})f\big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)} &\le C \Big\| \Big(\int_0^1 \Big| \sum_{j=1}^{N} r_j(t)\phi_j(\sqrt{\mathcal{H}_V})f \Big|^p dt \Big)^{\frac{1}{p}} \Big\|_{L^p(\Omega)} \\
&\le C \Big(\int_0^1 \Big\| \sum_{j=1}^{N} r_j(t)\phi_j(\sqrt{\mathcal{H}_V})f \Big\|^p_{L^p(\Omega)} dt \Big)^{\frac{1}{p}} \\
&\le C \Big(\int_0^1 \|f\|^p_{L^p(\Omega)}\, dt \Big)^{\frac{1}{p}} \\
&= C\|f\|_{L^p(\Omega)}.
\end{aligned}$$

Similarly, we get

$$\Big\| \Big(\sum_{j=-N}^{-1} \big|\phi_j(\sqrt{\mathcal{H}_V})f\big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)},$$

and hence,

$$\Big\| \Big(\sum_{j=-N}^{N} \big|\phi_j(\sqrt{\mathcal{H}_V})f\big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)} \quad \text{for any } N \in \mathbb{N}.$$

By taking the limit as $N \to \infty$ in the above inequality and Minkowski's inequality, we obtain

$$\|f\|_{\dot{B}^0_{p,2}(\mathcal{H}_V)} \le \Big\| \Big\| \big\{ \phi_j(\sqrt{\mathcal{H}_V})f \big\}_{j\in\mathbb{Z}} \Big\|_{l^2(\mathbb{Z})} \Big\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}.$$

Thus, we get the embedding (4.71).

We have to show (4.72). Let $F^{\pm}_N(t,\lambda)$, $N \in \mathbb{N}$, be functions on $[0,1]\times\mathbb{R}$ defined by

$$F^+_N(t,\lambda) := \sum_{j=1}^{N} r_j(t)\phi_j(\lambda), \qquad F^-_N(t,\lambda) := \sum_{j=-N}^{-1} r_{-j}(t)\phi_j(\lambda),$$

respectively. Let $m_0 \in \mathbb{N}$ with $m_0 > n/2$ and $\eta \in C_0^\infty((0,\infty))$. If we can prove that

$$\sup_{N,t}\sup_{\theta>0}\left\|\eta F_N^{\pm}(t,\theta\cdot)\right\|_{W^{m_0,\infty}(\mathbb{R})} < \infty, \tag{4.73}$$

then we get (4.72). Indeed, we have the Gaussian upper bound for the kernel of semigroup $e^{-t\mathcal{H}_V}$ from the assertion (iv) in Proposition 2.4 and the bound (4.73), which are just assumptions in Proposition 4.13. Hence all we have to do is to prove (4.73) and it is sufficient to handle only F_N^+, since the proof of the case of F_N^- is similar to F_N^+.

The bound (4.73) is equivalent to the following:

$$\sup_{\theta>0}\left\|\partial_\lambda^m\{\eta F_N^+(t,\theta\cdot)\}\right\|_{L^\infty(\mathbb{R})} < \infty \tag{4.74}$$

for all $m \in \{0,1,\dots,m_0\}$, and for any $t \in [0,1]$ and $N \in \mathbb{N}$. Obviously, we see that

$$\sup_{\theta>0}\left\|\eta F_N^+(t,\theta\cdot)\right\|_{L^\infty(\mathbb{R})} \le C\sup_{\lambda>0,\,\theta>0}\sum_{j=1}^{N}\phi_0(2^{-j}\theta\lambda) \le C\cdot 1 < \infty \tag{4.75}$$

for any $t \in [0,1]$ and $N \in \mathbb{N}$, which proves (4.74) for $m = 0$. Now, as to (4.74) for $m \in \{1,\dots,m_0\}$, since the support of $\eta(\lambda)$ is away from the origin, we find from (4.75) that

$$\begin{aligned}\left\|\partial_\lambda^m\{\eta F_N^+(t,\theta\cdot)\}\right\|_{L^\infty(\mathbb{R})} &\le \left\|\eta F_N^+(t,\theta\cdot)\right\|_{L^\infty(\mathbb{R})} + \sum_{k=1}^{m}\binom{m}{k}\left\|\eta^{(m-k)}\partial_\lambda^k F_N^+(t,\theta\cdot)\right\|_{L^\infty(\mathbb{R})}\\ &\le C + \sum_{k=1}^{m}\binom{m}{k}\left\|\eta^{(m-k)}\lambda^{-k}\right\|_{L^\infty(\mathbb{R})}\left\|\lambda^k\partial_\lambda^k F_N^+(t,\theta\cdot)\right\|_{L^\infty(\mathbb{R})}\end{aligned}$$

for any $N \in \mathbb{N}$, $\theta > 0$ and $t \in [0,1]$. Here, taking a real M satisfying $M > m_0$, we have the uniform bound for $\lambda^k\partial_\lambda^k F_N^+(t,\theta\lambda)$ with respect to $\lambda > 0$, $\theta > 0$, $N \in \mathbb{N}$ and $t \in [0,1]$, i.e.,

$$\begin{aligned}\lambda^k\left|\partial_\lambda^k F_N^+(t,\theta\lambda)\right| &\le \sum_{j=1}^{N}(2^{-j}\theta\lambda)^k\left|(\partial_\lambda^k\phi_0)(2^{-j}\theta\lambda)\right|\\ &\le C\sum_{j=1}^{\infty}(2^{-j}\theta\lambda)^k(1+2^{-j}\theta\lambda)^{-M}\\ &\le C\sum_{j=-\infty}^{\infty}(2^{-j})^k(1+2^{-j})^{-M}.\end{aligned}$$

We note here that in the last step of the above estimate, before summing up with respect to $j \in \mathbb{Z}$, for fixed $\theta > 0$ and $\lambda > 0$, we regarded $2^{-j}\theta\lambda$ as some number

near dyadic number or 0. Hence, combining the estimates obtained now, we get

$$\left\|\partial_\lambda^m\{\eta F_N^+(t,\theta\cdot)\}\right\|_{L^\infty(\mathbb{R})} \le C + C\sum_{k=1}^{m}\sum_{j=-\infty}^{\infty}(2^{-j})^k(1+2^{-j})^{-M} < \infty$$

for any $N \in \mathbb{N}$, $t \in [0,1]$ and $\theta > 0$. Therefore (4.74) is proved. Thus the proof of (4.71) is finished.

Next we prove the embedding

$$\dot{B}^0_{p,2}(\mathcal{H}_V) \hookrightarrow L^p(\Omega) \quad \text{for } 2 \le p < \infty. \tag{4.76}$$

Let p' be such that $1/p + 1/p' = 1$. Then the embedding (4.76) is an immediate consequence of $1 < p' \le 2$, $L^{p'}(\Omega) \hookrightarrow \dot{B}^0_{p',2}(\mathcal{H}_V)$, $L^{p'}(\Omega)^* = L^p(\Omega)$ and $\dot{B}^0_{p',2}(\mathcal{H}_V)^* = \dot{B}^0_{p,2}(\mathcal{H}_V)$.

Finally we prove the inhomogeneous case. By Proposition 2.1, there exists a real number $M \ge -\inf\sigma(\mathcal{H}_V)$ such that $\mathcal{H}_V + M$ is a non-negative and self-adjoint operator on $L^2(\Omega)$ satisfying Gaussian upper bound (4.69). Hence we can apply a similar argument to (4.71) in the homogeneous case to obtain

$$\|f\|_{B^0_{p,2}(\mathcal{H}_V+M)} \le C\|f\|_{L^p(\Omega)} \quad \text{for any } f \in C_0^\infty(\Omega), \tag{4.77}$$

provided that $1 < p \le 2$. If we show that

$$\|f\|_{B^0_{p,2}(\mathcal{H}_V)} \le C\|f\|_{B^0_{p,2}(\mathcal{H}_V+M)} \quad \text{for any } f \in C_0^\infty(\Omega), \tag{4.78}$$

then, by combining (4.77) and (4.78) with density argument, we obtain

$$L^p(\Omega) \hookrightarrow B^0_{p,2}(\mathcal{H}_V)$$

for any $1 < p \le 2$. By duality argument, we have

$$B^0_{p,2}(\mathcal{H}_V) \hookrightarrow L^p(\Omega)$$

for any $2 \le p < \infty$. Thus we conclude the inhomogeneous case. Hence all we have to do is to show (4.78).

For a real M, let $\mathcal{X}_{V,M}(\Omega)$ be a test function space $\mathcal{X}_V(\Omega)$ for $\mathcal{H}_V$ replaced by $\mathcal{H}_V + M$. Then we find that

$$\mathcal{X}_{V,M}(\Omega) = \mathcal{X}_V(\Omega), \tag{4.79}$$

since

$$\mathcal{D}(\mathcal{H}_V + M) = \mathcal{D}(\mathcal{H}_V)$$

by the Kato-Rellich theorem. Let $f \in C_0^\infty(\Omega)$. Since $f \in \mathcal{X}_V(\Omega)$, it follows from Proposition 4.5 and (4.79) that the identity

$$f = \psi(\mathcal{H}_V + M)f + \sum_{k=1}^{\infty} \phi_k(\sqrt{\mathcal{H}_V + M})f$$

holds true in $\mathcal{X}_V(\Omega)$. Hence we can write

$$\begin{aligned}
\|f\|_{B^0_{p,2}(\mathcal{H}_V)} \le{}& \|\psi(\mathcal{H}_V)\psi(\mathcal{H}_V + M)f\|_{L^p(\Omega)} + \sum_{k=1}^{\infty} \|\psi(\mathcal{H}_V)\phi_k(\mathcal{H}_V + M)f\|_{L^p(\Omega)} \\
&+ \left(\sum_{j=1}^{\infty} \|\phi_j(\mathcal{H}_V)\psi(\mathcal{H}_V + M)f\|^2_{L^p(\Omega)} \right)^{\frac{1}{2}} \\
&+ \left\{ \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \|\phi_j(\mathcal{H}_V)\phi_k(\mathcal{H}_V + M)f\|_{L^p(\Omega)} \right)^2 \right\}^{\frac{1}{2}} \\
=:{}& I + II + III + IV.
\end{aligned}$$

By (4.21) in Lemma 4.1, we estimate the first term as

$$I \le C\|\psi(\mathcal{H}_V + M)f\|_{L^p(\Omega)}.$$

As to the second term, writing

$$\psi(\mathcal{H}_V)\phi_k(\mathcal{H}_V + M)f = (\mathcal{H}_V + M)\psi(\mathcal{H}_V)(\mathcal{H}_V + M)^{-1}\phi_k(\mathcal{H}_V + M)f,$$

we see from (4.21) and (4.22) in Lemma 4.1 that

$$\begin{aligned}
\|\psi(\mathcal{H}_V)\phi_k(\mathcal{H}_V + M)f\|_{L^p(\Omega)} &\le C\|(\mathcal{H}_V + M)^{-1}\phi_k(\mathcal{H}_V + M)f\|_{L^p(\Omega)} \\
&\le C2^{-2k}\|\phi_k(\mathcal{H}_V + M)f\|_{L^p(\Omega)},
\end{aligned}$$

which implies that

$$II \le C\sum_{k=1}^{\infty} 2^{-2k}\|\phi_k(\mathcal{H}_V + M)f\|_{L^p(\Omega)} \le C\|f\|_{B^0_{p,2}(\mathcal{H}_V+M)}.$$

Similarly, we get

$$III \le C\|f\|_{B^0_{p,2}(\mathcal{H}_V+M)}.$$

As to the fourth term, putting

$$\Phi_k := \phi_{k-1} + \phi_k + \phi_{k+1},$$

we write

$$\phi_j(\mathcal{H}_V)\phi_k(\mathcal{H}_V + M)f = \mathcal{H}_V^{-1}\phi_j(\mathcal{H}_V)\big\{(\mathcal{H}_V + M) - M\big\}\Phi_k(\mathcal{H}_V + M)\phi_k(\mathcal{H}_V + M)f,$$

$\phi_j(\mathcal{H}_V)\phi_k(\mathcal{H}_V+M)f = (\mathcal{H}_V+M)\phi_j(\mathcal{H}_V)(\mathcal{H}_V+M)^{-1}\Phi_k(\mathcal{H}_V+M)\phi_k(\mathcal{H}_V+M)f.$
Then we see from (4.22) in Lemma 4.1 that

$$\begin{aligned}
&\|\phi_j(\mathcal{H}_V)\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\\
&=\|\mathcal{H}_V^{-1}\phi_j(\mathcal{H}_V)\big\{(\mathcal{H}_V+M)-M\big\}\Phi_k(\mathcal{H}_V+M)\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\\
&\le C2^{-2j}\|\big\{(\mathcal{H}_V+M)-M\big\}\Phi_k(\mathcal{H}_V+M)\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\\
&\le C2^{-2(j-k)}\|\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)},
\end{aligned}$$

$$\begin{aligned}
&\|\phi_j(\mathcal{H}_V)\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\\
&=\|(\mathcal{H}_V+M)\phi_j(\mathcal{H}_V)(\mathcal{H}_V+M)^{-1}\Phi_k(\mathcal{H}_V+M)\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\\
&\le C2^{2j}\|(\mathcal{H}_V+M)^{-1}\Phi_k(\mathcal{H}_V+M)\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\\
&\le C2^{2(j-k)}\|\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)},
\end{aligned}$$

for any $j,k\in\mathbb{N}$ and $1<p<\infty$. Combining the above inequalities, we obtain

$$\|\phi_j(\mathcal{H}_V)\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)} \le C2^{-2|j-k|}\|\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)} \tag{4.80}$$

for any $j,k\in\mathbb{N}$ and $1<p<\infty$. Then we deduce from (4.80) that

$$\begin{aligned}
IV &\le C\bigg\{\sum_{j=1}^{\infty}\bigg(\sum_{k=1}^{\infty}2^{-2|j-k|}\|\phi_k(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\bigg)^2\bigg\}^{\frac12}\\
&\le C\bigg\{\sum_{j=1}^{\infty}\bigg(\sum_{k'=-j+1}^{\infty}2^{-2|k'|}\|\phi_{j+k'}(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\bigg)^2\bigg\}^{\frac12}\\
&\le C\sum_{k'=1}^{\infty}2^{-2|k'|}\bigg\{\bigg(\sum_{j=1}^{\infty}\|\phi_{j+k'}(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\bigg)^2\bigg\}^{\frac12}\\
&\quad + C\sum_{k'=-\infty}^{0}2^{-2|k'|}\bigg\{\bigg(\sum_{j=-k'+1}^{\infty}\|\phi_{j+k'}(\mathcal{H}_V+M)f\|_{L^p(\Omega)}\bigg)^2\bigg\}^{\frac12}\\
&\le C\|f\|_{B^0_{p,2}(\mathcal{H}_V+M)}.
\end{aligned}$$

Hence, combining the estimates obtained now, we conclude (4.78). The proof of the assertion (iii) in Theorem 4.11 is finished. □

4.2.3 Equivalence relations

In this subsection we prove two results on isomorphisms. The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_V)$ are defined as subspaces of $\mathcal{Z}'_V(\Omega)$. The first result states that $\dot{B}^s_{p,q}(\mathcal{H}_V)$ are also regarded as subspaces of $\mathcal{X}'_V(\Omega)$ if indices s,p and q are restricted. Such characterization is known when $\Omega=\mathbb{R}^d$ (see, e.g., Kozono and Yamazaki [52]).

Theorem 4.14. *Suppose that the potential V satisfies assumptions* A *and* B. *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. If either $s < d/p$ or $(s,q) = (d/p, 1)$, then the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_V)$ are regarded as subspaces of $\mathcal{X}'_V(\Omega)$ according to the following isomorphism*:

$$\dot{B}^s_{p,q}(\mathcal{H}_V) \cong \Big\{ f \in \mathcal{X}'_V(\Omega) : \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} < \infty,\ f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_V}) f \text{ in } \mathcal{X}'_V(\Omega) \Big\}.$$

Proof. Putting

$$\dot{X}^s_{p,q}(\mathcal{H}_V) := \Big\{ f \in \mathcal{X}'_V(\Omega) : \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} < \infty,\ f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_V}) f \text{ in } \mathcal{X}'_V(\Omega) \Big\},$$

we see that

$$\dot{X}^s_{p,q}(\mathcal{H}_V) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}_V).$$

Hence it is sufficient to prove that

$$\dot{B}^s_{p,q}(\mathcal{H}_V) \hookrightarrow \dot{X}^s_{p,q}(\mathcal{H}_V). \tag{4.81}$$

Let $f \in \dot{B}^s_{p,q}(\mathcal{H}_V)$. Then $f \in \mathcal{Z}'_V(\Omega)$, and thanks to Lemma 4.5 (ii), f is written as

$$\begin{aligned} f &= \sum_{j=-\infty}^{0} \phi_j(\sqrt{\mathcal{H}_V}) f + \sum_{j=1}^{\infty} \phi_j(\sqrt{\mathcal{H}_V}) f \quad \text{in } \mathcal{Z}'_V(\Omega) \\ &=: I + II. \end{aligned} \tag{4.82}$$

For the low frequency part, it follows from (4.23) for $\alpha = 0$ that

$$\|I\|_{L^\infty(\Omega)} \le \sum_{j=-\infty}^{0} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^\infty(\Omega)} \le C \sum_{j=-\infty}^{0} 2^{\frac{d}{p} j} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)},$$

where the right member is finite when $(s,q) = (d/p, 1)$. In the case when $s < d/p$, we estimate

$$\begin{aligned} \|I\|_{L^\infty(\Omega)} &\le C \sum_{j=-\infty}^{0} 2^{(\frac{d}{p} - s) j} \sup_{k \le 0} 2^{sk} \|\phi_k(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} \\ &\le C \|f\|_{\dot{B}^s_{p,\infty}(\mathcal{H}_V)} \\ &\le C \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)}, \end{aligned}$$

where we used the embedding in Proposition 4.11 (ii-b) in the last step. Hence the above two estimates and Lemma 4.6 imply that I belongs to $\mathcal{X}'_V(\Omega)$. As to II, since the high frequency part of $q_{V,M}(\cdot)$ is equivalent to that of $p_{V,M}(\cdot)$, it follows that $II \in \mathcal{X}'_V(\Omega)$. Hence the identity (4.82) holds in the topology of $\mathcal{X}'_V(\Omega)$. Therefore, we get $f \in \dot{X}^s_{p,q}(\mathcal{H}_V)$. Thus we conclude the embedding (4.81). This completes the proof of Theorem 4.14. □

The second result states the equivalence relation among the Besov spaces generated by $\mathcal{H}_0$ and $\mathcal{H}_V$ with $V \in L^{\frac{d}{2},\infty}(\Omega)$. For the definition of the Lorentz space $L^{\frac{d}{2},\infty}(\Omega)$, see (4.84) below.

Theorem 4.15. *Suppose the same assumption on V as in Theorem 4.10. Let s, p, q be such that*

$$1 \le p, q \le \infty, \quad \begin{cases} -\min\left\{2, d\Big(1-\frac{1}{p}\Big)\right\} < s < \min\left\{\frac{d}{p}, 2\right\} & \text{if } d \ge 3, \\ -2+\frac{2}{p} < s < \frac{2}{p} & \text{if } d = 1, 2. \end{cases}$$

(i) *If V satisfies*

$$\begin{cases} V \in L^{\frac{d}{2},\infty}(\Omega) + L^{\infty}(\Omega) & \text{if } d \ge 3, \\ V \in K_d(\Omega) & \text{if } d = 1, 2, \end{cases}$$

then

$$B^s_{p,q}(\mathcal{H}_V) \cong B^s_{p,q}(\mathcal{H}_0).$$

(ii) *If V satisfies*

$$\begin{cases} V \in L^{\frac{d}{2},\infty}(\Omega) & \text{if } d \ge 3, \\ V \in L^1(\Omega) & \text{if } d = 2, \end{cases} \tag{4.83}$$

then

$$\dot{B}^s_{p,q}(\mathcal{H}_V) \cong \dot{B}^s_{p,q}(\mathcal{H}_0).$$

Let us give some remarks on Theorem 4.15.

- Theorem 4.15 implies not only the equivalence of norms, but also that of the following two approximations of the identity

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_0}) f \quad \text{in } \mathcal{Z}'_0(\Omega), \qquad f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_V}) f \quad \text{in } \mathcal{Z}'_V(\Omega),$$

  for f belonging to the homogeneous Besov spaces. Analogous approximations in $\mathcal{X}'_0(\Omega)$ and $\mathcal{X}'_V(\Omega)$ are also equivalent for the inhomogeneous Besov spaces.

- By considering the Lorentz spaces, it is possible to treat the potential V like

$$V(x) = c|x|^{-2}, \quad c > 0,$$

  which, in fact, $V \in L^{\frac{d}{2},\infty}(\Omega)$. On the other hand, if V is more singular, the range of the regularity s for the isomorphism becomes smaller, since $|x|^{-2-\varepsilon}$ ($\varepsilon > 0$) can not be controlled locally by the Laplacian for instance.

- If V is smooth more and more, then, s can be taken bigger and bigger so that the isomorphism holds. For instance, this comes from the following identity:

$$(-\Delta + V)^2 f = (-\Delta)^2 f + (-\Delta)(Vf) + V(-\Delta)f + V^2 f$$

when we consider the case $s = 4$. In fact, the term $(-\Delta)(Vf)$ requires the differentiability of V.

We use the theory of Lorentz spaces and introduce the following notations (see, e.g., [30, 89]). Let f be a measurable function on Ω. We define the non-increasing rearrangement of f as

$$f^*(t) := \inf\{c > 0 : d_f(c) \le t\},$$

where $d_f(c)$ is the distribution function of f which is defined by the Lebesgue measure of the set $\{x \in \Omega : |f(x)| > c\}$. We define a function $f^{**}(t)$ on $(0, \infty)$ as

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(t')\,dt'.$$

Lorentz spaces $L^{p,q}(\Omega)$ are defined by letting

$$L^{p,q}(\Omega) := \{f : \text{measurable on } \Omega : \|f\|_{L^{p,q}} < \infty\}, \tag{4.84}$$

where

$$\|f\|_{L^{p,q}(\Omega)} := \begin{cases} \left\{ \displaystyle\int_0^\infty \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} & \text{if } 1 \le p, q < \infty, \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & \text{if } 1 \le p \le \infty \text{ and } q = \infty. \end{cases}$$

Note that

$$\begin{gathered} L^{p,1}(\Omega) \hookrightarrow L^{p,q}(\Omega) \quad \text{if } 1 \le p, q \le \infty, \\ L^p(\Omega) = L^{p,\infty}(\Omega) \quad \text{if } p = 1, \infty, \\ L^{p,1}(\Omega) \hookrightarrow L^p(\Omega) = L^{p,p}(\Omega) \hookrightarrow L^{p,\infty}(\Omega) \quad \text{if } 1 < p < \infty. \end{gathered} \tag{4.85}$$

Let $1 < p < \infty$. We have the Hölder inequality and Young inequality in the Lorentz spaces:

$$\|fg\|_{L^{p,q}(\Omega)} \le C\|f\|_{L^{p_1,q_1}(\Omega)}\|g\|_{L^{p_2,q_2}(\Omega)} \quad \text{if } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{4.86}$$

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^{p_1,q_1}(\Omega)}\|g\|_{L^{p_2,q_2}(\Omega)} \quad \text{if } 1 = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{4.87}$$

$$\|f * g\|_{L^{p,q}(\mathbb{R}^d)} \le C\|f\|_{L^{p_1,q_1}(\mathbb{R}^d)}\|g\|_{L^{p_2,q_2}(\mathbb{R}^d)} \quad \text{if } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - 1, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{4.88}$$

where $1 \le p_1, p_2, q, q_1, q_2 \le \infty$. We often use the estimates in the Lorentz spaces on $\mathbb{R}^d$ for functions on Ω extending them by zero extension to the outside of Ω when the necessity arises. Recalling that the quasi-norm of $L^{p,\infty}(\Omega)$ is given by

$$\|f\|_{L^{p,\infty}(\Omega)} = \sup_{t>0} t d_f(t)^{\frac{1}{p}},$$

we have the following:

Lemma 4.16. *Let f be a measurable and nonnegative function on Ω. If $f \in L^{p_0,\infty}(\Omega)$ for some $1 < p_0 < \infty$, then*

$$\|f\|_{L^1(\{f>1\})} \le \frac{p_0}{p_0 - 1} \|f\|^{p_0}_{L^{p_0,\infty}(\Omega)}.$$

Proof. Let $f \in L^{p_0,\infty}(\Omega)$. Since

$$d_{\chi_{\{f>1\}} f}(t) = \begin{cases} d_f(t) & \text{if } t > 1, \\ d_f(1) & \text{if } 0 < t \le 1, \end{cases}$$

it follows that

$$\begin{aligned} \int_\Omega \chi_{\{f>1\}}(x) f(x)\, dx &= \int_1^\infty d_f(t)\, dt + d_f(1) \\ &= \int_1^\infty t^{-p_0} \left\{ t d_f(t)^{\frac{1}{p_0}} \right\}^{p_0} dt + \left\{ 1 \cdot d_f(1)^{\frac{1}{p_0}} \right\}^{p_0} \\ &\le \frac{1}{p_0 - 1} \|f\|^{p_0}_{L^{p_0,\infty}(\Omega)} + \|f\|^{p_0}_{L^{p_0,\infty}(\Omega)}, \end{aligned}$$

which proves the lemma. □

We prove Theorem 4.15 only for the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_V)$, since the inhomogeneous case is proved in an analogous way.

We prepare the following four lemmas.

Lemma 4.17. *Let $1 \le p_0 < p < \infty$ and $1 \le q \le \infty$. Suppose that the potential V satisfies assumptions* A *and* B. *Then there exists a constant $C > 0$ such that*

$$\|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^{p,q}(\Omega)} \le C 2^{d(\frac{1}{p_0} - \frac{1}{p})j} \|f\|_{L^{p_0}(\Omega)}$$

for any $j \in \mathbb{Z}$ and $f \in L^{p_0}(\Omega)$.

Proof. It is sufficient to consider the case $q = 1$ due to the embedding (4.85). Let p_1 be such that $1/p = 1/p_0 + 1/p_1 - 1$. Then it follows from the assertion (iv) in Proposition 2.4 and the Young inequality (4.88) that

$$\begin{aligned} \|\phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^{p,1}(\Omega)} &= \left\| e^{-2^{-2j}\mathcal{H}_V} \left\{ e^{2^{-2j}\mathcal{H}_V} \phi_j(\sqrt{\mathcal{H}_V}) \right\} f \right\|_{L^{p,1}(\Omega)} \\ &\le \|K_0(2^{-2j}, \cdot)\|_{L^{p_1,1}(\mathbb{R}^d)} \left\| \left\{ e^{2^{-2j}\mathcal{H}_V} \phi_j(\sqrt{\mathcal{H}_V}) \right\} f \right\|_{L^{p_0,\infty}(\Omega)} \\ &\le C(p_1) 2^{d(\frac{1}{p_0} - \frac{1}{p})j} \left\| \left\{ e^{2^{-2j}\mathcal{H}_V} \phi_j(\sqrt{\mathcal{H}_V}) \right\} f \right\|_{L^{p_0}(\Omega)} \\ &\le C(p_1) 2^{d(\frac{1}{p_0} - \frac{1}{p})j} \|f\|_{L^{p_0}(\Omega)}, \end{aligned}$$

where $K_0(t,x)$ is the function in the right member of (2.16), i.e.,

$$K_0(t,x) = Ct^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{8t}}, \quad t>0, \quad x\in\mathbb{R}^d,$$

and we used the fact that

$$\|K_0(2^{-2j},\cdot)\|_{L^{p_1,1}} = C(p_1)2^{d(\frac{1}{p_0}-\frac{1}{p})j} \quad \text{for } p_1>1.$$

Here we note that the above constant $C(p_1)$ is finite if and only if $p_1>1$ (i.e. $p_0<p$). Thus the proof of of Lemma 4.17 is complete. □

Lemma 4.18. *Let $1\le p\le\infty$. Suppose that the potential V satisfies assumptions* A *and* B. *Then*

$$\mathcal{H}_V^m\phi_j(\sqrt{\mathcal{H}_0})f\in\mathcal{Z}_V'(\Omega) \quad \textit{and} \quad \mathcal{H}_0^m\phi_j(\sqrt{\mathcal{H}_V})f\in\mathcal{Z}_0'(\Omega)$$

for any $j,m\in\mathbb{Z}$ and $f\in L^p(\Omega)$.

Proof. Let $j\in\mathbb{Z}$ be fixed. Since $\phi_j(\sqrt{\mathcal{H}_0})f\in L^p(\Omega)$ for any $f\in L^p(\Omega)$ by (4.23) in Lemma 4.1, it follows from Proposition 4.6 that $\phi_j(\sqrt{\mathcal{H}_0})f\in\mathcal{Z}_V'(\Omega)$. Hence, since $\mathcal{H}_V^m$ is a mapping from $\mathcal{Z}_V'(\Omega)$ to itself by (4.64), the first assertion is proved. In the same way, the second assertion holds. The proof of Lemma 4.18 is complete. □

Lemma 4.19. *Suppose that the potential V satisfies assumptions* A, B *and* (4.83). *Then the following assertions hold*:

(i) *Let $p=1$ for $d=2$ and $1\le p<d/2$ for $d\ge3$. Then*

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)}\le C2^{-2(j-k)}\|f\|_{L^p(\Omega)}, \tag{4.89}$$

$$\|\phi_k(\sqrt{\mathcal{H}_0})\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}\le C2^{-2(k-j)}\|f\|_{L^p(\Omega)} \tag{4.90}$$

for any $f\in L^p(\Omega)$, where $\Phi_j:=\phi_{j-1}+\phi_j+\phi_{j+1}$ for $j\in\mathbb{Z}$.

(ii) *Let $p=\infty$ for $d=2$ and $d/(d-2)<p\le\infty$ for $d\ge3$. Then*

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)}\le C2^{-2(k-j)}\|f\|_{L^p(\Omega)}, \tag{4.91}$$

$$\|\phi_k(\sqrt{\mathcal{H}_0})\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)}\le C2^{-2(j-k)}\|f\|_{L^p(\Omega)} \tag{4.92}$$

for any $f\in L^p(\Omega)$.

Proof. We prove only the assertion (i), since the estimates (4.91) and (4.92) are obtained by the duality argument of (4.89) and (4.90), respectively. Let us concentrate on the proof of (4.89) and (4.90) . We divide the proof into two cases: $d\ge3$ and $d=2$.

The case $d \geq 3$. Let $1 \leq p < d/2$ and $f \in C_0^\infty(\Omega)$. By the estimate (4.23) for $\alpha = 1$, we have

$$\begin{aligned} \|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} &= \|\phi_j(\sqrt{\mathcal{H}_V})\mathcal{H}_V^{-1}\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \\ &\leq C2^{-2j}\|\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \end{aligned} \tag{4.93}$$

for any $j, k \in \mathbb{Z}$. Here, we note that

$$\Phi_k(\sqrt{\mathcal{H}_0})f \in H_0^1(\Omega).$$

Furthermore, applying Lemma 4.16 for $p_0 = d/2$ to V_+, we see from (4.23) in Lemma 4.1 that

$$\begin{aligned} \|\sqrt{V_+}\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^2(\Omega)} &\leq \|V_+\|_{L^1(\{V_+>1\})}^{\frac{1}{2}}\|\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^\infty(\Omega)} + \|\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^2(\Omega)} \\ &\leq C\left(2^{\frac{d}{2}k}\sqrt{\frac{d}{d-2}}\,\|V_+\|_{L^{\frac{d}{2},\infty}}^{\frac{d}{4}} + 1\right)\|f\|_{L^2(\Omega)} \end{aligned}$$

for any $k \in \mathbb{Z}$. As a consequence, we find that

$$\sqrt{V_+}\Phi_k(\sqrt{\mathcal{H}_0})f \in L^2(\Omega). \tag{4.94}$$

Hence it follows from (4.94) that

$$\begin{aligned} {}_{L^2(\Omega)}(\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}_0})f, g)_{L^2(\Omega)} &= \int_\Omega \left(\nabla\Phi_k(\sqrt{\mathcal{H}_0})f \cdot \overline{\nabla g} + V\big(\Phi_k(\sqrt{\mathcal{H}_0})f\big)\,\overline{g}\right)dx \\ &= \int_\Omega \left(\mathcal{H}_0\Phi_k(\sqrt{\mathcal{H}_0})f + V\Phi_k(\sqrt{\mathcal{H}_0})f\right)\overline{g}\,dx \end{aligned}$$

for any $g \in \mathcal{D}(H_V)$. Therefore, since $\mathcal{D}(\mathcal{H}_V)$ is dense in $L^2(\Omega)$, we get

$$\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}_0})f(x) = \mathcal{H}_0\Phi_k(\sqrt{\mathcal{H}_0})f(x) + V\Phi_k(\sqrt{\mathcal{H}_0})f(x) \tag{4.95}$$

for almost every $x \in \Omega$, which implies that

$$\|\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \leq \|\mathcal{H}_0\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} + \|V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \tag{4.96}$$

for any $k \in \mathbb{Z}$. As to the first term, we estimate, by using (4.23) from Lemma 4.1,

$$\|\mathcal{H}_0\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \leq C2^{2k}\|f\|_{L^p(\Omega)} \tag{4.97}$$

for any $k \in \mathbb{Z}$. As to the second term, we use the following estimate: For any $1 \leq p < p_0 < \infty$ and $1 \leq q \leq \infty$, there exists a constant $C > 0$ such that

$$\|\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^{p_0,q}(\Omega)} \leq C2^{d(\frac{1}{p}-\frac{1}{p_0})k}\|f\|_{L^p(\Omega)} \tag{4.98}$$

for any $k \in \mathbb{Z}$ and $f \in L^{p_0}(\Omega)$ (see Lemma 4.17). Thanks to (4.98), we estimate

$$\begin{aligned} \|V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} &\le \|V\|_{L^{\frac{d}{2},\infty}(\Omega)}\|\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^{p_0,p}(\Omega)} \\ &\le C2^{2k}\|V\|_{L^{\frac{d}{2},\infty}(\Omega)}\|f\|_{L^p(\Omega)} \end{aligned} \tag{4.99}$$

for any $k \in \mathbb{Z}$, where p_0 is a real number with $1/p = 2/d + 1/p_0$. Hence, combining the estimates obtained now, we get

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \le C2^{-2(j-k)}\|f\|_{L^p(\Omega)}$$

for any $j, k \in \mathbb{Z}$. Therefore (4.89) is obtained by the density argument. In a similar way, we get (4.90). The proof of the case $d \ge 3$ is finished.

The case $d = 2$. We consider the case $d = 2$ and $p = 1$. We note from Lemma 4.18 that

$$\Phi_k(\sqrt{\mathcal{H}_0})f = \mathcal{H}_V^{-1}\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}_0})f \quad \text{in } \mathcal{Z}_V'(\Omega).$$

Thanks to the estimate (4.23) for $\alpha = 1$ and the assumption (4.83) on V, a formal calculation implies that

$$\begin{aligned} &\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)} \\ &= \|\phi_j(\sqrt{\mathcal{H}_V})\mathcal{H}_V^{-1}\mathcal{H}_V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)} \\ &\le C2^{-2j}\Big\{\|\mathcal{H}_0\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)} + \|V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)}\Big\} \\ &\le C2^{-2j}\Big\{2^{2k}\|f\|_{L^1(\Omega)} + \|V\|_{L^1(\Omega)}\|\phi_k(\sqrt{\mathcal{H}_0})f\|_{L^\infty(\Omega)}\Big\} \\ &\le C2^{-2j}2^{2k}\|f\|_{L^1(\Omega)}, \end{aligned}$$

which proves (4.89). As to the estimate (4.90), again by using (4.23) and the assumption (4.83) on V, we estimate

$$\begin{aligned} &\|\phi_k(\sqrt{\mathcal{H}_0})\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^1(\Omega)} \\ &= \|\phi_k(\sqrt{\mathcal{H}_0})\mathcal{H}_0^{-1}(\mathcal{H}_V - V)\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^1(\Omega)} \\ &\le C2^{-2k}\Big\{\|\mathcal{H}_V\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^1(\Omega)} + \|V\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^1(\Omega)}\Big\} \\ &\le C2^{-2k}\Big\{2^{2j}\|f\|_{L^1(\Omega)} + \|V\|_{L^1(\Omega)}\|\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^\infty(\Omega)}\Big\} \\ &\le C2^{-2k}2^{2j}\|f\|_{L^1(\Omega)}. \end{aligned}$$

This proves (4.90). Thus the estimate (i) for $d = 2$ and $p = 1$ is obtained. The proof of Lemma 4.19 is complete. □

Lemma 4.20. *Under the same assumptions as Lemma 4.19, the following assertions hold*:

(i) *Let* $1 \le p < \infty$ *and* $0 \le \alpha < \min\{2, d/p\}$. *Then*

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \le C2^{-\alpha(j-k)}\|f\|_{L^p(\Omega)}, \tag{4.100}$$

$$\|\phi_k(\sqrt{\mathcal{H}_0})\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \le C2^{-\alpha(k-j)}\|f\|_{L^p(\Omega)} \tag{4.101}$$

for any $j, k \in \mathbb{Z}$ *and* $f \in L^p(\Omega)$.

(ii) *Let* $1 < p \le \infty$ *and* $0 \le \alpha < \min\{2, d(1-1/p)\}$. *Then*

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \le C2^{-\alpha(k-j)}\|f\|_{L^p(\Omega)}, \tag{4.102}$$

$$\|\phi_k(\sqrt{\mathcal{H}_0})\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \le C2^{-\alpha(j-k)}\|f\|_{L^p(\Omega)} \tag{4.103}$$

for any $j, k \in \mathbb{Z}$ *and* $f \in L^p(\Omega)$.

Proof. The strategy of the proof is to apply the Riesz-Thorin interpolation theorem to the estimates in Lemma 4.19 and the following uniform estimates:

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)}, \tag{4.104}$$

$$\|\phi_k(\sqrt{\mathcal{H}_0})\Phi_j(\sqrt{\mathcal{H}_V})f\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)} \tag{4.105}$$

for any $j, k \in \mathbb{Z}$, which are proved by (4.23) for $\alpha = 0$.

Let $0 \le \alpha < \min\{2, d/p\}$. Then the proof of (4.100) for $1 \le p < d/2$ is performed by combining (4.89) and (4.104) with $q = p$. In fact, we estimate

$$\begin{aligned}
&\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} \\
&= \|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)}^{\frac{\alpha}{2}}\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)}^{1-\frac{\alpha}{2}} \\
&\le C\{2^{-2(j-k)}\}^{\frac{\alpha}{2}}\|f\|_{L^p(\Omega)} \\
&= C2^{-\alpha(j-k)}\|f\|_{L^p(\Omega)}.
\end{aligned} \tag{4.106}$$

This proves (4.100). In a similar way, by using (4.90) and (4.105), we get the estimate (4.101). When $d/2 \le p \le \infty$, we apply the Riesz-Thorin interpolation theorem to (4.104) with $q = \infty$ and the estimate (4.89) together with the argument (4.106).

Finally, estimates (4.102) and (4.103) are proved in analogous way as in (4.100) and (4.101), if we divide the cases into $d/(d-2) < p \le \infty$ and $1 \le p \le d/(d-2)$. The proof of Lemma 4.20 is complete. □

In what follows, we prove the isomorphism between $\dot{B}^s_{p,q}(\mathcal{H}_0)$ and $\dot{B}^s_{p,q}(\mathcal{H}_V)$ under the assumption on V in Theorem 4.15.

Proof of Theorem 4.15. First we prove the assertion (ii), i.e.,

$$\dot{B}^s_{p,q}(\mathcal{H}_V) \cong \dot{B}^s_{p,q}(\mathcal{H}_0). \tag{4.107}$$

The case $s > 0$. First we prove that

$$\dot{B}^s_{p,q}(\mathcal{H}_0) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}_V) \tag{4.108}$$

for any $s > 0$. To be more precise, for any $f \in \dot{B}^s_{p,q}(\mathcal{H}_0)$, we will regard f as an element of $\mathcal{Z}'_V(\Omega)$ by

$${}_{\mathcal{Z}'_V}\langle f, g\rangle_{\mathcal{Z}_V} = \sum_{j=-\infty}^{\infty} {}_{\mathcal{Z}'_V}\langle \phi_j(\sqrt{\mathcal{H}_0})f, g\rangle_{\mathcal{Z}_V}$$

and we will prove that

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} = \Big\| \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_0})f \Big\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} \le C\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}.$$

To begin with, for any $f \in \dot{B}^s_{p,q}(\mathcal{H}_0)$, we show that

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } \mathcal{Z}'_V(\Omega). \tag{4.109}$$

To see (4.109), we consider the formal identity

$${}_{\mathcal{Z}'_V}\langle f, g\rangle_{\mathcal{Z}_V} = \sum_{j=-\infty}^{\infty} {}_{\mathcal{Z}'_V}\langle f, \phi_j(\sqrt{\mathcal{H}_V})g\rangle_{\mathcal{Z}_V} = \sum_{j=-\infty}^{\infty} {}_{\mathcal{Z}'_V}\langle \phi_j(\sqrt{\mathcal{H}_V})f, g\rangle_{\mathcal{Z}_V}, \tag{4.110}$$

where the first identity is deduced from Lemma 4.5 (ii). Note that

$$f = \sum_{k=-\infty}^{\infty} \phi_k(\sqrt{\mathcal{H}_0})f \quad \text{in } \mathcal{Z}'_0(\Omega) \tag{4.111}$$

by Lemma 4.5 (ii). Plugging (4.111) into (4.110), we can write formally

$${}_{\mathcal{Z}'_V}\langle f, g\rangle_{\mathcal{Z}_V} = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} {}_{\mathcal{Z}'_V}\langle \phi_k(\sqrt{\mathcal{H}_0})f, \phi_j(\sqrt{\mathcal{H}_V})g\rangle_{\mathcal{Z}_V}.$$

Then it is sufficient to show that for any $g \in \mathcal{Z}_V(\Omega)$

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \big|\, {}_{\mathcal{Z}'_V}\langle \phi_k(\sqrt{\mathcal{H}_0})f, \phi_j(\sqrt{\mathcal{H}_V})g\rangle_{\mathcal{Z}_V}\big| \le C\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}\|g\|_{\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)}, \tag{4.112}$$

since

$$\mathcal{Z}_V(\Omega) \hookrightarrow \dot{B}^{-s}_{p',q'}(\mathcal{H}_V).$$

Let $\Phi_j := \phi_{j-1}+\phi_j+\phi_{j+1}$. By using $\phi_j = \phi_j\Phi_j$ and Hölder's inequality we estimate

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \big|\, {}_{\mathcal{Z}_V'}\langle \phi_k(\sqrt{\mathcal{H}_0})f, \phi_j(\sqrt{\mathcal{H}_V})g\rangle_{\mathcal{Z}_V}\big| \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \big|\, {}_{\mathcal{Z}_V'}\langle \phi_j(\sqrt{\mathcal{H}_V})\phi_k(\sqrt{\mathcal{H}_0})f, \Phi_j(\sqrt{\mathcal{H}_V})g\rangle_{\mathcal{Z}_V}\big| \\
&\le \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k=-\infty}^{\infty} \big\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})\phi_k(\sqrt{\mathcal{H}_0})f\big\|_{L^p(\Omega)}\Big)^q \Big\}^{\frac{1}{q}} \\
&\qquad \times \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{-sj}\big\|\Phi_j(\sqrt{\mathcal{H}_V})g\big\|_{L^{p'}}\Big)^{q'} \Big\}^{\frac{1}{q'}} \\
&=: I(s,f)\times II(s,g).
\end{aligned} \tag{4.113}$$

The estimate of the second factor $II(s,g)$ is an immediate consequence of the definition of norm of Besov spaces $\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)$, that is, we have

$$II(s,g) \le C\|g\|_{\dot{B}^{-s}_{p',q'}(\mathcal{H}_V)}. \tag{4.114}$$

As to the first factor $I(s,f)$, applying (4.100), we have, for any $j\in\mathbb{Z}$

$$\big\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})\phi_k(\sqrt{\mathcal{H}_0})f\big\|_{L^p(\Omega)} \le C \begin{cases} 2^{-\alpha(j-k)}\|\phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} & \text{if } k\le j, \\ \|\phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} & \text{if } k\ge j, \end{cases}$$

where α is a fixed constant such that $s<\alpha<\min\{2, d/p\}$. For the sake of simplicity, we put

$$a_k := \|\phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)}. \tag{4.115}$$

When $k\le j$, by using the above estimate, we estimate the first factor $I(s,f)$ in (4.113) as

$$\begin{aligned}
I(s,f) &\le C\Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj}\sum_{k=-\infty}^{j} 2^{-\alpha(j-k)}a_k\Big)^q\Big\}^{\frac{1}{q}} \\
&= C\Big\{ \sum_{j=-\infty}^{\infty} \Big(\sum_{k'=0}^{\infty} 2^{-(\alpha-s)k'}2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\sum_{k'=0}^{\infty} 2^{-(\alpha-s)k'}\Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{s(j-k')}a_{j-k'}\Big)^q\Big\}^{\frac{1}{q}} \\
&\le C\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)},
\end{aligned} \tag{4.116}$$

and when $k \geq j$, we have

$$\begin{aligned} I(s,f) &\leq C\Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k=j}^{\infty} a_k\Big)^q \Big\}^{\frac{1}{q}} \\ &= C\Big\{ \sum_{j=-\infty}^{\infty} \Big(\sum_{k'=-\infty}^{0} 2^{sk'} 2^{s(j-k')} a_{j-k'} \Big)^q \Big\}^{\frac{1}{q}} \\ &\leq C \sum_{k'=-\infty}^{0} 2^{sk'} \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{s(j-k')} a_{j-k'}\Big)^q \Big\}^{\frac{1}{q}} \\ &\leq C\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}. \end{aligned} \tag{4.117}$$

Summarizing (4.114)–(4.117), we conclude that the series (4.110) is absolutely convergent, and hence, the identity (4.109) is justified. Also, as a consequence of (4.116) and (4.117), we obtain

$$\begin{aligned} \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} &\leq \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k=-\infty}^{\infty} \big\|\phi_j(\sqrt{\mathcal{H}_V})\phi_k(\sqrt{\mathcal{H}_0})f\big\|_{L^p(\Omega)}\Big)^q \Big\}^{\frac{1}{q}} \\ &\leq C\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}. \end{aligned}$$

Therefore, the embedding (4.108) holds.

It is also possible to show the embedding

$$\dot{B}^s_{p,q}(\mathcal{H}_V) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}_0)$$

by the same argument as above, if we apply (4.101) instead of (4.100). The proof of isomorphism (4.107) for $s > 0$ is complete.

The case $s < 0$. In this case, the argument for $s > 0$ works well. The only difference is to obtain estimates corresponding to (4.116) and (4.117), so that we concentrate on proving that

$$\Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k=-\infty}^{\infty} \big\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})\phi_k(\sqrt{\mathcal{H}_0})f\big\|_{L^p(\Omega)}\Big)^q \Big\}^{\frac{1}{q}} \leq C\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}. \tag{4.118}$$

It follows from (4.100) that for any $j \in \mathbb{Z}$

$$\big\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})\phi_k(\sqrt{\mathcal{H}_0})f\big\|_{L^p(\Omega)} \leq C \begin{cases} \|\phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} & \text{if } k \leq j, \\ 2^{-\alpha(k-j)}\|\phi_k(\sqrt{\mathcal{H}_0})f\|_{L^p(\Omega)} & \text{if } k \geq j, \end{cases}$$

where α is a fixed constant such that $|s| < \alpha < \min\{2, d(1-1/p)\}$. Then, by using

the above estimate and recalling the definition (4.115) of a_k, we have for $k \le j$,

$$
\begin{aligned}
&\Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k=-\infty}^{j} \big\| \phi_j(\sqrt{\mathcal{H}_V}) \Phi_k(\sqrt{\mathcal{H}_0}) \phi_k(\sqrt{\mathcal{H}_0}) f \big\|_{L^p(\Omega)} \Big)^q \Big\}^{\frac{1}{q}} \\
\le & C \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k=-\infty}^{j} a_k \Big)^q \Big\}^{\frac{1}{q}} \\
= & C \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k'=0}^{\infty} a_{j-k'} \Big)^q \Big\}^{\frac{1}{q}} \\
= & C \Big\{ \sum_{j=-\infty}^{\infty} \Big(\sum_{k'=0}^{\infty} 2^{sk'} 2^{s(j-k')} a_{j-k'} \Big)^q \Big\}^{\frac{1}{q}} \\
\le & C \sum_{k'=0}^{\infty} 2^{sk} \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{s(j-k')} a_{j-k'} \Big)^q \Big\}^{\frac{1}{q}} \\
\le & C \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)},
\end{aligned}
$$

and in the case when $k \ge j$, we estimate

$$
\begin{aligned}
&\Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k=j}^{\infty} \big\| \phi_j(\sqrt{\mathcal{H}_V}) \Phi_k(\sqrt{\mathcal{H}_0}) \phi_k(\sqrt{\mathcal{H}_0}) f \big\|_{L^p(\Omega)} \Big)^q \Big\}^{\frac{1}{q}} \\
\le & C \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k=j}^{\infty} 2^{-\alpha(k-j)} a_k \Big)^q \Big\}^{\frac{1}{q}} \\
= & C \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{sj} \sum_{k'=-\infty}^{0} 2^{\alpha k'} a_{j-k'} \Big)^q \Big\}^{\frac{1}{q}} \\
= & C \Big\{ \sum_{j=-\infty}^{\infty} \Big(\sum_{k'=-\infty}^{0} 2^{(\alpha+s)k'} 2^{s(j-k')} a_{j-k'} \Big)^q \Big\}^{\frac{1}{q}} \\
\le & C \sum_{k'=-\infty}^{0} 2^{(\alpha+s)k'} \Big\{ \sum_{j=-\infty}^{\infty} \Big(2^{s(j-k')} a_{j-k'} \Big)^q \Big\}^{\frac{1}{q}} \\
\le & C \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}.
\end{aligned}
$$

Therefore, the estimate (4.118) is verified, and the proof of the isomorphism (4.107) for $s < 0$ is finished.

The case $s = 0$**.** In this case we have only to show the corresponding estimates to (4.118). Since $1 < p < \infty$, Lemma 4.20 implies that

$$
\begin{aligned}
\|\phi_j(\sqrt{\mathcal{H}_V}) \Phi_k(\sqrt{\mathcal{H}_0}) f\|_{L^p(\Omega)} &\le C 2^{-\alpha|j-k|} \|f\|_{L^p(\Omega)}, \\
\|\phi_k(\sqrt{\mathcal{H}_0}) \Phi_j(\sqrt{\mathcal{H}_V}) f\|_{L^p(\Omega)} &\le C 2^{-\alpha|j-k|} \|f\|_{L^p(\Omega)},
\end{aligned}
$$

where $0 < \alpha < \min\{2, d/p, d(1-1/p)\}$. Then it follows from Young's inequality that

$$\begin{aligned}
&\Big\{ \sum_{j=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} \big\| \phi_j(\sqrt{\mathcal{H}_V}) \Phi_k(\sqrt{\mathcal{H}_0}) \phi_k(\sqrt{\mathcal{H}_0}) f \big\|_{L^p(\Omega)} \Big)^q \Big\}^{\frac{1}{q}} \\
\le & C \Big\{ \sum_{j=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} 2^{-\alpha|j-k|} \big\| \phi_k(\sqrt{\mathcal{H}_0}) f \big\|_{L^p(\Omega)} \Big)^q \Big\}^{\frac{1}{q}} \\
\le & C \Big(\sum_{j=-\infty}^{\infty} 2^{-\alpha|j|} \Big) \Big\{ \sum_{k=-\infty}^{\infty} \big\| \phi_k(\sqrt{\mathcal{H}_0}) f \big\|_{L^p(\Omega)}^q \Big\}^{\frac{1}{q}} \\
\le & C \|f\|_{\dot{B}^0_{p,q}(\mathcal{H}_0)}.
\end{aligned}$$

Therefore, the case $s = 0$ also holds. Thus the proof of isomorphism (4.107) for homogeneous case is finished. □

Finally, let us prove the assertion (i), i.e.,

$$B^s_{p,q}(\mathcal{H}_V) \cong B^s_{p,q}(\mathcal{H}_0). \tag{4.119}$$

When assumption B is not imposed on V, which is the assumption on the inhomogeneous Besov spaces, the same estimates in Lemmas 4.17–4.20 also hold for $j, k \in \mathbb{N}$, since the proof is done analogously by applying (2.13) and (4.22) instead of (2.16) and (4.23), respectively. The proof of (4.119) is similar to the homogeneous case. The only difference is to handle potentials V to get the estimates as in Lemma 4.19. Hence let us prove only the estimates (4.89) in Lemma 4.19. We divide the proof into the two cases: $d \ge 3$ and $d = 1, 2$.

The case $d \ge 3$. We write

$$V = V_1 + V_2, \quad V_1 \in L^{\frac{d}{2},\infty}(\Omega), \quad V_2 \in L^\infty(\Omega).$$

To prove the estimates (4.89), it is sufficient to show that

$$\|V_1 \Phi_k(\sqrt{\mathcal{H}_0}) f\|_{L^p(\Omega)} \le C 2^{2k} \|V_1\|_{L^{\frac{n}{2},\infty}(\Omega)} \|f\|_{L^p(\Omega)}, \tag{4.120}$$

$$\|V_2 \Phi_k(\sqrt{\mathcal{H}_0}) f\|_{L^p(\Omega)} \le C 2^{2k} \|V_2\|_{L^\infty(\Omega)} \|f\|_{L^p(\Omega)} \tag{4.121}$$

for any $k \in \mathbb{N}$. The estimates (4.120) are obtained in the same way as (4.99). As to the estimates (4.121), we deduce from Lemma 4.1 that

$$\begin{aligned}
\|V_2 \Phi_k(\sqrt{\mathcal{H}_0}) f\|_{L^p(\Omega)} &\le 2^{2k} \|V_2\|_{L^\infty(\Omega)} \|\Phi_k(\sqrt{\mathcal{H}_0}) f\|_{L^p(\Omega)} \\
&\le C 2^{2k} \|V_2\|_{L^\infty(\Omega)} \|f\|_{L^p(\Omega)}
\end{aligned}$$

for any $k \in \mathbb{N}$. The proof of the estimates (4.89) in the case $d \ge 3$ is finished.

The case $d = 1, 2$**.** Under the assumption that $V \in K_d(\Omega)$, we prove the estimate (4.89) for $p = 1$, i.e.,

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)} \le C2^{-2(j-k)}\|f\|_{L^1(\Omega)} \tag{4.122}$$

for any $j, k \in \mathbb{N}$. Noting that $V \in K_d(\Omega)$, we see from (i) in Proposition 2.1 and (i) in Lemma 2.3 that

$$\mathcal{D}(\mathcal{H}_V) = \{f \in H_0^1(\Omega) : \mathcal{H}_V f \in L^2(\Omega)\}.$$

Hence we obtain the identity (4.95) by the same argument as Lemma 4.19. By the same argument as (4.93), (4.96) and (4.97) in the proof of Lemma 4.19, we have

$$\|\phi_j(\sqrt{\mathcal{H}_V})\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)} \le C2^{-2j}\big\{2^{2k}\|f\|_{L^1(\Omega)} + \|V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)}\big\}.$$

The second term in the right hand side of the above is estimated as

$$\begin{aligned}\|V\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)} &\le \|V(I+\mathcal{H}_0)^{-1}\|_{\mathscr{B}(L^1(\Omega))}\|(I+\mathcal{H}_0)\Phi_k(\sqrt{\mathcal{H}_0})f\|_{L^1(\Omega)}\\ &\le C\|V(I+\mathcal{H}_0)^{-1}\|_{\mathscr{B}(L^1(\Omega))}\|f\|_{L^1(\Omega)}.\end{aligned}$$

for any $k \in \mathbb{N}$. If

$$\|V(I+\mathcal{H}_0)^{-1}\|_{\mathscr{B}(L^1(\Omega))} < \infty, \tag{4.123}$$

then (4.122) is obtained, Hence let us concentrate on the proof of (4.123). We utilize (2.13) in Proposition 2.4, i.e., there exist $C > 0$ and $\omega \ge -\inf\sigma(\mathcal{H}_V)$ such that

$$0 \le e^{-t\mathcal{H}_0}(x,y) \le Ce^{\omega t}e^{2t\Delta}(x,y) = Ce^{\omega t}(8\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{8t}} \text{ a.e. } x, y \in \Omega,$$

for any $t > 0$. Let $M > \omega$ and $f \in L^1(\Omega) \cap L^2(\Omega)$. We see that

$$\begin{aligned}|(M+\mathcal{H}_0)^{-1}f(x)| &\le \int_0^\infty |e^{-Mt}e^{-t\mathcal{H}_V}f(x)|\,dt\\ &\le C\int_0^\infty e^{-Mt}e^{\omega t}e^{2t\Delta}|\tilde{f}(x)|dt\\ &= C(M-\omega-2\Delta)^{-1}|\tilde{f}(x)|\end{aligned}$$

for almost every $x \in \Omega$, where $\tilde{f}$ is the zero extension of f to $\mathbb{R}^d$. We also denote by $\tilde{V}$ the zero extension of V to $\mathbb{R}^d$. Since $\tilde{V} \in K_d(\mathbb{R}^d)$, it follows from Proposition A.2.3 by Simon [76] that

$$\|V(M+\mathcal{H}_0)^{-1}f\|_{L^1(\Omega)} \le C\|(M-\omega-2\Delta)^{-1}|\tilde{f}|\|_{L^1(\mathbb{R}^d)} \le C\|\tilde{f}\|_{L^1(\mathbb{R}^d)} = C\|f\|_{L^1(\Omega)},$$

which implies that (4.123). Therefore (4.122) is proved under the assumption that $V \in K_d(\Omega)$ with $d = 1, 2$. Thus we conclude (4.119). □

4.2.4 A lemma on convergence in Besov spaces

In this subsection, we discuss the convergence in Besov spaces, which is used in the latter part of proof of Theorem 5.1 in chapter 5.

Lemma 4.21. *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Suppose that the potential V satisfies assumptions* A *and* B. *Assume that $\{f_N\}_{N\in\mathbb{N}}$ is a bounded sequence in $\dot{B}^s_{p,q}(\mathcal{H}_V)$, and that there exists an $f \in \mathcal{X}'_V(\Omega)$ such that*

$$f_N \to f \quad \text{in } \mathcal{X}'_V(\Omega) \text{ as } N \to \infty. \tag{4.124}$$

Then $f \in \dot{B}^s_{p,q}(\mathcal{H}_V)$ and

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)} \le \liminf_{N\to\infty} \|f_N\|_{\dot{B}^s_{p,q}(\mathcal{H}_V)}. \tag{4.125}$$

Before going to the proof, let us give a remark on the idea of proof of the lemma. When $1 < p, q < \infty$, $\dot{B}^s_{p,q}(\mathcal{H}_V)$ are reflexive for any $s \in \mathbb{R}$. This fact and the limiting properties of the weak convergence imply the inequality (4.125). Otherwise, we need the pointwise convergence of $\phi_j(\sqrt{\mathcal{H}_V})f_N$, which is obtained directly with a property of the kernel $\phi(\mathcal{H}_V)(x, y)$ of the operator $\phi(\mathcal{H}_V)$. Let us investigate the property of the kernel.

Lemma 4.22. *Let $1 \le p \le \infty$ and T be a bounded linear operator from $L^p(\Omega)$ to $L^\infty(\Omega)$, and $T(x, y)$ the kernel of T. Then*

$$\|T\|_{\mathscr{B}(L^p(\Omega),L^\infty(\Omega))} = \sup_{x\in\Omega} \|T(x,\cdot)\|_{L^{p'}(\Omega)},$$

where p' is the conjugate exponent of p.

Proof. We have:

$$\|T\|_{\mathscr{B}(L^p(\Omega),L^\infty(\Omega))} \le \sup_{x\in\Omega} \|T(x,\cdot)\|_{L^{p'}(\Omega)} \tag{4.126}$$

for any $1 \le p \le \infty$. In fact, let $f \in L^p(\Omega)$. Then it follows from Hölder's inequality that

$$\begin{aligned} |Tf(x)| &= \Big| \int_\Omega T(x,y) f(y)\, dy \Big| \\ &\le \|T(x,\cdot)\|_{L^{p'}(\Omega)} \|f\|_{L^p(\Omega)} \end{aligned}$$

for a.e. $x \in \Omega$. Hence we obtain

$$\|Tf\|_{L^\infty(\Omega)} \le \sup_{x\in\Omega} \|T(x,\cdot)\|_{L^{p'}(\Omega)} \|f\|_{L^p(\Omega)},$$

which implies (4.126). Therefore it suffices to prove the converse:

$$\|T(\cdot,\cdot)\|_{L^\infty(\Omega;L^{p'}(\Omega))} \le \|T\|_{\mathscr{B}(L^p(\Omega),L^\infty(\Omega))} \tag{4.127}$$

for any $1 \le p \le \infty$. When $1 \le p < \infty$, we estimate

$$\begin{aligned}
\|T(x,\cdot)\|_{L^{p'}(\Omega)} &= \sup_{f\in L^p(\Omega),\, \|f\|_{L^p(\Omega)}=1} \Big| \int_\Omega T(x,y) f(y)\, dy \Big| \\
&= \sup_{f\in L^p(\Omega),\, \|f\|_{L^p(\Omega)}=1} \big| Tf(x) \big| \\
&\le \sup_{f\in L^p(\Omega),\, \|f\|_{L^p(\Omega)}=1} \|T\|_{\mathscr{B}(L^p(\Omega),L^\infty(\Omega))} \|f\|_{L^p(\Omega)} \\
&\le \|T\|_{\mathscr{B}(L^p(\Omega),L^\infty(\Omega))}
\end{aligned}$$

for any $x \in \Omega$. This proves (4.127) for $1 \le p < \infty$. When $p = \infty$, fixing $x_0 \in \Omega$, we estimate

$$\begin{aligned}
\|T(x_0,\cdot)\|_{L^1(\Omega)} &= \int_\Omega |T(x_0,y)|\, dy \\
&= \int_\Omega T(x_0,y) e^{-i \arg\{T(x_0,y)\}}\, dy \\
&\le \sup_{x\in\Omega} \Big| \int_\Omega T(x,y) e^{-i \arg\{T(x_0,y)\}}\, dy \Big| \\
&= \sup_{x\in\Omega} \big| T e^{-i \arg\{T(x_0,\cdot)\}}(x) \big| \\
&\le \|T\|_{\mathscr{B}(L^\infty(\Omega))} \big\| e^{-i \arg\{T(x_0,\cdot)\}} \big\|_{L^\infty(\Omega)} \\
&= \|T\|_{\mathscr{B}(L^\infty(\Omega))},
\end{aligned}$$

which proves (4.127) for $p = \infty$. The proof of Lemma 4.22 is finished. □

The following lemma states that the kernel of $\phi(\mathcal{H}_V)$ belongs to $\mathcal{X}_V(\Omega)$. More precisely, we have:

Lemma 4.23. *Let Ω be an open set of $\mathbb{R}^d$. Suppose that the potential V satisfies assumption* A. *Then for any $\phi \in \mathscr{S}(\mathbb{R})$, we have*

$$\phi(\mathcal{H}_V)(x,\cdot) = \phi(\mathcal{H}_V)(\cdot,x) \in \mathcal{X}_V(\Omega) \quad \textit{for each } x \in \Omega. \tag{4.128}$$

Proof. Note from Lemma 4.22 that

$$\sup_{x\in\Omega} \|\phi(\mathcal{H}_V)(x,\cdot)\|_{L^{p'}(\Omega)} = \|\phi(\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega),L^\infty(\Omega))}$$

for any $1 \le p \le \infty$, where p' is the conjugate exponent of p. Hence, since

$$\|\phi(\mathcal{H}_V)\|_{\mathscr{B}(L^p(\Omega),L^\infty(\Omega))} < \infty$$

for any $1 \le p \le \infty$ by Lemma 4.1, we have

$$\phi(\mathcal{H}_V)(x,\cdot) \in L^{p'}(\Omega) \quad \text{for each } x \in \Omega. \tag{4.129}$$

In particular, we have

$$\mathcal{H}_V^M(\phi(\mathcal{H}_V)(x,\cdot)) \in \mathcal{X}_V'(\Omega)$$

for any $M \in \mathbb{N}$, since $L^{p'}(\Omega) \hookrightarrow \mathcal{X}_V'(\Omega)$ and $\mathcal{H}_V^M$ maps $\mathcal{X}_V'(\Omega)$ to itself. We denote by $K_{\mathcal{H}_V^M\phi(\mathcal{H}_V)}(x,y)$ the kernel of $\mathcal{H}_V^M\phi(\mathcal{H}_V)$. Then, for any $f \in \mathcal{X}_V(\Omega)$, we have

$$\begin{aligned}
{}_{\mathcal{X}_V'(\Omega)}\langle \mathcal{H}_V^M(\phi(\mathcal{H}_V)(x,\cdot)), f\rangle_{\mathcal{X}_V(\Omega)} &= {}_{\mathcal{X}_V'(\Omega)}\langle \phi(\mathcal{H}_V)(x,\cdot), \mathcal{H}_V^M f\rangle_{\mathcal{X}_V(\Omega)} \\
&= \phi(\mathcal{H}_V)\mathcal{H}_V^M\overline{f}(x) \\
&= \mathcal{H}_V^M\phi(\mathcal{H}_V)\overline{f}(x) \\
&= {}_{\mathcal{X}_V'(\Omega)}\langle K_{\mathcal{H}_V^M\phi(\mathcal{H}_V)}(x,\cdot), f\rangle_{\mathcal{X}_V(\Omega)}
\end{aligned}$$

for any $x \in \Omega$, which implies that

$$\mathcal{H}_V^M(\phi(\mathcal{H}_V)(x,\cdot))(y) = K_{\mathcal{H}_V^M\phi(\mathcal{H}_V)}(x,y) \quad \text{a.e. } y \in \Omega$$

for any $x \in \Omega$. Since $\lambda^M\phi(\lambda) \in \mathscr{S}(\mathbb{R})$ for any $M \in \mathbb{N}$, it follows from (4.129) for $p' = 1$ and $p' = 2$ that

$$K_{\mathcal{H}_V^M\phi(\mathcal{H}_V)}(x,\cdot) \in L^1(\Omega) \cap L^2(\Omega)$$

for any $M \in \mathbb{N}$ and $x \in \Omega$. Hence we obtain

$$\mathcal{H}_V^M(\phi(\mathcal{H}_V)(x,\cdot)) \in L^1(\Omega) \cap L^2(\Omega)$$

for any $M \in \mathbb{N}$ and $x \in \Omega$. Thus we conclude (4.128). The proof of Lemma 4.23 is finished. □

We are now in a position to prove Lemma 4.21.

Proof of Lemma 4.21. First, we show that

$$\phi_j(\sqrt{\mathcal{H}_V})f_N(x) \to \phi_j(\sqrt{\mathcal{H}_V})f(x) \quad \text{a.e. } x \in \Omega \text{ as } N \to \infty \tag{4.130}$$

for each $j \in \mathbb{Z}$. Put $\Phi_j = \phi_{j-1} + \phi_j + \phi_{j+1}$ for $j \in \mathbb{Z}$. Then, noting from the assertion (i) in Lemma 4.5 that

$$\Phi_j(\sqrt{\mathcal{H}_V})f_N \in L^\infty(\Omega),$$

and from Lemma 4.23 that

$$\phi_j(\sqrt{\mathcal{H}_V})(x,\cdot) \in \mathcal{X}_V(\Omega) \quad \text{for each } x \in \Omega,$$

we write

$$\begin{aligned}
\phi_j(\sqrt{\mathcal{H}_V})f_N(x) &= \phi_j(\sqrt{\mathcal{H}_V})\Phi_j(\sqrt{\mathcal{H}_V})f_N(x) \\
&= {}_{\mathcal{X}_V'(\Omega)}\langle \Phi_j(\sqrt{\mathcal{H}_V})f_N, \phi_j(\sqrt{\mathcal{H}_V})(x,\cdot)\rangle_{\mathcal{X}_V(\Omega)}
\end{aligned} \tag{4.131}$$

for each $j \in \mathbb{Z}$ and $x \in \Omega$. In a similar way, we have

$$\phi_j(\sqrt{\mathcal{H}_V})f(x) = {}_{\mathcal{X}_V'(\Omega)}\langle \Phi_j(\sqrt{\mathcal{H}_V})f, \phi_j(\sqrt{\mathcal{H}_V})(x,\cdot)\rangle_{\mathcal{X}_V(\Omega)} \tag{4.132}$$

for each $j \in \mathbb{Z}$ and $x \in \Omega$. Since

$$\Phi_j(\sqrt{\mathcal{H}_V})f_N \to \Phi_j(\sqrt{\mathcal{H}_V})f \quad \text{in } \mathcal{X}_V'(\Omega) \text{ as } N \to \infty$$

for each $j \in \mathbb{Z}$ by assumption (4.124) and the continuity of $\Phi_j(\sqrt{\mathcal{H}_V})$ from $\mathcal{X}_V'(\Omega)$ into itself, we deduce that

$$\begin{aligned} &{}_{\mathcal{X}_V'(\Omega)}\langle \Phi_j(\sqrt{\mathcal{H}_V})f_N, \phi_j(\sqrt{\mathcal{H}_V})(x,\cdot)\rangle_{\mathcal{X}_V(\Omega)} \\ &\qquad \to {}_{\mathcal{X}_V'(\Omega)}\langle \Phi_j(\sqrt{\mathcal{H}_V})f, \phi_j(\sqrt{\mathcal{H}_V})(x,\cdot)\rangle_{\mathcal{X}_V(\Omega)} \end{aligned} \tag{4.133}$$

for each $j \in \mathbb{Z}$ and $x \in \Omega$ as $N \to \infty$. Hence, combining (4.131) and (4.132) with (4.133), we get the pointwise convergence (4.130).

Let us turn to the proof of the inequality (4.125). To begin with, given $1 \le p \le \infty$, we claim that

$$\|\phi_j(\sqrt{\mathcal{H}_V})f\|_{L^p(\Omega)} \le \liminf_{N\to\infty} \|\phi_j(\sqrt{\mathcal{H}_V})f_N\|_{L^p(\Omega)} \tag{4.134}$$

for each $j \in \mathbb{Z}$. When $1 \le p < \infty$, the inequality (4.134) is a consequence of (4.130) and Fatou's lemma. We have to prove the case when $p = \infty$. In this case, thanks to (4.130), the inequality (4.134) is true for $p = \infty$, since $\{\phi_j(\sqrt{\mathcal{H}_V})f_N\}_{N\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\Omega)$. Finally, multiplying by 2^{sj} to the both sides of (4.134), we conclude the required inequality (4.125). The proof of Lemma 4.21 is finished. □

Chapter 5

Bilinear estimates

The bilinear estimates in Sobolev spaces or Besov spaces are of great importance to study the well-posedness for the Cauchy problem to nonlinear partial differential equations. In this chapter we study the bilinear estimates in Besov spaces, which were proved in Iwabuchi, Matsuyama and Taniguchi [42]. These estimates are also called the fractional Leibniz rule or the Kato-Ponce inequality. The basis of proving the bilinear estimates is to use frequency decomposition called the Bony paraproduct formula (see Bony [4]) and the boundedness of Fourier multipliers (see Bourgain and Li [5], D'Ancona [17], Fujiwara, Georgiev and Ozawa [24], Grafakos and Oh [31] and references therein).

Our goal is to prove the bilinear estimates in Besov spaces generated by the Dirichlet Laplacian $\mathcal{H}_0$. It will be revealed that the bilinear estimates hold in the Besov spaces generated by $\mathcal{H}_0$ for small regularity number, and it is possible to construct a counter-example for high regularity. These estimates are proved by using the gradient estimates for heat semigroup together with the Bony paraproduct formula and the boundedness of spectral multipliers. As a by-product, we obtain these estimates in Besov spaces generated by Schrödinger operators $\mathcal{H}_V$ with potentials such that

$$B^s_{p,q}(\mathcal{H}_V) \cong B^s_{p,q}(\mathcal{H}_0) \quad \text{or} \quad \dot{B}^s_{p,q}(\mathcal{H}_V) \cong \dot{B}^s_{p,q}(\mathcal{H}_0).$$

In this chapter we always assume that Ω is a domain of $\mathbb{R}^d$ for the technical reason.

5.1 Bilinear estimates in Besov spaces

Let Ω be a domain such that the following gradient estimate

$$\|\nabla e^{-t\mathcal{H}_0}\|_{\mathscr{B}(L^\infty(\Omega))} \le Ct^{-\frac{1}{2}} \tag{5.1}$$

holds either for any $t \in (0,1]$ or for any $t > 0$, where $\{e^{-t\mathcal{H}_0}\}_{t>0}$ is the semigroup generated by $\mathcal{H}_0$.

We shall prove here the following:

Theorem 5.1. *Let $0 < s < 2$ and p, p_1, p_2, p_3, p_4 and q be such that*

$$1 \le p, p_1, p_2, p_3, p_4, q \le \infty \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then the following assertions hold:

(i) *Let Ω be a domain of $\mathbb{R}^d$ such that (5.1) holds for any $t \in (0, 1]$. Then there exists a constant $C > 0$ such that*

$$\|fg\|_{B^s_{p,q}(\mathcal{H}_0)} \le C\big(\|f\|_{B^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)} + \|f\|_{L^{p_3}(\Omega)}\|g\|_{B^s_{p_4,q}(\mathcal{H}_0)}\big) \tag{5.2}$$

for any $f \in B^s_{p_1,q}(\mathcal{H}_0) \cap L^{p_3}(\Omega)$ and $g \in B^s_{p_4,q}(\mathcal{H}_0) \cap L^{p_2}(\Omega)$.

(ii) *Let Ω be a domain of $\mathbb{R}^d$ such that (5.1) holds for any $t > 0$. Then there exists a constant $C > 0$ such that*

$$\|fg\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} \le C\big(\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)} + \|f\|_{L^{p_3}(\Omega)}\|g\|_{\dot{B}^s_{p_4,q}(\mathcal{H}_0)}\big) \tag{5.3}$$

for any $f \in \dot{B}^s_{p_1,q}(\mathcal{H}_0) \cap L^{p_3}(\Omega)$ and $g \in \dot{B}^s_{p_4,q}(\mathcal{H}_0) \cap L^{p_2}(\Omega)$.

Let us give two remarks; the first one is concerned with the regularity number s such that the bilinear estimates hold, and the second is about necessity of the assumption on the gradient estimate (5.1). As is well known, in the case when Ω is the whole space $\mathbb{R}^d$, one does not need to impose any restriction on the regularity number $s > 0$ of Besov spaces. However, when we consider these estimates for functions whose regularity is measured by the Dirichlet Laplacian $\mathcal{H}_0$ on domains, a restriction is required on the regularity. In fact, it is possible to construct a counter-example for high regularity (see section 5.2). This is because $\mathcal{H}_0(fg)$ does not necessarily belong to $\mathcal{D}(\mathcal{H}_0)$ even if f and g belong to $\mathcal{D}(\mathcal{H}_0^2)$. This can be seen from the following observation: Let Ω be a domain with smooth boundary. Applying the Leibniz rule to $\mathcal{H}_0(fg)$, we are confronted with the term $\nabla f \cdot \nabla g$ which does not belong to $\mathcal{D}(\mathcal{H}_0)$, since it does not in general vanish on the boundary. Here, we refer to Iwabuchi [39] in which the one dimensional differential operator ∂_x maps functions involved with the Dirichlet boundary condition into those with the Neumann one, and vice versa. Hence, in general, it is impossible to get the estimates in high regularity.

As to the second remark, as far as our proof of main theorem is concerned, we need to estimate the derivative of functions. Therefore, the gradient estimates for heat semigroup in L^∞ or even L^p are required.

When Ω is the whole space $\mathbb{R}^d$ or the half space $\mathbb{R}^d_+$ with $d \ge 1$, we observe from the explicit representation formula of the heat kernels that the estimate (5.1) holds for any $t > 0$. Let us give examples of domains such that (5.1) holds, and other examples of domains where the bilinear estimates still hold for p in some restricted ranges.

(i) When Ω is a domain with uniform $C^{2,\alpha}$-boundary for some $\alpha \in (0,1)$, (5.1) holds for any $t \in (0,1]$ (see Fornaro, Metafune and Priola [22]). Hence, the bilinear estimate (5.2) in Theorem 5.1 holds in such a domain. In particular, when Ω is bounded, (5.1) holds for any $t > 0$, since the infimum of the spectrum is strictly positive. Hence, the bilinear estimate (5.3) in Theorem 5.1 holds.

(ii) Let Ω be an open set in $\mathbb{R}^d$. Then there exists an exponent $p_0 = p_0(\Omega) \in [2,\infty]$ depending on Ω such that if $p \in [1,p_0]$, then

$$\|\nabla e^{-t\mathcal{H}_0}\|_{\mathscr{B}(L^p(\Omega))} \le Ct^{-\frac{1}{2}}, \quad t > 0. \tag{5.4}$$

Here we note that (5.4) was proved for $p \in [1,2]$ in Theorem 3.2. In this case, it should be mentioned that we can prove the estimates (5.2) and (5.3) for $1 \le p, p_1, p_2, p_3, p_4 \le p_0$ by performing some trivial modifications of the proof of Theorem 5.1.

Finally, let us mention some domains and the range of p such that (5.4) holds.

(a) Let $d \ge 3$. Assume that Ω is the exterior domain of a compact set with $C^{1,1}$-boundary. Then (5.4) holds for any $p \in [1,d]$ (see Theorem 6.1 in chapter 6). In this case we may take $p_0 = p_0(\Omega) = d$.

We are able to take domains and p such that the Riesz transform is bounded, namely, L^p-boundedness of $\nabla \mathcal{H}_0^{-\frac{1}{2}}$ implies the gradient estimate:

$$\|\nabla e^{-t\mathcal{H}_0} f\|_{L^p(\Omega)} = t^{-\frac{1}{2}} \|\nabla \mathcal{H}_0^{-\frac{1}{2}} (t\mathcal{H}_0)^{\frac{1}{2}} e^{-t\mathcal{H}_0} f\|_{L^p(\Omega)} \le Ct^{-\frac{1}{2}} \|f\|_{L^p(\Omega)}$$

for $t > 0$. Hence, the following results are immediate consequences of (5.4) with $p = 1$ and L^p-boundedness of the Riesz transform for some $p = p_0$ in [15, 46] (see also [74, 75, 87]).

(b) Let $d \ge 2$. If Ω is a bounded domain with C^1-boundary, then (5.4) holds for any $p \in [1,\infty)$. In this case we may take p_0 as any finite number.

(c) Let $d \ge 2$. If Ω is a bounded and Lipschitz domain, then (5.4) holds for any $p \in [1,p_0]$, where $p_0 = 3$ for $d \ge 3$ and $p_0 = 4$ for $d = 2$.

As a consequence of Theorems 4.15 and 5.1, we have the bilinear estimates in the case of Schrödinger operators.

Corollary 5.2. *Let p, p_1, p_2, p_3, p_4 and q be such that*

$$1 \le p, p_1, p_2, p_3, p_4, q \le \infty \quad \textit{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

and let s be such that

$$0 < s < \min\left\{\frac{d}{p_1}, \frac{d}{p_4}, 2\right\} \quad \text{if } d \geq 3; \quad 0 < s < \min\left\{\frac{2}{p_1}, \frac{2}{p_4}\right\} \quad \text{if } d = 1, 2.$$

Then, under the same assumption on V in Theorem 4.15, the assertions (i) *and* (ii) *in Theorem 5.1 hold for $B^s_{p,q}(\mathcal{H}_V)$ and $\dot{B}^s_{p,q}(\mathcal{H}_V)$, respectively.*

In the rest of this section, let us give a proof of Theorem 5.1. For this purpose, we prepare five lemmas.

Based on Lemma 4.1, we have:

Lemma 5.3. *Let Ω be an open set of $\mathbb{R}^d$. Then for any $1 \leq p \leq \infty$ and $\alpha \geq 0$ there exists a constant $C > 0$ such that*

$$\Big\| \mathcal{H}_0^\alpha \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}_0}) \Big\|_{\mathscr{B}(L^p(\Omega))} \leq C 2^{2\alpha j} \tag{5.5}$$

for any $j \in \mathbb{Z}$.

Proof. When $\alpha > 0$, the estimate (5.5) follows from the estimate (4.23). In fact, we estimate

$$\begin{aligned} \Big\| \mathcal{H}_0^\alpha \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}_0}) \Big\|_{\mathscr{B}(L^p(\Omega))} &\leq \sum_{k=-\infty}^{j} \| \mathcal{H}_0^\alpha \phi_k(\sqrt{\mathcal{H}_0}) \|_{\mathscr{B}(L^p(\Omega))} \\ &\leq C \sum_{k=-\infty}^{j} 2^{2\alpha k} \\ &\leq C 2^{2\alpha j}. \end{aligned}$$

Let us now prove the case when $\alpha = 0$. It follows from the identities (4.36) and (4.40) that

$$\sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}_0}) f = \psi(2^{-2j}\mathcal{H}_0) f \quad \text{in } L^2(\Omega)$$

for any $j \in \mathbb{Z}$ and $f \in L^2(\Omega)$, which implies that

$$\Big\| \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}_0}) g \Big\|_{L^p(\Omega)} = \big\| \psi(2^{-2j}\mathcal{H}_0) g \big\|_{L^p(\Omega)} \leq C \|g\|_{L^p(\Omega)}$$

for any $j \in \mathbb{Z}$ and $g \in L^p(\Omega) \cap L^2(\Omega)$. Thus, when $1 \leq p < \infty$, the estimate (5.5) for $\alpha = 0$ is proved by the density argument, and the case $p = \infty$ is obtained from L^1-estimate by the duality argument. Thus the estimate (5.5) for $\alpha = 0$ is proved. The proof of Lemma 5.3 is finished. □

Based on Theorem 3.2 and the gradient estimate (5.1), we have:

Lemma 5.4. *Let $1 \le p \le \infty$. Then the following assertions hold:*

(i) *Assume that Ω is an open set of $\mathbb{R}^d$ such that (5.1) holds for any $t \in (0,1]$. Then for any $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$\|\nabla \mathcal{H}_0^m \psi(2^{-2j}\mathcal{H}_0)\|_{\mathscr{B}(L^p(\Omega))} \le C 2^{(2m+1)j}, \tag{5.6}$$

$$\|\nabla \mathcal{H}_0^\alpha \phi_j(\sqrt{\mathcal{H}_0})\|_{\mathscr{B}(L^p(\Omega))} \le C 2^{(2\alpha+1)j} \tag{5.7}$$

for any $j \in \mathbb{N}$.

(ii) *Assume that Ω is an open set of $\mathbb{R}^d$ such that (5.1) holds for any $t > 0$. Then the estimates (5.6) and (5.7) hold for any $j \in \mathbb{Z}$. Furthermore, for any $\alpha \ge 0$ there exists a constant $C > 0$ such that*

$$\Big\|\nabla \mathcal{H}_0^\alpha \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}_0})\Big\|_{\mathscr{B}(L^p(\Omega))} \le C 2^{(2\alpha+1)j} \tag{5.8}$$

for any $j \in \mathbb{Z}$.

Proof. We prove the assertion (i). The case $p = 1$ is an immediate consequence of Theorem 3.2 for $\theta = 2^{-2j}$, since

$$\lambda^m \psi(\lambda) \in C_0^\infty(\mathbb{R}), \quad \lambda^\alpha \phi_0(\sqrt{\lambda}) \in C_0^\infty((0,\infty)).$$

Hence it suffices to show the case $p = \infty$. In fact, once the case $p = \infty$ is proved, the Riesz-Thorin interpolation theorem allows us to conclude the estimates (5.6) and (5.7) for any $1 \le p \le \infty$.

Let $f \in L^\infty(\Omega)$. Then it follows from the estimate (5.1) for $0 < t \le 1$ that

$$\begin{aligned}
\big\|\nabla \mathcal{H}_0^m \psi(2^{-2j}\mathcal{H}_0) f\big\|_{L^\infty(\Omega)} &= \big\|\nabla e^{-2^{-2j}\mathcal{H}_0} e^{2^{-2j}\mathcal{H}_0} \mathcal{H}_0^m \psi(2^{-2j}\mathcal{H}_0) f\big\|_{L^\infty(\Omega)} \\
&\le C 2^j \big\|e^{2^{-2j}\mathcal{H}_0} \mathcal{H}_0^m \psi(2^{-2j}\mathcal{H}_0) f\big\|_{L^\infty(\Omega)} \\
&= C 2^{(2m+1)j} \big\|e^{2^{-2j}\mathcal{H}_0} (2^{-2j}\mathcal{H}_0)^m \psi(2^{-2j}\mathcal{H}_0) f\big\|_{L^\infty(\Omega)}
\end{aligned} \tag{5.9}$$

for any $j \in \mathbb{N}$. Since

$$e^\lambda \lambda^m \psi(\lambda) \in C_0^\infty(\mathbb{R}),$$

it follows from the estimate (3.1) for $p = \infty$ in Theorem 3.1 that

$$\big\|e^{2^{-2j}\mathcal{H}_0} (2^{-2j}\mathcal{H}_0)^m \psi(2^{-2j}\mathcal{H}_0) f\big\|_{L^\infty(\Omega)} \le C \|f\|_{L^\infty(\Omega)}. \tag{5.10}$$

Thus the required estimate (5.6) for $p = \infty$ is an immediate consequence of (5.9) and (5.10). In a similar way, we get (5.7). Thus the assertion (i) is proved.

Next we prove the assertion (ii). We can prove the estimates (5.6) and (5.7) for any $j \in \mathbb{Z}$ in the same way as (i). Furthermore, the estimate (5.8) is proved by using (5.7) in the same way as the proof of (5.5) for $\alpha > 0$ in Lemma 5.3. Hence we may omit the details. The proof of Lemma 5.4 is finished. □

The following lemma is about the approximation of the identity for functions in $L^p(\Omega)$.

Lemma 5.5. *Let $1 \le p < \infty$. Then for any $f \in L^p(\Omega)$, we have*

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_0})f \quad \text{in } \mathcal{X}_0'(\Omega). \tag{5.11}$$

Proof. Since $L^2(\Omega) \hookrightarrow \mathcal{X}_0'(\Omega)$, the identity (5.11) holds for any $f \in L^p(\Omega) \cap L^2(\Omega)$. Then the identity (5.11) holds for any $f \in L^p(\Omega)$ by the density argument, since $1 \le p < \infty$. The proof of Lemma 5.5 is finished. □

The following lemma states the Leibniz rule for the Dirichlet Laplacian.

Lemma 5.6. *Assume that Ω is an open set of $\mathbb{R}^d$ such that (5.1) holds for any $t \in (0,1]$. Let $\Phi, \Psi \in C_0^\infty(\mathbb{R})$. Then for any $f, g \in \mathcal{X}_0'(\Omega)$, we have*

$$\begin{aligned}\mathcal{H}_0\big(\Phi(\mathcal{H}_0)f \cdot \Psi(\mathcal{H}_0)g\big) &= \mathcal{H}_0\Phi(\mathcal{H}_0)f \cdot \Psi(\mathcal{H}_0)g \\ &\quad - 2\nabla\Phi(\mathcal{H}_0)f \cdot \nabla\Psi(\mathcal{H}_0)g + \Phi(\mathcal{H}_0)f \cdot \mathcal{H}_0\Psi(\mathcal{H}_0)g \quad \text{in } \mathcal{X}_0'(\Omega).\end{aligned} \tag{5.12}$$

Proof. To begin with, we note from Lemma 4.5 that $\Phi(\mathcal{H}_0)f$ and $\Psi(\mathcal{H}_0)g$ are regarded as elements in $L^\infty(\Omega)$:

$$\Phi(\mathcal{H}_0)f,\ \Psi(\mathcal{H}_0)g \in L^\infty(\Omega). \tag{5.13}$$

Noting that the assumption (5.1) is necessary for Lemma 5.4, we apply Lemmas 5.3 and 5.4 for $p = \infty$. Then we see that

$$\mathcal{H}_0\Phi(\mathcal{H}_0)f,\ \mathcal{H}_0\Psi(\mathcal{H}_0)g,\ \nabla\Phi(\mathcal{H}_0)f,\ \nabla\Psi(\mathcal{H}_0)g \in L^\infty(\Omega). \tag{5.14}$$

Hence, all terms on the right hand side of (5.12) belong to $L^\infty(\Omega)$. Therefore, it suffices to show that (5.12) holds in $\mathscr{D}'(\Omega)$, where $\mathscr{D}'(\Omega)$ is the space consisting of distributions on Ω, i.e., the dual space of $\mathscr{D}(\Omega)$. In fact, if (5.12) holds in $\mathscr{D}'(\Omega)$, then (5.12) holds almost everywhere on Ω. Thus we conclude that (5.12) holds in $\mathcal{X}_0'(\Omega)$.

Since

$$\mathcal{H}_0 h = -\Delta h \quad \text{for } h \in \mathscr{D}(\Omega),$$

we write, by using (5.13),

$${}_{\mathscr{D}'(\Omega)}\langle \mathcal{H}_0\big(\Phi(\mathcal{H}_0)f \cdot \Psi(\mathcal{H}_0)g\big), h\rangle_{\mathscr{D}(\Omega)} = {}_{L^\infty(\Omega)}\langle \Psi(\mathcal{H}_0)g, \overline{\Phi(\mathcal{H}_0)f}(-\Delta h)\rangle_{L^1(\Omega)} \tag{5.15}$$

for any $h \in \mathscr{D}(\Omega)$. Here, noting that

$$-\Delta\Phi(\mathcal{H}_0)f = \mathcal{H}_0\Phi(\mathcal{H}_0)f \quad \text{in } \mathscr{D}'(\Omega),$$

we observe from the Leibniz rule that

$$\begin{aligned}&\overline{\Phi(\mathcal{H}_0)f}(-\Delta h)\\&\qquad=-\Delta(\overline{\Phi(\mathcal{H}_0)f}\cdot h)-(\overline{\mathcal{H}_0\Phi(\mathcal{H}_0)f})h+2\overline{\nabla\Phi(\mathcal{H}_0)f}\cdot\nabla h\quad\text{in }\mathscr{D}'(\Omega).\end{aligned}\tag{5.16}$$

Since all the terms in (5.16) belong to $L^1(\Omega)$ by (5.13) and (5.14), multiplying (5.16) by $\Psi(\mathcal{H}_0)g$, and using (5.15), we write

$$\begin{aligned}&{}_{\mathscr{D}'(\Omega)}\langle\mathcal{H}_0\big(\Phi(\mathcal{H}_0)f\cdot\Psi(\mathcal{H}_0)g\big),h\rangle_{\mathscr{D}(\Omega)}\\&={}_{L^\infty(\Omega)}\langle\Psi(\mathcal{H}_0)g,-\Delta(\overline{\Phi(\mathcal{H}_0)f}\cdot h)\rangle_{L^1(\Omega)}\\&\quad-{}_{L^\infty(\Omega)}\langle(\mathcal{H}_0\Phi(\mathcal{H}_0)f)\Psi(\mathcal{H}_0)g,h\rangle_{L^1(\Omega)}+2{}_{L^\infty(\Omega)}\langle\Psi(\mathcal{H}_0)g,\overline{\nabla\Phi(\mathcal{H}_0)f}\cdot\nabla h\rangle_{L^1(\Omega)}.\end{aligned}\tag{5.17}$$

As to the first term in the right member of (5.17), integrating by parts, we get

$${}_{L^\infty(\Omega)}\langle\Psi(\mathcal{H}_0)g,-\Delta(\overline{\Phi(\mathcal{H}_0)f}\cdot h)\rangle_{L^1(\Omega)}={}_{L^\infty(\Omega)}\langle-\Delta\Psi(\mathcal{H}_0)g,\overline{\Phi(\mathcal{H}_0)f}\cdot h\rangle_{L^1(\Omega)}.$$

Here, we note that

$$-\Delta\Psi(\mathcal{H}_0)g=\mathcal{H}_0\Psi(\mathcal{H}_0)g\quad\text{in }\mathscr{D}'(\Omega).\tag{5.18}$$

Since $\mathcal{H}_0\Psi(\mathcal{H}_0)g$ belongs to $L^\infty(\Omega)$ by (5.13) and Lemma 5.3 for $p=\infty$, the identity (5.18) holds almost everywhere on Ω. Hence we have

$${}_{L^\infty(\Omega)}\langle-\Delta\Psi(\mathcal{H}_0)g,\overline{\Phi(\mathcal{H}_0)f}\cdot h\rangle_{L^1(\Omega)}={}_{L^\infty(\Omega)}\langle\Phi(\mathcal{H}_0)f\cdot\mathcal{H}_0\Psi(\mathcal{H}_0)g,h\rangle_{L^1(\Omega)},$$

since $\overline{\Phi(\mathcal{H}_0)f}\cdot h\in L^1(\Omega)$. Therefore, the first term is written as

$${}_{L^\infty(\Omega)}\langle\Psi(\mathcal{H}_0)g,-\Delta(\overline{\Phi(\mathcal{H}_0)f}\cdot h)\rangle_{L^1(\Omega)}={}_{L^\infty(\Omega)}\langle\Phi(\mathcal{H}_0)f\cdot\mathcal{H}_0\Psi(\mathcal{H}_0)g,h\rangle_{L^1(\Omega)}.$$

In a similar way, the third term in the right member of (5.17) is written as

$$\begin{aligned}&{}_{L^\infty(\Omega)}\langle\Psi(\mathcal{H}_0)g,\overline{\nabla\Phi(\mathcal{H}_0)f}\cdot\nabla h\rangle_{L^1(\Omega)}\\&=-{}_{\mathscr{D}'(\Omega)}\langle\Delta\Phi(\mathcal{H}_0)f\cdot\Psi(\mathcal{H}_0)g,h\rangle_{\mathscr{D}(\Omega)}-{}_{\mathscr{D}'(\Omega)}\langle\nabla\Phi(\mathcal{H}_0)f\cdot\nabla\Psi(\mathcal{H}_0)g,h\rangle_{\mathscr{D}(\Omega)}\\&={}_{\mathscr{D}'(\Omega)}\langle\mathcal{H}_0\phi(\mathcal{H}_0)f\cdot\Psi(\mathcal{H}_0)g,h\rangle_{\mathscr{D}(\Omega)}-{}_{\mathscr{D}'(\Omega)}\langle\nabla\Phi(\mathcal{H}_0)f\cdot\nabla\Psi(\mathcal{H}_0)g,h\rangle_{\mathscr{D}(\Omega)}.\end{aligned}\tag{5.19}$$

Therefore, summarizing (5.17) and (5.19), we conclude that (5.12) holds in $\mathscr{D}'(\Omega)$. The proof of Lemma 5.6 is finished. □

The space $\mathcal{P}_0(\Omega)$ in (4.44) for $V=0$ is explicitly written. More precisely, we have the following:

Lemma 5.7. *If Ω is a domain such that (5.1) holds for any $t>0$, then*

$$\mathcal{P}_0(\Omega)=\textit{either}\quad\{0\}\quad\textit{or}\quad\{f=c\ \textit{on}\ \Omega:c\in\mathbb{C}\}.$$

Proof. Let $f \in \mathcal{P}_0(\Omega)$. Then we prove that

$$f = \psi(2^{-2k}\mathcal{H}_0)f \quad \text{in } \mathcal{X}_0'(\Omega) \tag{5.20}$$

for any $k \in \mathbb{Z}$. Indeed, replacing λ in the identity (4.14) by $2^{-k}\lambda$, we see that

$$\psi(2^{-2k}\lambda^2) + \sum_{j=k+1}^{\infty} \phi_j(\lambda) = \psi((2^{-k}\lambda)^2) + \sum_{j=1}^{\infty} \phi_j(2^{-k}\lambda) = 1$$

for $\lambda \geq 0$ and $k \in \mathbb{Z}$. Hence, we deduce from the identity (4.34) in Proposition 4.5 that

$$f = \psi(2^{-2k}\mathcal{H}_0)f + \sum_{j=k+1}^{\infty} \phi_j(\sqrt{\mathcal{H}_0})f \quad \text{in } \mathcal{X}_0'(\Omega) \tag{5.21}$$

for any $k \in \mathbb{Z}$. Here, it follows from part (i-b) in Proposition 4.8 that

$$\phi_j(\sqrt{\mathcal{H}_0})f = 0$$

for any $j \in \mathbb{Z}$. Hence, we conclude from these equations and (5.21) that (5.20) holds true.

Since the gradient estimate (5.1) holds for $t = 2^{-2k}$, applying (5.6) from Lemma 5.4 to (5.20), we get

$$\|\nabla f\|_{L^\infty(\Omega)} = \|\nabla \psi(2^{-2k}\mathcal{H}_0)f\|_{L^\infty(\Omega)} \leq C2^k \|f\|_{L^\infty(\Omega)}$$

for any $k \in \mathbb{Z}$, which implies that $\nabla f = 0$ in Ω. Since Ω is connected, f is a constant in Ω. Summarizing the above argument, we deduce that

$$\{0\} \subset \mathcal{P}_0(\Omega) \subset \{f = c \text{ on } \Omega : c \in \mathbb{C}\}.$$

Since $\mathcal{P}_0(\Omega)$ is a linear space, we conclude that if $\mathcal{P}_0(\Omega) \neq \{0\}$, then $\mathcal{P}_0(\Omega)$ is the space of all constant functions on Ω. This proves (iii). The proof of Lemma 5.7 is finished. □

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. It is sufficient to prove the homogeneous case (ii), since one can reduce the argument of the proof of (i) to that of (ii). Therefore, we shall concentrate on proving the case (ii).

For the sake of simplicity, we use the following notations:

$$f_j := \phi_j(\sqrt{\mathcal{H}_0})f, \quad S_j(f) = S_j(\sqrt{\mathcal{H}_0})(f) := \sum_{k=-\infty}^{j} \phi_k(\sqrt{\mathcal{H}_0})f, \quad j \in \mathbb{Z}.$$

We have to divide the proof into two cases:

$$\text{“}1 \leq p_2, p_3 < \infty\text{” and “}p_2 = \infty \text{ or } p_3 = \infty\text{”},$$

since the approximation by the Littlewood-Paley dyadic decomposition is available only for $p_2, p_3 < \infty$ (see Lemma 5.5) and a constant function in $\mathcal{P}_0(\Omega)$ appears only in the case when $p_2 = \infty$ or $p_3 = \infty$.

The case: $1 \le p_2, p_3 < \infty$**.** Let $f \in \dot{B}^s_{p_1,q}(\mathcal{H}_0) \cap L^{p_3}(\Omega)$ and $g \in \dot{B}^s_{p_4,q}(\mathcal{H}_0) \cap L^{p_2}(\Omega)$. Referring to the Bony paraproduct formula (see [4]), we write fg formally as

$$fg = \sum_{k=-\infty}^{\infty} f_k S_{k-3}(g) + \sum_{k=-\infty}^{\infty} S_{k-3}(f) g_k + \sum_{k=-\infty}^{\infty} \sum_{l=k-2}^{k+2} f_k g_l.$$

Then we shall estimate $\dot{B}^s_{p,q}(\mathcal{H}_0)$-norm of each term in the right member as

$$\|fg\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} \le I + II + III + IV + V + VI,$$

where we put

$$I := \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{|k-j|\le 2} \left\| \phi_j(\sqrt{\mathcal{H}_0}) \Big(f_k S_{k-3}(g) \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$II := \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{|k-j|> 2} \left\| \phi_j(\sqrt{\mathcal{H}_0}) \Big(f_k S_{k-3}(g) \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$III := \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{|k-j|\le 2} \left\| \phi_j(\sqrt{\mathcal{H}_0}) \Big(S_{k-3}(f) g_k \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$IV := \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{|k-j|> 2} \left\| \phi_j(\sqrt{\mathcal{H}_0}) \Big(S_{k-3}(f) g_k \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$V := \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{k-j\ge -4} \left\| \phi_j(\sqrt{\mathcal{H}_0}) \Big(\sum_{l=k-2}^{k+2} f_k g_l \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}},$$

$$VI := \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{k-j< -4} \left\| \phi_j(\sqrt{\mathcal{H}_0}) \Big(\sum_{l=k-2}^{k+2} f_k g_l \Big) \right\|_{L^p(\Omega)} \right)^q \right\}^{\frac{1}{q}}.$$

We note that when $\Omega = \mathbb{R}^d$, the terms II, IV and VI vanish. Indeed, observing that

$$\phi_j(\sqrt{-\Delta}) \Big(f_k S_{k-3}(g) \Big) = \mathscr{F}^{-1} \Big[\phi_j(|\xi|) \Big\{ \big(\phi_k(|\xi|) \mathscr{F} f \big) * \big(S_{k-3}(|\xi|) \mathscr{F} g \big) \Big\} \Big],$$

and that

$$\operatorname{supp} \phi_j \cap \operatorname{supp} \ \big(\big(\phi_k(|\xi|) \mathscr{F} f \big) * \big(S_{k-3}(|\xi|) \mathscr{F} g \big) \big) = \emptyset$$

for $|k-j|>2$, we deduce that

$$\phi_j(\sqrt{-\Delta})\Big(f_k S_{k-3}(g)\Big)=0,$$

provided $|k-j|>2$. Thus we get $II=0$. In a similar way we find that IV also vanishes. As to the term VI, observing that

$$\operatorname{supp}\phi_j\cap\operatorname{supp}\left(\sum_{l=k-2}^{k+2}\big(\phi_k(|\xi|)\mathscr{F}f\big)*\big(\phi_l(|\xi|)\mathscr{F}g\big)\right)=\emptyset$$

for $k-j<-4$, we deduce that

$$\phi_j(\sqrt{\mathcal{H}_0})\Big(\sum_{l=k-2}^{k+2}f_k g_l\Big)=0,$$

provided $k-j<-4$, which implies that $VI=0$.

However, when $\Omega\neq\mathbb{R}^d$, the situation is different. In fact, if II, IV and VI vanish, the bilinear estimates hold for all positive regularity s by the argument of Case A below. However it contradicts the counter-example constructed in section 5.2. It should be noted that the assumption (5.1) on the gradient estimate plays an essential role in the estimation of these terms II, IV and VI.

Thus we estimate separately as follows:

Case A: Estimates for I, III and V,

Case B: Estimates for II, IV and VI.

Case A: Estimates for I, III and V. These terms can be estimated in the same way as in the case when $\Omega=\mathbb{R}^d$. Since similar arguments also appear for II, IV and VI, we give the proof in a self-contained way. First we estimate the term I. Noting from the assertion (ii) in Lemma 5.3 that $f_k\in L^{p_1}(\Omega)$ and $S_{k-3}(g)\in L^{p_2}(\Omega)$ for each $k\in\mathbb{Z}$, we deduce from Hölder's inequality and the estimate (5.5) for $\alpha=0$ in Lemma 5.3 that

$$\begin{aligned}\Big\|\phi_j(\sqrt{\mathcal{H}_0})\Big(f_k S_{k-3}(g)\Big)\Big\|_{L^p(\Omega)}&\le C\|f_k\|_{L^{p_1}(\Omega)}\|S_{k-3}(g)\|_{L^{p_2}(\Omega)}\\&\le C\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)},\end{aligned}$$

since $1/p = 1/p_1 + 1/p_2$. Thus we conclude from the above estimate that

$$\begin{aligned}
I &\le C\Big\{\sum_{j=-\infty}^{\infty}\Big(2^{sj}\sum_{|k-j|\le 2}\|f_k\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&= C\Big\{\sum_{j=-\infty}^{\infty}\Big(\sum_{|k'|\le 2}2^{-sk'}\cdot 2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&\le C\sum_{|k'|\le 2}2^{-sk'}\Big\{\sum_{j=-\infty}^{\infty}\Big(2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\
&\le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)},
\end{aligned}$$

where we used Minkowski's inequality in the third step. As to the term III, interchanging the role of f and g in the above argument, we get

$$III \le C\|f\|_{L^{p_3}(\Omega)}\|g\|_{\dot{B}^s_{p_4,q}(\mathcal{H}_0)},$$

where $1/p = 1/p_3 + 1/p_4$. As to the term V, we estimate

$$\begin{aligned}
&\Big\{\sum_{j=-\infty}^{\infty}\Big(2^{sj}\sum_{|k-j|\le 4}\Big\|\phi_j(\sqrt{\mathcal{H}_0})\Big(\sum_{l=k-2}^{k+2}f_kg_l\Big)\Big\|_{L^p(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\Big\{\sum_{j=-\infty}^{\infty}\Big(2^{sj}\sum_{|k-j|\le 4}\|f_k\|_{L^{p_1}(\Omega)}\Big(\sum_{l=k-2}^{k+2}\|g_l\|_{L^{p_2}(\Omega)}\Big)\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\Big\{\sum_{j=-\infty}^{\infty}\Big(2^{sj}\sum_{|k-j|\le 4}\|f_k\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}.
\end{aligned}$$

Here, by applying Minkowski's inequality to the right member in the above inequality, we find that

$$\begin{aligned}
&\Big\{\sum_{j=-\infty}^{\infty}\Big(2^{sj}\sum_{|k-j|\le 4}\|f_k\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&= \Big\{\sum_{j=-\infty}^{\infty}\Big(\sum_{|k'|\le 4}2^{-sk'}\cdot 2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\sum_{|k'|\le 4}2^{-sk'}\Big\{\sum_{j=-\infty}^{\infty}\Big(2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\Big)^q\Big\}^{\frac{1}{q}}\\
&\le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}.
\end{aligned}$$

Hence, combining the above two estimates, we conclude that

$$V \le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)}.$$

Case B: Estimates for II, IV and VI. First let us estimate the term II. When $k-j>2$, we deduce from the same argument as in I that

$$\left\{\sum_{j=-\infty}^{\infty}\left(2^{sj}\sum_{k-j>2}\left\|\phi_j(\sqrt{\mathcal{H}_0})\Big(f_kS_{k-3}(g)\Big)\right\|_{L^p(\Omega)}\right)^q\right\}^{\frac{1}{q}} \le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)}.$$

Hence all we have to do is to prove the case when $k-j<-2$, i.e.,

$$\left\{\sum_{j=-\infty}^{\infty}\left(2^{sj}\sum_{k-j<-2}\left\|\phi_j(\sqrt{\mathcal{H}_0})\Big(f_kS_{k-3}(g)\Big)\right\|_{L^p(\Omega)}\right)^q\right\}^{\frac{1}{q}} \le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)}. \quad (5.22)$$

In fact, noting from Lemma 4.5 that $f_k, S_{k-3}(g) \in L^\infty(\Omega)$ and from (4.18) that $L^\infty(\Omega) \hookrightarrow \mathcal{X}_0'(\Omega)$, we have

$$f_kS_{k-3}(g) \in \mathcal{X}_0'(\Omega).$$

Then we write

$$\phi_j(\sqrt{\mathcal{H}_0})\big(f_kS_{k-3}(g)\big) = \mathcal{H}_0^{-1}\phi_j(\sqrt{\mathcal{H}_0})\mathcal{H}_0\big(f_kS_{k-3}(g)\big) \quad \text{in } \mathcal{X}_0'(\Omega). \quad (5.23)$$

Here it should be noted that the operator $\mathcal{H}_0^{-1}$ in (5.23) is well-defined, since

$$\mathcal{H}_0^{-1}\phi_j(\sqrt{\mathcal{H}_0})h \in \mathcal{X}_0'(\Omega)$$

for any $h \in \mathcal{X}_0'(\Omega)$. Hence, applying the Leibniz rule in Lemma 5.6 to the identities (5.23), we have:

$$\begin{aligned}&\phi_j(\sqrt{\mathcal{H}_0})\big(f_kS_{k-3}(g)\big)\\ &=\mathcal{H}_0^{-1}\phi_j(\sqrt{\mathcal{H}_0})\Big\{(\mathcal{H}_0f_k)S_{k-3}(g) - 2\nabla f_k\cdot\nabla S_{k-3}(g) + f_k\big(\mathcal{H}_0S_{k-3}(g)\big)\Big\}\end{aligned} \quad (5.24)$$

in $\mathcal{X}_0'(\Omega)$. Thanks to (3.1) from Lemma 4.1 and (5.5) from Lemma 5.3, the first term in the right member in (5.24) is estimated as

$$\begin{aligned}\left\|\mathcal{H}_0^{-1}\phi_j(\sqrt{\mathcal{H}_0})\Big\{(\mathcal{H}_0f_k)S_{k-3}(g)\Big\}\right\|_{L^p(\Omega)} &\le C2^{-2j}\big\|(\mathcal{H}_0f_k)S_{k-3}(g)\big\|_{L^p(\Omega)}\\ &\le C2^{-2j}\|\mathcal{H}_0f_k\|_{L^{p_1}(\Omega)}\|S_{k-3}(g)\|_{L^{p_2}(\Omega)}\\ &\le C2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)}.\end{aligned}$$

In a similar way, we estimate the third term as

$$\Big\|\mathcal{H}_0^{-1}\phi_j(\sqrt{\mathcal{H}_0})\Big\{f_k\mathcal{H}_0 S_{k-3}(g)\Big\}\Big\|_{L^p(\Omega)} \le C2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)}.$$

As to the second, thanks to (5.7) and (5.8) from Lemma 5.4, we estimate

$$\begin{aligned}\Big\|\mathcal{H}_0^{-1}\phi_j(\sqrt{\mathcal{H}_0})\Big\{\nabla f_k\cdot\nabla S_{k-3}(g)\Big\}\Big\|_{L^p(\Omega)} &\le C2^{-2j}\big\|\nabla f_k\cdot\nabla S_{k-3}(g)\big\|_{L^p(\Omega)}\\ &\le C2^{-2j}\|\nabla f_k\|_{L^{p_1}(\Omega)}\|\nabla S_{k-3}(g)\|_{L^{p_2}(\Omega)}\\ &\le C2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)}.\end{aligned}$$

Hence, combining the identity (5.24) with the above three estimates, we get

$$\Big\|\phi_j(\sqrt{\mathcal{H}_0})\Big(f_k S_{k-3}(g)\Big)\Big\|_{L^p(\Omega)} \le C2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)}$$

for any $j,k\in\mathbb{Z}$. Therefore, we conclude from this estimate that

$$\begin{aligned}&\left\{\sum_{j=-\infty}^{\infty}\left(2^{sj}\sum_{k-j<-2}\Big\|\phi_j(\sqrt{\mathcal{H}_0})\Big(f_k S_{k-3}(g)\Big)\Big\|_{L^p(\Omega)}\right)^q\right\}^{\frac{1}{q}}\\ &\le C\left\{\sum_{j=-\infty}^{\infty}\left(2^{sj}\sum_{k-j<-2}2^{-2(j-k)}\|f_k\|_{L^{p_1}(\Omega)}\right)^q\right\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\ &= C\left\{\sum_{j=-\infty}^{\infty}\left(\sum_{k'<-2}2^{(2-s)k'}\cdot 2^{s(j+k')}\|f_{j+k'}\|_{L^{p_1}(\Omega)}\right)^q\right\}^{\frac{1}{q}}\|g\|_{L^{p_2}(\Omega)}\\ &\le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)},\end{aligned}$$

since $s<2$, which proves (5.22). Thus we conclude that

$$II \le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)}.$$

Similarly, we estimate

$$IV \le C\|f\|_{L^{p_3}(\Omega)}\|g\|_{\dot{B}^s_{p_4,q}(\mathcal{H}_0)},$$

$$VI \le C\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_0)}\|g\|_{L^{p_2}(\Omega)}.$$

Summarizing cases A and B, we arrive at the required estimate (5.3). The proof of the case when $1\le p_2,p_3<\infty$ is finished.

It remains to prove the case when $p_2=\infty$ or $p_3=\infty$.

The case: $p_2 = \infty$ or $p_3 = \infty$. We may prove only the case when $p_2 = p_3 = \infty$, since the other cases are proved in a similar way. In this case, we note that $p_1 = p_4 = p$. Let $f, g \in \dot{B}^s_{p,q}(\mathcal{H}_0) \cap L^\infty(\Omega)$. Then it follows from Lemma 5.3 that

$$\Big\| \sum_{j=k}^{\infty} f_j \Big\|_{L^\infty(\Omega)} \le C \|f\|_{L^\infty(\Omega)} \tag{5.25}$$

for any $k \in \mathbb{Z}$. Hence there exist a subsequence

$$\Big\{ \sum_{j=k_l}^{\infty} f_j \Big\}_{l \in \mathbb{N}}$$

and a function $F \in L^\infty(\Omega)$ such that

$$\sum_{j=k_l}^{\infty} f_j \rightharpoonup F \quad \text{weakly* in } L^\infty(\Omega) \tag{5.26}$$

as $l \to \infty$, which also yields the convergence in $\mathcal{X}_0'(\Omega)$ and $\mathcal{Z}_0'(\Omega)$ by the embedding

$$L^\infty(\Omega) \hookrightarrow \mathcal{X}_0'(\Omega) \hookrightarrow \mathcal{Z}_0'(\Omega).$$

On the other hand, it follows from Lemma 4.5 that

$$\sum_{j=k_l}^{\infty} f_j \to f \quad \text{in } \mathcal{Z}_0'(\Omega)$$

as $l \to \infty$. Hence we see that $F = f$ in $\mathcal{Z}_0'(\Omega)$, which implies that

$$P_f := f - F \in \mathcal{P}_0(\Omega).$$

Therefore we conclude from (5.26) that

$$\sum_{j=k_l}^{\infty} f_j \rightharpoonup f - P_f \quad \text{weakly* in } L^\infty(\Omega) \tag{5.27}$$

as $l \to \infty$. In a similar way, there exist a subsequence

$$\Big\{ \sum_{j=k_{l'}}^{\infty} g_j \Big\}_{l' \in \mathbb{N}}$$

and $P_g \in \mathcal{P}_0(\Omega)$ such that

$$\sum_{j=k_{l'}}^{\infty} g_j \rightharpoonup g - P_g \quad \text{weakly* in } L^\infty(\Omega) \tag{5.28}$$

as $l' \to \infty$. Hence, by (5.27) and (5.28), there exists a subsequence $\{l'(l)\}_{l=1}^{\infty}$ of $\{l'\}_{l'=1}^{\infty}$ such that

$$\Big(\sum_{j=k_l}^{\infty} f_j\Big)\Big(\sum_{j=k_{l'(l)}}^{\infty} g_j\Big) \rightharpoonup (f-P_f)(g-P_g) \quad \text{weakly* in } L^\infty(\Omega)$$

as $l \to \infty$. Hence we have

$$\Big(\sum_{j=k_l}^{\infty} f_j\Big)\Big(\sum_{j=k_{l'(l)}}^{\infty} g_j\Big) \to (f-P_f)(g-P_g) \quad \text{in } \mathcal{X}_0'(\Omega) \tag{5.29}$$

as $l \to \infty$, since $L^\infty(\Omega) \hookrightarrow \mathcal{X}_0'(\Omega)$. Now, the estimate of $\dot{B}^s_{p,q}$-norm of the left member in (5.29) is obtained by the argument as in the previous case $1 \le p_2, p_3 < \infty$. Hence, there exists a constant $C > 0$ such that

$$\begin{aligned} &\Big\|\Big(\sum_{j=k_l}^{\infty} f_j\Big)\Big(\sum_{j=k_{l'(l)}}^{\infty} g_j\Big)\Big\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} \\ &\le C\big(\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}\big) \end{aligned} \tag{5.30}$$

for any $l \in \mathbb{N}$. Here, we note that P_f and P_g are constants by Lemma 5.7. As a consequence of (5.29) and (5.30), we conclude from Lemma 4.21 that

$$\begin{aligned} \|fg\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} &\le \liminf_{l\to\infty} \Big\|\Big(\sum_{j=k_l}^{\infty} f_j\Big)\Big(\sum_{j=k_{l'(l)}}^{\infty} g_j\Big)\Big\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} \\ &\quad + \|fP_g\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} + \|P_f g\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} + \|P_f P_g\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} \\ &\le C\big(\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}\big) \\ &\quad + \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}|P_g| + |P_f|\|g\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} + \|P_f P_g\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)}. \end{aligned}$$

Here, we deduce from part (c) in (i) from Proposition 4.8 that

$$\|P_f P_g\|_{\dot{B}^s_{p,q}(\mathcal{H}_0)} = 0.$$

Noting (5.27), and using (5.25), we estimate

$$|P_f| \le \|f\|_{L^\infty(\Omega)} + \liminf_{l\to\infty} \Big\|\sum_{j=k_l}^{\infty} f_j\Big\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)}.$$

In a similar way, we have

$$|P_g| \le C\|g\|_{L^\infty(\Omega)}.$$

Combining the last four inequalities, we conclude the required estimate (5.3) in the case when $p_2 = p_3 = \infty$. The proof of Theorem 5.1 is finished. □

5.2 A remark on high regularity case

In this section we show that the bilinear estimates do not necessarily hold for some $s \geq 2$. For the sake of simplicity, let us consider the case when

$$\Omega = \{x \in \mathbb{R}^3 : |x| > 1\}$$

and

$$p = \frac{3}{2}, \quad p_1 = p_2 = p_3 = p_4 = 3, \quad q = 2, \quad f = g,$$

namely,

$$\|f^2\|_{\dot{B}^s_{\frac{3}{2},2}(\mathcal{H}_0)} \leq C\|f\|_{\dot{B}^s_{3,2}(\mathcal{H}_0)}\|f\|_{L^3(\Omega)} \tag{5.31}$$

for any $f \in \dot{B}^s_{3,2}(\mathcal{H}_0) \cap L^3(\Omega)$. We note that the estimate (5.31) is already proved for any $0 < s < 2$ (see the case (ii-a) in section 5.1). We shall show that the estimate (5.31) holds for $s = 2$ and does not hold for $s > 2$. In the proof we use the following facts:

(a) The following gradient estimates hold for any $1 \leq p \leq 3$:

$$\|\nabla e^{-t\mathcal{H}_0} f\|_{L^p(\Omega)} \leq Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}, \quad t > 0$$

for any $f \in L^p(\Omega)$ (see Theorem 6.1 below).

(b) The gradient estimate

$$\|\nabla e^{-t\mathcal{H}_0} f\|_{L^{p_0}(\Omega)} \leq Ct^{-1}\|f\|_{L^{\frac{3}{2}}(\Omega)}, \quad t > 1, \quad f \in L^{\frac{3}{2}}(\Omega)$$

with $p_0 > 3$ is sharp in the sense that it is not possible to replace the time decay rate t^{-1} with $t^{-1-\gamma}$ for any $\gamma > 0$ (see Theorem 6.7 below).

5.2.1 The case $s = 2$

In this subsection we show the following:

Proposition 5.8. *The estimate* (5.31) *holds for* $s = 2$, *namely,*

$$\|f^2\|_{\dot{B}^2_{\frac{3}{2},2}(\mathcal{H}_0)} \leq C\|f\|_{\dot{B}^2_{3,2}(\mathcal{H}_0)}\|f\|_{L^3(\Omega)} \tag{5.32}$$

for any $f \in \dot{B}^2_{3,2}(\mathcal{H}_0) \cap L^3(\Omega)$.

Proof. The proof is based on the method of proof of Proposition 3.6 in [14]. Let $f \in \dot{B}^2_{3,2}(\mathcal{H}_0) \cap L^3(\Omega)$. Using the formula

$$\nabla(\theta\mathcal{H}_0 + I)^{-1} = \int_0^\infty e^{-t}\nabla e^{-t\theta\mathcal{H}_0}\, dt,$$

we estimate

$$\begin{aligned}\|\nabla(\theta\mathcal{H}_0+I)^{-1}f\|_{L^3(\Omega)} &\le \int_0^\infty e^{-t}\|\nabla e^{-t\theta\mathcal{H}_0}f\|_{L^3(\Omega)}\,dt\\ &\le C\theta^{-\frac12}\|f\|_{L^3(\Omega)}\int_0^\infty t^{-\frac12}e^{-t}\,dt\\ &\le C\theta^{-\frac12}\|f\|_{L^3(\Omega)}\end{aligned}$$

for any $\theta>0$, where we used the fact (a) in the second step. Hence we have

$$\|\nabla f\|_{L^3(\Omega)} \le C\theta^{-\frac12}\|(\theta\mathcal{H}_0+I)f\|_{L^3(\Omega)} \le C\theta^{-\frac12}\big(\theta\|\mathcal{H}_0 f\|_{L^3(\Omega)}+\|f\|_{L^3(\Omega)}\big).$$

Taking $\theta=\|f\|_{L^3(\Omega)}\|\mathcal{H}_0 f\|^{-1}_{L^3(\Omega)}$, we obtain the interpolation inequality

$$\|\nabla f\|^2_{L^3(\Omega)} \le C\|\mathcal{H}_0 f\|_{L^3(\Omega)}\|f\|_{L^3(\Omega)}. \tag{5.33}$$

Noting from the Leibniz rule that

$$\mathcal{H}_0(f^2)=2(\mathcal{H}_0 f)\cdot f-2|\nabla f|^2 \quad \text{in } \mathscr{D}'(\Omega),$$

we deduce from (ii-b) and (iii) in Theorem 4.11, Hölder's inequality and the estimate (5.33) that

$$\begin{aligned}\|f^2\|_{\dot{B}^2_{\frac32,2}(\mathcal{H}_0)} &\le C\|\mathcal{H}_0(f^2)\|_{L^{\frac32}(\Omega)} \le C\big(\|(\mathcal{H}_0 f)\cdot f\|_{L^{\frac32}(\Omega)}+\|\nabla f\|^2_{L^3(\Omega)}\big)\\ &\le C\big(\|\mathcal{H}_0 f\|_{L^3(\Omega)}\|f\|_{L^3(\Omega)}\big) \le C\|f\|_{\dot{B}^2_{3,2}(\mathcal{H}_0)}\|f\|_{L^3(\Omega)}.\end{aligned}$$

This proves Proposition 5.8. □

5.2.2 The case $s>2$

We can show the following claim.

Claim 5.9. Let $\varepsilon>0$ and $\varepsilon<\delta\le 2$. If the estimate (5.31) holds for $s=2+\delta-\varepsilon, 2+\delta+\varepsilon$, then there exists a constant $C>0$ such that

$$\|\nabla e^{-t\mathcal{H}_0}f\|_{L^{p_0}(\Omega)} \le Ct^{-1-\frac{\delta-\varepsilon}{4}}\|f\|_{L^{\frac32}(\Omega)}, \quad t>1 \tag{5.34}$$

for any $f\in L^{\frac32}(\Omega)$, where the exponent $p_0>3$ is given by $\delta=2(1-3/p_0)$.

However (5.34) contradicts the fact (b). Hence, if Claim 5.9 is proved, then we conclude the bilinear estimate (5.31) does not hold for $s>2$.

Let us concentrate on the proof of Claim 5.9. Let $f\in C_0^\infty(\Omega)$. By the Leibniz rule, we have

$$\mathcal{H}_0(e^{-t\mathcal{H}_0}f)^2=2(\mathcal{H}_0 e^{-t\mathcal{H}_0}f)(e^{-t\mathcal{H}_0}f)-2|\nabla e^{-t\mathcal{H}_0}f|^2 \quad \text{in } \mathscr{D}'(\Omega),$$

and hence,

$$\begin{aligned}\|\nabla e^{-t\mathcal{H}_0}f\|_{L^{p_0}(\Omega)}^2 &\le \|\mathcal{H}_0(e^{-t\mathcal{H}_0}f)^2\|_{L^{\frac{p_0}{2}}(\Omega)} + \|(\mathcal{H}_0e^{-t\mathcal{H}_0}f)(e^{-t\mathcal{H}_0}f)\|_{L^{\frac{p_0}{2}}(\Omega)} \\ &=: I + II.\end{aligned} \tag{5.35}$$

As to the term II, we see from Hölder's inequality and L^p-L^q-estimates for $e^{-t\mathcal{H}_0}$ (see Lemma 6.2 of chapter 6) that

$$II \le \|\mathcal{H}_0e^{-t\mathcal{H}_0}f\|_{L^{p_0}(\Omega)}\|e^{-t\mathcal{H}_0}f\|_{L^{p_0}(\Omega)} \le Ct^{-2-\frac{\delta}{2}}\|f\|_{L^{\frac{3}{2}}(\Omega)}^2,$$

where we recall that $\delta = 2(1-3/p_0)$. As to the term I, applying the estimate (5.31) for $s = 2+\delta-\varepsilon, 2+\delta+\varepsilon$, and again using Lemma 6.2, we obtain

$$\begin{aligned}I &\le C\big(\|(e^{-t\mathcal{H}_0}f)^2\|_{\dot{B}^{2+\delta-\varepsilon}_{\frac{3}{2},2}(\mathcal{H}_0)} + \|(e^{-t\mathcal{H}_0}f)^2\|_{\dot{B}^{2+\delta+\varepsilon}_{\frac{3}{2},2}(\mathcal{H}_0)}\big) \\ &\le C\big(\|e^{-t\mathcal{H}_0}f\|_{\dot{B}^{2+\delta-\varepsilon}_{3,2}(\mathcal{H}_0)}\|e^{-t\mathcal{H}_0}f\|_{L^3(\Omega)} + \|e^{-t\mathcal{H}_0}f\|_{\dot{B}^{2+\delta+\varepsilon}_{3,2}(\mathcal{H}_0)}\|e^{-t\mathcal{H}_0}f\|_{L^3(\Omega)}\big)\end{aligned} \tag{5.36}$$

By Lemma 6.2, we have

$$\|e^{-t\mathcal{H}_0}f\|_{L^3(\Omega)} \le Ct^{-1}\|f\|_{L^{\frac{3}{2}}(\Omega)}.$$

Noting from Theorem 3.1 and Lemma 6.2 that

$$\|\mathcal{H}_0^se^{-t\mathcal{H}_0}\phi_j(\sqrt{\mathcal{H}_0})f\|_{L^3(\Omega)} \le t^{-\frac{s}{2}-1}\|\phi_j(\sqrt{\mathcal{H}_0})f\|_{L^{\frac{3}{2}}(\Omega)}, \quad j \in \mathbb{Z},$$

we deduce from (ii-b) and (iii) in Theorem 4.11 that

$$\|e^{-t\mathcal{H}_0}f\|_{\dot{B}^s_{3,2}(\mathcal{H}_0)} \le Ct^{-\frac{s}{2}-1}\|f\|_{\dot{B}^0_{\frac{3}{2},2}(\mathcal{H}_0)} \le Ct^{-\frac{s}{2}-1}\|f\|_{L^{\frac{3}{2}}(\Omega)}$$

for $s = 2+\delta-\varepsilon, 2+\delta-\varepsilon$. Hence, by combining the estimates obtained now, we get

$$I \le C\big(t^{-2-\frac{\delta-\varepsilon}{2}} + t^{-2-\frac{\delta+\varepsilon}{2}}\big)\|f\|_{L^{\frac{3}{2}}(\Omega)}^2 \le Ct^{-2-\frac{\delta-\varepsilon}{2}}\|f\|_{L^{\frac{3}{2}}(\Omega)}^2, \quad t > 1.$$

Therefore we find from (5.36) and the above estimate that

$$\|\nabla e^{-t\mathcal{H}_0}f\|_{L^{p_0}(\Omega)} \le Ct^{-1-\frac{\delta-\varepsilon}{4}}\|f\|_{L^{\frac{3}{2}}(\Omega)}, \quad t > 1.$$

Thus we conclude Claim 5.9.

Chapter 6

Gradient estimates for heat equation

In this chapter we consider the gradient estimates for the Dirichlet problem of heat equation in an exterior domain Ω of $\mathbb{R}^d$:

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = 0, & t \in (0,\infty), \quad x \in \Omega, \\ u(t,x) = 0, & t \in (0,\infty), \quad x \in \partial\Omega, \\ u(0,x) = f(x), & x \in \Omega. \end{cases} \tag{6.1}$$

When Ω is the whole space $\mathbb{R}^d$ or a half space $\mathbb{R}^d_+$, the following gradient estimates

$$\|\nabla u(t)\|_{L^p(\Omega)} \le Ct^{-\frac{1}{2}} \|f\|_{L^p(\Omega)}, \quad t > 0 \tag{6.2}$$

hold for any $1 \le p \le \infty$. The estimates (6.2) follow immediately from the explicit representation formula of solution u to the problem (6.1). These estimates are also true when Ω is a bounded domain. In this case we can replace the decay rate $t^{-1/2}$ of (6.2) by an exponential decay rate.

We are concerned with the question whether the estimates (6.2) in exterior domains are true for any $1 \le p \le \infty$ or not. These estimates are always true for any $1 \le p \le 2$ (see Theorem 3.2). On the other hand, the situation of the case $p > 2$ is more complicated (see [26, 37, 38, 43, 51]). In this case, the question seems to remain without complete answer due to our knowledge.

Surprisingly, an answer to this question and more information can be found in the case of Stokes equations. Maremonti and Solonnikov revealed that the following estimates hold for solutions to the Dirichlet problem of Stokes equations in exterior domain of $\mathbb{R}^d$, $d \ge 3$, with sufficiently smooth boundary:

$$\|\nabla u(t)\|_{L^p(\Omega)} \le \begin{cases} Ct^{-\frac{1}{2}} \|f\|_{L^p(\Omega)} & \text{for } 0 < t \le 1, \\ Ct^{-\mu} \|f\|_{L^p(\Omega)} & \text{for } t \ge 1, \end{cases} \tag{6.3}$$

where

$$\mu = \begin{cases} \frac{1}{2} & \text{if } 1 \le p \le d, \\ \frac{d}{2p} & \text{if } d < p \le \infty. \end{cases}$$

(see [57]). The case $d = 2$ is studied in Dan and Shibata [16]. The optimality of the estimate (6.3) is discussed in [57], where the authors show that the estimate

$$\|\nabla u(t)\|_{L^p(\Omega)} \le Ct^{-\mu-\delta}\|f\|_{L^p(\Omega)}, \quad t > 1, \quad \delta > 0 \tag{6.4}$$

is not true for $d \ge 3$ and $p > d$.

The first purpose is to prove the gradient estimate (6.3) for solutions to the problem (6.1) in exterior domains. The second purpose is to show that (6.2) is not fulfilled when Ω is the exterior of a ball. In this case, denoting by $u(t; f)$ the solution to (6.1) with initial data f, we can show that

$$0 < \sup_{t>0,\, f\in L^p(\Omega),\, \|f\|_{L^p(\Omega)}=1} t^{\mu}\|\nabla u(t; f)\|_{L^p(\Omega)} < \infty$$

for any $1 \le p \le \infty$. The right inequality follows from the gradient estimate (6.3). The left inequality, i.e., the positivity of the supremum, gives variational characterization of the best constant in (6.3) and implies the optimality of (6.3) (see Definition 6.6 and Theorem 6.7 below). The results in this chapter are based on Georgiev and Taniguchi [27].

6.1 Gradient estimates for the Dirichlet problem

In this section we shall prove the following:

Theorem 6.1. *Let $d \ge 2$ and Ω be the exterior domain in $\mathbb{R}^d$ of a compact set with $C^{1,1}$-boundary. Then, for any $1 \le p < \infty$, there exists a constant $C > 0$ such that the solution u of* (6.1) *satisfies*

$$\|\nabla u(t)\|_{L^p(\Omega)} \le \begin{cases} Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)} & \text{for } 0 < t \le 1, \\ Ct^{-\mu}\|f\|_{L^p(\Omega)} & \text{for } t > 1 \end{cases} \tag{6.5}$$

for any $f \in L^p(\Omega)$, where the exponent μ is given by

$$\mu = \begin{cases} \frac{1}{2} & \text{if } 1 \le p \le d, \\ \frac{d}{2p} & \text{if } d < p < \infty. \end{cases} \tag{6.6}$$

Let us give a few remarks on the theorem.

- Since we consider boundary with weak regularity, it is not clear whether the gradient estimate (6.5) with $p = \infty$ is true for any $f \in L^\infty(\Omega)$ due to our knowledge. However the gradient estimate is true for classical solutions. In fact, the classical bounded solutions to Dirichlet problem of parabolic equations in bounded or unbounded domains with sufficiently smooth boundary satisfy the local (in time) gradient estimate

$$\|\nabla u(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{1}{2}}\|f\|_{L^\infty(\Omega)}, \quad 0 < t \le 1$$

(see [22, 56], and references therein). One can establish the global estimate

$$\|\nabla u(t)\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)}, \quad t > 1,$$

by combining the above local gradient estimate with L^∞-estimate for u in Lemma 6.2 below.

- In the case of Neumann boundary condition, the estimate (6.2) holds in exterior domains. We note that (6.2) is stronger than the estimate (6.5) (see, e.g., [36, 78]).

- The supremum

$$\sup_{t>0,\, f\in L^p(\Omega),\, \|f\|_{L^p(\Omega)}=1} t^\mu \|\nabla u(t;f)\|_{L^p(\Omega)}$$

is a well-defined positive number (see Definition 6.6 and Theorem 6.7 below). This gives an optimality of (6.5) and a variational characterization of the best constant $C = C(\Omega, p)$ in (6.5).

- The estimate (6.5) is sharp in the context discussed in [57]. In other words, (6.4) is not true for any $\delta > 0$. This is weaker than the above optimality, i.e., the optimality of Definition 6.6.

For the purpose, we prepare key estimates for solutions of heat equations (6.1). The first one is the result on L^p-L^q-estimates which is an immediate consequence of (iii) in Proposition 2.4.

Lemma 6.2. *Let Ω be an open set in $\mathbb{R}^d$ and $1 \le p \le q \le \infty$. Then there exists a constant $C > 0$ such that*

$$\|u(t)\|_{L^q(\Omega)} \le Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}$$

for any $t > 0$ and $f \in L^p(\Omega)$.

The second one is the result on the gradient estimates for $1 \le p \le 2$ which is an immediate consequence of (ii) in Theorem 3.2.

Lemma 6.3. *Let Ω be an open set in $\mathbb{R}^d$ and $1 \le p \le 2$. Then*

$$\|\nabla u(t)\|_{L^p(\Omega)} \le Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

for any $t > 0$ and $f \in L^p(\Omega)$.

Furthermore, we prepare two fundamental inequalities. The first one is the special case of the Gagliardo-Nirenberg inequality (see [25, 61]).

Lemma 6.4. *Let Ω be a bounded domain in $\mathbb{R}^d$ having the cone property. Then, for any $1 < p < \infty$, there exist constants $C_1, C_2 > 0$ such that*

$$\|\nabla f\|_{L^p(\Omega)} \le C_1 \Bigg(\sum_{|\alpha|=2} \|\partial_x^\alpha f\|_{L^p(\Omega)}^{\frac{1}{2}} \Bigg) \|f\|_{L^p(\Omega)}^{\frac{1}{2}} + C_2 \|f\|_{L^p(\Omega)}$$

for any $f \in W^{2,p}(\Omega)$.

The second one is the global $W^{2,p}$-estimate (see Theorem 9.13 in [29]).

Lemma 6.5. *Let Ω be a domain in $\mathbb{R}^d$ with $C^{1,1}$ boundary. Then, for any $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$\|f\|_{W^{2,p}(\Omega')} \le C\big(\|\Delta f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega'')} \big)$$

for any $f \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, where Ω' and Ω'' are bounded domains in $\mathbb{R}^d$ such that

$$\Omega' \subset \Omega'' \subset \Omega \quad \textit{and} \quad \operatorname{dist}(\partial\Omega, \Omega'' \setminus \Omega') > 0.$$

Proof of Theorem 6.1. The case $p = 1$ is proved in Lemma 6.3. Hence, in order to obtain (6.5) for any $1 \le p < \infty$, it suffices to prove the case $d \le p < \infty$ by density and interpolation argument: For any $d \le p < \infty$, there exists a constant $C > 0$ such that

$$\|\nabla u(t)\|_{L^p(\Omega)} \le \begin{cases} Ct^{-\frac{1}{2}} \|f\|_{L^p(\Omega)} & \text{for } 0 < t \le 1, \\ Ct^{-\frac{d}{2p}} \|f\|_{L^p(\Omega)} & \text{for } t > 1 \end{cases} \tag{6.7}$$

for any $f \in C_0^\infty(\Omega)$. Let us choose $L > 0$ such that $\mathbb{R}^d \setminus \Omega \subset \{|x| < L\}$. Putting

$$\Omega_{L+2} := \Omega \cap \{|x| < L + 2\},$$

we estimate

$$\|\nabla u(t)\|_{L^p(\Omega)} \le \|\nabla u(t)\|_{L^p(\Omega_{L+2})} + \|\nabla u(t)\|_{L^p(\{|x| \ge L+2\})}. \tag{6.8}$$

As to the first term, we can obtain

$$\|\nabla u(t)\|_{L^p(\Omega_{L+2})} \le \begin{cases} Ct^{-\frac{1}{2}} \|f\|_{L^p(\Omega)} & \text{for } 0 < t \le 1, \\ Ct^{-\frac{d}{2p}} \|f\|_{L^p(\Omega)} & \text{for } t > 1 \end{cases} \tag{6.9}$$

by using Lemmas 6.4 and 6.5. In fact, noting that

$$u(t) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

for any $t > 0$ and $f \in C_0^\infty(\Omega)$, we can apply Lemmas 6.4 and 6.5 to estimate

$$\begin{aligned} &\|\nabla u(t)\|_{L^p(\Omega_{L+2})} \\ &\le C_1 \Bigg(\sum_{|\alpha|=2} \|\partial_x^\alpha u(t)\|_{L^p(\Omega_{L+2})}^{\frac{1}{2}} \Bigg) \|u(t)\|_{L^p(\Omega_{L+2})}^{\frac{1}{2}} + C_2 \|u(t)\|_{L^p(\Omega_{L+2})} \\ &\le C\big(\|\Delta u(t)\|_{L^p(\Omega)}^{\frac{1}{2}} \|u(t)\|_{L^p(\Omega)}^{\frac{1}{2}} + \|u(t)\|_{L^p(\Omega_{L+4})} \big). \end{aligned} \tag{6.10}$$

Since

$$\|\Delta u(t)\|_{L^p(\Omega)}^{\frac{1}{2}}\|u(t)\|_{L^p(\Omega)}^{\frac{1}{2}} \le Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

and

$$\|u(t)\|_{L^p(\Omega_{L+4})} \le C\|u(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{d}{2p}}\|f\|_{L^p(\Omega)}$$

for any $t > 0$ by Lemma 6.2, the right hand side in (6.10) is estimated as

$$\|\Delta u(t)\|_{L^p(\Omega)}^{\frac{1}{2}}\|u(t)\|_{L^p(\Omega)}^{\frac{1}{2}} + \|u(t)\|_{L^p(\Omega_{L+4})} \le C\max(t^{-\frac{1}{2}}, t^{-\frac{d}{2p}})\|f\|_{L^p(\Omega)}$$

for any $t > 0$. Therefore we obtain the required estimates (6.9). Thus all we have to do is to estimate the second term in (6.8) as follows:

$$\|\nabla u(t)\|_{L^p(\{|x|>L+2\})} \le \begin{cases} Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)} & \text{for } 0 < t \le 1, \\ Ct^{-\frac{d}{2p}}\|f\|_{L^p(\Omega)} & \text{for } t > 1. \end{cases} \tag{6.11}$$

We divide the proof of (6.11) into two cases: $0 < t \le 1$ and $t > 1$.

The case $0 < t \le 1$**.** We denote by χ_L a smooth function on $\mathbb{R}^d$ such that

$$\chi_L(x) = \begin{cases} 1 & \text{for } |x| \ge L+1, \\ 0 & \text{for } |x| \le L, \end{cases} \tag{6.12}$$

and have

$$u(t,x) = \chi_L(x)u(t,x), \quad |x| \ge L+2.$$

Let us decompose $\chi_L u(t)$ into

$$\chi_L u(t) = v_1(t) - v_2(t) \tag{6.13}$$

for $0 < t \le 1$. Here $v_1(t)$ is the solution to the Cauchy problem of heat equation in $\mathbb{R}^d$:

$$\begin{cases} \partial_t v_1(t,x) - \Delta v_1(t,x) = 0, & t \in (0,1], \quad x \in \mathbb{R}^d, \\ v_1(0,x) = \chi_L(x)f(x), & x \in \mathbb{R}^d, \end{cases}$$

and $v_2(t)$ is the solution to the Cauchy problem of heat equation in $\mathbb{R}^d$:

$$\begin{cases} \partial_t v_2(t,x) - \Delta v_2(t,x) = F(t,x), & t \in (0,1], \quad x \in \mathbb{R}^d, \\ v_2(0,x) = 0, & x \in \mathbb{R}^d, \end{cases}$$

where

$$F(t,x) = -2\nabla\chi_L(x)\cdot\nabla u(t,x) + (\Delta\chi_L(x))u(t,x). \tag{6.14}$$

It is easily proved that

$$\|\nabla v_1(t)\|_{L^p(\{|x|>L+2\})} \le Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)} \tag{6.15}$$

for any $0 < t \le 1$. Hence it is sufficient to show that

$$\|\nabla v_2(t)\|_{L^p(\{|x|>L+2\})} \le Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)} \tag{6.16}$$

for any $0 < t \le 1$. Letting $e^{t\Delta}$ be the semigroup generated by $-\Delta$ on $\mathbb{R}^d$, we write $v_2(t)$ as

$$v_2(t,x) = \int_0^t e^{(t-s)\Delta}F(s,x)\,ds$$

for $0 < t \le 1$ and $x \in \mathbb{R}^d$. Recalling that

$$e^{t\Delta}(x,y) = (4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}},$$

we estimate

$$\begin{aligned}
&\|\nabla v_2(t)\|_{L^p(\{|x|>L+2\})}\\
&\le \int_0^t \|\nabla e^{(t-s)\Delta}F(s,\cdot)\|_{L^p(\{|x|>L+2\}}\,ds\\
&\le \int_0^t \int_{L<|y|\le L+1} \|\nabla_x e^{(t-s)\Delta}(x,y)\|_{L^p(\{|x|>L+2\})}|F(s,y)|\,dy\,ds.
\end{aligned} \tag{6.17}$$

Here we note that

$$\begin{aligned}
\left|\nabla_x e^{t\Delta}(x,y)\right| &\le Ct^{-\frac{n+1}{2}}\frac{|x-y|}{2\sqrt{t}}e^{-\frac{|x-y|^2}{4t}}\\
&\le Ct^{-\frac{n+1}{2}}\left(1+\frac{|x-y|^2}{t}\right)^{-\frac{n+1}{2}}\\
&= C(t+|x-y|^2)^{-\frac{n+1}{2}}
\end{aligned}$$

for any $t > 0$ and $x \in \mathbb{R}^d$. In particular, if $|x| \ge L+2$ and $|y| \le L+1$, then

$$|x-y| \ge |x| - |y| \ge |x| - \frac{L+1}{L+2}|x| = \frac{1}{L+2}|x|,$$

and hence,

$$|\nabla_x e^{(t-s)\Delta}(x,y)| \le C\{(t-s)+|x|^2\}^{-\frac{d+1}{2}}$$

for any $0 < s < t$. Therefore we deduce that

$$\|\nabla_x e^{(t-s)\Delta}(x,y)\|_{L^p(\{|x|>L+2\})} \le C\{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}} \tag{6.18}$$

for any $0 < s < t$ and $L < |y| \le L+1$. Combining (6.17) and (6.18), we obtain

$$\begin{aligned}
\|\nabla v_2(t)\|_{L^p(\{|x|>L+2\})} &\le \int_0^t \int_{L<|y|\le L+1} \{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}}|F(s,y)|\,dy\,ds\\
&= \int_0^t \{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}}\|F(s,\cdot)\|_{L^1(\{L<|y|\le L+1\})}\,ds.
\end{aligned} \tag{6.19}$$

Recalling the definition (6.14) of $F(s,x)$, and using (6.9) and Lemma 6.2, we estimate

$$\begin{aligned}\|F(s,\cdot)\|_{L^1(\{L<|y|\le L+1\})} &\le C\big(\|\nabla u(s)\|_{L^1(\{L<|y|\le L+1\})} + \|u(s)\|_{L^1(\{L<|y|\le L+1\})}\big)\\ &\le C\big(\|\nabla u(s)\|_{L^p(\{L<|y|\le L+1\})} + \|u(s)\|_{L^p(\{L<|y|\le L+1\})}\big)\\ &\le Cs^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}\end{aligned}$$

for any $0 < s \le 1$. Combining the above two estimates, we deduce that

$$\begin{aligned}\|\nabla v_2(t)\|_{L^p(\{|x|>L+2\})} &\le C\int_0^t \{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}} s^{-\frac{1}{2}}\,ds\cdot\|f\|_{L^p(\Omega)}\\ &\le C\int_0^t s^{-\frac{1}{2}}\,ds\cdot\|f\|_{L^p(\Omega)}\\ &\le C\|f\|_{L^p(\Omega)}\end{aligned}$$

for any $0 < t \le 1$, which proves (6.16). Therefore the estimate (4.56) for any $0 < t \le 1$ is proved by (6.15) and (6.16).

The case $t > 1$. In a similar way to (6.13) in the previous case, we decompose $\chi_L u(t)$ into

$$\chi_L u(t) = w_1(t) - w_2(t)$$

for $t \ge 1$. Here $w_1(t)$ is the solution to the Cauchy problem of heat equation in $\mathbb{R}^d$:

$$\begin{cases}\partial_t w_1(t,x) - \Delta w_1(t,x) = 0, & t \in (1,\infty), \quad x \in \mathbb{R}^d,\\ w_1(1,x) = \chi_L(x)u(1,x), & x \in \mathbb{R}^d,\end{cases}$$

and $w_2(t)$ is the solution to the Cauchy problem of heat equation in $\mathbb{R}^d$:

$$\begin{cases}\partial_t w_2(t,x) - \Delta w_2(t,x) = F(t,x), & t \in (1,\infty), \quad x \in \mathbb{R}^d,\\ w_2(1,x) = 0, & x \in \mathbb{R}^d,\end{cases}$$

where we recall (6.12) and (6.14). It is easily proved that

$$\|\nabla w_1(t)\|_{L^p(\{|x|>L+2\})} \le Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)} \tag{6.20}$$

for any $t > 1$. Hence it is sufficient to show that

$$\|\nabla w_2(t)\|_{L^p(\{|x|>L+2\})} \le Ct^{-\frac{d}{2p}}\|f\|_{L^p(\Omega)} \tag{6.21}$$

for any $t > 1$. Writing $w_2(t)$ as

$$w_2(t,x) = \int_1^t e^{(t-s)\Delta}F(s,x)\,ds$$

for $t > 1$ and $x \in \mathbb{R}^d$, we estimate, in a similar way to (6.19),

$$\|\nabla w_2(t)\|_{L^p(\{|x|>L+2\})} \le C \int_1^t \{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}} \|F(s,\cdot)\|_{L^1(\{L<|y|\le L+1\})}\, ds.$$

Recalling the definition (6.14) of $F(s,x)$, and using (6.9) and Lemma 6.2, we estimate

$$\begin{aligned}\|F(s,\cdot)\|_{L^1(\{L<|y|\le L+1\})} &\le C\big(\|\nabla u(s)\|_{L^1(\{L<|y|\le L+1\})} + \|u(s)\|_{L^1(\{L<|y|\le L+1\})}\big)\\ &\le C\big(\|\nabla u(s)\|_{L^p(\{L<|y|\le L+1\})} + \|u(s)\|_{L^\infty(\{L<|y|\le L+1\})}\big)\\ &\le C s^{-\frac{d}{2p}} \|f\|_{L^p(\Omega)}\end{aligned}$$

for any $s > 1$. Combining the above two estimates, we deduce that

$$\|\nabla w_2(t)\|_{L^p(\{|x|>L+2\})} \le C \int_1^t \{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}} s^{-\frac{d}{2p}}\, ds \cdot \|f\|_{L^p(\Omega)}.$$

for any $t > 1$. For $1 < t < 2$ we use the inequality

$$\int_1^t \{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}} s^{-\frac{d}{2p}}\, ds \le \int_1^2 s^{-\frac{d}{2p}}\, ds \le C t^{-\frac{1}{2}}.$$

For $t > 2$ and $p \ge n$, we have

$$\int_1^{\frac{t}{2}} \{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}} s^{-\frac{d}{2p}}\, ds \le C t^{-\frac{d+1}{2}+\frac{d}{2p}} \int_1^{\frac{t}{2}} s^{-\frac{d}{2p}}\, ds \le C t^{-\frac{1}{2}}$$

and

$$\int_{\frac{t}{2}}^t \{1+(t-s)\}^{-\frac{d+1}{2}+\frac{d}{2p}} s^{-\frac{d}{2p}}\, ds \le C\big(t^{-\frac{d}{2p}} + t^{-\frac{n-1}{2}}\big) \le C t^{-\frac{d}{2p}}.$$

Hence we obtain the estimate (6.21) for any $t > 1$. Therefore the estimate (4.56) for any $t > 1$ is proved by (6.20) and (6.21).

Thus, combining (6.8) with (6.9) and (4.56), we conclude the estimates (6.7). The proof of Theorem 6.1 is complete. □

6.2 A remark on optimality of time decay rates

To state the result, let us give the definition of optimality of time decay rates.

Definition 6.6. We say that the gradient estimate (6.5) is optimal if there exist sequences $\{f_m\}_{m\in\mathbb{N}} \subset L^p(\Omega)$ and $\{t_m\}_{m\in\mathbb{N}}$ such that

$$t_m > 0 \quad \text{for } m \in \mathbb{N}, \quad t_m \to \infty \quad \text{as } m \to \infty$$

and

$$\limsup_{m\to\infty} \frac{t_m^\mu \|\nabla u_m(t_m)\|_{L^p(\Omega)}}{\|f_m\|_{L^p(\Omega)}} > 0,$$

where u_m is a solution to (6.1) with initial data f_m and the exponent μ is given by (6.6).

Our result on the optimality is the following. To simplify the proof, we shall fix the space dimension $d = 3$.

Theorem 6.7. *Let $d = 3$ and Ω be the exterior domain of a ball. Then, for any $1 \le p \le \infty$, the gradient estimate* (6.5) *is optimal in the sense of Definition 6.6.*

Proof. Let f be a radial function on Ω. Since $u(t)$ is also radial, we write

$$F(r) := f(x), \quad U(t,r) := u(t,x)$$

for $t > 0$ and $r = |x|$. We rewrite the problem (6.1) to the following problem by the polar coordinates and making change $v(t,r) = (r+1)U(t,r+1)$:

$$\begin{cases} \partial_t v(t,r) - \partial_r^2 v(t,r) = 0, & t \in (0,\infty), \quad r \in (0,\infty), \\ v(t,r) = 0, & t \in (0,\infty), \quad r = 0, \\ v(0,r) = g(r), & r \in (0,\infty), \end{cases} \tag{6.22}$$

where $g(r) = (r+1)F(r+1)$ and $r = |x|$. Then solutions v to (6.22) and the derivative $\partial_r v$ can be represented as

$$v(t,r) = (4\pi t)^{-\frac{1}{2}} \int_0^\infty \big\{ e^{-\frac{(r-s)^2}{4t}} - e^{-\frac{(r+s)^2}{4t}} \big\} g(s)\, ds, \tag{6.23}$$

$$\partial_r v(t,r) = (4\pi t)^{-\frac{1}{2}} \int_0^\infty \Big\{ -\frac{r-s}{2t} e^{-\frac{(r-s)^2}{4t}} + \frac{r+s}{2t} e^{-\frac{(r+s)^2}{4t}} \Big\} g(s)\, ds \tag{6.24}$$

for $t > 0$ and $r > 0$. Furthermore, noting that $u(t,x) = U(t,r) = r^{-1} v(t, r-1)$, we write

$$\|\nabla u(t)\|_{L^p(\Omega)} = (4\pi)^{\frac{1}{p}} \big\| -(r+1)^{-2+\frac{2}{p}} v(t) + (r+1)^{-1+\frac{2}{p}} \partial_r v(t) \big\|_{L^p(0,\infty)}. \tag{6.25}$$

In order to prove the optimality, we choose appropriate initial data f_m and estimate from below the quantity from Definition 6.6:

$$\frac{t_m^\mu \|\nabla u_m(t_m)\|_{L^p(\Omega)}}{\|f_m\|_{L^p(\Omega)}}$$

for $m \in \mathbb{N}$, where the exponent μ is defined in (6.6). We divide the proof into two cases: $1 \le p \le 3$ and $3 < p \le \infty$.

The case $1 \le p \le 3$. We take $t_m = m^2$ for $m \in \mathbb{N}$, and define the initial data as follows

$$f_m(x) := \begin{cases} C_m |x|^{-1}, & r \in (m+1, 2m+1], \\ 0, & \text{otherwise.} \end{cases} \tag{6.26}$$

Here we choose the constant C_m such that

$$C_m > 0 \quad \text{and} \quad \|f_m\|_{L^p(\Omega)} = 1. \tag{6.27}$$

Then we have

$$g_m(r) = \begin{cases} C_m, & r \in (m, 2m], \\ 0, & \text{otherwise}, \end{cases} \tag{6.28}$$

and

$$C_m \sim m^{1-\frac{3}{p}} \tag{6.29}$$

as $m \to \infty$. Here the notation $A_m \sim B_m$ as $m \to \infty$ means that there exist constants $C_1, C_2 > 0$ such that

$$C_1 \le \frac{A_m}{B_m} \le C_2 \quad \text{as } m \to \infty.$$

Let us denote by u_m and v_m the solutions to (6.1) and (6.22) with initial data f_m and g_m, respectively. By the equality (6.25), we write

$$\|\nabla u_m(t)\|_{L^p(\Omega)} = (4\pi)^{\frac{1}{p}} \big\| -(r+1)^{-2+\frac{2}{p}} v_m(t) + (r+1)^{-1+\frac{2}{p}} \partial_r v_m(t) \big\|_{L^p(0,\infty)}.$$

Letting $t > 0$ and $s > 0$ be fixed, we see that the function

$$e^{-\frac{(r-s)^2}{4t}} - e^{-\frac{(r+s)^2}{4t}}, \quad r > 0,$$

is monotonically decreasing with respect to $r \in [\sqrt{2t} + s, \infty)$. Hence, noting from (6.28) that $g_m \ge 0$ and $m \le s \le 2m$, we have

$$v_m(t, r) \ge 0 \quad \text{and} \quad \partial_r v_m(t, r) \le 0$$

for any $r \in [\sqrt{2t} + 2m, \infty)$. Thanks to this observation, we estimate from below

$$\|\nabla u_m(t)\|_{L^p(\Omega)} \ge \big\| (r+1)^{-2+\frac{2}{p}} v_m(t) \big\|_{L^p(\sqrt{2t}+2m,\infty)}.$$

Taking $t = t_m = m^2$, we write

$$\|\nabla u_m(t_m)\|_{L^p(\Omega)} \ge \big\| (r+1)^{-2+\frac{2}{p}} v_m(m^2) \big\|_{L^p(c_0 m,\infty)}, \tag{6.30}$$

where $c_0 = 2 + \sqrt{2}$. From the representation (6.23) and definition (6.28) of g_m, the right hand side is estimated as

$$\begin{aligned} &\big\| (r+1)^{-2+\frac{2}{p}} v_m(m^2) \big\|_{L^p(c_0 m,\infty)} \\ &\ge C \cdot C_m m^{-1} \left\| (r+1)^{-2+\frac{2}{p}} \int_m^{2m} \left\{ e^{-\frac{(r-s)^2}{4m^2}} - e^{-\frac{(r+s)^2}{4m^2}} \right\} ds \right\|_{L^p(c_0 m,\infty)}. \end{aligned} \tag{6.31}$$

Since

$$e^{-\frac{(r-s)^2}{4m^2}} - e^{-\frac{(r+s)^2}{4m^2}} = e^{-\frac{(r-s)^2}{4m^2}}(1 - e^{-\frac{rs}{m^2}}) \geq (1 - e^{-c_0})e^{-\frac{c_0^2}{4}}e^{-\frac{s^2}{4m^2}}$$

for any $r > c_0 m$ and $m \leq s \leq 2m$, the integral in the right hand side in (6.31) is estimated from below as

$$\int_m^{2m} \left\{ e^{-\frac{(r-s)^2}{4m^2}} - e^{-\frac{(r+s)^2}{4m^2}} \right\} ds \geq C \int_m^{2m} e^{-\frac{s^2}{4m^2}}\, ds = Cm \int_1^2 e^{-\frac{s^2}{4}}\, ds. \tag{6.32}$$

Hence, by combining (6.30)–(6.32), we estimate from below

$$\left\| (r+1)^{-2+\frac{2}{p}} v_m(m^2) \right\|_{L^p(c_0 m, \infty)} \geq C \cdot C_m \left\| (r+1)^{-2+\frac{2}{p}} \right\|_{L^p(c_0 m, 2c_0 m)}. \tag{6.33}$$

Hence, noting from (6.29) that

$$C_m \left\| (r+1)^{-2+\frac{2}{p}} \right\|_{L^p(c_0 m, 2c_0 m)} \sim m^{1-\frac{3}{p}} m^{-2+\frac{3}{p}} = m^{-1} = t_m^{-\frac{1}{2}}$$

as $m \to \infty$, we deduce from (6.30)–(6.33) that

$$\|\nabla u_m(t_m)\|_{L^p(\Omega)} \geq C t_m^{-\frac{1}{2}} \tag{6.34}$$

for sufficiently large $m \in \mathbb{N}$, where the constant $C > 0$ is independent of m. By combining (6.27) and (6.34), we conclude that

$$\limsup_{m \to \infty} \frac{t_m^{\frac{1}{2}} \|\nabla u_m(t_m)\|_{L^p(\Omega)}}{\|f_m\|_{L^p(\Omega)}} > 0.$$

Thus the optimality for $1 \leq p \leq 3$ is proved.

The case $3 < p \leq \infty$**.** Recalling the equality (6.25) and representations (6.23) and (6.24), we write

$$\begin{aligned} \|\nabla u(t)\|_{L^p(\Omega)} &\geq \left\| -(r+1)^{-2+\frac{2}{p}} v(t) + (r+1)^{-1+\frac{2}{p}} \partial_r v(t) \right\|_{L^p(0,\infty)} \\ &= (4\pi t)^{-\frac{1}{2}} \left\| (r+1)^{-1+\frac{2}{p}} \int_0^\infty K(t,r,s) g(s)\, ds \right\|_{L^p(0,\infty)}, \end{aligned} \tag{6.35}$$

where

$$K(t,r,s) = \left[\left\{ -(r+1)^{-1} - \frac{r-s}{2t} \right\} e^{-\frac{(r-s)^2}{4t}} + \left\{ (r+1)^{-1} + \frac{r+s}{2t} \right\} e^{-\frac{(r+s)^2}{4t}} \right].$$

Again we take $t = t_m = m^2$ and denote by u_m and v_m the solutions to (6.1) and (6.22) with initial data f_m in (6.26) and g_m in (6.28), respectively.

To begin with, we prove the following: For sufficiently large $m \in \mathbb{N}$, there exists a constant $C > 0$, independent of m, such that

$$K(m^2, r, s) \geq \frac{C}{m} \tag{6.36}$$

for any $10 \le r \le m^{1/4}$ and $m \le s \le 2m$. Writing

$$e^{-\frac{(r-s)^2}{4t}} = e^{-\frac{s^2}{4m^2}+\frac{r}{2m}-\frac{r^2}{4m^2}} = e^{-\frac{s^2}{4m^2}}\Big\{1+\frac{r}{2m}+\mathrm{O}\Big(\frac{r^2}{m^2}\Big)\Big\},$$
$$e^{-\frac{(r+s)^2}{4t}} = e^{-\frac{s^2}{4m^2}-\frac{r}{2m}-\frac{r^2}{4m^2}} = e^{-\frac{s^2}{4m^2}}\Big\{1-\frac{r}{2m}+\mathrm{O}\Big(\frac{r^2}{m^2}\Big)\Big\},$$

we calculate

$$\begin{aligned} K(m^2,r,s) &= e^{-\frac{s^2}{4m^2}}\bigg[\Big\{-(r+1)^{-1}-\frac{r-s}{2m^2}\Big\}\Big\{1+\frac{r}{2m}+\mathrm{O}\Big(\frac{r^2}{m^2}\Big)\Big\} \\ &\qquad +\Big\{(r+1)^{-1}+\frac{r+s}{2m^2}\Big\}\Big\{1-\frac{r}{2m}+\mathrm{O}\Big(\frac{r^2}{m^2}\Big)\Big\}\bigg] \\ &= e^{-\frac{s^2}{4m^2}}\Big\{\frac{s}{m^2}-(r+1)^{-1}\frac{r}{m}-\frac{r^2}{2m^3}+\mathrm{O}\Big(\frac{r^2}{m^2}\Big)\Big\} \\ &\ge e^{-1}\Big\{\frac{1}{11m}+\mathrm{O}\Big(\frac{r^2}{m^2}\Big)\Big\}, \end{aligned} \tag{6.37}$$

where we used in the last step

$$\frac{s}{m^2}-(r+1)^{-1}\frac{r}{m} \ge \frac{1}{m}-\frac{10}{11m} \ge \frac{1}{11m}$$

for $10 \le r \le m^{1/4}$ and $m \le s \le 2m$. Since we can neglect the remainder terms in (6.37) if m is sufficiently large, we obtain (6.36).

Let us turn to estimate form below of L^p-norm of $\nabla u_m(t_m)$. By combining (6.35) and (6.36), we estimate

$$\begin{aligned} \|\nabla u_m(t_m)\|_{L^p(\Omega)} &\ge Cm^{-2}\bigg\|(r+1)^{-1+\frac{2}{p}}\int_m^{2m} C_m\,ds\bigg\|_{L^p(10,m^{\frac14})} \\ &= C\cdot C_m m^{-1}\big\|(r+1)^{-1+\frac{2}{p}}\big\|_{L^p(10,m^{\frac14})} \\ &\ge C\cdot C_m m^{-1} \end{aligned} \tag{6.38}$$

for sufficiently large $m \in \mathbb{N}$. Noting from (6.29) that

$$C_m m^{-1} \sim m^{1-\frac{3}{p}}m^{-1} = m^{-\frac{3}{p}}$$

as $m \to \infty$, we conclude from (6.38) that

$$\|\nabla u_m(t_m)\|_{L^p(\Omega)} \ge Cm^{-\frac{3}{p}} = Ct_m^{-\frac{3}{2p}}$$

for sufficiently large $m \in \mathbb{N}$, where the constant $C > 0$ is independent of m. This proves that

$$\limsup_{m\to\infty} \frac{t_m^{\frac{3}{2p}}\|\nabla u_m(t_m)\|_{L^p(\Omega)}}{\|f_m\|_{L^p(\Omega)}} > 0,$$

since $\|f_m\|_{L^p(\Omega)} = 1$ by (6.26). Thus the optimality for $3 < p \le \infty$ is proved. The proof of Theorem 6.7 is finished. □

Chapter 7

The case of the Neumann Laplacian

The purpose in this chapter is to give definitions of Besov spaces generated by the Neumann Laplacian on a domain, and prove their fundamental properties, which were proved in Taniguchi [78]. The results in this chapter would be applicable to the study of the Neumann problem to partial differential equations.

We assume that Ω is a Lipschitz domain in $\mathbb{R}^d$ with $d \geq 1$. Namely, it can be represented, locally near the boundary, as the region above the graph of a Lipschitz function. We consider the Neumann Laplacian $\mathcal{H}_N = -\Delta$ on $L^2(\Omega)$. More precisely, $\mathcal{H}_N$ is the non-negative self-adjoint operator on $L^2(\Omega)$ associated with the following quadratic form:

$$\mathcal{Q}(f,g) := \int_\Omega \nabla f(x) \cdot \overline{\nabla g(x)}\, dx$$

for any $f, g \in H^1(\Omega)$. It follows from Lemma 2.2 in section 2.2 that the domain $\mathcal{D}(\mathcal{H}_N)$ can be written as

$$\mathcal{D}(\mathcal{H}_N) = \{f \in H^1(\Omega) : \exists h_f \in L^2(\Omega) \text{ such that } \mathcal{Q}(f,g) = (h_f, g)_{L^2(\Omega)} \text{ for any } g \in H^1(\Omega)\}.$$

Thanks to the spectral theorem, there exists a spectral resolution $\{E_{\mathcal{H}_N}(\lambda)\}_{\lambda\in\mathbb{R}}$ of the identity for $\mathcal{H}_N$, and we write

$$\mathcal{H}_N = \int_0^\infty \lambda\, dE_{\mathcal{H}_N}(\lambda).$$

For a Borel measurable function ϕ on $\mathbb{R}$, an operator $\phi(\mathcal{H}_N)$ is defined by

$$\phi(\mathcal{H}_N) = \int_0^\infty \phi(\lambda)\, dE_{\mathcal{H}_N}(\lambda).$$

When $\mathrm{vol}(\Omega) = \infty$, the situation is similar to that of the Dirichlet Laplacian, since zero is not an eigenvalue of $\mathcal{H}_N$. However, if $\mathrm{vol}(\Omega) < \infty$, the situation is different. In particular case when Ω is a bounded and Lipschitz domain, the spectrum of $\mathcal{H}_N$ is discrete and zero is an eigenvalue of $\mathcal{H}_N$. Thus, in this case, let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues of $\mathcal{H}_N$ such that

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = \infty. \tag{7.1}$$

We denote by $\mathcal{E}$ the eigenspace associated with zero eigenvalue. It is well known that $\mathcal{E}$ is the space consisting of all constant functions on Ω. Its orthogonal complement $\mathcal{E}^{\perp}$ is the space

$$\mathcal{E}^{\perp} = \left\{ f \in L^2(\Omega) : \int_{\Omega} f(x)\, dx = 0 \right\}.$$

Then the space $L^2(\Omega)$ is decomposed as the direct sum of $\mathcal{E}$ and $\mathcal{E}^{\perp}$:

$$L^2(\Omega) = \mathcal{E} \oplus \mathcal{E}^{\perp}.$$

Let us define test function spaces on Ω and Besov spaces generated by $\mathcal{H}_N$ in a similar way to the Dirichlet case (see sections 4.1.1 and 4.2.1)

Definition (Test functions and distributions on Ω).

(i) (Linear topological spaces $\mathcal{X}_N(\Omega)$ and $\mathcal{X}'_N(\Omega)$). A linear topological space $\mathcal{X}_N(\Omega)$ is defined by letting

$$\mathcal{X}_N(\Omega) := \left\{ f \in L^1(\Omega) \cap \mathcal{D}(\mathcal{H}_N) : \mathcal{H}_N^M f \in L^1(\Omega) \cap \mathcal{D}(\mathcal{H}_N) \text{ for any } M \in \mathbb{N} \right\}$$

equipped with the family of semi-norms $\{p_M(\cdot)\}_{M=1}^{\infty}$ given by

$$p_M(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{N}} 2^{Mj} \|\phi_j(\sqrt{\mathcal{H}_N}) f\|_{L^1(\Omega)}.$$

Furthermore, $\mathcal{X}'_N(\Omega)$ denotes the topological dual of $\mathcal{X}_N(\Omega)$.

(ii) (Linear topological spaces $\mathcal{Z}_N(\Omega)$ and $\mathcal{Z}'_N(\Omega)$). A linear topological space $\mathcal{Z}_N(\Omega)$ is defined by letting

$$\begin{aligned} &\mathcal{Z}_N(\Omega) \\ &:= \{ f \in \mathcal{X}_N(\Omega) : q_M(f) < \infty \text{ for any } M \in \mathbb{N} \} \\ &= \left\{ f \in \mathcal{X}_N(\Omega) \cap \mathcal{E}^{\perp} : \sup_{j\le 0} 2^{M|j|} \left\| \phi_j(\sqrt{\mathcal{H}_N}) f \right\|_{L^1(\Omega)} < \infty \text{ for any } M \in \mathbb{N} \right\} \end{aligned}$$

equipped with the family of semi-norms $\{q_M(\cdot)\}_{M=1}^{\infty}$ given by

$$q_M(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{Z}} 2^{M|j|} \left(|f_0| + \|\phi_j(\sqrt{\mathcal{H}_N}) f\|_{L^1(\Omega)} \right),$$

where $f = f_0 + f_0^{\perp}$ with $f_0 \in \mathcal{E}$ and $f_0^{\perp} \in \mathcal{E}^{\perp}$. Furthermore, $\mathcal{Z}'_N(\Omega)$ denotes the topological dual of $\mathcal{Z}_N(\Omega)$.

Definition 7.1 (Besov space)**.** Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Then the Besov spaces are defined as follows:

(i) The inhomogeneous Besov spaces $B^s_{p,q}(\mathcal{H}_N)$ are defined by letting

$$B^s_{p,q}(\mathcal{H}_N) := \left\{ f \in \mathcal{X}'_N(\Omega) : \|f\|_{B^s_{p,q}(\mathcal{H}_N)} < \infty \right\},$$

where

$$\|f\|_{B^s_{p,q}(\mathcal{H}_N)} := \|\psi(\mathcal{H}_N) f\|_{L^p(\Omega)} + \left\| \left\{ 2^{sj} \|\phi_j(\sqrt{\mathcal{H}_N}) f\|_{L^p(\Omega)} \right\}_{j \in \mathbb{N}} \right\|_{l^q(\mathbb{N})}. \quad (7.2)$$

(ii) The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_N)$ are defined by letting

$$\dot{B}^s_{p,q}(\mathcal{H}_N) := \left\{ f \in \mathcal{Z}'_N(\Omega) : \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_N)} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_N)} := \left\| \left\{ 2^{sj} \|\phi_j(\sqrt{\mathcal{H}_N}) f\|_{L^p(\Omega)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})}. \quad (7.3)$$

This chapter is organized as follows. In section 7.1 we prove L^p-L^q-estimates for spectral multipliers for $\mathcal{H}_N$, which play a crucial role in studying the Besov spaces. In section 7.2 we prove fundamental properties of the test function spaces and the spaces of distributions on Ω. In section 7.3 some results on Besov spaces generated by $\mathcal{H}_N$ are introduced. In section 7.4 we discuss the bilinear estimates in Besov spaces generated by $\mathcal{H}_N$.

7.1 Boundedness of spectral multipliers

This section is devoted to proving L^p-L^q-estimates for spectral multipliers for $\mathcal{H}_N$. Introducing the characteristic function $\chi_{(0,\infty)}(\lambda)$ of $(0, \infty)$, we write for brevity a projection as

$$P := \chi_{(0,\infty)}(\mathcal{H}_N). \quad (7.4)$$

Throughout this section, we assume that Ω is a Lipschitz domain in $\mathbb{R}^d$ with a compact boundary, where $d \geq 3$ if Ω is unbounded, and $d \geq 1$ if Ω is bounded. This assumption is necessary for developing functional calculus.

Then we have the following:

Proposition 7.2. *Let $1 \leq p \leq q \leq \infty$, and let $\{\psi\} \cup \{\phi_j\}_j$ be functions given by* (4.12), (4.13) *and* (4.14)*. Then for any $m \in \mathbb{N}_0$, there exists a constant $C > 0$ such that*

$$\|\mathcal{H}_N^m \psi(\mathcal{H}_N)\|_{\mathscr{B}(L^p(\Omega), L^q(\Omega))} \leq C, \quad (7.5)$$

and for any $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that

$$\|\mathcal{H}_N^{\alpha}\phi_j(\sqrt{\mathcal{H}_N})\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \le C2^{d(\frac{1}{p}-\frac{1}{q})j+2\alpha j} \tag{7.6}$$

for any $j \in \mathbb{Z}$. In particular, if Ω is bounded, then for any $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$ there exist two constants $\mu > 0$ and $C > 0$ such that

$$\|\mathcal{H}_N^{m}\psi(2^{-2j}\mathcal{H}_N)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \le \begin{cases} C2^{d(\frac{1}{p}-\frac{1}{q})j+2mj} & \text{for } j \ge 1, \\ C2^{d(\frac{1}{p}-\frac{1}{q})j+2mj}e^{-\mu 2^{-j}} & \text{for } j \le 0, \end{cases} \tag{7.7}$$

$$\|\mathcal{H}_N^{\alpha}\phi_j(\sqrt{\mathcal{H}_N})\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \le \begin{cases} C2^{d(\frac{1}{p}-\frac{1}{q})j+2\alpha j} & \text{for } j \ge 1, \\ C2^{d(\frac{1}{p}-\frac{1}{q})j+2\alpha j}e^{-\mu 2^{-j}} & \text{for } j \le 0. \end{cases} \tag{7.8}$$

Proposition 7.2 is an immediate consequence of the following.

Lemma 7.3. *Let $\phi \in \mathscr{S}(\mathbb{R})$. Then $\phi(\mathcal{H}_N)$ is extended to a bounded linear operator from $L^p(\Omega)$ to $L^q(\Omega)$ provided that $1 \le p \le q \le \infty$. Furthermore, we have the uniform estimates*:

(i) *If Ω is unbounded, then there exists a constant $C > 0$ such that*

$$\|\phi(\theta\mathcal{H}_N)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \tag{7.9}$$

for any $\theta > 0$.

(ii) *If Ω is bounded, then the estimate (7.9) holds for any $0 < \theta \le 1$. In particular, if $\phi \in C_0^{\infty}((0,\infty))$, then there exist two constants $\mu > 0$ and $C > 0$ such that*

$$\|\phi(\theta\mathcal{H}_N)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\mu\theta} \tag{7.10}$$

for any $\theta > 0$.

To prove Lemma 7.3, we need the Gaussian upper bounds for semigroup $\{e^{-t\mathcal{H}_N}\}_{t>0}$ generated by $\mathcal{H}_N$.

Lemma 7.4. *Let $e^{-t\mathcal{H}_N}(x,y)$ be the kernel of the semigroup $e^{-t\mathcal{H}_N}$. Then the following assertions hold*:

(i) *If Ω is unbounded, then there exist two constants $C_1 > 0$ and $C_2 > 0$ such that*

$$0 \le e^{-t\mathcal{H}_N}(x,y) \le C_1 t^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{C_2 t}} \tag{7.11}$$

for any $t > 0$ and $x, y \in \Omega$.

(ii) *If Ω is bounded, then there exist two constants $C_3 > 0$ and $C_4 > 0$ such that*

$$0 \le e^{-t\mathcal{H}_N}(x,y) \le C_3 \max\{t^{-\frac{d}{2}}, 1\} e^{-\frac{|x-y|^2}{C_4 t}} \tag{7.12}$$

for any $t > 0$ and $x, y \in \Omega$. Furthermore, let $(Pe^{-t\mathcal{H}_N})(x,y)$ be the kernel of $Pe^{-t\mathcal{H}_N}$. Then there exist three constants $\mu > 0$, $C_5 > 0$ and $C_6 > 0$ such that

$$\left|(Pe^{-t\mathcal{H}_N})(x,y)\right| \le C_5 t^{-\frac{d}{2}} e^{-\mu t - \frac{|x-y|^2}{C_6 t}} \tag{7.13}$$

for any $t > 0$ and $x, y \in \Omega$.

Proof. The estimate (7.11) is proved by Chen, Williams and Zhao (see [11]), and the estimate (7.12) is proved by Choulli, Kayser and Ouhabaz (see [12]). Hence it suffices to prove the estimate (7.13).

Since the spectrum of $\mathcal{H}_N$ satisfies (7.1), it follows that

$$\|Pe^{-t\mathcal{H}_N} f\|^2_{L^2(\Omega)} = \int_{\lambda_2}^{\infty} e^{-2t\lambda}\, d\|E_{\mathcal{H}_N}(\lambda) f\|^2_{L^2(\Omega)} \le e^{-2\lambda_2 t}\|f\|^2_{L^2(\Omega)} \tag{7.14}$$

for any $t > 0$ and $f \in L^2(\Omega)$. Next, we claim that

$$\|e^{-t\mathcal{H}_N} f\|_{L^\infty(\Omega)} \le \begin{cases} Ct^{-\frac{d}{4}}\|f\|_{L^2(\Omega)} & \text{for } 0 < t \le 1, \\ Ct^{\frac{d}{4}}\|f\|_{L^2(\Omega)} & \text{for } t \ge 1 \end{cases} \tag{7.15}$$

for any $f \in L^2(\Omega)$. In fact, put

$$K_t(x) := e^{-\frac{|x|^2}{C_2 t}},$$

Letting $\tilde{f}$ be the zero extension of f from Ω to $\mathbb{R}^d$, we estimate, by using (7.12), Young's inequality,

$$\begin{aligned} \|e^{-t\mathcal{H}_N} f\|_{L^\infty(\Omega)} &\le C_3 \max\{t^{-\frac{d}{2}}, 1\} \|K_t * |\tilde{f}|\|_{L^\infty(\mathbb{R}^d)} \\ &\le C_3 \max\{t^{-\frac{d}{2}}, 1\} \|K_t\|_{L^2(\mathbb{R}^d)} \|\tilde{f}\|_{L^2(\mathbb{R}^d)} \\ &= C_3 \left(\frac{C_2\pi}{2}\right)^{\frac{d}{4}} \max\{t^{-\frac{d}{2}}, 1\} t^{\frac{d}{4}} \|f\|_{L^2(\Omega)}, \end{aligned}$$

which proves (7.15). Hence, when $t > 1$, combining (7.14) and (7.15), we find that

$$\begin{aligned} \|Pe^{-t\mathcal{H}_N} f\|_{L^\infty(\Omega)} &= \|e^{-\frac{t}{2}\mathcal{H}_N} Pe^{-\frac{t}{2}\mathcal{H}_N} f\|_{L^\infty(\Omega)} \\ &\le Ct^{\frac{d}{4}} \|Pe^{-\frac{t}{2}\mathcal{H}_N} f\|_{L^2(\Omega)} \\ &\le Ct^{\frac{d}{4}} e^{-\frac{\lambda_2}{2}t} \|f\|_{L^2(\Omega)} \end{aligned}$$

for any $f \in L^2(\Omega)$, which implies that by duality argument,

$$\|Pe^{-t\mathcal{H}_N} f\|_{L^2(\Omega)} \le Ct^{\frac{d}{4}} e^{-\frac{\lambda_2}{2}t} \|f\|_{L^1(\Omega)}$$

for any $t > 1$ and $f \in L^1(\Omega)$. Hence, combining the estimates obtained now, we get

$$\begin{aligned} \|Pe^{-t\mathcal{H}_N} f\|_{L^\infty(\Omega)} &= \|Pe^{-\frac{t}{2}\mathcal{H}_N} Pe^{-\frac{t}{2}\mathcal{H}_N} f\|_{L^\infty(\Omega)} \\ &\le Ct^{\frac{d}{4}} e^{-\frac{\lambda_2}{4}t} \|Pe^{-\frac{t}{2}\mathcal{H}_N} f\|_{L^2(\Omega)} \\ &\le C\big(t^{\frac{d}{4}} e^{-\frac{\lambda_2}{4}t}\big)^2 \|f\|_{L^1(\Omega)} \end{aligned} \tag{7.16}$$

for any $t > 1$ and $f \in L^1(\Omega)$. Here we note from the standard argument that

$$\sup_{x\in\Omega} \|Pe^{-t\mathcal{H}_N}(x,\cdot)\|_{L^\infty(\Omega)} = \|Pe^{-t\mathcal{H}_N}\|_{\mathscr{B}(L^1(\Omega),L^\infty(\Omega))}$$

Then, putting $L = \mathrm{diam}(\Omega)$, we deduce from (7.16) that

$$\big|Pe^{-t\mathcal{H}_N}(x,y)\big| \le Ct^{\frac{d}{2}} e^{-\frac{\lambda_2}{2}t} \le Ct^{\frac{d}{2}} e^{-\frac{\lambda_2}{2}t} e^{-\frac{|x-y|^2}{C_2 t}}$$

for any $t > 1$ and $x, y \in \Omega$. Thus we conclude the estimate (7.13). The proof of Lemma 7.4 is finished. □

Proof of Lemma 7.3. We prove only the estimates (7.10) for any $\theta > 1$ in the assertion (ii), since the proof of other assertions is similar to that of Theorem 3.1.

Since the support of ϕ is away from the origin, we write

$$\phi(\theta\mathcal{H}_N) = P\phi(\theta\mathcal{H}_N).$$

Let $f \in L^1(\Omega) \cap L^2(\Omega)$. Then, by using the estimate (7.14) and the above identity, we deduce that

$$\begin{aligned} \|\phi(\theta\mathcal{H}_N) f\|_{L^1(\Omega)} &\le |\Omega|^{\frac{1}{2}} \|\phi(\theta\mathcal{H}_N) f\|_{L^2(\Omega)} \\ &= |\Omega|^{\frac{1}{2}} \|Pe^{-\theta\mathcal{H}_N} e^{2\theta\mathcal{H}_N} \phi(\theta\mathcal{H}_N) e^{-\theta\mathcal{H}_N} f\|_{L^2(\Omega)} \\ &\le C|\Omega|^{\frac{1}{2}} e^{-\lambda_2\theta} \|e^{2\theta\mathcal{H}_N} \phi(\theta\mathcal{H}_N) e^{-\theta\mathcal{H}_N} f\|_{L^2(\Omega)}. \end{aligned} \tag{7.17}$$

Since the support of ϕ is compact, it follows that

$$e^{2\lambda}\phi(\lambda) \in L^\infty(\mathbb{R}),$$

and hence,

$$\|e^{2\theta\mathcal{H}_N} \phi(\theta\mathcal{H}_N) e^{-\theta\mathcal{H}_N} f\|_{L^2(\Omega)} \le C\|e^{-\theta\mathcal{H}_N} f\|_{L^2(\Omega)}. \tag{7.18}$$

Therefore, we deduce from (7.17) and (7.18) that

$$\|\phi(\theta\mathcal{H}_N) f\|_{L^1(\Omega)} \le C|\Omega|^{\frac{1}{2}} e^{-\lambda_2\theta} \|e^{-\theta\mathcal{H}_N} f\|_{L^2(\Omega)}. \tag{7.19}$$

On the other hand, it follows from the estimate (7.15) for $t = \theta > 1$ that

$$\|e^{-\theta \mathcal{H}_N} f\|_{L^\infty(\Omega)} \leq C\theta^{\frac{d}{4}} \|f\|_{L^2(\Omega)},$$

and hence, by duality argument we deduce that

$$\|e^{-\theta \mathcal{H}_N} f\|_{L^2(\Omega)} \leq C\theta^{\frac{d}{4}} \|f\|_{L^1(\Omega)}. \tag{7.20}$$

Hence, combining (7.19) and (7.20), we obtain

$$\big\|\phi(\theta \mathcal{H}_N)\big\|_{\mathscr{B}(L^1(\Omega))} \leq C|\Omega|^{\frac{1}{2}} \theta^{\frac{d}{4}} e^{-\lambda_2 \theta}$$

for any $\theta > 1$. Thus, performing the previous argument, we conclude the estimate (7.10) in the assertion (ii). The proof of Theorem 7.3 is finished. □

7.2 Fundamental properties of test function and distribution spaces

In this section we discuss the fundamental properties of $\mathcal{X}_N(\Omega)$, $\mathcal{Z}_N(\Omega)$ and their dual spaces. The results in this section form the basis for the proofs of the theorems in the next section.

The first result is the following.

Proposition 7.5. *Let Ω be as in section 7.1. Then $\mathcal{X}_N(\Omega)$ and $\mathcal{Z}_N(\Omega)$ are complete.*

Proof. We can prove the completeness of $\mathcal{X}_N(\Omega)$ in a similar way as in Proposition 4.2, regardless of unboundedness or boundedness of Ω. Also, when Ω is unbounded, the proof of completeness of $\mathcal{Z}_N(\Omega)$ is similar to that lemma. So we omit the details in these cases. Based on this consideration, we prove the completeness of $\mathcal{Z}_N(\Omega)$ in the case when Ω is the bounded domain.

Let $\{f_m\}_{m=1}^\infty$ be a Cauchy sequence in $\mathcal{Z}_N(\Omega)$. Since $\mathcal{Z}_N(\Omega)$ is a subspace of $\mathcal{X}_N(\Omega)$, and since $\mathcal{X}_N(\Omega)$ is complete, $\{f_m\}_{m=1}^\infty$ is also a Cauchy sequence in $\mathcal{X}_N(\Omega)$, and hence, there exists an element $f \in \mathcal{X}_N(\Omega)$ such that f_m converges to f in $\mathcal{X}_N(\Omega)$ as $m \to \infty$. Then we can check that f satisfies

$$\sup_{j \leq 0} 2^{M|j|} \big\|\phi_j(\sqrt{\mathcal{H}_N}) f\big\|_{L^1(\Omega)} < \infty \text{ for any } M \in \mathbb{N}$$

in the same way as in the latter part of proof of Proposition 4.2. Furthermore, since $\mathcal{E}^\perp$ is a closed subspace of $L^2(\Omega)$ and f_m converges to f in $L^2(\Omega)$ as $m \to \infty$, we have $f \in \mathcal{E}^\perp$. Hence $f \in \mathcal{Z}_N(\Omega)$. Thus we conclude that $\mathcal{Z}_N(\Omega)$ is complete. The proof of Proposition 7.5 is finished. □

The following propositions are proved in the completely same arguments as Propositions 4.3 and 4.4 in subsection 4.1.2, respectively. So we may omit the proofs.

Proposition 7.6. *Let Ω be as in section 7.1. Then we have the following assertions*:

(i) *For any $f \in \mathcal{X}'_N(\Omega)$, there exist a number $M_0 \in \mathbb{N}$ and a constant $C_f > 0$ such that*

$$|{}_{\mathcal{X}'_N(\Omega)}\langle f, g\rangle_{\mathcal{X}_N(\Omega)}| \leq C_f p_{M_0}(g)$$

for any $g \in \mathcal{X}_N(\Omega)$.

(ii) *For any $f \in \mathcal{Z}'_N(\Omega)$, there exist a number $M_1 \in \mathbb{N}$ and a constant $C_f > 0$ such that*

$$|{}_{\mathcal{Z}'_N(\Omega)}\langle f, g\rangle_{\mathcal{Z}_N(\Omega)}| \leq C_f p_{M_1}(g)$$

for any $g \in \mathcal{Z}_N(\Omega)$.

Proposition 7.7. *Let Ω be as in section 7.1. Then we have the following assertions*:

(i) *For any $\phi \in C_0^\infty(\mathbb{R})$, $\phi(\mathcal{H}_N)$ maps continuously from $\mathcal{X}_N(\Omega)$ into itself, and from $\mathcal{X}'_N(\Omega)$ into itself.*

(ii) *For any $\phi \in C_0^\infty((0,\infty))$, $\phi(\mathcal{H}_N)$ maps continuously from $\mathcal{Z}_N(\Omega)$ into itself, and from $\mathcal{Z}'_N(\Omega)$ into itself.*

Next we introduce approximations of identity in $\mathcal{X}_N(\Omega)$ and $\mathcal{Z}_N(\Omega)$. More precisely, we have the following.

Proposition 7.8. *Let Ω be as in section 7.1. Then we have the following assertions*:

(i) *For any $f \in \mathcal{X}_N(\Omega)$, we have*

$$f = \psi(\mathcal{H}_N)f + \sum_{j=1}^{\infty} \phi_j(\sqrt{\mathcal{H}_N})f \quad \text{in } \mathcal{X}_N(\Omega). \tag{7.21}$$

Furthermore, for any $f \in \mathcal{X}'_N(\Omega)$, the identity (7.21) holds in $\mathcal{X}'_N(\Omega)$, and $\psi(\mathcal{H}_N)f$ and $\phi_j(\sqrt{\mathcal{H}_N})f$ are regarded as elements of $L^\infty(\Omega)$.

(ii) *For any $f \in \mathcal{Z}_N(\Omega)$, we have*

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_N})f \quad \text{in } \mathcal{Z}_N(\Omega). \tag{7.22}$$

Furthermore, for any $f \in \mathcal{Z}'_N(\Omega)$, the identity (7.22) holds in $\mathcal{Z}'_N(\Omega)$, and $\phi_j(\sqrt{\mathcal{H}_N})f$ are regarded as elements of $L^\infty(\Omega)$.

Proof. We prove the assertion (ii) in the case when Ω is the bounded domain, since the unbounded case are proved in the same way as in Proposition 4.5 in subsection 4.1.2. Let $f \in \mathcal{Z}_N(\Omega)$. Since $\mathcal{Z}_N(\Omega) \subset \mathcal{E}^\perp$, it follows that $f \in \mathcal{E}^\perp$, and hence, we have

$$f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_N})f \quad \text{in } L^2(\Omega). \tag{7.23}$$

On the other hand, we find from the estimates (7.6) for $p = q = 1$ in Proposition 7.2 that

$$q_M(\phi_j(\sqrt{\mathcal{H}_N})f) \le C2^{2j} q_M(\mathcal{H}_N^{-1}\phi_j(\sqrt{\mathcal{H}_N})f) \le C2^{2j} q_{M+2}(f),$$

which implies that

$$\sum_{j=-\infty}^{0} q_M(\phi_j(\sqrt{\mathcal{H}_N})f) \le C q_{M+2}(f) \sum_{j=-\infty}^{0} 2^{2j} < \infty$$

for any $M \in \mathbb{N}$. This means that the series in the right member of (7.23) converges absolutely in $\mathcal{Z}_N(\Omega)$. Thus (7.22) is proved. The latter part is proved by combining the Hahn-Banach theorem with

$$\left| {}_{\mathcal{Z}'_N(\Omega)}\langle \phi_j(\sqrt{\mathcal{H}_N})f, h\rangle_{\mathcal{Z}_N(\Omega)} \right| \le C\|h\|_{L^1(\Omega)}$$

for any $f \in \mathcal{Z}'_N(\Omega)$ and $h \in \mathcal{Z}_N(\Omega)$. For more details, see the proof of Proposition 4.5 in subsection 4.1.2.

Similarly, the assertion (i) is proved by using the estimate (7.5) instead of (7.6). The proof of Proposition 7.8 is finished. □

The following result states the relations among Lebesgue spaces and the spaces of test functions and distributions on Ω.

Proposition 7.9. *Let Ω be as in section 7.1. Then*

$$\mathcal{Z}_N(\Omega) \hookrightarrow \mathcal{X}_N(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{X}'_N(\Omega) \hookrightarrow \mathcal{Z}'_N(\Omega) \tag{7.24}$$

for any $1 \le p \le \infty$. Furthermore, we have

$$\mathcal{Z}_N(\Omega) \subset \mathcal{X}_N(\Omega) \subset C^\infty(\Omega). \tag{7.25}$$

Proof. For the proof of (7.24), see Proposition 4.6 in subsection 4.1.2. The inclusion (7.25) is an immediate consequence of the interior elliptic regularity. In fact, it follows from the interior elliptic regularity that

$$\bigcap_{m=1}^{\infty} \mathcal{D}(\mathcal{H}_N^m) \subset C^\infty(\Omega).$$

The proof of Proposition 7.9 is complete. □

In the rest of this section we shall characterize the space $\mathcal{Z}'_N(\Omega)$ by the quotient space of $\mathcal{X}'_N(\Omega)$. Let us recall that $\mathcal{X}'_N(\Omega)$ and $\mathcal{Z}'_N(\Omega)$ correspond to $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}'_0(\mathbb{R}^d)$, respectively. Let us define a space $\mathcal{P}_N(\Omega)$ by

$$\mathcal{P}_N(\Omega) := \left\{ f \in \mathcal{X}'_N(\Omega) : {}_{\mathcal{Z}'_N(\Omega)}\langle J(f), g\rangle_{\mathcal{Z}_N(\Omega)} = 0 \text{ for any } g \in \mathcal{Z}_N(\Omega) \right\}, \tag{7.26}$$

where $J(f)$ is the restriction of f on the subspace $\mathcal{Z}_N(\Omega)$ of $\mathcal{X}_N(\Omega)$. It is readily checked that $\mathcal{P}_N(\Omega)$ is a closed subspace of $\mathcal{X}'_N(\Omega)$, and hence, the quotient space $\mathcal{X}'_N(\Omega)/\mathcal{P}_N(\Omega)$ is a linear topological space endowed with the quotient topology.

Proposition 7.10. *Let Ω be as in section 7.1. Then*

$$\mathcal{Z}'_N(\Omega) \cong \mathcal{X}'_N(\Omega)/\mathcal{P}_N(\Omega).$$

The proof of Proposition 7.10 is done by using Theorem in p.126 from Schaefer [72] and Propositions 35.5 and 35.6 from Tréves [79] (see also Theorem 1.1 in Sawano [70]).

The space $\mathcal{P}_N(\Omega)$ enjoys the following.

Proposition 7.11. *Let Ω be as in section 7.1. Then the following assertions hold*:

(i) *Let $f \in \mathcal{X}'_N(\Omega)$. Then the following assertions are equivalent*:

  (a) $f \in \mathcal{P}_N(\Omega)$;

  (b) $\phi_j(\sqrt{\mathcal{H}_N})f = 0$ *in* $\mathcal{X}'_N(\Omega)$ *for any* $j \in \mathbb{Z}$;

  (c) $\|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_N)} = 0$ *for any* $s \in \mathbb{R}$ *and* $1 \le p, q \le \infty$.

(ii) *If we further assume that Ω is a domain such that the gradient estimate* (7.31) *in section 7.3 holds for any $t > 0$, then*

$$\mathcal{P}_N(\Omega) = \textit{either} \quad \{0\} \quad \textit{or} \quad \{f = c \text{ on } \Omega : c \in \mathbb{C}\}. \tag{7.27}$$

In addition, if Ω is a bounded domain, then

$$\mathcal{P}_N(\Omega) = \mathcal{E}. \tag{7.28}$$

Proof. The proof of the assertion (i) is the same as that of Proposition 4.6 in subsection 4.1.2. Hence it is sufficient to prove the assertion (ii).

Let $f \in \mathcal{P}_N(\Omega)$. We claim that $f \in L^\infty(\Omega)$. In fact, by the same argument as the proof of (5.21), we find from the identity (7.21) in Proposition 7.8 and the assertion (i-b) that

$$f = \psi(2^{-2j}\mathcal{H}_N)f + \sum_{k=j+1}^{\infty} \phi_k(\sqrt{\mathcal{H}_N})f = \psi(2^{-2j}\mathcal{H}_N)f \quad \text{in } \mathcal{X}'_N(\Omega) \tag{7.29}$$

for any $j \in \mathbb{Z}$. Hence it follows from the latter part of the assertion (i) in Proposition 7.8 that $f \in L^\infty(\Omega)$. Then, thanks to (7.29), recalling that Ω is a smooth domain, we find from (7.33) in Proposition 7.16 below that

$$\|\nabla f\|_{L^\infty(\Omega)} = \|\nabla \psi(2^{-2j}\mathcal{H}_N) f\|_{L^\infty(\Omega)} \le C 2^j \|f\|_{L^\infty(\Omega)}$$

for any $j \in \mathbb{Z}$, which implies that $\nabla f = 0$ in Ω. Then f is a constant on Ω. Hence we have the inclusion

$$\{0\} \subset \mathcal{P}_N(\Omega) \subset \{f = c \text{ on } \Omega : c \in \mathbb{C}\}. \tag{7.30}$$

Since $\mathcal{P}_N(\Omega)$ is a linear space, we conclude that if $\mathcal{P}_N(\Omega) \neq \{0\}$, then $\mathcal{P}_N(\Omega)$ is the space of all constant functions on Ω. This proves (7.27).

Finally, we consider the case when Ω is a bounded domain. Then it follows from (7.30) that

$$\mathcal{P}_N(\Omega) \subset \mathcal{E}.$$

To prove the converse, since $\mathcal{Z}_N(\Omega) \subset \mathcal{E}^\perp$ by the definition of $\mathcal{Z}_N(\Omega)$, we see from the definition (7.26) of $\mathcal{P}_N(\Omega)$ that

$$\mathcal{E} = (\mathcal{E}^\perp)^\perp \subset \mathcal{Z}_N(\Omega)^\perp \subset \mathcal{P}_N(\Omega).$$

This proves (7.28). The proof of Proposition 7.11 is finished. □

7.3 Fundamental properties of Besov spaces generated by the Neumann Laplacian

In this section we state results on fundamental properties of Besov spaces generated by $\mathcal{H}_N$. The proofs are similar to those of the Dirichlet case. So we may omit the details.

The first result is concerned with completeness of Besov spaces and the relations among Besov spaces, test function spaces and the spaces of distributions.

Theorem 7.12. *Assume that Ω is as in section 7.1. Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Then the following assertions hold*:

(i) (*Inhomogeneous Besov spaces*)

(a) $B^s_{p,q}(\mathcal{H}_N)$ *is independent of the choice of* $\{\psi\} \cup \{\phi_j\}_{j\in\mathbb{N}}$ *satisfying* (4.12), (4.13) *and* (4.14), *and enjoys the following*:

$$\mathcal{X}_N(\Omega) \hookrightarrow B^s_{p,q}(\mathcal{H}_N) \hookrightarrow \mathcal{X}'_N(\Omega).$$

(b) $B^s_{p,q}(\mathcal{H}_N)$ *is a Banach space.*

(ii) (*Homogeneous Besov spaces*)

(a) $\dot{B}^s_{p,q}(\mathcal{H}_N)$ *is independent of the choice of* $\{\phi_j\}_{j\in\mathbb{Z}}$ *satisfying* (4.12) *and* (4.13)*, and enjoys the following*:

$$\mathcal{Z}_N(\Omega) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}_N) \hookrightarrow \mathcal{Z}'_N(\Omega).$$

(b) $\dot{B}^s_{p,q}(\mathcal{H}_N)$ *is a Banach space.*

The following result states the fundamental properties of the Besov spaces such as duality, lifting properties, and embedding relations.

Theorem 7.13. *Assume that* Ω *is as in section 7.1. Let* $s, s_0 \in \mathbb{R}$ *and* $1 \le p, q, q_0, r \le \infty$*. Then the following assertions hold*:

(i) *If* $1 \le p, q < \infty$*,* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$*, then the dual spaces of* $B^s_{p,q}(\mathcal{H}_N)$ *and* $\dot{B}^s_{p,q}(\mathcal{H}_N)$ *are* $B^{-s}_{p',q'}(\mathcal{H}_N)$ *and* $\dot{B}^{-s}_{p',q'}(\mathcal{H}_N)$*, respectively.*

(ii) (a) *The inhomogeneous Besov spaces enjoy the following properties*:

$$(I + \mathcal{H}_N)^{\frac{s_0}{2}} f \in B^{s-s_0}_{p,q}(\mathcal{H}_N) \quad \textit{for any } f \in B^s_{p,q}(\mathcal{H}_N);$$

$$B^{s+\varepsilon}_{p,q}(\mathcal{H}_N) \hookrightarrow B^s_{p,q_0}(\mathcal{H}_N) \quad \textit{for any } \varepsilon > 0;$$

$$B^s_{p,q}(\mathcal{H}_N) \hookrightarrow B^{s_0}_{p,q}(\mathcal{H}_N) \quad \textit{if } s \ge s_0;$$

$$B^{s+d(\frac{1}{r}-\frac{1}{p})}_{r,q}(\mathcal{H}_N) \hookrightarrow B^s_{p,q_0}(\mathcal{H}_N) \quad \textit{if } 1 \le r \le p \le \infty \textit{ and } q \le q_0.$$

(b) *The homogeneous Besov spaces enjoy the following properties*:

$$\mathcal{H}_N^{\frac{s_0}{2}} f \in \dot{B}^{s-s_0}_{p,q}(\mathcal{H}_N) \quad \textit{for any } f \in \dot{B}^s_{p,q}(\mathcal{H}_N);$$

$$\dot{B}^{s+d(\frac{1}{r}-\frac{1}{p})}_{r,q}(\mathcal{H}_N) \hookrightarrow \dot{B}^s_{p,q_0}(\mathcal{H}_N) \quad \textit{if } 1 \le r \le p \le \infty \textit{ and } q \le q_0.$$

(iii) *We have*

$$L^p(\Omega) \hookrightarrow B^0_{p,2}(\mathcal{H}_N), \dot{B}^0_{p,2}(\mathcal{H}_N) \quad \textit{if } 1 < p \le 2;$$

$$B^0_{p,2}(\mathcal{H}_N), \dot{B}^0_{p,2}(\mathcal{H}_N) \hookrightarrow L^p(\Omega) \quad \textit{if } 2 \le p < \infty.$$

The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{H}_N)$ are the subspaces of $\mathcal{Z}'_N(\Omega)$ by the definition. When Ω is unbounded, $\dot{B}^s_{p,q}(\mathcal{H}_N)$ are also regarded as subspaces of $\mathcal{X}'_N(\Omega)$ if indices s, p and q are appropriately restricted. On the other hand, when Ω is bounded, $\dot{B}^s_{p,q}(\mathcal{H}_N)$ are always regarded as subspaces of $\mathcal{X}'_N(\Omega)$. Summarizing the above considerations, we have the following.

Theorem 7.14. *Let* $1 \le p, q \le \infty$*. Then we have the following*:

(i) *Let Ω be an unbounded Lipschitz domain in $\mathbb{R}^d$ with compact boundary, where $d \geq 3$. If either $s < d/p$ or $(s,q) = (d/p, 1)$, then*

$$\dot{B}^s_{p,q}(\mathcal{H}_N) \cong \left\{ f \in \mathcal{X}'_N(\Omega) : \|f\|_{\dot{B}^s_{p,q}(\mathcal{H}_N)} < \infty,\ f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{\mathcal{H}_N}) f \text{ in } \mathcal{X}'_N(\Omega) \right\}.$$

(ii) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 1$. Then the isomorphism in* (i) *holds also for any $s \in \mathbb{R}$.*

7.4 Bilinear estimates in Besov spaces generated by the Neumann Laplacian

In this section, we discuss the bilinear estimates in Besov spaces generated by the Neumann Laplacian. Following the argument in chapter 5, we see that the gradient estimates play an important role in proving the bilinear estimates.

Let us consider the domain Ω such that the following estimate holds:

$$\|\nabla e^{-t\mathcal{H}_N}\|_{\mathscr{B}(L^\infty(\Omega))} \leq Ct^{-\frac{1}{2}} \tag{7.31}$$

either for any $0 < t \leq 1$, or for any $t > 0$, where $C > 0$ is the constant independent of t. When Ω is an exterior domain in $\mathbb{R}^d$ with compact and smooth boundary, the estimate (7.31) for $t > 0$ is proved by Ishige (see [36]). As to the case when Ω is a bounded domain, we have the following:

Proposition 7.15. *Let Ω be a bounded and smooth domain in $\mathbb{R}^d$ with $d \geq 1$. Then the estimate* (7.31) *holds for any $t > 0$.*

Proof. When Ω is bounded and smooth, the estimate (7.31) for $0 < t \leq 1$ holds (see, e.g., section 1 in [36]). Hence it is sufficient to prove (7.31) for $t \geq 1$. We recall the definition (7.4) of P:

$$P := \chi_{(0,\infty)}(\mathcal{H}_N).$$

We note that

$$\nabla e^{-t\mathcal{H}_N} g = \nabla e^{-t\mathcal{H}_N} g_0^\perp = \nabla e^{-t\mathcal{H}_N} Pg, \tag{7.32}$$

where $g = g_0 + g_0^\perp$ with $g_0 \in \mathcal{E}$ and $g_0^\perp \in \mathcal{E}^\perp$. Then, writing

$$\|\nabla e^{-t\mathcal{H}_N} f\|_{L^\infty(\Omega)} = \|\nabla e^{-\frac{1}{2}\mathcal{H}_N} e^{-\frac{1}{2}\mathcal{H}_N} P e^{-(t-1)\mathcal{H}_N} f\|_{L^\infty(\Omega)},$$

and applying (7.31) for $t = 1/2$ to the right member of the above equation, we get

$$\|\nabla e^{-t\mathcal{H}_N} f\|_{L^\infty(\Omega)} \leq C \|e^{-\frac{1}{2}\mathcal{H}_N} P e^{-(t-1)\mathcal{H}_N} f\|_{L^\infty(\Omega)}.$$

Hence, applying (7.15) to the right member of the above estimate, we find that

$$\|\nabla e^{-t\mathcal{H}_N} f\|_{L^\infty(\Omega)} \le C\|Pe^{-(t-1)\mathcal{H}_N} f\|_{L^2(\Omega)}$$

for any $t > 1$ and $f \in L^\infty(\Omega)$. Here, thanks to L^2-estimate (7.14) and Hölder's inequality, there exists a constant $\mu > 0$ such that

$$\begin{aligned}\|Pe^{-(t-1)\mathcal{H}_N} f\|_{L^2(\Omega)} &\le Ce^{-\mu t}\|f\|_{L^2(\Omega)} \\ &\le C|\Omega|^{\frac{1}{2}} e^{-\mu t}\|f\|_{L^\infty(\Omega)}\end{aligned}$$

for any $t > 1$ and $f \in L^\infty(\Omega)$. Hence, combining two estimates obtained now, we get the estimate (7.31) for any $t > 1$. □

We shall prove here the following.

Proposition 7.16. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^d$ with compact boundary, where $d \ge 3$ if Ω is unbounded, and $d \ge 1$ if Ω is bounded. Let $1 \le p \le \infty$, and let $\{\psi\} \cup \{\phi_j\}_j$ be functions given by* (4.12), (4.13) *and* (4.14). *Then the following assertions hold*:

(i) *Assume further that Ω is a domain such that the gradient estimate* (7.31) *holds for any $0 < t \le 1$. Then there exists a constant $C > 0$ such that*

$$\|\nabla\psi(2^{-2j}\mathcal{H}_N)\|_{\mathscr{B}(L^p(\Omega))} \le C2^j, \tag{7.33}$$

$$\|\nabla\phi_j(\sqrt{\mathcal{H}_N})\|_{\mathscr{B}(L^p(\Omega))} \le C2^j \tag{7.34}$$

for any $j \in \mathbb{N}$.

(ii) *Assume further that Ω is a domain such that the gradient estimate* (7.31) *holds for any $t > 0$. Then the estimates* (7.33) *and* (7.34) *hold for any $j \in \mathbb{Z}$.*

For the proof of Proposition 7.16, we need the following.

Lemma 7.17. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^d$ with compact boundary, where $d \ge 3$ if Ω is unbounded, and $d \ge 1$ if Ω is bounded. Let $\phi \in C_0^\infty(\mathbb{R})$. Then $\phi(\mathcal{H}_N)$ is extended to a bounded linear operator from $L^p(\Omega)$ to $W^{1,p}(\Omega)$ provided that $1 \le p \le 2$. Furthermore, there exists a constant $C > 0$ such that*

$$\|\nabla\phi(\theta\mathcal{H}_N)\|_{\mathscr{B}(L^p(\Omega))} \le C\theta^{-\frac{1}{2}}$$

for any $\theta > 0$.

Proof. Since

$$\begin{aligned}\|\nabla\phi(\theta\mathcal{H}_N)f\|^2_{L^2(\Omega)} &= (\mathcal{H}_N\phi(\theta\mathcal{H}_N)f, \phi(\theta\mathcal{H}_N)f)_{L^2(\Omega)}\\ &\le \|\mathcal{H}_N\phi(\theta\mathcal{H}_N)f\|_{L^2(\Omega)}\|\phi(\theta\mathcal{H}_N)f\|_{L^2(\Omega)},\end{aligned}$$

by using

$$\mathcal{H}_N\phi(\theta\mathcal{H}_N)f = \int_{-\infty}^{\infty}\lambda\phi(\theta\lambda)\,dE_{\mathcal{H}_N}(\lambda)f,$$

we readily see that

$$\|\nabla\phi(\theta\mathcal{H}_N)\|_{\mathscr{B}(L^2(\Omega))} \le C\theta^{-\frac{1}{2}}$$

for any $\theta > 0$. Hence, taking account of the Riesz-Thorin theorem, we have only to prove that

$$\|\nabla\phi(\theta\mathcal{H}_N)\|_{\mathscr{B}(L^1(\Omega))} \le C\theta^{-\frac{1}{2}}. \tag{7.35}$$

When Ω is unbounded, the estimate (7.35) for any $\theta > 0$ is proved in a similar way to the proof of Theorem 3.2. When Ω is bounded, the estimate (7.35) for $0 < \theta \le 1$ is obtained in a similar way to the unbounded case. Hence all we have to do is to prove (7.35) for $\theta > 1$ in the case when Ω is bounded.

By the same argument as in (7.32), we deduce from (7.14) that

$$\begin{aligned}\|\nabla e^{-\theta\mathcal{H}_N}g\|^2_{L^2(\Omega)} &= \|\nabla e^{-\theta\mathcal{H}_N}Pg\|^2_{L^2(\Omega)}\\ &\le \|\mathcal{H}_N e^{-\theta\mathcal{H}_N}Pg\|_{L^2(\Omega)}\|e^{-\theta\mathcal{H}_N}Pg\|_{L^2(\Omega)}\\ &\le \theta^{-1}e^{-2\lambda_2\theta}\|g\|^2_{L^2(\Omega)}\end{aligned} \tag{7.36}$$

for any $g \in L^2(\Omega)$. Now we estimate

$$\begin{aligned}\|\nabla\phi(\theta\mathcal{H}_N)f\|_{L^1(\Omega)} &\le |\Omega|^{\frac{1}{2}}\|\nabla\phi(\theta\mathcal{H}_N)f\|_{L^2(\Omega)}\\ &= |\Omega|^{\frac{1}{2}}\|\nabla e^{-\theta\mathcal{H}_N}\phi(\theta\mathcal{H}_N)e^{2\theta\mathcal{H}_N}e^{-\theta\mathcal{H}_N}f\|_{L^2(\Omega)}\end{aligned} \tag{7.37}$$

for any $f \in L^1(\Omega)\cap L^2(\Omega)$. Then, by using (7.36), we estimate the right member of (7.37) as

$$\begin{aligned}\|\nabla e^{-\theta\mathcal{H}_N}\phi(\theta\mathcal{H}_N)e^{2\theta\mathcal{H}_N}e^{-\theta\mathcal{H}_N}f\|_{L^2(\Omega)} &\le \theta^{-\frac{1}{2}}e^{-\lambda_2\theta}\|\phi(\theta\mathcal{H}_N)e^{2\theta\mathcal{H}_N}e^{-\theta\mathcal{H}_N}f\|_{L^2(\Omega)}\\ &\le C\theta^{-\frac{1}{2}}e^{-\lambda_2\theta}\|e^{-\theta\mathcal{H}_N}f\|_{L^2(\Omega)}\\ &\le C\theta^{-\frac{1}{2}}\theta^{\frac{d}{4}}e^{-\lambda_2\theta}\|f\|_{L^1(\Omega)}\end{aligned} \tag{7.38}$$

for any $f \in L^1(\Omega)\cap L^2(\Omega)$, where we used (7.20) in the last step. Thus, combining (7.37) and (7.38), we conclude the desired L^1-estimate by density argument. The proof of Lemma 7.17 is finished. □

We are now in a position to prove Proposition 7.16.

Proof of Proposition 7.16. We prove only the assertion (ii), since the proof of assertion (i) is similar to that of (ii). Thanks to Lemma 7.17 for $p = 1$ and the Riesz-Thorin interpolation theorem, it suffice to show that

$$\|\nabla\psi(2^{-2j}\mathcal{H}_N)\|_{\mathscr{B}(L^\infty(\Omega))} \le C2^j, \tag{7.39}$$

$$\|\nabla\phi_j(\sqrt{\mathcal{H}_N})\|_{\mathscr{B}(L^\infty(\Omega))} \le C2^j \tag{7.40}$$

for any $j \in \mathbb{Z}$.

When Ω is unbounded, these estimates are immediate consequences of the gradient estimate (7.31) for $t > 0$ and the assertion (i) in Lemma 7.3. In a similar way, when Ω is bounded, the estimate (7.40) is proved by combining the estimate (7.31) with the latter part of the assertion (i) in Lemma 7.3. We have to prove (7.39) for bounded domain case. Let $f \in L^\infty(\Omega)$. Then we see that $f \in L^2(\Omega)$, and hence, following the idea of derivation of (7.32), we write

$$\nabla\psi(2^{-2j}\mathcal{H}_N)f = \nabla\psi(2^{-2j}\mathcal{H}_N)F(2^{-2j}\mathcal{H}_N)f$$

for any $j \in \mathbb{Z}$, where F is a smooth and non-negative function on $\mathbb{R}$ such that

$$F(\lambda) = \begin{cases} 1 & \text{for } \lambda \ge \lambda_2, \\ 0 & \text{for } \lambda \le \dfrac{\lambda_2}{2}. \end{cases}$$

Then, combining the estimate (7.31) with the estimate (7.10) in Lemma 7.3, we deduce that

$$\begin{aligned} \|\nabla\psi(2^{-2j}\mathcal{H}_N)f\|_{L^\infty(\Omega)} &= \|\nabla e^{-2^{-2j}\mathcal{H}_N} e^{2^{-2j}\mathcal{H}_N}\psi(2^{-2j}\mathcal{H}_N)F(2^{-2j}\mathcal{H}_N)f\|_{L^\infty(\Omega)} \\ &\le C2^j\|e^{2^{-2j}\mathcal{H}_N}\psi(2^{-2j}\mathcal{H}_N)F(2^{-2j}\mathcal{H}_N)f\|_{L^\infty(\Omega)} \\ &\le C2^j\|f\|_{L^\infty(\Omega)} \end{aligned}$$

for any $j \in \mathbb{Z}$ and $f \in L^\infty(\Omega)$, since

$$e^\lambda\psi(\lambda)F(\lambda) \in C_0^\infty((0,\infty)).$$

Thus we obtain the estimate (7.33) for $p = \infty$. The proof of Proposition 7.16 is now finished. □

Our final result is as follows.

Theorem 7.18. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^d$ with compact boundary, where $d \ge 3$ if Ω is unbounded, and $d \ge 1$ if Ω is bounded. Let $0 < s < 2$ and p, p_1, p_2, p_3, p_4 and q be such that*

$$1 \le p, p_1, p_2, p_3, p_4, q \le \infty \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then the following assertions hold:

(i) *Assume further that* Ω *is a domain such that the gradient estimate* (7.31) *holds for any* $0 < t \leq 1$. *Then there exists a constant* $C > 0$ *such that*

$$\|fg\|_{B^s_{p,q}(H_N)} \leq C \left(\|f\|_{B^s_{p_1,q}(\mathcal{H}_N)} \|g\|_{L^{p_2}(\Omega)} + \|f\|_{L^{p_3}(\Omega)} \|g\|_{B^s_{p_4,q}(\mathcal{H}_N)} \right)$$

for any $f \in B^s_{p_1,q}(\mathcal{H}_N) \cap L^{p_3}(\Omega)$ *and* $g \in B^s_{p_4,q}(\mathcal{H}_N) \cap L^{p_2}(\Omega)$.

(ii) *Assume further that* Ω *is a domain such that the gradient estimate* (7.31) *holds for any* $t > 0$. *Then there exists a constant* $C > 0$ *such that*

$$\|fg\|_{\dot{B}^s_{p,q}(\mathcal{H}_N)} \leq C \left(\|f\|_{\dot{B}^s_{p_1,q}(\mathcal{H}_N)} \|g\|_{L^{p_2}(\Omega)} + \|f\|_{L^{p_3}(\Omega)} \|g\|_{\dot{B}^s_{p_4,q}(\mathcal{H}_N)} \right)$$

for any $f \in \dot{B}^s_{p_1,q}(\mathcal{H}_N) \cap L^{p_3}(\Omega)$ *and* $g \in \dot{B}^s_{p_4,q}(\mathcal{H}_N) \cap L^{p_2}(\Omega)$.

Proof. Since the gradient estimates are established in Proposition 7.16, the proof is performed by a similar argument as in chapter 5. So we may omit the details. □

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