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Reeb components of leafwise complex foliations and their symmetries III

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Abstract

The automorphisms group of the 3-dimensional Reeb component with complex leaves is computed in the case where the component is obtained by the Hopf construction and the holonomy of the boundary leaf is not tangent to the identity to the infinite order. Combined with a previous work, for 3-dimensional Reeb components obtained by the Hopf construction, we have an almost complete description of the groups of leafwise holomorphic smooth automorphisms.

0 Introduction

This is a sequel to the series of works [HM] and [Ho] in which the groups of leafwise holomorphic smooth automorphisms of Reeb complexes with complex leaves are studied. In this paper we compute the automorpshism groups for three dimensional Reeb components (*i.e.*, with complex one dimensional leaves) obtained by the Hopf construction, in [HM] we computed the automorphism groups in the case where the holonomy $\varphi \in \mathit{Diff}^{\infty}([0,\infty))$ of the boundary leaf is infinitely tangent to the identity at $x = 0$, while the case where the holonomy is tangent to the identity only to a finite degree is left. In this paper we settle down this case, including the case where the holonomy has non-tirivial linear term.

The principal tools in this paper are, Sternberg's linearization [St], Takens' normal forms [Ta], and a classical theory on Fourier expansion of smooth periodic functions. These solve the functional equation $\beta \circ \varphi = \lambda \cdot \beta$ that we treated in [HM] with φ of different type. The result on this equation is identical to that in [HM] while the arguments are quite different and each of them is independent of the other.

After we reviewed the Hopf construction and the basic properties of the automorphism groups in Section 1, we proceed to solving the functional equation in Section 2, which is the main part of the present paper.

In [Ho] the first author computed the automorphism groups of Leafwise complex Reeb component of real dimension 5 with holonomy infinitely tangent to the identity, based on the Kodaira's classification of Hopf surfaces. Basically it must be possible to obtain similar results for the case where the holonomy only finitely tangent to the identity, while the classification and computations might be much more complicated.

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In this paper we follow the notations and the basic arguments in [HM].

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1 Hopf construction and automorphism groups

In this section we review the basic and relevant notions and results mainly from [HM] and also set up the situation which is particular for this paper. Most of basic results in [HM] are valid also under the situation in this paper.

A 3-dimensional Reeb component is a compact 3-manifold $R = D^2 \times S^1$ with a (smooth) foliation of codimension one, where the boundary torus is a compact leaf and the interior is foliated by planes and they spiral around the boundary leaf in a simple way. In this paper we deal with Reeb components obtained by the Hopf construction which gives leafwise complex structure as well. From only the differentiable view point every Reeb component comes from the Hopf construction. If we take the leafwise complex structure into account it is not plausible.

Let $\tilde{R}$ be $\mathbb{C} \times \mathbb{R}_{\geq 0} \setminus \{(0,0)\}$, take $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, and $\varphi \in \mathit{Diff}^{\infty}(\mathbb{R}_{\geq 0})$ be a diffeomorphism of the half line $\mathbb{R}_{\geq 0} = [0, +\infty)$ satisfying $\varphi(x) - x > 0$ for $x > 0$. The origin is the unique and hyperbolic fixed point of φ.

Let $G : \mathbb{C} \to \mathbb{C}$ be the multiplication by λ and $T : \tilde{R} \to \tilde{R}$ be $T = G \times \varphi$. Then we obtain a Reeb component $(R, \mathcal{F}, J) = (\tilde{R}, \tilde{\mathcal{F}}, J_{\mathrm{std}})/T^{\mathbb{Z}}$ as the quotient, as well as the boundary elliptic curve $H = \mathbb{C} \setminus \{0\}/G^{\mathbb{Z}}$. Here, on the upstairs the leaves of the foliation $\tilde{\mathcal{F}} = \{\mathbb{C} \times \{x\} | x > 0\} \sqcup \{\mathbb{C}^* \times \{0\}\}$ are equipped with the natural complex structure J_{std} which is inherited by those of $\mathcal{F}$. The usual modulus of H is $(-2\pi i)^{-1} \log \lambda$.

Now we treat the case where the holonomy φ is tangent to the identity only to a finite order. Namely, we assume

(i) $\varphi'(0) = \mu > 1$

or

(ii) $\varphi'(0) = 1$, $\varphi''(0) = \cdots = \varphi^{(n-1)}(0) = 0$, $\varphi^{(n)}(0) > 0$ for some $n > 1$.

Sternberg for the case (i) and Takens for (ii) showed (see Theorem 2.1 and 2.2) φ is the time one map of the flow generated by a smooth vector field X on $[0, \infty)$ and therefore the centralizer Z_φ of φ in $\mathit{Diff}^{\infty}([0,\infty))$ is the set of exponential maps, namely $Z_\varphi = \{\exp(tX)\,;\, t \in \mathbb{R}\} \cong \mathbb{R}$.

Let $AutR = Aut(R, \mathcal{F}, J)$ be the group of automorphisms of $(R, \mathcal{F}, J)$, namely, the set of diffeomorphisms of R which preserve the foliation $\mathcal{F}$ and are holomorphic on each leaves. Also let $AutH$ denote the group of holomorphic automorphisms of the boundary elliptic curve H and Aut_0H its connected component including the identity. Aut_0H consists exactly of the translations as two dimensional torus.

Propoition 1.1 (Proposition 4.1, [HM]) *The image of the restriction map $AutR \to AutH$ is Aut_0H.*

The proof in [HM] holds. The kernel of the restriction map is denoted by $Aut(R,H)$. Then, as remarked above, in our case the centralizer Z_φ of the holonomy φ in $Diff^\infty([0,\infty))$ consists of the smooth one parameter family $Z_\varphi = \{\exp(tX)\,;\,t\in\mathbb{R}\}$ and it implies the following.

Theorem 1.2 (Theorem 4.8, [HM]) The restriction map $AutR \twoheadrightarrow Aut_0H$ splits smoothly. Namely, it has a natural right inverse $Aut_0H \to Aut(R,\mathcal{F},J)$ which is a homomorphism and is continuous in the smooth sense.

The kernel is denoted by $Aut(R,H)$. We have the following split exact sequence

$$1 \to Aut(R,H) \to AutR \to AutH \to 1\,.$$

Therefore $AutR$ is the semi-direct product $AutR \cong Aut(R,H) \rtimes Aut_0H$.

Also the group $AutR$ is naturally isomorphic to the quotient group $Aut(\tilde{R}\,;T)/T^{\mathbb{Z}}$, where $Aut(\tilde{R}) = Aut(\tilde{R},\tilde{\mathcal{F}},J_{\mathrm{std}})$ denotes the group of automorphims of the universal covering $(\tilde{R},\tilde{\mathcal{F}},J_{\mathrm{std}})$ and $Aut(\tilde{R}\,;T)$ the centralizer of T in $Aut\tilde{R}$, while the subgroup $Aut(R,H)$ is isomorphic to the stabilizer $Aut(\tilde{R},\tilde{H};T)$ of the boundary $\tilde{H} = \mathbb{C}\setminus\{\mathrm{O}\}$ in $Aut(\tilde{R}\,;T)$ (we do not have to take the quotient) through this correspondence.

An element $f \in Aut(\tilde{R},\tilde{H};T)$ admits a presentation $f(z,x) = (z + b(x), \eta(x))$ ($b(0)=0$). Note that the first component is an isomorphism from $\mathbb{C}$ to $\mathbb{C}$ so that it is a linear function depending on x, while the assumption that it is identical on the boundary implies that the linear term is identical regardless to x. Therefore taking the vertical component $\eta \in Z_\varphi$ of f, we obtain a surjective homomorphism $Aut(\tilde{R},\tilde{H};T) \twoheadrightarrow Z_\varphi \cong \mathbb{R}$. Apparently $\eta \to f = (\mathrm{id}_{\mathbb{C}},\eta)$ is a right inverse of this surjection. $Aut(\tilde{R},\tilde{H};T)$ is again a semi-direct product. Let $\mathcal{K}$ denote the kernel. $\mathcal{K}$ consists of $f(z,x) = (z+\beta(x),x)$ such that β is a smooth $\mathbb{C}$-valued function on $[0,\infty)$ satisfying the equation

$$(\mathrm{I}) \ :\ \beta(\varphi(x)) = \lambda\beta(x).$$

This equation is nothing but the condition for f to commute with T. Then the the groups $Aut(R,H) \cong Aut(\tilde{R},\tilde{H};T)$ is presented as

$$Aut(R,H) \cong \mathcal{K} \rtimes Z_\varphi\,.$$

The right component η acts on the left β's as $\beta \mapsto \beta\circ\eta$. The space $\mathcal{K}$ is a complex vector space and is quite different between the case (i) and (ii).

Summarizing above all, we have the next.

Theorem 1.3 (Theorem 4.7 and 4.8 in [HM]) The automorphism group $AutR$ admits the following description;

$$\begin{aligned} AutR &\cong (\mathcal{K}\rtimes Z_\varphi)\rtimes Aut_0H &&\cong \mathcal{K}\rtimes\{(Z_\varphi\times\mathbb{C}^*)/T^{\mathbb{Z}}\} \\ &\cong (\mathcal{K}\rtimes\mathbb{R})\rtimes\mathbb{R}^2/\mathbb{Z}^2 &&\cong \mathcal{K}\rtimes((\mathbb{R}\times\mathbb{C}^*)/T^{\mathbb{Z}}) \end{aligned}$$

where $\mathbb{C}^*$ acts on $\mathcal{K}$ by multiplication of the inverse on the value. In an abstract sense it is also presented as follows.

$$AutR \cong \mathcal{K} \rtimes (\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}) \cong \{\mathcal{K} \rtimes (\mathbb{R}/\mathbb{Z} \times \mathbb{R})\} \times \mathbb{R}/\mathbb{Z}$$

Now the principal aim of this paper is to detrmine the space $\mathcal{K}$ of solutions to (I). It will be done in the next sections.

2 Functional equations and ordinary equations

The goal of this section is to determine the space of solutions to the equation (I) under the condition (i) or (ii). We also pay attention to the system of equations (II) on β_1, $\beta_2 \in C^\infty([0,\infty);\mathbb{C})$, which is needed for the symmetries of 5 or higher-dimensional Reeb components with complex leaves.

Equation (I) : $\beta(\varphi(x)) = \lambda\beta(x)$.

Equation (II) : $\beta_1(\varphi(x)) = \lambda\beta_1(x) + \beta_2(x), \quad \beta_2(\varphi(x)) = \lambda\beta_2(x)$.

We review the works by Sternberg and by Takens which play crucial roles in solving the equation. In fact it enables us to reduce the problem into a linear homogeneous first order ordinary differential equation.

2.1 Normal forms due to Sternberg and Takens

Theorem 2.1 (Sternberg, [St]) Under the condition (i) there exists a diffeomorphism $h \in \mathit{Diff}^\infty([0,\infty))$ which conjugates φ into a linear diffeomorphism $\psi(x) = \mu x$ where $\mu = \varphi'(0)$. Namely,

$$\psi = h^{-1} \circ \varphi \circ h$$

holds.

Theorem 2.2 (Takens, [Ta]) Under the condition (ii) there exists a diffeomorphism $h \in \mathit{Diff}^\infty([0,\infty))$ which conjugates φ into a diffeomorphism $\psi \in \mathit{Diff}^\infty([0,\infty))$ of the following polynomial type on $[0, x_1]$ $(\exists x_1 > 0)$

$$\psi(x) = x + x^n + \alpha x^{2n-1} \quad \text{and} \quad \psi = h^{-1} \circ \varphi \circ h$$

where the coefficient $\alpha \in \mathbb{R}$ is determined by the $(2n-1)$-jet of φ at $x = 0$. As the time one map of the flow generated by a vector field

$$X = \rho(x)\frac{d}{dx}, \quad \rho(x) = x^n + ax^{2n-1} \text{ on } [0, x_2]\ (\exists x_2 > 0)$$

for $a \in \mathbb{R}$ is conjugated to $x \mapsto x + x^n + (a + n/2)x^{2n-1}$ near $x = 0$, there also exists a diffeomorphism $k \in \mathit{Diff}^\infty([0,\infty))$ which conjugates φ into such an exponential for $a = \alpha - n/2$, namely

$$\exp X = k^{-1} \circ \varphi \circ k\,.$$

In both cases of (i) and (ii) we can assume that our φ is in such normal forms. The results in two cases are quite different.

2.2 Case (i): Linearizable holonomy

In the case (i) we have very few solutions to the equation (I).

Propoition 2.3 1) (Resonant case) If $\lambda = \mu^n$ is satisfied for some $n \in \mathbb{N}$, then the solution β to the equation (I) is a monomial of order n. Namely we have the 1-dimensional space $\mathcal{K} = \{c \cdot x^n \,;\, c \in \mathbb{C}\}$ of solutions.
2)(Non-resonant case) If no positive integer $n \in \mathbb{N}$ satisfies $\lambda = \mu^n$, then there exists no solution to (I) but $\beta(x) \equiv 0$, and we have $\mathcal{K} = \{0\}$.

Corollary 2.4 If the holonomy φ has the non-trivial linear part, the automorphism group $AutR$ is a solvable Lie group of dimension 3 or 5, depending on the resonance condition, $\log \lambda / \log \mu \notin \mathbb{N}$ or $\in \mathbb{N}$.

Proof. From Sternberg's linearization theorem, we are allowed to assume '$\varphi(x) = \mu x$' for some real number $\mu > 1$. Therefore the equation (I) takes the following form.

$$\beta(\mu x) = \lambda \beta(x) \qquad \text{for } x \in [0, \infty)\,.$$

Differentiating this equation for arbitrary many times at $x = 0$, we see that the Taylor expansion at $x = 0$ can be non-trivial only at the degree $n = \log \lambda / \log \mu$. Therefore in the resonant case, the possibity is $\beta(x) = c \cdot x^n + f(x)$ where $f(x)$ is a flat function. Then, as $c \cdot x^n$ is a solution to (I), so is $f(x)$. In the non-resonant case, only flat functions are not yet excluded.

However, in both cases, if we had a nontrivial flat solution $f(x)$, it would contradict as follows. Take $x_0 \in (0, \infty)$ with $f(x_0) \neq 0$ and look at $f(\mu^{-k} x_0) = \lambda^{-k} f(x_0)$ for $k \in \mathbb{N}$. On the other hand, as f is flat we have $\lim_{x \to 0} f(x)/x^l = 0$ for any $l \in \mathbb{N}$. So large enough l ($\geq |\log \lambda / \log \mu|$) gives rise to a contradiction. □

Remark 2.5 1) Above argument is nothing but the well-known proof for the fact that weighted homogeneous functions wich are smooth at the origin is an monomial.
2) For the equation (II), of course in the non-resonant case we only have $(0,0)$ as a solution. In the resonant case, only $(\beta_1, \beta_2) = (c \cdot x^n, 0)$ is possible. So the space of solution is of dimension one.

2.3 Case (ii) : Higher order tangency

Let us consider the case (ii). Thanks to Takens' theorem, it is allowed to assume the holonomy φ is of the form

$$\varphi = \exp X_{n,a}\,, \quad X_{n,a} = \rho_{n,a}(x) \frac{d}{dx}\,, \quad \rho_{n,a}(x) = x^n + a x^{2n-1} \text{ on } [0, x_0]$$

for some $n \geq 2$, $a \in \mathbb{R}$, and $x_0 > 0$.

The method employed in [HM] seems not to work in this case. So we adopt a different way which consists of two steps.

First we modify the functional equation (I) into an ordinary differential equation for each choice of $\log \lambda \in \mathbb{C}$. Of course we have the ambiguity of $2\pi i\mathbb{Z}$ ($i = \sqrt{-1}$) in fixing $\log \lambda \in \mathbb{C}$ so that we have countably many ODE's for each of which the space of solutions is of $\dim_{\mathbb{C}} = 1$.

Next we take the some of these solutions for a λ and obtain the complete space of solutions to (I). This process is nothing but the Fourier expansion of the solution of (I).

After all the result is the same as in the case treated in [HM]. Let us first state it before proving. Recall we are considering the equation (I) for $\varphi \in \mathit{Diff}^{\infty}(\mathbb{R}_{\geq 0})$ which is tangent to the identity exactly to the $(n-1)$-th order at $x = 0$ and satisfies $\varphi(x) - x > 0$ for $x > 0$ and a complex number λ with $|\lambda| > 1$. Let us also recall the equations(s) on on β, β_1 and β_2 $\in C^{\infty}(\mathbb{R}_{\geq 0}; \mathbb{C})$.

Equation (I) : $\quad \beta(\varphi(x)) = \lambda\beta(x).$

Equation (II) : $\quad \beta_1(\varphi(x)) = \lambda\beta_1(x) + \beta_2(x), \quad \beta_2(\varphi(x)) = \lambda\beta_2(x).$

First consider these equations on $(0, \infty)$. We fix any non-zero solution $\beta^*(x) \in C^{\infty}((0, \infty); \mathbb{C}^*)$ to (I). (Later in Step 2, we will make a more specific choice of β^*.) Then each solution corresponds to a smooth function on $S^1 = (0, \infty)/\varphi^{\mathbb{Z}}$ by taking $\beta \mapsto \beta/\beta^*$. This gives a bijective correspondence between the space $\mathcal{Z} = \mathcal{Z}_{\varphi,\lambda}$ of solutions to (I) considered on $(0, \infty)$ and $C^{\infty}(S^1; \mathbb{C})$ as vector spaces.

Also take the space $\mathcal{S} = \mathcal{S}_{\varphi,\lambda}$ of solutions to Equation (II) on $(0, \infty)$. The kernel of the projection $P_2 : \mathcal{S} \to \mathcal{Z}$ assigning β_2 to a solution (β_1, β_2) coincides with $\mathcal{Z}$. The projection P_2 is surjective because for any $\beta_2 \in \mathcal{Z}$

$$\beta_1(x) = \frac{1}{\lambda \log \lambda} \beta_2(x) \log \beta^*(x)$$

gives a solution $(\beta_1, \beta_2) \in \mathcal{S}$, where for $\log \beta^*(x)$ any smooth branch can be taken. Therefore $\mathcal{S}$ admits a the short exact sequence of vector spaces;

$$0 \to \mathcal{Z} \to \mathcal{S} \to \mathcal{Z} \to 0\,.$$

Theorem 2.6 1) Any solution $\beta \in \mathcal{Z}$ extends to $\mathbb{R}_{\geq 0}$ so as to be a smooth function which is flat at $x = 0$, *i.e.*, k-th jet satisfies $j^k\beta(0) = 0$ for any $k = 0, 1, 2, \cdots$.
2) The same applies to any solution $(\beta_1, \beta_2) \in \mathcal{S}$.

In particular, the space $\mathcal{Z} = \mathcal{Z}_{\varphi,\lambda}$ is nothing but the kernel $\mathcal{K}$ in Theorem 1.3, where it is taken that the solutions β is extended to $x = 0$.

Step 1. Let Λ be one of the values of $\log \lambda$ so that $e^{\Lambda} = \lambda$. We consider the following ordinary equation on $(0, \infty)$ instead of (I).

$$(\mathrm{I} - \Lambda) : \qquad \beta'(x) = \frac{\Lambda}{\rho(x)} \beta(x)$$

This is of course equivalent to the following ODE in the variable t.

$$\left.\frac{d}{dt}\right|_{t=0} \beta(\exp(tX)(x)) = \Lambda \cdot \beta(x)$$

Therefore any solution β is presented as $\beta(\exp(tX)(x_0)) = e^{\Lambda t} \cdot \beta(x_0) = C \cdot e^{\Lambda t}$ for a constant $C \in \mathbb{C}$. It is also clear that β satisfies the equation (I) on $(0, \infty)$. In the variable x, $\beta(x)$ is presented as $\beta(x) = C \cdot \exp\left(\Lambda \int_{x_0}^{x} \frac{1}{\rho(x)}\right) dx$. In particular on $(0, x_0)$, we have

$$\beta(x) = C \cdot \exp\left((R + i\theta_l) \int_{x_0}^{x} \frac{1}{x^n(1 + ax^{n-1})} dx\right)$$

where we choose and fix a branch of $\log \lambda$ to be $R + i\theta_0$ where the real part $R = |\log \lambda|$ is positive. Then other general branches Λ are given as $\Lambda = R + i\theta_l$, $\theta_l = \theta_0 + 2l\pi$ for $l \in \mathbb{Z}$. It is easy to compute the integration but we only need to remark that for some $\delta > 0$ and any $x \in (0, \delta)$ we have

$$\left|\int_{x_0}^{x} \frac{1}{x^n(1 + ax^{n-1})} dx\right| \geq \left|\frac{1}{2x}\right| .$$

Propoition 2.7 The solution $\beta(x)$ to (I-Λ) is extended to $[0, \infty)$ as $\beta(0) = 0$, and then $\beta(x)$ is smooth and flat at $x = 0$.

Proof. The derivative of β of any order k is a multiplication of β and some rational function in the variable x. Therefore for any $k \in \mathbb{N}$ we have

$$|\beta^{(k)}(x)| \leq |\text{a rational function}| \times \exp(-\frac{1}{2x}) \to 0 \ (x \to 0)$$

which suffices to show the smoothness and flatness of β at $x = 0$. □

In the next step we will need to take a slightly closer look at those rational functions.

Now for each $l \in \mathbb{Z}$, let $\beta_{\langle l \rangle}$ denote the solution to the homogeneous linear ODE (I-$(R + i\theta_l)$) of the first order which satisfies $\beta_{\langle l \rangle}(x_0) = 1$ and it is taken that $\beta_{\langle l \rangle}$ is extended to $[0, \infty)$ as above. So far we have obtained the followings.

Propoition 2.8 The vector space over $\mathbb{C}$ spanned by $\beta_{\langle l \rangle}$'s ($l \in \mathbb{Z}$) is contained in the space of smooth solutions of the functional equation (I) for λ. It is also contained in $\mathcal{Z}_{\varphi,\lambda}$.

Step 2. By taking appropriate completion of the span of $\beta_{\langle l \rangle}$'s ($l \in \mathbb{Z}$), we show that $\mathcal{Z}_{\varphi,\lambda}$ coincides with the space of all solutions to (I) (Theorem 2.6).

First we make the correspondence between $\mathcal{Z}_{\varphi,\lambda}$ and $C^\infty(S^1; \mathbb{C})$ clearer. Take $\beta_{\langle 0 \rangle}$ as β^* in defining the correspondence. Also take t-coordinate instead of $x \in (0, \infty)$ by putting $x = \exp(tX)(x_0)$ ($t \in \mathbb{R}$) and regard $\check{\beta}(t) =$

$\beta(\exp(tX)(x_0))/\beta_{\langle 0\rangle}(\exp(tX)(x_0))$ as the element of $C^\infty(S^1;\mathbb{C})$ where S^1 is regarded as $\mathbb{R}/\mathbb{Z}$. From direct computations we see

$$\check{\beta}_{\langle l\rangle}(t) = e^{2\pi lt\cdot i} \qquad \text{for } \ l \in \mathbb{Z}$$

so that $\check{\beta}_{\langle l\rangle}$'s ($l \in \mathbb{Z}$) form a Fourier basis for $C^\infty(S^1;\mathbb{C})$. The following well-known fact well fits into our situation.

Theorem 2.9 (see *e.g.*, [Ka]) The infinite sum with coefficients $c_l \in \mathbb{C}$

$$\sum_{l=-\infty}^{\infty} c_l \cdot e^{i\theta}$$

defines a smooth function on $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ if and only if the sequence of coefficients $\{c_k\}_{k\in\mathbb{Z}}$ is rapidly decreasing, namely it satisfies

$$\sum_{l=-\infty}^{\infty} |l|^d |c_l| < \infty \quad \text{for any } \ d \in \mathbb{N}\,.$$

Therefore any $\check{\beta} \in \mathcal{Z}_{\varphi,\lambda}$ is given as an infinite sum

$$\check{\beta} = \sum_{l=-\infty}^{\infty} c_l \cdot \check{\beta}_{\langle l\rangle}$$

with a rapidly decreasing sequence of coefficients $\{c_k\}_{k\in\mathbb{Z}}$. Our main theorem is stated as follows.

Theorem 2.10 The infinite sum

$$\beta(x) = \sum_{l=-\infty}^{\infty} c_l \cdot \beta_{\langle l\rangle}(x)$$

with a rapidly decreasing sequence of coefficients $\{c_k\}_{k\in\mathbb{Z}}$ is smooth and flat at $x = 0$.

Now we investigate the derivatives of $\beta_{\langle l\rangle}(x)$ in slightly more detail. Recall that n, a, and λ are already fixed.

Lemma 2.11 For $k \in \mathbb{N}$ and $j = 1, \cdots, k$, there exists a fixed polynomial $Q_{k,j}(x)$ which satisfies on $(0, x_0)$

$$\beta^{(k)}_{\langle l\rangle}(x) = \left\{ \frac{1}{P(x)^k} \sum_{j=1}^{k} Q_{k,j}(x)(R + i\theta_l)^j \right\} \beta_{\langle l\rangle}(x), \quad P(x) = x^n + ax^{2n-1}$$

and $Q_{k,j}(x)$ is a linear combination of multiplications of $(k-j)$-many of $P(x)$, $P'(x)$, $\cdots$, $P^{(k-j)}(x)$, with total degree of differentiation $(k-j)$.

For example, $Q_{k,k}(x) = 1$, $Q_{k,k-1}(x) = (1-k)P'(x)$, and so on. The lemma is easily proved by induction on k.

Let us develop $(R + i\theta_l)^j$ into a polynomial of l as follows.

$$(R + i\theta_l)^j = (R + i(\theta_0 + 2\pi l))^j = \sum_{d=0}^{j} R_{j,d} l^d$$

Here the constants $R_{j,d}$ ($j \in \mathbb{N}$, $d = 0, \cdots, j$) are determined by R and θ_0.

Proof of Theorem 2.10. For a rapidly decreasing sequence

$$\{c_k\}_{k\in\mathbb{Z}} \quad \text{with} \quad \sum_{l=-\infty}^{\infty} |l|^d |c_l| = M_d < \infty \ \text{ for } \forall d \in \mathbb{N} \cup \{0\}$$

take $\beta(x) = \sum_{l=-\infty}^{\infty} c_l \cdot \beta_{\langle l\rangle}(x)$. Then we have the following estimate;

$$\begin{aligned}
|\beta^{(k)}(x)| &= \left| \sum_{l=-\infty}^{\infty} c_l \cdot \beta_{\langle l\rangle}^{(k)}(x) \right| \\
&= \left| \sum_{l=-\infty}^{\infty} c_l \cdot \left\{ \frac{1}{P(x)^k} \sum_{j=1}^{k} Q_{k,j}(x)(R + i\theta_l)^j \right\} \beta_{\langle l\rangle}(x) \right| \\
&\le \left| \frac{1}{P(x)^k} \right| \left| \sum_{j=1}^{k} Q_{k,j}(x) \sum_{l=-\infty}^{\infty} c_l \left(\sum_{d=0}^{j} R_{j,d} l^d \right) \beta_{\langle l\rangle}(x) \right| \\
&= \left| \frac{1}{P(x)^k} \right| \left| \sum_{j=1}^{k} \sum_{d=0}^{j} Q_{k,j}(x) R_{j,d} \left(\sum_{l=-\infty}^{\infty} c_l \cdot l^d \right) \beta_{\langle l\rangle}(x) \right| \\
&\le \left\{ \left| \frac{1}{P(x)^k} \right| \sum_{j=1}^{k} \sum_{d=0}^{j} |Q_{k,j}(x)||R_{j,d}| M_d \right\} |\beta_{\langle 0\rangle}(x)| \to 0 \quad (x \to 0+0)
\end{aligned}$$

because the last $\{\cdots\}$ is a rational function when x is close to 0. Also this computation shows the validity of the first equality. □

Remark 2.12 For the equation (II), the smoothness and flatness of $\beta \log \beta_{\langle 0\rangle}$ for a solution β to (I) follow from more or less the same arguments, because $\log \beta_{\langle 0\rangle} = (R + i\theta_0) \int_{x_0}^{x} \frac{1}{P(x)} dx$.

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