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Geometry and dynamics of Engel structures

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The aim of this paper is to extend basic understanding of Engel structures through developing geometric constructions which are canonical to a certain degree and the dynamics of Cauchy characteristics in the transverse spaces which may exhibit elliptic, parabolic, or hyperbolic natures in typical cases.

Contents

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Subsections with * deal with issues which are new at least to the author.

0 Introduction

This article is devoted to extending our basic understandings of Engel structures from two points of view. As basic constructions, the Cartan prolongation from the contact 3-manifolds and the Lorentz prolongation from the Lorentzian 3-manifolds are known since Elie Cartan [C]. We review these constructions and introduce a new one among several other methods of basic constructions, which is called the *pre-quantum prolongation*. Not only review the formality of these constructions, we give many examples of these constructions. One big class of examples are coming from surfaces with metrics. The second aim is to discuss the nature of the dynamics of the Cauchy characteristic acting on the even contact structure modulo the Cauchy characteristic. Kotschick and Vogel proposed the notion of (weak-)hyperbolicity for this action and raised very illustrative examples. We will develop slightly further this notion and formulate the ellipticity, parabolicity, or hyperbolicity of this action.

The Cartan prolongation provides elliptic structures in this sense. Looking at hyperbolic structures or negatively curved metrics on surfaces, the Lorentz prolongation provides a bunch of hyperbolic ones. Therefore looking for "parabolic" structures coming from to a certain degree canonical constructions is one of the motivation and in fact a strong driving force of this study.

Recently from the point of view of construction of Engel structures, after the break through by Vogel [V], an h-principle oriented study has been developed . However the notion of Engel structure has not yet been widely common. So we start with the article with some standard definitions and then proceed to provide basic constructions including classical ones, *i.e.*, the Cartan and Lorentz prolongations, as well as a new one. Then in later sections after some basic notions on the dynamics of Engel structures have been introduced, more detailed observations are given to the Lorentz prolongations coming from surfaces with definite or non-definite Riemannian metrics and also to the new construction. In this article we assume that manifolds, distributions, and any other objects are smooth unless otherwise stated.

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1 Basic concepts and constructions in the study of Engel structures

1.1 Basic definitions

An Engel structure $\mathcal{D}$ on a 4-manifold M is a 2-dimensional distribution which satisfies the following non-integrability condition on the derived distributions;

$$\text{(D1)} \quad : \operatorname{rank} \mathcal{E} \equiv 3 \quad \text{where} \quad \mathcal{E} = [\mathcal{D}, \mathcal{D}],$$

and

$$\text{(D2)} \quad : \operatorname{rank} [\mathcal{E}, \mathcal{E}] \equiv 4\,, \ \textit{i.e.}, \ [\mathcal{E}, \mathcal{E}] \equiv TM.$$

Here by abuse of notation, the distributions also denote the sheaves of germs of their smooth sections. It is well-known and easy to show that in general for any smooth distribution $\mathcal{A}$ of constant rank, $\mathcal{A} \subset [\mathcal{A}, \mathcal{A}]$ and the Lie bracket $[\,,\,] : \mathcal{A} \otimes \mathcal{A} \to [\mathcal{A}, \mathcal{A}]/\mathcal{A}$ is skew symmetric and tensorial. Therefore the surjectivity of $[\,,\,] : \mathcal{E} \otimes \mathcal{E} \to [\mathcal{E}, \mathcal{E}]/\mathcal{E} = TM/\mathcal{E}$ implies the annihilator $\mathcal{W} \subset \mathcal{E}$ is of constant rank one. Now we see that $\mathcal{W}$ is contained in $\mathcal{D}$ becuase otherwise we have $\mathcal{D} \cong \mathcal{E}/\mathcal{W}$ which implies $[\,,\,] : \mathcal{D} \otimes \mathcal{D} \to TM/\mathcal{E}$ is surjective while we have $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$.

This line field $\mathcal{W}$ is called the *Cauchy characteristic* of $\mathcal{D}$, or *characteristic foliation*, *characteristic line field*, *etc*... and plays a fundamental role in the study of Engel structures. Let $\mathcal{F}_{\mathcal{W}}$ denote the 1 dimensional foliation which $\mathcal{W}$ defines.

Let us take a flow box neighborhood U of any point $P \in M$, *i.e.*, a coordinate neighborhood $U \equiv D^4$ with local coordinate (X, Y, Z, W) such that the Cauchy characteristic foliation $\mathcal{F}_{\mathcal{W}}$ coincides with the local foliation $\{X \equiv Y \equiv Z \equiv \text{constant}\}$. By definition $\mathcal{E}$ is invariant under the flow along $\mathcal{W}$ and induces an invariant plane field on the transverse spaces to this foliation. Therefore on the local 3-dimensional quotient space $U/\mathcal{F}_{\mathcal{W}} = \{(x, y, z)\}$ a plane field ξ is induced. This plane field is in fact a contact plane field because of (D2). For this reason, we call $\mathcal{E}$ the *even contact structure* associated with the Engel structure $\mathcal{D}$.

We can modify the local coordinate into (x, y, z, w) as follows. First if we need we take U smaller such that $\xi = \ker [dy - z dx]$ on $U/\mathcal{F}_{\mathcal{W}}$ and at the point P $\mathcal{D}_P = \ker dy \cap \ker dz$ hold, thanks to the conatct Darboux theorem (See [G] for the fundamental theory for contact structures). If necessary we take U further smaller and we can define the function w so as to satisfy $\mathcal{D} = \ker [dz - w dx] \cap \mathcal{E}$. The condition (D1) implies dw never vanishes on U and dx, dy, dz, and dw are linearly independent. Therefore we can take the local coordinate (x, y, z, w) such that $\mathcal{E}$ is defined by the 1-form $dy - zdx$ and $\mathcal{D}$ is defined by the pair of 1-forms

$$dy - zdx\,, \qquad dz - wdx.$$

We call this coordinate neighborhood the *Engel-Darboux* coordinate neighborhood (or 'chart' for short).

As the Engel-Darboux chart suggests, the most important and fundamental example of Engel manifolds is the space $J^2(1,1)$ of 2-jets of smooth functions of one variable. As a space $J^2(1,1)$ is nothing but $\mathbb{R}^4 = \{(x, y, z, w)\}$. Let us consider smooth functions $y = f(x)$ and its derivatives $z = f'(x)$ and $w = f''(x)$. Then its 2-jet graph $\Gamma J^2(f) = \{(x, y, z, w) = (x, f(x), f(x)', f(x)'')\}$ is tangent to the canonical plane field $\mathcal{D} = \mathcal{D}_0$ defined by the pair of 1-forms $dy - zdx$ and $dz - wdx$. Let us fix a particular frame

$$[\mathrm{EF}]: \qquad X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}, \quad W = \frac{\partial}{\partial w}$$

adapted to the Engel-Darboux coordinate. Of course the Cauchy cahracteristic of this Engel structure is spanned by W and the even contact structure $\mathcal{E}$ is defined by the 1-form $dy - zdx$. Therefore the full flag associated with the Engel stucture is indicated as follows.

$$\begin{array}{ccccccccc} 0 & \subset & \mathcal{W} & \subset & \mathcal{D} & \subset & \mathcal{E} & \subset & T\mathbb{R}^4 \\ & & \| & & \| & & \| & & \| \\ & & \langle W\rangle & & \mathcal{W}\oplus\langle X\rangle & & \mathcal{D}\oplus\langle Z\rangle & & \mathcal{E}\oplus\langle Y\rangle \end{array}$$

Remark 1.1 The vector field X looks just complementary, while in the Lorentz prolongation it has a certain importance. See Subsection 1.3.

The coordinates (x, y, z, θ) with the pair of 1-forms

$$dy - zdx \quad \text{and} \quad \cos\theta\, dz - \sin\theta\, dx$$

also define an Engel structure D_l on $\mathbb{R}^4$. Its restriction $\mathcal{D}_l|_{\mathbb{R}^3\times(-\pi/2,\pi/2)}$ is isomorphic to the standard Engel structure $\mathcal{D}_0$ through the identification $w = \tan\theta$. In a Engel manifold we can take local coordinates as above. Then we call them a *long Engel-Darboux* coordinate, no matter how long we can take θ along $\mathcal{W}$ curves.

Once we have an Engel structure $\mathcal{D}$ on M, it gives rise to a full flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$. We have four real line bundles none of which is necessarily oriented, while their orientabilities are related to a certain degree. First of all, $TM/\mathcal{W}$ is oriented because $\mathcal{E}/\mathcal{W}$ is a contact structure on this 3-dimensional space. $\mathcal{W}$ is not necessarily oriented, but if we give locally an orientation to $\mathcal{W}$, through the movement of $\mathcal{D}/\mathcal{W}$ in $\mathcal{E}/\mathcal{W}$ along $\mathcal{W}$, it also defines an orientation of $\mathcal{E}/\mathcal{W}$ locally. Now $TM/\mathcal{E}$ is spanned by the Reeb vector field of the contact structure $\mathcal{E}/\mathcal{W}$ on $M/\mathcal{W}$. Therefore the orientation sheaf of $TM/\mathcal{E}$ is also coherent to those of $\mathcal{E}/\mathcal{W}$ and of $\mathcal{W}$. The orientability of $\mathcal{D}/\mathcal{W}$ and hence that of $\mathcal{E}/\mathcal{D}$ are quite independent of others. Summarizing above all in other words, the two vector bundles of rank three, *i.e.*, $\mathcal{E}$ and $TM/\mathcal{W}$, are oriented and there is no more constraint.

1.2 Cartan prolongation

In this and the next subsection, we review two classical constructions of Engel structures from certain geometric structures on 3-manifolds, both of which Elie Cartan understood well ([C]).

The first procedure which we introduce here is called the *prolongation* or *Cartan prolongation*. Let us take a contact plane field ξ on a 3-manifold N and the projectification $\pi : M = \mathbb{R}P(\xi) \to N$ of ξ which is an $\mathbb{R}P^1$-bundle over N. Instead of taking $\mathbb{R}P^1$-bundle, also we can take the associated S^1-bundle or if ξ is a topologically trivial R^2-bundle we can take its infinite cyclic covering which is a principal $\mathbb{R}^1$-bundle. Let (n, ℓ) denote a point in M, namely, n is a point in N and ℓ is a line in ξ_n. We define the plane field $\mathcal{D}$ on M as $\mathcal{D}_{(n,\ell)} = (D\pi)^{-1}(\ell) \subset T_{(n,\ell)}M$.

In this construction, it is straightforward to see from the definition that the even contact structure $\mathcal{E}$ is the pull-back of the contact structure ξ on N by the projection π and the Cauchy characteristic $\mathcal{W}$ is the line field tangent to the fibres of π. All the characteristic lines are closed and the (transverse) holonomy of the 1-dimensional foliation $\mathcal{F}_{\mathcal{W}}$ along any closed leaf is trivial.

1.3 Lorentz prolongation

The *Lorentz prolongation* is reviewed here. Let us take a Lorentzian 3-manifold (V, dg), *i.e.*, a smooth three manifold N and a non-definite inner product on TV. We assume dg has the signature $(2, 1)$. A null line in T_vV is a line which is null with respect to dg. It is a line on the 'light cone' and thus the set of all such lines is a circle, which we call the *null circle* and is denoted by $NC(T_vV)$. It is worth noting that in $\mathbb{R}P(T_vV)$ it is a priori not a linear circle but a quadratic one. Then we consider the null-circle bundle $\pi : M = NC(TV) \to V$ on which the Engel plane field $\mathcal{D}$ is defined to be $\mathcal{D}_{(v,\ell)} = (D\pi)^{-1}(\ell) \subset T_{(v,\ell)}M$. Here (v, ℓ) denotes the point in $NC(T_vV)$ indicating the null line $l \subset T_vV$.

Theorem 1.2 (Lorentz prolongation, E. Cartan [C]) 1) The first derived distribution $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is given as $\mathcal{E}_{(v,\ell)} = (D\pi)^{-1}(\ell^{\perp})$.
2) The second derived $[\mathcal{E}, \mathcal{E}]$ is TM.
3) Therefore $\mathcal{D}$ is an Engel structure on $M = NC(TV)$.

The Cauchy characteristic of the Lorentz prolongation exhibits importance in various senses.

Theorem 1.3 (Cauchy characteristics and null-geodesics, E. Cartan [C]) The Cauchy characteristic $\mathcal{W}$ of the Engel structure $\mathcal{D}$ is the line field given by the null-geodesic flow, namely, the natural lifts of null-geodesics on V to $NC(TV)$.

Corollary 1.4 Up to parametrization, the null-geodesics are invariant under conformal change of Lorentzian metrics. Namely, if $\gamma(t)$ is a null geodesic of Lore tzian manifold (M^3, dg), then for any smoooth positive function f on M $\gamma(t(s))$ is also a null-geodesic for the Lorentzian metric $f dg$ with some reparametrization $t(s)$.

Remark 1.5 This fact is well-known even for Lorentzian manifolds of any dimension. In our dimension, we can understand it from the process of Lorentz prolongation, because it only depends on the conformal class.

For the sake of being self-contained and for understanding constructions in this article, we give a proof of these theorems. For Theorem 1.2 another proof is given in 3.3 which is based on a characterization of Cauchy characteristic curves in terms of infinitesimal deformation whose formulation is given in 3.2.

Proof.
The arguments can be done locally, so take a small open set U in V and $NC(TU)$ is considered. First take a light-like vector field, namely a smooth vector field k satisfying $dg(k, k) \equiv -1$. Next take a smooth orthonormal frame $\langle e, f \rangle$ of the orthogonal complement $\langle k \rangle^{\perp}$ which is positive definite with respect to the Lorentzian metric dg. Then the null-line ℓ is indicated as $\ell = \langle \cos\theta\, e + \sin\theta\, f + k \rangle$. So the null circle at each point $n \in N$ parameterized by $\theta \in S^1$. Therefore $NC(TU)$ is now identified with $U \times S^1$ in this sense. By definition, $\mathcal{D}$ is spanned by two vector fields $F = \frac{\partial}{\partial\theta}$ and $L = \cos\theta\, e + \sin\theta\, f + k$. Clearly we have $Y = [F, L] = -\sin\theta\, e + \cos\theta\, f$ which spans the orthonormal complement $\langle L \rangle^{\perp}$ together with L. This explains 1). Now we have also $[F, Y] = -(\cos\theta\, e + \sin\theta\, f) = -X$. As X,Y,and $k = L - X$ span TU, and $T(NC(TU))$ together with F, we see that $[\mathcal{E}, \mathcal{E}] = TM$. This proves 2) and thus 3).

To prove the second theorem we need more precise computation. On a Lorentzian manifold, exactly the same in the case of Riemannian manifolds, there exists a unique connection ∇, the (Lorentzian) Levi-Civita connection, which is compatible with the Lorentzian metric dg and torsion free,, namely, for any vector fields A, B, C, the two properties

$$A\, dg(B, C) = dg(\nabla_A B, C) + dg(B, \nabla_A C),$$
$$\nabla_A B - \nabla_B A = [A, B]$$

are satisfied. A curve $\gamma(t)$ on V is a geodesic iff $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$. A geodesic is a null-geodesic iff $dg(\dot\gamma(t), \dot\gamma(t)) = 0$. Of course $dg(\dot\gamma(t), \dot\gamma(t)) = 0$ for some t implies the same holds for any t. We take a small neighborhood U of point in V as above and prove the second the second theorem on $U \times S^1 \subset M$.

Remark 1.6 For a null-geodesic $\gamma(t)$ with $\dot\gamma(t) \neq 0$ $\gamma(\varphi(t))$ is again a geodesic iff $\varphi(t)$ is a constant. On the other hand, it is in general impossible to normalize the velocity globally. In other words, even though a null geodesic $\gamma(t)$ comes back to an initial point on a closed trajectory, namely $\gamma(0) = \gamma(1)$ and $\dot\gamma(0) = c\dot\gamma(1)$ for some $c > 0$, it does not imply $\dot\gamma(0) = \dot\gamma(1)$.

Take every null-geodesics $\gamma(t)$ which pass through a point in U and consider their natural lift $(\dot\gamma(t), \langle\dot\gamma(t)\rangle)$ to $NC(TU)$. By taking a smaller space-like disk $D \cong D^2$ and the orbits inside U which meet the disk D, we can assume that such orbits fill up a neighborhood $\tilde U$ in $NC(TU)$ of the fibre $\pi^{-1}(v)$ of the center v of D and each orbit is a segment. Here U is also modified to be the union of such orbits. Therefore we can smoothly assign the parameterization of each geodesics in this neighborhood $\tilde U$. Then we have a local vector field Γ which generates the local null-geodesic flow on $\tilde U$. Using the local framing which we used for the above proof, $\tilde U$ can be regarded as the product $U \times S^1$. Then with respect to this product structure, we have $\Gamma(v,\theta) = \Gamma_V + f(v,\theta)F$ where f is a smooth function.

According to this product structure, the connection $\tilde\nabla$ on $T(NC(TU))$ is also defined as the product $\tilde\nabla = \nabla^L \oplus \nabla^{S^1}$ of the Levi-Civita connection ∇^L on $T\tilde U$ associated with the Lorentzian metric on U and the trivial connection ∇^{S^1} on TS^1. The connection $\tilde\nabla$ is a symmetric, *i.e.*, torsion free, and compatible with the product metric. It is also compatible with the partial metric π^*dg.

In this formulation, the condition that Γ generates a local null-geodesic flow is described that the V-component of $\tilde\nabla_\Gamma\Gamma_V$ is trivial. This is equivalent to that $\tilde\nabla_\Gamma\Gamma$ has trivial V-component because F is a parallel field.

The statement 4) is nothing but $[\Gamma,\mathcal{E}] \subset \mathcal{E}$, where $\mathcal{E}_{(v,\ell)} = (D\pi_{(v,\ell)})^{-1}(\ell^\perp)$. As we have seen that $[\Gamma,(D\pi)^{-1}(\ell)] \subset \mathcal{E}$, what we have to show is $[\Gamma, Y] \subset \mathcal{E}$, namely, $\pi^*dg(\Gamma,[\Gamma,Y]) = 0$, which is computed as follows.

$$\begin{aligned}
\pi^*dg(\Gamma,[\Gamma,Y]) &= \pi^*dg(\Gamma, \tilde\nabla_\Gamma Y - \tilde\nabla_Y\Gamma) \\
&= \pi^*dg(\Gamma,\tilde\nabla_\Gamma Y) - \pi^*dg(\Gamma,\tilde\nabla_Y\Gamma) \\
&= \Gamma\pi^*dg(\Gamma,Y) - \pi^*dg(\tilde\nabla_\Gamma\Gamma, Y) - \frac{1}{2}Y\pi^*dg(\Gamma,\Gamma) \\
&= -\pi^*dg(\tilde\nabla_\Gamma\Gamma,Y) = 0.
\end{aligned}$$

Q.E.D. □

Remark 1.7 1) As the second theorem is very important, we give an alternative proof in a later section, which is much shorter and relies on a rigidity property of Cauchy characteristic. It is related to the causality of the Lorentz structure.
2) An Engel structure obtained by Lorentz prolongation is equipped with an extra line field $\langle F\rangle \subset \mathcal{D}$ which is transverse to the Cauchy characteristic $\mathcal{W}$ in $\mathcal{D}$. It is not true even locally that an Engel structure with arbitrary line field inside $\mathcal{D}$ which is transverse in $\mathcal{D}$ is obtained by Lorentz prolongation. This fact is studied by Chern in [Ch] along an equivalence problem of 3rd order ODE's. The author learned this from Robert Bryant. The Chern's work was initiated by Wünschmann in his thesis [W] under the supervision by F. Engel. After Chern, through the Tanaka theory, Sato-Yoshikawa [SY] gave a geometrically clear formulation of the Wünschmann invariant which is well-adapted to our context. For more historical informations, see *e.g.*, [GN] and [NP].

The obstruction is geometrically understood as follows. Locally we can take the quotient by collapsing integral curves of $\langle F \rangle$ to poiints so that we obtain a 3-dimensional space. In the porjective plane of the tangent space of each point of this 3-dimensional space, the Engel plane asigns a point. The trace of this point along each $\langle F \rangle$-curve should be a small circle if it is obtained (locally) by Lorentz prolongation. Conversely if it is the case, the small circle defines a conformal class of Lorentzian metric on the 3-dimensional space and locally the Engel structure is interpreted as obtained from the Lorentzian prolongation.
3) If we start from the flat Lorentzian space, *i.e.*, the Minkowski 3-space $\mathbb{R}^{2,1}$ the fibre direction $\langle F \rangle \subset \mathcal{D}$ coincides with $\langle X \rangle$ of the particular framing [EF] in the Engel-Darboux coordinate in Subsection 1.1.

We will see various constructions by the Lorentz prolongation starting from surfaces with metrics.

1.4 Pre-quantum prolongation*

We start from richer data to construct Engel structures. For this, we use the construction of a complex line bundle with $U(1)$-connection, or equivalently an S^1-bundle with an S^1-connection which is well-known as the *pre-quantization* or *pre-quantum bundle*. See *e.g.*, [Kos] for fundamentals.

Lemma 1.8 (Pre-quantization) For a smooth manifold V. an integral cohomology class $\alpha \in H^2(V;\mathbb{Z})$, and and a closed 2-form ω which represents α mod torsion in $H^2(V;\mathbb{R})$, there exists an S^1-bundle with an S^1-connection ∇ such that its curvature 2-form Ω_∇ exactly coincides with $2\pi\omega$ and the euler class coincides with α.

Let $\xi = \ker\alpha$ be a contact structure on a 3-manifold V with a non-singular Legendrian vector field $\underline{W}$ which is volume preserving with respect to a smooth volume $d\mathrm{vol}$ and whose asymptotic cycle presents an integral 1st homology class. The last condition is also stated as the Poincaré dual closed 2-form $\iota_{\underline{W}} d\mathrm{vol}$ presents an integral 2nd cohomology class in $H^2(V;\mathbb{Z})/\mathrm{Torsion} \subset H^2(V;\mathbb{R})$.

We take the pre-quantum S^1-bundle $\pi : M^4 \to V^3$ with a connection ∇ for the closed 2-form $\omega = \iota_{\underline{W}} d\mathrm{vol}$. The connection defines the horizontal hyperplane $H_\nabla \subset T_m M$ at each point $m \in M$.

Theorem 1.9 (Pre-quantum prolongation) 1) The the plane field $\mathcal{D}$ on M defined as $\mathcal{D}_m = H_\nabla \cap (D\pi_m)^{-1}\xi$ is an Engel structure.
2) The even contact structure is exactly the horizontal distribution $\mathcal{E} = H_\nabla$ and the Cauchy characteristic is the horizontal lift $\mathcal{W} = H_\nabla \cap (D\pi_m)^{-1}\langle \underline{W} \rangle$ of $\langle \underline{W} \rangle$.

Proof.
The proof is divided into two parts. The first part shows the existence of local coordinates (x, z, w) on V which are well-adapted to ξ, $\langle \underline{\mathrm{W}} \rangle$, and ω.

Then in the second part under the preparation of the first part we show that the fibre coordinate y can be so nicely chosen that (x, y, z, w) forms an Engel-Darboux coordinate system.

Part I.

In general for a non-singular vector field X and an invariant smooth volume $d$vol on a manifold, if we take a smooth positive function f, $f^{-1}d\mathrm{vol}$ is also invariant under fX and we have $\iota_{fX} f^{-1} d\mathrm{vol} = \iota_X d\mathrm{vol}$. Therefore $\iota_X d\mathrm{vol}$, which is regarded as the transverse invariant volume to the foliation spanned by $\langle X \rangle$, is more essential and is closed.

On a neighborhood U of any point of V it is easy to choose smooth functions x and z on U such that $dx \wedge dz = \omega|_U$ and that ξ never coincides with $\ker dz$. Then on U, the relation $\xi = \ker [dz - w dx]$ defines a function w on U. From the construction we see that (x, z, w) gives a local coordinate on U. Also we have $\left\langle \frac{\partial}{\partial w} \right\rangle = \langle \underline{W} \rangle$, while $\frac{\partial}{\partial w}$ does not necessarily coincides with $\underline{W}$ even modulo multiplication by constant.

Part II.

Fix a local trivialization of the pre-quantum S^1-bundle over U and give the coordinate (x, z, w, θ) $(\theta \in S^1)$. Then on the total space the connection 1-form Θ^∇ is indicated as $\Theta^\nabla = d\theta + \beta(x, z, w)$ where $\beta(x, z, w)$ is a 1-form on the base space V with $d\beta(x, z, w) = \omega = dx \wedge dz$. As $d(-z dx) = dx \wedge dz$ and by the Poincaré lemma, there exists a function φ of (x, z, w) such that $\beta = d\varphi - z dx$. Therefore we obtain $\Theta^\nabla = d\theta + d\varphi - z dx = d(\theta + \varphi(x, z, w)) - z dx$. This implies by a gauge transformation $\theta \mapsto \theta + \varphi(x, z, w)$ (mod 2π) or by change of th local trivialization by $\theta^* = \theta + \varphi(x, z, w)$, the connection form is indicated as $\Theta^\nabla = d\theta - z dx$.

For a certain part of the fibre S^1, we can give a real valued coordinate y in place of θ and then we naturally obtain the followings;

- $\mathcal{E} = H_\nabla = \ker[dy - z dx]$, $\quad (D\pi)^{-1}\xi = \ker[dz - w dx]$,
- $\mathcal{D} = H_\nabla \cap (D\pi)^{-1}\xi$,
- $\mathcal{W} = \left\langle \frac{\partial}{\partial w} \right\rangle = (D\pi)^{-1} \langle \underline{\mathrm{W}} \rangle \cap H_\nabla$.

Q.E.D. $\square$

Remark 1.10 1) From a similar argument, the space of S^1-connections of a given S^1-bundle is regarded as not a vector space but the affine space of 1-forms. Then once the curvature form is specified, then it is the affine space of closed 1-forms. The difference of an exact 1-form is absorbed as in the above proof by a gauge transformation which is isotopic to the identity. Those which correspond to integral 1-st cohomologies are also absorbed by the gauge tansformations $M \to S^1$. Therefore in the above construction the structures might have $H^1(V; \mathbb{R}/2\pi\mathbb{Z})$ as its moduli. Of course it is more complicated to consider how this moduli reduces considering diffeomorphisms of the total space M.

2) If $H^2(V; \mathbb{Z})$ has torsion, the integral euler class is determined up to the

torsion from the real class $[\omega]$. Any of the S^1-bundles associated with such an integral class the pre-quantization works. Therefore provided that V is compact we might have non-uniqueness and finitely many possibilities for the manifold M from the given data V, ξ, $d$vol, and $\underline{W}$.

This happens when we take the unit tangent bundle $V = S^1T\Sigma$ of a closed hyperbolic surface Σ of genus g. As $\operatorname{Tor} H^2(V;\mathbb{Z}) \cong \mathbb{Z}/_{2g-2}$, we have $(2g-2)$-many possibilities for M.
3) It might be worth remarking that in the pre-quantum prolongation, contrary to the Cartan prolongation, the even contact structure comes the second.

Examples of pre-quantum prolongation are given in Subsection 4.4.

1.5 Suspension by contact diffeomorphism

Here we review a fairy general method to construct or to modify Engel structures.

Let (V, ξ) be a contact structure and ℓ be a non-singular Legendrian line field. Also we assume that ξ is oriented plane field. Then we put some/any Euclidean metric on ξ so that the angle between two Legendrian lines are defined. Let φ be a contact morphism of (V, ξ) which preserves the orientation of ξ. Also we assume that the oriented angle $d(v)$ of $(\varphi_*\ell)_v$ from ℓ_v $(v \in V)$ is continuously well-defined and bounded from above. For example if φ is isotopic to the identity among contactmorphisms and V is compact, this is the case. Also note that this condition is independent of the choice of metric.

Let us consider the mapping torus $M_\varphi = \mathbb{R} \times V/\sim$ of φ, where $\sim$ identifies $(t+n, v)$ and $(t, \varphi^n(v))$ for $n \in \mathbb{Z}$. The contact structure ξ it is pulled back to $\mathbb{R} \times V$ as a hyperplane field $\widetilde{\mathcal{E}} = \tilde{\xi}$. Because ξ is invariant under φ, It is also well-defined as a hyperplane field on $M = M_\varphi$ which is denoted by $\mathcal{E}$ and is going to be $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$. Also let $\mathcal{W}$ be the suspension direction, *i.e.*, the natural projection image of $\widetilde{\mathcal{W}} = \left\langle \dfrac{\partial}{\partial t} \right\rangle$ on $\mathbb{R} \times V$.

Take a smooth metric (conformal structure) on $\mathcal{E}/\mathcal{W}$ and pull it back to $\widetilde{\mathcal{E}}/\widetilde{\mathcal{W}}$. Consider the continuous twisting function $d(v)$ for this metric restricted to $\{0\} \times V$ and take an integer K so that $d(v) < K\pi$ for any $v \in V$. On $[0,1] \times V$ let us define $\widetilde{\mathcal{D}}_{(t,v)} = R(\rho(t,v))(\ell_v)$ where $R(\rho)$ is an rotation of $\widetilde{\mathcal{D}}_{(t,v)} = \xi_v$ by the angle ρ in such a way that the smooth function $\rho(t,v)$ satisfies

$$\rho(0,v) \equiv 0, \quad \rho(1,v) \equiv K\pi - d(v), \text{ and } \frac{\partial \rho}{\partial t} > 0$$

and also that the deck transformations $(t+n, v) \sim (t, \varphi^n(v))$ extend $\widetilde{\mathcal{D}}$ on $[0,1] \times V$ to the whole $\widetilde{\mathcal{D}}$ on $\mathbb{R} \times V$ as an smooth Engel structure which is invariant under the deck transformations. As a result we obtain an Engel structure $\mathcal{D}$ on $M = M_\varphi$ naturally.

We can perform a similar modification to a given Engel structure $(M, \mathcal{D})$. Let us consider a transversely embedded hypersurface $V \subset M$ to the Cauchy characteristic $\mathcal{W}$ with a neighborhood $U \cong \cup_{v \in V}[-\alpha(v), \beta(v)] \times \{v\} \supset \{0\} \times V = V$. Here, α and β are positive smooth function on V, each curve $[-\alpha(v), \beta(v)] \times \{v\}$ is a Cauchy characteristic curve, and the first coordinate t is a twisting angle between $(\mathcal{D}/\mathcal{W})_{(t,v)}$ and $(\mathcal{D}/\mathcal{W})_(0, v)$ with respect to some metric. V need not be closed. On V a contact structure $\xi = \mathcal{E}|_V \cup TV$ is induced. Consider a contactmorphism φ of (V, ξ). W cut the manifold M along V and paste it again by φ to obtain a new manifold which is also denoted by M_φ. More precisely, remove $\{0\} \times V$, complete $M \setminus \{0\} \times V$ by the points $(0-0, v)$ and $(0+0, v)$ separately, and identify $(0-0, v)$ with $(0+0, \varphi(v))$

Because the even contact structure $\mathcal{E}$ is well-preserved by this operation, a new even contact structure $\mathcal{E}_\varphi$ is induced on M_φ. The new Cauchy characteristic $\mathcal{W}_\varphi$ is also naturally defined. Now consider the twisting function $d(v)$ for φ with respect to $(\mathcal{D}/\mathcal{W})_{(-\alpha(t),v)} \subset \xi_v$. If we can take $d(v)$ continuously so as to satisfies $d(v) < \alpha(v) + \beta(v)$ and $d(v) = 0$ on $V \setminus \operatorname{supp} \varphi$, there is an Engel structure on M_φ which coinsids with $\mathcal{D}$ on $M \setminus U$ and whose even contact structure is $\mathcal{E}_\varphi$. It is unique up to isotopy along $\mathcal{W}_\varphi|_{U_\varphi}$.

For any transversal V to $\mathcal{W}$, we can find some positive functions α and β. Then contactmorphism which is sufficiently C^1-close to the identity with regard to α and β, the above construction is possible. By this modification, we can always perturb the dynamics of $\mathcal{W}$.

Example 1.11 (Cartan prolongation) If the contact structure ξ is trivial as an $\mathbb{R}^2$-bundle, it admits a non-singular Legendrian vector field ℓ. Then its Cartan prolongation is considered to be the result of the suspension construction by the identity, $K = 1$, and $\rho(v, t) = \pi t$.

Example 1.12 (Bi-Engel structure, [KV]) Let ϕ_t be a contact Anosov flow on a 3-manifold V , namely, X is a Reeb vector field for a contact structure ξ and generates an Anosov flow ϕ_t. Then there is associated a bi-contact structure (ξ_+, ξ_-), where we have $\xi_+ \cup \xi_- = \langle X \rangle$ and $\xi_\pm$ are twisted by $(\phi_t)_*$ in the opposite directions. Fix any $T > 0$ and consider the suspension of (V, ξ) by the time T map $\varphi = \phi_T$. Then we obtain a pair $\mathcal{D}_\pm$ of Engel structures associated with $\xi_\pm$ for the same even contact structure. The twisting directions are opposite to each other exactly like the bi-contact structure. This example will appear repeatedly.

2 Action of Cauchy characteristics

The transverse dynamics of the 1-dimensional foliation $\mathcal{F}_\mathcal{W}$ spanned by the Cauchy characteristic $\mathcal{W}$ is an important character of Engel structures.

In particular, as $[\mathcal{W}, \mathcal{E}] = \mathcal{E}$, we can look at the action of $\mathcal{W}$ on the 2-dimensional space (vector bundle) $\mathcal{E}/\mathcal{W}$ and the movement of $\mathcal{D}/\mathcal{W}$ inside $\mathcal{E}/\mathcal{W}$ along $\mathcal{W}$.

For example, in the case of Cartan prolongation of a contact structure, each orbit of $\mathcal{W}$ is the fibre circle and the holonomy of $\mathcal{F}_{\mathcal{W}}$ is identical along any closed leaf, while $\mathcal{D}/\mathcal{W}$ rotates by angle 2π along each $\mathcal{W}$-orbit. Many more examples are studied in later sections.

First we review the projective structure defined on each orbit of $\mathcal{W}$. This is introduced by Bryant-Hsu([BH]) and also studied by Inaba([I]).

Next, we will consider to extend such properties of each orbit to the whole system $\mathcal{W} \curvearrowright \mathcal{E}/\mathcal{W}$. This is initiated by Kotschick-Vogel([KV]).

2.1 Projective structure on Cauchy characteristic lines

2.1.1 Review of projective structure on 1-manifolds

Roughly speaking, a projective structure on 1-manifold Λ is a geometric structure modeled on $(PGL(2;\mathbb{R}), \mathbb{R}P^1)$ as a (G, X)-manifold, namely, there exists an atlas of Λ whose charts take value in $\mathbb{R}P^1$ and the coordinate changes are given by elements of $PGL(2;\mathbb{R})$.

We do not consider non-orientable projective structure so that we take $(PGL^+(2;\mathbb{R}) = PSL(2;\mathbb{R}), \mathbb{R}P^1)$. More precisely or more formally, taking the universal covering $(\widetilde{PSL(2;\mathbb{R})}, \widetilde{\mathbb{R}P^1})$ as the model and consider the developing map $\Phi : \tilde{\Lambda} \to \widetilde{\mathbb{R}P^1}$ which is an immersion and is equivariant with respect to $\pi_1(\Lambda)$. On $\tilde{\Lambda}$ $\pi_1(\Lambda)$ acts as the covering transformation and on $\widetilde{\mathbb{R}P^1}$ through the *holonomy* homomorphism $\varphi : \pi_1(\Lambda) \to \widetilde{PSL(2;\mathbb{R})}$. As Λ is 1-dimensional, $\pi_1(\Lambda)$ is trivial or isomorphic to $\mathbb{Z}$. In the latter case, often we identify φ with $\varphi(1) \in \widetilde{PSL(2;\mathbb{R})}$. In the case of general (G, X)-manifolds, the developing map is not necessarily injective, however, so is it in our case. Even then it may not be surjective.

Two projective structure on the same manifold Λ is isomorphic (or called simply 'the same') if they are united to define one atlas. This condition is equivalent to saying that if the developing maps Φ_1 and Φ_2 are related by a single element of $\widetilde{PSL(2;\mathbb{R})}$, namely, for some $g \in \widetilde{PSL(2;\mathbb{R})}$ $\Phi_2 = g \circ \Phi_1$ holds.

Two projective manifolds Λ_1 and Λ_2 are isomorphic or diffeomorphic as projective manifolds if there exists a diffeomorphism $\psi : \Lambda_1 \to \Lambda_2$ through which two structures are isomorphic.

2.1.2 Projective structure on Cauchy characteristic lines

The projective structure of each orbit $\gamma(t)$ of $\mathcal{W}$ is defined as follows. First take the universal covering $\tilde{\gamma}$ if necessary and fix a trivialization $(\mathcal{E}/\mathcal{W})|_{\tilde{\gamma}} \cong \mathbb{R} \times \mathbb{R}^2$ by using the action of $\mathcal{W}$. For example, using a parameterization $\tilde{\gamma}(t)$ $(t \in \mathbb{R})$ of the orbit, we can identify $(\mathcal{E}/\mathcal{W})|_{\tilde{\gamma}(t)}$ with $(\mathcal{E}/\mathcal{W})|_{\tilde{\gamma}(0)}$ for any $t \in \mathbb{R}$ by the differential of the holonomy transformation of $\mathcal{F}_{\mathcal{W}}$ along γ. As a foliation of codimension 3, a priori the linearized holonomy between 3-dimensional normal spaces is defined. In the case of the Cauchy characteristic foliation, in the normal space $TM/\mathcal{W}|_{\tilde{\gamma}(t)}$,

$\mathcal{E}/\mathcal{W}|_{\tilde{\gamma}(t)}$ is 2-dimensional and invariant under the holonomy along $\mathcal{F}_{\mathcal{W}}$. Therefore the above identification is defined.

Then we have the tautological developing map $\check{\Phi} : \tilde{\gamma} \to \mathbb{R}P^1 = P((\mathcal{E}/\mathcal{W})|_{\gamma(0)})$ defined by $\check{\Phi}(\tilde{\gamma}(t)) = (\mathcal{D}/\mathcal{W})|_{\tilde{\gamma}(t)} \in P((\mathcal{E}/\mathcal{W})|_{\tilde{\gamma}(t)} \cong P((\mathcal{E}/\mathcal{W})|_{\tilde{\gamma}(0)})$. The fact that $\check{\Phi}$ is a submersion is a direct conclusion of the definition of Engel structure. This defines a projective structure on the $\mathcal{W}$-orbit γ. Under this setting we do not need to take the developing map to $\widetilde{\mathbb{R}P^1}$. Thus the projective structure of the each orbit is well-defined.

Remark 2.1 In [BH] Bryant and Hsu define the projective structure inside an Engel-Darboux coordinate by $(x, y, z, w(t)) \mapsto w \in \mathbb{R} \cong \mathbb{R}P^1 \setminus \{\infty\}$. Inside the Engel-Darboux coordinate a $\mathcal{W}$-orbit is exactly tangent to $\frac{\partial}{\partial w}$ and x, y, and z are constant along it. Then they checked that by any Engel-Darboux chart, the coordinate change gives a new w which defines the same projective structure.

Following our definition and taking $\dfrac{\partial}{\partial x} + z\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial z}$ as the invariant frame of $\mathcal{E}/\mathcal{W}$ along the $\mathcal{W}$ orbit $(x, y, z, w(t))$ with x, y, and z constant, we see that it coincides with the projective structure by Bryant-Hsu.

In [I] Inaba adopted a long Engel-Darboux coordinate (x, y, z, θ) in order to treat longer $\mathcal{W}$ orbit, which is not necessarily rigid. There, the Engel structure is defined as $\mathcal{D} = \ker[dy - zdx] \cap \ker[\cos\theta dz - \sin\theta dx]$ and of course the developing map $\theta \in \widetilde{P(\mathcal{E}/\mathcal{W})}$ defines the projective structure.

Remark also that inside an Engel-Darboux chart, we do not have to go up to the universal covering.

2.2 Projective structure on closed orbits and length

As a typical case, here we consider the projective structures on closed orbits of the Cauchy characteristic in an Engel manifold.

2.2.1 Review of projective structure on circle

Let us review the projective structures on a circle Λ. The classification was first given by Kuiper ([Kui]).

Consider the developing map between the universal coverings $\Phi : \tilde{\Lambda} \to \widetilde{\mathbb{R}P^1} \cong \mathbb{R}$ and its image. Then the holonomy $\varphi \in \widetilde{PSL(2;\mathbb{R})}$ determines the projective structure on Λ as $\Phi(\tilde{\Lambda})/\varphi^{\mathbb{Z}}$.

To classify them, first we need to list up all the pairs of connected subspace of the whole line $\widetilde{\mathbb{R}P^1}$ on which the holonomy $\varphi^{\mathbb{Z}}$ acts freely. In an abstract sense, the holonomy φ determines the structure. First let us recall that elements $A \in PSL(2;\mathbb{R}) \setminus \{\mathrm{id}\}$ are classified into three categories:

1)(elliptic) $|\mathrm{tr}\, A| < 2$, no fixed point in $\mathbb{R}P^1$, conjugate to $\pm\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

2)(parabolic) $|\mathrm{tr}\, A| = 2$, one fixed point in $\mathbb{R}P^1$, conjugate to $\pm\begin{pmatrix}1 & \mp 1\\ 0 & 1\end{pmatrix}$.

3)(hyperbolic) $|\mathrm{tr}\, A| > 2$, two hyperbolic fixed points in $\mathbb{R}P^1$, contracting and expanding, conjugate to $\pm\begin{pmatrix}e^a & 0\\ 0 & e^{-a}\end{pmatrix}$ $(a > 0)$.

Now we proceed to the classification of projective structures on a circle Λ. There are two cases, in one of which the image of the developing map is the whole line $\widetilde{\mathbb{R}P^1}$, while in the other case it is an interval of finite length. It is natural to regard $\mathbb{R}P^1 = \mathbb{R}/\pi\mathbb{Z}$. The classification of the holonomy $\varphi \in \widetilde{PSL(2;\mathbb{R})}$ and the projective structure on $\Lambda \cong S^1$ is given as follows. Let $\langle\varphi\rangle$ denote the class of φ in $PSL(2;\mathbb{R})$.

1) (elliptic case) $|\mathrm{tr}\,\langle\varphi\rangle| < 2$, no fixed point in $\widetilde{\mathbb{R}P^1}$, conjugate to the translation $t \mapsto t + (n\pi + \theta)$ for some $n \in \mathbb{Z}$, where $\theta \in (0, \pi)$ is as above. Basically we assume $n > 0$. The developing image is the whole $\mathbb{R} = \widetilde{\mathbb{R}P^1}$, the projective length of Λ is $n\pi + \theta$. The case $\langle\varphi\rangle = E$ can be also included here as the case of $\theta = 0$, provided that φ is just the translation by $n\pi$ for some $n \in \mathbb{N}$. Apparently the rotations are projective symmetries, thus the structure is homogeneous.

2) (parabolic case) $|\mathrm{tr}\,\langle\varphi\rangle| = 2$ and there are fixed points in $\widetilde{\mathbb{R}P^1}$. Then after a conjugation, it takes the following form; $\langle\varphi\rangle = \pm\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$, the fixed point set Fix(φ) is $\pi\mathbb{Z}$ and the developing image is $(0, \pi)$. The projective length of $\tilde{\Lambda}$ is π, while that of Λ has no meaning. The action of $\{\langle\varphi\rangle = \pm\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix} \mid t \in \mathbb{R}\}$ on $\mathbb{R}P^1$ descends to Λ as rotational symmetries. Therefore the structure is homogeneous. $\pm\begin{pmatrix}1 & +1\\ 0 & 1\end{pmatrix}$ is eliminated because it is just the inverse of the above and it is preferable that the action is taken in the positive way in the angle coordinate.

3) (hyperbolic case) $|\mathrm{tr}\,\langle\varphi\rangle| > 2$ and there are fixed points in $\widetilde{\mathbb{R}P^1}$. After conjugation, it takes the following form; $\langle\varphi\rangle = \pm\begin{pmatrix}e^a & 0\\ 0 & e^{-a}\end{pmatrix}$ $(a > 0)$, the fixed point set in $\widetilde{\mathbb{R}P^1}$ is Fix$(\varphi) = \frac{1}{2}\pi\mathbb{Z}$, and the developing image is $(\pi/2, \pi)$. Projectively the length of $\tilde{\Lambda}$ has no meaning, because it can take any value in $(0, \pi)$ by conjugation. On the other hand, $|\mathrm{tr}\,\langle\varphi\rangle| > 2$ or the derivative $|\log(\varphi)'|$ at fixed points (as a function $\varphi : \mathbb{R} \to \mathbb{R}$) is a projective invariant.

Because the projective symmetry $\{\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix}, \mid t \in \mathbb{R}\}$ induces the rotational symmetry on Λ, the structure is homogeneous.

4) (trans-parabolic case) $|\mathrm{tr}\,\langle\varphi\rangle| = 2$ and there are no fixed points in $\mathbb{R} = \widetilde{\mathbb{R}P^1}$. After conjugation let $\check{\varphi} \in \widetilde{PSL(2;\mathbb{R})}$ denote the one in the parabolic case with $\langle\varphi\rangle = \langle\check{\varphi}\rangle = \pm\begin{pmatrix}1 & \mp 1\\ 0 & 1\end{pmatrix}$. Then φ acts on $\mathbb{R}$ as $\varphi(\theta) = \check{\varphi}(\theta) + n\pi$ for some $n \in \mathbb{N}$, after taking the inverse if necessary. Thus *e.g.*, $[0, n\pi)$ is a fundamental domain of the action.

Depending on ∓ 1 in the off diagonal component of $\langle\varphi\rangle$, $n\pm$ is the projective invariant. The length of Λ should be understood as $n\pi \pm 0$.

On $\mathbb{R}$ the action of $\{\varphi \in \widetilde{PSL(2;\mathbb{R})} \mid \langle\varphi\rangle = \pm\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}, t \in \mathbb{R}\}$ including those by the center $\theta \mapsto \theta + k\pi$ $k \in \mathbb{Z}$ is the symmetry. Therefore the decomposition $\Lambda = \{[\theta] \in \Lambda \mid \theta \in \pi\mathbb{Z}\} \sqcup \{[\theta] \in \Lambda \mid \theta \notin \pi\mathbb{Z}\}$ gives the orbit decomposition of the projective symmetry action. The structure is not homogeneous.

5) (trans-hyperbolic case) $|\mathrm{tr}\,\langle\varphi\rangle| > 2$ and there are no fixed points in $\mathbb{R} = \widetilde{\mathbb{R}P^1}$. After conjugation let $\check{\varphi} \in \widetilde{PSL(2;\mathbb{R})}$ denote the one in the hyperbolic case with $\langle\varphi\rangle = \langle\check{\varphi}\rangle = \pm\begin{pmatrix}e^a & 0\\ 0 & e^{-a}\end{pmatrix}$. Then φ acts on $\mathbb{R}$ as $\varphi(\theta) = \check{\varphi}(\theta) + n\pi$ for some $n \in \mathbb{N}$, after taking the inverse if necessary. Thus *e.g.*, $[0, n\pi)$ is a fundamental domain of the action and $n \in \mathbb{N}$ and $|\mathrm{tr}\,\langle\varphi\rangle|$ or equivalently the derivative $|\log(\varphi)'|$ at $(\pi/2)\mathbb{Z}$ are the projective invariants. The length of Λ should be understood as $n\pi \pm 0$.

On $\mathbb{R}$ the action of $\{\varphi \in \widetilde{PSL(2;\mathbb{R})} \mid \langle\varphi\rangle = \pm\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix}, t \in \mathbb{R}\}$ including those by the center $\theta \mapsto \theta + k\pi$ $k \in \mathbb{Z}$ is the symmetry. Therefore the decomposition $\Lambda = \{[\theta] \in \Lambda \mid \theta \in \pi\mathbb{Z}\} \sqcup \{[\theta] \in \Lambda \mid \theta \in \pi(\mathbb{Z} + 1/2)\} \sqcup \{[\theta] \in \Lambda \mid \theta \in (0, \pi/2)\} \sqcup \{[\theta] \in \Lambda \mid \theta \in (\pi/2, \pi)\mathbb{Z}\}$ gives the orbit decomposition of the projective symmetry action. The structure is not homogeneous.

2.2.2 Projective structure on closed Cauchy characteristic lines

Take a simple closed Cauchy characteristic curve. Then after fixing the orientation of $\mathcal{E}/\mathcal{W}$ and $\mathcal{W}$ so that along the curve $\mathcal{D}/\mathcal{W}$ is moving in a positive angular direction. Then it naturally admits one of projective structures classified in 2.2.1.

Take a parameterization $\Gamma : [a, b] \to M$ of this simple closed curve ($\Gamma(a) = \Gamma(b)$) a trivialization of $\mathcal{E}/\mathcal{W}|_\Gamma$ which is invariant under the action of $\mathcal{W}$ as mentioned in 2.1.2, in such a way that they are respecting the above orientations. The trivialization identifies $\mathcal{E}/\mathcal{W}|_{\Gamma(t)}$ with $\mathcal{E}/\mathcal{W}|_{\Gamma(a)} \equiv \mathbb{R}^2$ for $a \le t \le b$.

The differential of the first return map gives the holonomy $\langle\varphi\rangle \in PSL(2;\mathbb{R})$. The (reduced) developing map $\langle\mathcal{W}/\mathcal{D}\rangle|_{\Gamma(t)} : [a, b] \to \mathbb{R}P^1 = P(\mathcal{E}/\mathcal{W}|_{\Gamma(a)})$ lifts to the genuine developing map to $\mathbb{R} = \widetilde{\mathbb{R}P^1}$, which tells how many times it turns around and what is $\varphi \in \widetilde{PSL(2;\mathbb{R})}$. All the ambiguities which might appear in above construction stay in the equivalence explained in the previous sections.

Example 2.2 1) (Elliptic orbit) The Cartan prolongation $(M, \mathcal{D})$ of a contact 3-manifold (V, ξ) has all Cauchy characteristic closed, namely, they are the fibres of $\pi : M \to V$. Therefore the holonomy is trivial, it follow from the above definition that it is the elliptic case of length π.

Elliptic ones with other (arbitrary) length naturally appear in the Lorentz prolongations and the pre-quantum ones as well. We will see them in later sections.

2) (Parabolic orbits) In the Lorentz prolongation of the flat Lorentzian 3-torus $(T^3, dg = dx^2 + dy^2 - dz^2)$, there are lots of closed Cauchy characteristics corresponding to the closed geodesics of (T^3, dg). All of them are parabolic. Many more example will come later.

3) Hyperbolic, trans-parabolic, or trans-hyperbolic orbits are easily realized by the suspension construction reviewed in 1.5. Of course so are elliptic or parabolic ones as well.

Remark 2.3 In contrast with trans-hyperbolic or trans-parabolic ones, we may call hyperbolic or parabolic ones *genuine-hyperbolic* or *genuine-parabolic*. In the following subsection, the terminology will have more meaning.

•review from Bryant-Hsu on rigidity and projective structure and interpretation

•projective length

2.3 Action of Cauchy characteristic on $\mathcal{E}/\mathcal{W}^*$

In order to understand the global structure of an Engel structure $\mathcal{D}$, in particular on a closed 4-manifold M, the behavior of the Cauchy characteristic $\mathcal{W}$ as 1-dimensional foliation and its dynamics on M as well as on $\mathcal{E}/\mathcal{W}$ are important view points. Based on this the behavior of $\mathcal{D}/\mathcal{W}$ in $\mathcal{E}/\mathcal{W}$ along $\mathcal{W}$ is more clearly seen.

We reviewed in the preceding subsections that closed orbits of $\mathcal{W}$ have their own characters. This does not always apply to non-closed orbits nor to the whole structure, however, here we consider a very limited case where the whole structure still admits such a character, while it seems to have certain importance. This is similar to the differential geometric study of surfaces where it is not always true that the curvature has a single sign or is vanishing everywhere, while such cases have importance in various senses.

Let $(M, \mathcal{E})$ be an even contact structure on a closed 4-manifold M. We take and fix a fiberwise metric on 2-dimensional vector bundle $\mathcal{E}/\mathcal{W}$. We assume that $\mathcal{W}$ and $\mathcal{E}/\mathcal{W}$ are oriented. Then take a non-singular vector field W which spans $\mathcal{W}$ and its flow $\phi_t = \exp tW$, whose lift to $\mathcal{E}/\mathcal{W}$ is denoted by φ_t. First we define the character of the even contact structure $\mathcal{E}$ in special cases. Remark that we do not have to start with an Engel structure.

Definition 2.4 1) (elliptic) This is the case where the conformal distortion of φ_t is uniformly bounded. Precisely it is formulated as follows. For each point $P \in M$ take an oriented orthonormal basis of $(\mathcal{E}/\mathcal{W})_P$, present the linear holonomy $\varphi_{tP} : (\mathcal{E}/\mathcal{W})_P \to (\mathcal{E}/\mathcal{W})_{\phi_t(P)}$ with respect to these bases, and let $\langle\varphi\rangle(t, P)$ denote its class in $\widetilde{PGL}^+(2;\mathbb{R}) = \widetilde{PSL}(2;\mathbb{R})$. Then the uniform boundedness of the conformal distortion of the linear holonomy in $\mathcal{E}/\mathcal{W}$ is stated that the set $\{\langle\varphi\rangle(t, P) \mid t \in \mathbb{R}, P \in M\} \subset \widetilde{PSL}(2;\mathbb{R})$ is bounded. If this is the case, we call $\mathcal{E}$ is of *elliptic* type.
2) (parabolic) Let us assume there exists a real trivial sub-line bundle l^h of $\mathcal{E}/\mathcal{W}$ and if necessary change the orientation of $\mathcal{E}/\mathcal{W}$. Then take an oriented orthonormal frame $\langle \ell_1, \ell_2 \rangle$ of $\mathcal{E}/\mathcal{W}$ so that ℓ_1 lies in l^h. If there exist positive constants c and T_0 so that

$$\pm\frac{\langle \phi_{\pm t}\ell_{1P}, \phi_{\pm t}\ell_{2P}\rangle}{\langle \phi_{\pm t}\ell_{1P}, \phi_{\pm t}\ell_{1P}\rangle} \geq c \cdot t \quad \text{for } \forall P \in M,\ \forall t \geq T_0$$

the even contact structure $\mathcal{E}$ is said to be of *parabolic* type.
3) (hyperbolic) If there exist two independent sub-line bundles l^u and l^s of $\mathcal{E}/\mathcal{W}$ which are invariant under the action of φ_t and positive constants c, c' and T_0 such that the following is satisfied.

$$\frac{\|\varphi_t v^u\|}{\|\varphi_t v^s\|} \geq c' \exp(ct)\frac{\|v^u\|}{\|v^s\|} \quad \text{for } \forall P \in M,\ \forall v^u \in l^u_P,\ \forall v^s \neq 0 \in l^s_P,\ \forall t \geq T_0$$

Then the even contact structure is of *hyperbolic* type. It is also called *weakly hyperbolic* or sometimes *projectively hyperbolic*. Remark here that under the compactness of M this notion is independent of the choice of fiberwise metric on $\mathcal{E}/\mathcal{W}$.

Remark 2.5 1) Note that these notions are independent of fiberwise metric. For the ellipticity (1), it is also independent of the choice of oriented orthonormal bases.
2) Moreover, in the parabolic case (2) or in the hyperbolic case (3), we can easily modify the fiberwise metric so that we can take and $T_0 = 0$ for (2) and (3) and $c' = 1$ for (3) (see *e.g.*, [KV] or [Mi]).
3) Also note that in the parabolic case or in the hyperbolic case, there is no other $\mathcal{W}$-invariant (continuous) sub-line bundles of $\mathcal{E}/\mathcal{W}$ other than l^h, l^u, or l^s.

Definition 2.6 Let $\mathcal{D}$ be an Engle structure $\mathcal{D}$ on a closed connected 4-manifold M.

(1) (elliptic) $\mathcal{D}$ is of *elliptic* type just if its even contact structure $\mathcal{E}$ is of elliptic type, .

(2) (parabolic) Let us assume the even contact structure $\mathcal{E}$ to be parabolic. If moreover $\mathcal{D}/\mathcal{W}$ does not intersect with l^h, $\mathcal{D}$ is called of *genuine-parabolic* type, or just of *parabolic* type. If there exists a constant $T_1 > 0$ such that the forward orbit $\{\phi_t(P) \mid t \in [0, T_1]$ of any point $P \in M$ (*i.e.*, a portion of a $\mathcal{W}$-curve) contains a point on which $\mathcal{D}/\mathcal{W}$ and l^h intersect with each other , $\mathcal{D}$

is called of *trans-parabolic* type, Otherwise it is called of *incomplete-parabolic* type.

(3) (hyperbolic) Let us assume the even contact structure $\mathcal{E}$ to be hyperbolic. If moreover $\mathcal{D}/\mathcal{W}$ does not intersect with l^u nor with l^s, $\mathcal{D}$ is called of *genuine-hyperbolic* type, or just of *hyperbolic* type. If there exists a constant $T_1 > 0$ such that the forward orbit $\{\phi_t(P) \mid t \in [0, T_1]$ of any point $P \in M$ (*i.e.*, a portion of a $\mathcal{W}$-curve) contains a point on which $\mathcal{D}/\mathcal{W}$ and $l^u \cup l^s$ intersect with each other, $\mathcal{D}$ is called of *trans-hyperbolic* type, Otherwise it is called of *incomplete-hyperbolic* type.

Remark 2.7 In parabolic or hyperbolic case, we do not know if there do exists incomplete ones. In particular, we do not know if there exists an Engel structure with parabolic even contact structure, which admits $\mathcal{W}$-orbits of two types, one contains a point on which $\mathcal{D}/\mathcal{W}$ and $l^u \cup l^s$ intersect with each other, and the other does not. These are fundamental problems to be considered.

In the case of the Cartan prolongation of a contact 3-manifold, if the contact structure has trivial Euler class as plane field, then it is considered to be obtained by suspension construction by the identity. Even if the Euler class is not trivial, locally it is considered similarly and the resulting Engel structure is of elliptic type. See the following example.

If there exists a closed $\mathcal{W}$-curve Γ in an Engel structure of one of the above types, Γ it self has the same type.

Proposition 2.8 Let $\mathcal{E}$ be a suitably oriented even contact structure on a closed 4-manifold M which is of parabolic or hyperbolic type. Then there exists an Engel structure D on M with whose even contact structure coincides with the given one and which is of genuine-parabolic or genuine-hyperbolic type. In the elliptic case, an even contact structure may fail to admit a compatible Engel structure.

Proof. In both cases, take the metric on $\mathcal{E}/\mathcal{W}$ as in Remark 2.5 2). Then it suffices to take $\mathcal{D}$ to be $\langle \ell_2 \rangle \oplus \mathcal{W}$ in the parabolic case, and $\mathcal{D}_\pm$ to be $\langle \ell_1 \pm \ell_2 \rangle \oplus \mathcal{W}$ in the hyperbolic case. Eventually in the hyperbolic case we obtain a (genuine-hyperbolic) bi-Engel structure in the sense of Kotschick and Vogel [KV]. Q.E.D. □

For the elliptic case, see the following examples.

Example 2.9 Let ξ be an oriented contact structure on $S^2 \times S^1$. Take an S^1-bundle $H = h \times \mathrm{id}_{S^1} : S^3 \times S^1 \to S^2 \times S^1$ where $h : S^3 \to S^2$ is the Hopf fibration. Then the even contact structure $\mathcal{E} = (DH)^{-1}\xi$ on $S^3 \times S^1$ admits a compatible Engel structure if and only if $e(\xi) \neq 0 \in H^2(S^2 \times S^1; \mathbb{Z}) \cong H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$.

In particular the standard tight contact structure on $S^2 \times S^1$, we do not obtain an Engel structure in this sense.

Proof. For the sake of simplicity, if n is negative, we change the orientation of ξ and assume that $n \geq 0$. If $e(\xi) = nin\mathbb{Z} \cong H^2(S^2 \times S^1; \mathbb{Z})$, its Cartan prolongation is defined on $L(2n,1) \times S^1$ whose even contact structure is given as $\mathcal{E}_{2n} = DH_{2n}^{-1}(\xi)$ where $h_{2n} : L(2n,1) \to S^2$ is the S^1-bundle of Euler class $2n$ and $H_{2n} : L(2n,1) \times S^1 \to S^2 \times S^1$ is defined as $H_{2n} = h_{2n} \times \mathrm{id}_{S^1}$. Then taking $2n$-fold covering in $L(2n,1)$ direction, we obtain what we want.

On the other hand, if $n = 0$, by contradiction, we assume that there exists an Engel structure $\mathcal{D}$ on $S^3 \times S^1$ compatible with $\mathcal{E}$. Because $e(\xi) = 0$, there exists non-singular Legendrian vector field ℓ on $S^2 \times S^1$. By abuse of notation let ℓ also denote the pull-back of ℓ to $\mathcal{E}/\mathcal{W}$ on $S^3 \times S^1$. Then, from the definition of Engel structures $\Sigma = \{P \in S^3 \times S^1 \mid (\mathcal{D}/\mathcal{W})_P = \ell_P\}$ is a non-singular closed hypersurface of $S^3 \times S^1$ which is transverse to $\mathcal{W}$. Along any fibre $H^{-1}(x)$ ($x \in S^2 \times S^1$) if we trace the movement of $\mathcal{D}/\mathcal{W}$ in ξ_x, it is clear that each fibre intersects with Σ. Namely Σ is a multi-section of H. Therefore its Euler class is at most of torsion. This is a contradiction. Q.E.D. $\square$

3 Accessible set, causality, and rigidity

Bryant and Hsu showed that W-curves inside an Engel-Darboux coordinate neighborhood exhibit a rigidity property among D-curves. Inaba improved their computation and established the notion of accessible set, which seems perfectly fits into the causality property of Lorentz manifolds.

In Subsection 3.2 we propose an infinitesimal version of the rigidity which characterizes W-curves. This notion is valid for any W-curves of any length and the mechanism of the rigidity is very simple. Moreover it is well-adapted to give a fairy simple proof of Theorem 1.3.

3.1 Rigidity of Cauchy characteristic curves and accessible sets

Bryant and Hsu found the following rigidity phenomena on the Cauchy characteristic curve among $\mathcal{D}$-curves in an Engel manifold $(M, \mathcal{D})$, where $\mathcal{D}$-curve is a smooth curve which is everywhere tangent to $\mathcal{D}$.

Let $\gamma : [a,b] \to M$ be an embedded regular curve which is tangent to the Cauchy characteristic $\mathcal{W}$ and is included in an Engel-Darboux coordinate. Therefore after changing the coordinates of the Engel-Darboux chart and $[a,b]$ if necessary we may assume that $\gamma(t) = (0,0,0,t)$ for $t \in$: $[0,T]$ in an Engel-Darboux chart $\{(x,y,z,w)\}$ with $\mathcal{D} = \ker[dy - zdx] \cap \ker[dz - wdx]$. We consider $\mathcal{D}$-curves which are C^1 close to γ.

Theorem 3.1 (Bryant-Hsu, [BH]) Let $\omega(t) = (x(t), y(t), z(t), t)$ ($0 \le t \le T$) be a $\mathcal{D}$-curve which satisfies $\omega(0) = (0,0,0,0)$ and $\omega(T) = (0,0,0,T)$. Then ω coincides with γ, *i.e.*, $\omega(t) = (0,0,0,t)$ for $0 \le t \le T$.

Proof. We follow Inaba's computation ([I]) which is easier to see. First let us forget the condition $\omega(T) = \gamma(T)$ and compute $y(T)$:

$$y(T) = \int_0^T \frac{dy}{dt} dt = \int_0^T z \frac{dx}{dt} dt = \int_0^T z \frac{1}{w} \frac{dz}{dt} dt = \int_0^T \frac{1}{2w} \frac{d(z^2)}{dt} dt\,.$$

Here the integral is ordinary, because $\frac{z}{w} \to 0$ and $\frac{1}{w}\frac{d(z^2)}{dt} \to 0$ when $t \to 0+0$. By integrating by parts, we have

$$y(T) = \frac{z^2}{2w} - \int_0^T \frac{z^2}{2} \frac{d(w^{-1})}{dt} dt = \frac{z^2}{2w} + \int_0^T \frac{z^2}{2w^2} dt dt\,.$$

Therefore if we only impose $y(T) = 0$ we can conclude that $z(t) \equiv 0$ and thus $y(t) \equiv 0$ and $x(t) \equiv 0$ for $t \in [0, T]$. Q.E.D. □

The above computation enabled Inaba to define the *accessible set* A in the Engel-Darboux chart.

Definition 3.2 (Accessible set, [I]) Let A be the following subset of the Engel-Darboux neighborhood $\mathbb{R}^4 = \{(x, y, z, w)\}$.

$$A = A_+ \cup A_- \cup A_{\mathcal{W}} \quad \text{where} \quad A_{\mathcal{W}} = \{x = y = z = 0\}\,,$$
$$A_+ = \{y > \frac{z^2}{2w},\ w > 0\}, \quad A_- = \{y < \frac{z^2}{2w},\ w < 0\}.$$

A is called the *accessible set* from the origin. Note that $A_\pm$ is irrelevant to the x-coordinate.

Theorem 3.3 (Inaba, [I]) 1) If a curve $\gamma : [0, T] \to \mathbb{R}^4$ of the form $\gamma(t) = (x(t), y(t), z(t), w = t)$ $(0 \le t \le T)$ in the Engel-Darboux chart is a $\mathcal{D}$-curve which starts at the origin (*i.e.*, $\gamma(0) = (0, 0, 0, 0)$), then the other end point $\gamma(T)$ lies in A_+ or $x(t) \equiv y(t) \equiv z(t) \equiv 0$ namely γ itself stays in $A_{\mathcal{W}}$ (*i.e.*, is a $\mathcal{W}$-curve).

Similarly, a $\mathcal{D}$-curve $\gamma : [-T, 0] \to \mathbb{R}^4$ of the form $\gamma(t) = (x(t), y(t), z(t), w = t)$ $(-T \le t \le 0)$ which ends at the origin (*i.e.*, $\gamma(0) = (0, 0, 0, 0)$), then the other end point $\gamma(-T)$ lies in A_- or $x(t) \equiv y(t) \equiv z(t) \equiv 0$ namely γ itself stays in $A_{\mathcal{W}}$ (*i.e.*, is a $\mathcal{W}$-curve).

2) Conversely any point in $A_\pm$ can be joind to the origin by such a D-curve in the Engel-Darboux chart. If the curve touches $A_{\mathcal{W}}$, *i.e.*, $\gamma(t) = (0, 0, 0, t)$ for some $t \neq 0$, on $[t, 0]$ or on $[0, t]$ (depending non the sign of t), γ stays in $A_{\mathcal{W}}$.

Inaba computed the accessiboe set in the long Engel-Darboux chart. Then at the critical length π the set is the natural continuation of what is described above. In the usual coordinate it is also noteworthy that the shape of the accessiboe set is the right cone in (y, z, w)-space, becuase $0 = z^2 - 2yw = z^2 + (\frac{1}{\sqrt{2}}(y - w))^2 - (\frac{1}{\sqrt{2}}(z + w))^2$.

3.2 Infinitesimal rigidity*

Let us introduce an infinitesimal version of the rigidity of Cauchy characteristic curves.

Definition 3.4 For a nonsingular $\mathcal{D}$-curve $\gamma : [a,b] \to M$ is called *linearly strongly flexible* (*LSF* for short) iff there exists a smooth deformation $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ for some $\varepsilon > 0$ satisfying

$$\begin{array}{rl} \text{(i)} & \Gamma(s,\cdot) = \gamma_s \ : \text{non}-\text{singular } \mathcal{D}-\text{curve } \ \forall s \in (-\varepsilon,\varepsilon), \\ \text{(ii)} & \Gamma(s,t) = \gamma(t) \text{ on } \exists\text{neighborhood of } (-\varepsilon,\varepsilon) \times \{a,b\}, \\ \text{(iii)} & \qquad \dfrac{\partial \Gamma}{\partial s}(0,t) \notin \mathcal{E}_{\Gamma(0,t)} \quad \exists t \in (a,b). \end{array}$$

γ is called *infinitesimally weakly rigid* (*IWR* for short) if it is not LSF.

Proposition 3.5 (LSF)
1) If a non-singular $\mathcal{D}$-curve $\gamma : [a,b] \to M$ is not a $\mathcal{W}$-curve, it is LSF.
2) A non-singular $\mathcal{D}$-curve $\gamma : [a,b] \to M$ is IWR iff it is a $\mathcal{W}$-curve, in particular, regardless to its length as projective structure.

Of course 2) implies 1) and eventually 1) can be also stated as "iff".

Proof. 1) follows from the Lemma below. Therefore in order to prove 2) it is enough to show the sufficiency. .

Take a $\mathcal{W}$-curve $\gamma : [a,b] \to M$. Then we can find a long Engel-Darboux coordinates (x,y,z,θ) on a neighborhood of $\gamma([a,b])$ with $\mathcal{D} = \ker[dy - zdx] \cap \ker[\cos\theta\, dz - \sin\theta\, dx]$ in such a way that $\gamma(a) = (0,0,0,0)$, $\gamma(b) = (0,0,0,\Theta)$, $x(t) \equiv y(t) \equiv z(t) \equiv 0$ for $t \in [a,b]$, are satisfied. So γ is identified with the curve $(0,0,0,\theta)$ for $\theta \in [0,\Theta]$. We have to show that any deformation Γ with the properties (i) and (ii) in Definition 3.4 does not satisfies (iii). As the even contact structure $\mathcal{E} = \ker[dy - zdx]$ coincides with the (x,z,θ)-hyperplane along γ, (iii) is equivalent to $\dfrac{\partial y}{\partial s}(0,\theta) \neq 0$.

In order to compute $\dfrac{\partial y}{\partial s}(0,\theta)$, we divide $[0,\Theta]$ into (possibly shorter) closed intervals so that on each interval $\tan\theta$ or $\cot\theta$ is well defined. On the intervals on which $\cot\theta$ is well-defined, from Inaba's computation, we have

$$y(s,\theta_1) - y(s,\theta_0) = \left[\frac{1}{2} z(s,\theta)^2 \cot\theta\right]_{\theta_0}^{\theta_1} + \int_{\theta_0}^{\theta_1} \frac{1}{2} z(s,\theta)^2 (1 + \cot^2\theta) d\theta$$

while on the intervals on which $\tan\theta$ is well-defined, we can exchange the roll of x and z by integration by parts and then we obtain

$$\begin{aligned} y(s,\theta_1) - y(s,\theta_0) &= [z(s,\theta)x(s,\theta)]_{\theta_0}^{\theta_1} \\ &\quad - \left[\frac{1}{2} x(s,\theta)^2 \tan\theta\right]_{\theta_0}^{\theta_1} + \int_{\theta_0}^{\theta_1} \frac{1}{2} x(s,\theta)^2 (1 + \tan^2\theta) d\theta \,. \end{aligned}$$

In any case, on the right hand side each term is quadratic with respect to x and z. Therefore if we apply $\left.\frac{\partial}{\partial s}\right|_{s=0}$ we obtain $\frac{\partial y}{\partial s}(0,\theta_1) = \frac{\partial y}{\partial s}(0,\theta_0)$ because $x(0,\theta) \equiv z(0,\theta) \equiv 0$. Starting from $\frac{\partial y}{\partial s}(0,0) = 0$ and repeatedly applying the above, we obtain $\frac{\partial y}{\partial s}(0,\theta) = 0$ for any $\theta \in [0,\Theta]$. Q.E.D. □

Remark 3.6 1) We see from the above argument that in the definition of IWR, we do not have to fix the both end of γ to deform into Γ, *e.g.*, the boundary condition $\Gamma(s,a) = \gamma(a)$ for $s \in (-\varepsilon,\varepsilon)$ is enough.
2) A better computation without dividing into intervals must be found for the proof of 2) if we carefully translate the computation in Lorentzian spaces which is done in the next subsection. This is left to the readers.

Lemma 3.7 (Normal form for $\mathcal{D}$-curves transverse to $\mathcal{W}$)
1) If a non-singular $\mathcal{D}$-curve $\gamma : [c,d] \to M$ is no where tangent to $\mathcal{W}$, for any $t_0 \in (c,d)$ there is a neighborhood $[c',d']$ of t_0 in $[c,d]$ and an Engel-Darboux chart around $\gamma([c',d'])$ such that $\gamma([c',d'])$ is an segment $[-\varepsilon,\varepsilon]$ in the x-axis for some $\varepsilon > 0$ and $\gamma(t_0)$ is the origin.
2) Take a smooth function $f(x)$ of a single variable x which is supported in $(-\varepsilon,\varepsilon)$ and satisfies $f(0) \neq 0$. Then the family of functions $F(s,x) = sf(x)$ for $s \in (-\varepsilon,\varepsilon)$ gives rise to a deformation which shows that γ in 1) is LSF.

Proof. The condition implies the projection of γ to the local quotient space $M/\mathcal{F}_{\mathcal{W}}$ is an immersion. Take a smaller part if necessary so that it is an embedding. Then fix a Darboux coordinate (x,y,z) for $\mathcal{E}/\mathcal{W}$ on this small neighborhood such that $\ker[dy - zdx] = \mathcal{E}/\mathcal{W}$ holds and the image of is as in the statement 1). Then naturally the statement follows. Q.E.D. □

3.3 Null-geodesic in Lorentzian 3-manifold*

Here we give an alternative proof for Theorem 1.3 by using 1) in Proposition 3.5.

A priori, we have to show that a natural lift of null-geodesic is a $\mathcal{W}$-curve and the converse implication, while in fact it is sufficient to show only the first because of the following reason. At any point of $(v,l) \in M = NC(TV)$, l being represented by a non-zero null vector $\boldsymbol{\ell}$, $(\gamma(0),\dot\gamma(0)) = (v,\boldsymbol{\ell})$ gives an initial condition of a null-geodesic $\gamma(t)$. Then the equation of (null)-geodesics admits at least locally a unique solution. It implies the natural lifts of null-geodesics defines a 1-dimensional smooth non-singular foliation on $NC(TV)$.

Let $\beta : [a,b] \to V$ is a non-constant null-geodesic of a Lorentzian 3-manifold (V,dg) and (M,D) be its Lorentz prolongation. By definition, the natural lift γ of β to $M = NC(TV)$ is a non-singular D-curve. Take a deformation Γ as in Definition 3.4 with properties (i) and (ii) and then consider if the property (iii) $\frac{\partial\Gamma}{\partial s}(0,t) \notin \mathcal{E}_{\Gamma(0,t)} \quad \exists t \in (a,b)$ is realizable or not. Now the condition (iii) is equivalent to

$$\pi^* dg\left(\frac{\partial\Gamma}{\partial t}(0,t), \frac{\partial\Gamma}{\partial s}(0,t)\right) \neq 0 \text{ for some } t \in (a,b)\,.$$

Here $\pi^* dg(\cdot, \cdot)$ denotes the pull-back of the Lorentzian metric tensor to M by the projection $\pi : M \to V$. On the other hand apparently we have

$$\pi^* dg\left(\frac{\partial\Gamma}{\partial t}(0,t), \frac{\partial\Gamma}{\partial s}(0,t)\right) = dg\left(\dot\beta(t), \frac{\partial B}{\partial s}(0,t)\right)$$

where $B = \pi \circ \Gamma$ so that we can compute them on V.

For any such deformation and $t \in [a,b]$, using the Levi-Civita connection ∇, we have

$$dg\left(\dot\beta(t), \frac{\partial B}{\partial s}(0,t)\right) = \int_0^t \frac{\partial}{\partial t} dg\left(\dot\beta(\tau), \frac{\partial B}{\partial s}(0,\tau)\right) d\tau$$
$$= \int_0^t dg\left(\nabla_{\dot\beta(\tau)}\dot\beta(\tau), \frac{\partial B}{\partial s}(0,\tau)\right) d\tau + dg\left(\dot\beta(\tau), \nabla_{\dot\beta(\tau)}\frac{\partial B}{\partial s}(0,\tau)\right) d\tau\,.$$

The first term vanishes because β is a geodesic. As we have the map $B : (-\varepsilon,\varepsilon) \times [a,b] \to V$, $\dfrac{\partial B}{\partial s}$ and $\dfrac{\partial B}{\partial t}$ commute to each other. Therefore we can compute the second term as follows.

$$\int_0^t dg\left(\dot\beta(\tau), \nabla_{\dot\beta(\tau)}\frac{\partial B}{\partial s}(0,\tau)\right) d\tau = \int_0^t dg\left(\frac{\partial B}{\partial t}(0,\tau), \left(\nabla_{\frac{\partial B}{\partial s}}\frac{\partial B}{\partial t}\right)(0,\tau)\right) d\tau$$
$$= \int_0^t \frac{1}{2}\left.\frac{\partial}{\partial s}\right|_{s=0} dg\left(\frac{\partial B}{\partial t}(s,\tau), \frac{\partial B}{\partial t}(s,\tau)\right) d\tau\,.$$

The final term is nothing but $\displaystyle\frac{1}{2}\int_0^t \left.\frac{\partial}{\partial s}\right|_{s=0} dg\left(\dot\beta_s(\tau), \dot\beta_s(\tau)\right) d\tau$ and vanishes because β_s is a null-curve for any $s \in (-\varepsilon,\varepsilon)$. Q.E.D. □

3.4 Causality

We do not discuss about global causal structure but take a look at only local problems. For a general Lorentzian manifold V and on a small neighborhood U of a point $P \in V$, fix a future/past orientation. Namely, as the set of time-like vectors in T_vV has two components continuous choice of one of them for each point v is the future orientation. Then non-zero light-like(=null) vectors are also split into future or past oriented ones.

Now we consider curves joining a point P to another one Q in a small neighborhood U whose velocity is future oriented. It is not difficult to see that if P is joined to Q by such a curve, we can always find another curve ω joining P to Q whose velocity is always future oriented light-like.

Let us fix the starting point P and consider the local accessible set $A_P = \{Q \in U\,;\, \gamma(0) = P,\, \gamma(1) = Q,\, \dot{\gamma}(t)(\leq 0) : \text{future oriented}\}$ from P. It is locally a closed cone, whose interior consists of points to which time-like curves can reach.

the accessible set A_P associated with P is defined as the set of points at which a positive time-like curve starting from P arrives. Its closure $\overline{A_P}$ is the set of points at which (time positive) light-like (*i.e.*, null-)curves arrive. If the accessible sets define a strict partial order on V, the global causality is established.

If we consider the causality in a local sense, it is known that the boundary of accessible set is achieved only by null-geodesics. Together with the Bryant-Hsu rigidity, this fact should give one more proof of Theorem 1.3. This is left to the readers, while this motivated the proof in the previous subsection and the notion of IWR.

4 Geometry and dynamics of basic examples

In this section we look at examples of Lorentz prolongation and pre-quantum prolongation, for which the dynamical property of $\mathcal{E}/\mathcal{W}$ is also investigated.

4.1 Lorentz prolongation-I : Product extension*

For a surface Σ with a Riemannian metric dh, consider the Lorentzian 3-manifold $(V, dg) = (\Sigma, dh) \times (S^1, -d\theta^2)$ which is just the direct product with a circle with negative metric. We call this construction the *product* (*Lorentzian*) *extension*. Then we obtain an Engel manifold $(M = NC(TV), \mathcal{D})$ as explained in 1.3. Here we follow the notations there. We consider the the case where Σ is complete and practically closed.

The Cauchy characteristic curves are the natural lifts of the null-geodesics, while each null-geodesic of V is just the combination of a geodesic on Σ and that on S^1 with same speed. The unit tangent circle bundle $S^1(T\Sigma)$ admits the geodesic flow ϕ_t, which is defined as $\phi_t((\sigma, v)) = (\gamma(t), \dot{\gamma}(t))$, where γ is the unique geodesic on Σ with the initial condition $\gamma(0) = \sigma$ and $\dot{\gamma}(0) = v$.

Even though we do not need detailed description of $S^1(T\Sigma)$ and the geodesic flow, for later use, we fix some notations here. Let X be the vector field which is the horizontal (with respect to the Levi-Civita connection) lift of the tautological vectors, namely, $\pi_* X_{(\sigma,v)} = v \in T\Sigma$. Also Y denotes the horizontal unit vector field, so that X and Y form an oriented orthonormal basis of the horizontal space. Let Z denote the unit tangent vector field along the fibres. Therefore they satisfy the following commutation relations

$$[Z, X] = Y\,, \quad [Z, Y] = -X\,, \quad \textit{and} \quad [X, Y] = \kappa \circ \pi \cdot Z$$

where κ denotes the curvature function of Σ and $\pi : S^1(T\Sigma) \to \Sigma$ is the bundle projection. The projection $M = NC(TV) \to V$ is also denoted by π by abuse of notation, because through the identification explained below they correspond to each other. The geodesic flow ϕ_t is generated by X, namely $X = \dot{\phi}_t \circ (\phi_t)^{-1}$ holds.

On each point $(\sigma, \theta) \in V = \Sigma \times S^1$, $NC(TV_{(\sigma,\theta)})$ is identified with the unit tangent circle $S^1(T_\sigma\Sigma)$ through the identification $v \in S^1(T_\sigma\Sigma) \leftrightarrow \langle v + \dfrac{\partial}{\partial\theta}\rangle \in NC(TV_{(\sigma,\theta)})$. Therefore $M = NC(TV)$ is identified with $S^1(T\Sigma) \times S^1$. Under this identification, the Cauchy characteristic is generated by the vector field $X + \dfrac{\partial}{\partial\theta}$. Note that $\dfrac{\partial}{\partial\theta}$ commutes with any of X, Y, and Z.

The first one X^* from of the dual frame X^*, Y^*, Z^* for $T^*S^1(T\Sigma)$ is, under the identification $S^1(T^*\Sigma) = S^1(T\Sigma)$ by the Riemannian metric dh, nothing but the tautological 1-form and defines the Liouville contact structure ξ_0, whose Reeb vector field is nothing but X.

The Cauchy characteristic is given by $\mathcal{W} = \left\langle W = X + \dfrac{\partial}{\partial\theta} \right\rangle$ and the Engel structure is given as the span $\mathcal{D} = \langle W, Z\rangle$.

Proposition 4.1 The Engel manifold $(M = NC(TV), \mathcal{D})$ obtained as the Lorentz prolongation of $(\Sigma, dh) \times (S^1, -d\theta^2)$ is isomorphic to the one given by the suspension construction (see 1.5) starting from the contact manifold $(S^1(T\Sigma), \xi_0 = \langle Y, Z\rangle)$, the Legendrian field Y or Z either of which will do, and the contact diffeomorphism given as the time 2π map $\phi_{2\pi}$ of the geodesic flow, with an appropriate twisting.

Proof. The one by the suspension construction is given as follows. On the mapping cylinder $M' = S^1(T\Sigma) \times \mathbb{R}/\sim$ where $\sim$ is the identification $((\sigma, v), t + 2\pi) \sim (\phi_{2\pi}((\sigma, v)), t)$, the Cauchy characteristic $\mathcal{W}' = \left\langle \dfrac{\partial}{\partial t} \right\rangle$ and the even contact structure $\mathcal{E}' = \mathcal{W} \oplus \xi_0$ is automatically fixed. The Engel structure is defined as $\mathcal{D}'_{((\sigma,v),t)} = \langle W' \oplus \phi_{-t*}(\langle Z\rangle)\rangle$. On the cyclic covering $\tilde{M}' = S^1(T\Sigma) \times \mathbb{R}$, $\tilde{\mathcal{W}}'$, $\tilde{\mathcal{D}}'$, and $\tilde{\mathcal{E}}'$ are defined as well and they are invariant under the deck transformation $T' : ((\sigma, v), t) \mapsto (\phi_{-2\pi}((\sigma, v)), t + 2\pi)$. and thus we obtain $\mathcal{D}'$. We can check $[\mathcal{D}', \mathcal{D}'] = \mathcal{E}'$ by the commutation relation $[-X, Z] = Y$ directly. But instead of doing it we show that $(M', \mathcal{W}', \mathcal{D}', \mathcal{E}')$ is isomorphic to the Lorentz prolongation $(M', \mathcal{W}', \mathcal{D}', \mathcal{E}')$.

Let us follow the identification $M = S^1(T\Sigma) \times S^1$ given above and consider its cyclic covering. $\tilde{M} = S^1(T\Sigma) \times \mathbb{R}^1$ where everything is lifted and indicated with $\tilde{}$. The deck transformation is $T : ((\sigma, v), \theta) \mapsto ((\sigma, v), \theta + 2\pi)$. Consider the diffeomorphism

$$\tilde{\Phi} : \tilde{M}' \to \tilde{M}, \quad \tilde{\Phi}((\sigma, v), t) = (\phi_t(\sigma, v), \theta).$$

It is clear from the construction that we have $\tilde{\Phi} \circ T' = T \circ \tilde{\Phi}$, $\tilde{\Phi}_* \tilde{\mathcal{W}}' = \tilde{\mathcal{W}}$, $\tilde{\Phi}_*\{\phi_{-t} * Z\} = Z$, and thus $\tilde{\Phi}_* \tilde{\mathcal{D}}' = \tilde{\mathcal{D}}$ as well. The fact that ϕ_t preserves the Liouville contact structure ξ_0 implies $\tilde{\Phi}_*\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}$. Therefore $\tilde{\Phi}$ descends to the diffeomorphism $\Phi : (M', \mathcal{W}', \mathcal{D}', \mathcal{E}') \to (M, \mathcal{W}, \mathcal{D}, \mathcal{E})$. Q.E.D. $\square$

Let us take a look at the dynamics of the Cauchy characteristic $\mathcal{W}$. $\mathcal{W}$ is spanned by the vector field $W = X + \dfrac{\partial}{\partial\theta}$ Therefore the holonomy action on $TM/\mathcal{E}$ is trivial in this construction, because it comes from $\dfrac{\partial}{\partial\theta}$ on S^1. The action on $\mathcal{E}/\mathcal{W}$ is nothing but that of the geodesic flow and the curvature of the surface Σ is directly reflected.

Proposition 4.2 In the case where the curvature κ of (Σ, dh) is positive, the action of $\mathcal{W}$ on $\mathcal{E}/\mathcal{W}$ is of elliptic type. In the case $\kappa \equiv 0$, *i.e.*, Σ is flat, it is of parabolic type. I the case $\kappa < 0$, it is of hyperbolic type. Trans-hyperbolic nor trans-parabolic case never happens in this construction.

Remark 4.3 If we start from a surface with negative curvature, what we obtain is one of the bi-Engel structure of Example 1.9 that Kotschick and Vogel obtained in [KV]. In this case the bi-Engel structure corresponds to the bi-contact structure $(\xi_+ = \langle X, Y\rangle = \langle h, l\rangle$, $\xi_- = \langle X, Z\rangle = \langle h, k\rangle)$ associated with the geodesic Anosov flow $\exp tX$. (See the next subsectioj for the notation h, l, and k.) In particular, the Engel structure we obtained here corresponds to ξ_-.

A natural construction which gives rise to the other one corresponding to ξ_+ is given in Subsection 4.3.

4.2 Lorentz prolongation-II : Magnetic extension*

We start from a Riemannian surface (Σ, dh) like in the previous subsection while a slightly different constrction is adopted to obtain a 3-dimensional Lorentzian manidfold (V, dg). Let V be the unit tangent circle bundle $S^1(T\Sigma)$, so that it admits the unique Levi-Civita connection ∇^Σ. At each point $(\sigma, v) \in S^1(T\Sigma)$, the tangent space admits the horizontal/vertical splitting $T_{(\sigma,v)}S^1(T\Sigma) = H_{(\sigma,v)} \oplus V_{(\sigma,v)}$, where we have the natural identification $H_{(\sigma,v)} \cong T_\sigma\Sigma$ and $V_{(\sigma,v)} \cong T_v S^1(T_\sigma\Sigma)$. The Lorentzian metric $dg = \langle , \rangle$ on $V = S^1(T\Sigma)$ is defined as $dg = dh \oplus (-d\theta^2)$ with respect to the splitting. Of course $d\theta^2$ denotes the canonical metric of the unit circle $S^1(T_\sigma T\Sigma)$. We call this construction *magnetic (Lorentzian) extension* of a Riemannian surface Σ. In this subsection we are particularly interested in the Lorentz prolongation of the magnetic extension of compact srufaces with constant curvature.

Before getting into special examples, let us fix some notations which are valid throughout this subsction. Let X, Y, and Z denote the same vector fields on $S^1(T\Sigma)$, so that they form a Lorentzian orthonormal frame, namely, $\langle X, X\rangle = \langle Y, Y\rangle = -\langle Z, Z\rangle = 1$ and $\langle X, Y\rangle = \langle Y, Z\rangle = \langle Z, X\rangle = 0$. At each point $(\sigma, v) \in V = S^1(T\Sigma)$, the horizontal lift of $v \in T\Sigma$ to $H_{(\sigma,v)}$ is $X_{(\sigma,v)}$ by definition. $NC(T_{(\sigma,v)}V)$ is identified with $S^1 \cong S^1(H_{(\sigma,v)})$ by assigning $X(\theta) \mapsto l = \langle X(\theta) + Z\rangle$ where $X(\theta)$ and $Y(\theta)$ denote $\cos\theta\, X +$

$\sin\theta\, Y$ and $-\sin\theta\, X + \cos\theta\, Y$ respectively. Therefore $M = NC(TV)$ is naturally identified with $S^1(T\Sigma) \times S^1$. With respect to this product structure, let us introduce three horizontal vector fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and a vertical vectorfield Θ as

$$\tilde{X}|_{V\times\{\theta\}} = X(\theta), \ \ \tilde{Y}|_{V\times\{\theta\}} = Y(\theta), \ \ \tilde{Z}|_{V\times\{\theta\}} = Z, \ \ \Theta = \frac{\partial}{\partial\theta}.$$

which form a global frame of TM and satisfy the commutation relations

$$[\tilde{Z}, \tilde{X}] = \tilde{Y}, \ [\tilde{Z}, \tilde{Y}] = -\tilde{X}, \ [\tilde{X}, \tilde{Y}] = \kappa\tilde{Z},$$

$$[\Theta, \tilde{X}] = \tilde{Y}, \ [\Theta, \tilde{Y}] = -\tilde{X}, \ [\Theta, \tilde{Z}] = 0$$

and the metric relations

$$\langle \tilde{X}, \tilde{X} \rangle = \langle \tilde{Y}, \tilde{Y} \rangle = -\langle \tilde{Z}, \tilde{Z} \rangle = 1, \quad \langle \tilde{X}, Y \rangle = \langle \tilde{Y}, \tilde{Z} \rangle = \langle \tilde{Z}, \tilde{X} \rangle = 0,$$

and

$$\langle \Theta, \cdot \rangle = 0.$$

Here again κ denotes the (pull-back of) curvature of Σ and by abuse of notation, $\langle \cdot, \cdot \rangle$ denotes also the pull-back of itself by the projection $\pi : S^1(T\Sigma) \times S^1 \to S^1(T\Sigma)$.

If the surface is a flat torus, the magnetic extension is the same as taking product with $(S^1, -d\theta^2)$. If the curvature κ is not identically zero, a priori the resullt is different from the product extension.

Particularly interesting is the case of hyperbolic surfaces, *i.e.*, $\kappa \equiv -1$. Let us first look at this case because this case can be described in a special and totally different way and also because even κ is negative constant only the case $\kappa \equiv -1$ exhibits quite a different feature.

Example 4.4 (Magnetic Lorentzian extension of hyperbolic surface) A hyperbolic surface is a quotient $\Gamma\backslash\mathbb{H}^2$ where $\pi_1(\Sigma) \cong \Gamma \subset \mathrm{Isom}^+(\mathbb{H}^2) = PSL(2;\mathbb{R})$ is a torsion free co-compact discrete subgroup. The hyperbolic plane $\mathbb{H}^2$ is described as $\mathbb{H}^2 = PSL(2;\mathbb{R})/PSO(2;\mathbb{R})$ and the unit tangent bundles are described as $S^1(T\mathbb{H}^2) = PSL(2;\mathbb{R})$ and $S^1(T\Sigma) = \Gamma\backslash PSL(2;\mathbb{R})$. In its Lie algebra $psl(2;\mathbb{R})$, take the basis $h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $l = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $k = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that each of them generates the 1-parameter subgroups $\left\{\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}\right\}$, $\left\{\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\right\}$, and $\left\{\begin{pmatrix} \cos t/2 & -\sin t/2 \\ \sin t/2 & \cos t/2 \end{pmatrix}/\{\pm 1\}\right\}$ respectively. As the elemets of $psl(2;\mathbb{R})$ are considered to be left-invariant vector fields on $PSL(2;\mathbb{R})$, they descend to $S^1(T\Sigma) = \Gamma\backslash PSL(2;\mathbb{R})$. In this context h, l, and k correspond to X, Y, and Z respectively.

Then the left invariant vector field $L = X + Z \in psl(2;\mathbb{R})$ canonically assigns a null-vector to any point $(\sigma, v) \in S^1(T\Sigma)$.

Proposition 4.5 The null-vector field L fills up $S^1(T\Sigma)$ with the null-geodesic orbits.

This fact is understood in many ways. For example, the Lorentzian metric on $PSL(2;\mathbb{R})$ in this case is not only left-invariant but also right-invariant. Then any 1-parameter subgroup is a geodesic even respecting the parameter. Then any left translation of it is also a geodesic.

Verifying $\nabla_L L = 0$ by computing $\langle \nabla_L L, \cdot \rangle = 0$ for a global framing, *e.g.*, X, Y, and L is another way. A similar comlutation in more general setting will be done in the next example. A computation of this type also proves that a 1-parameter subgroup is a geodesic for a bi-invariant metric.

Proposition 4.5 implies that the natural lift of the null-geodesic which is an orbit of L on $S^1(T\Sigma)$ lies exactly on $S^1(T\Sigma) \times \{\theta = 0\}$. Here remark that if we regard Y generating a geodesic flow of Σ insead of X, Y is of Anosov and $\langle L \rangle$ is eactly its stable foliation, because $[Y, L] = L$.

For $\theta \in S^1$ put $L(\theta) = X(\theta) + Z$ and extend L to $\tilde{L} = \{L(\theta)\}_\theta$ on $M = NC(T(S^1(T\Sigma))) = S^1(T\Sigma) \times S^1$. $L(\theta)$ is the right-translation (or the Adjoint image) of L by $\exp(\theta Z)$. Thus we obtain all null-geodesics in this way. An orbit of $L(\theta)$ lies on $S^1(T\Sigma) \times \{\theta\}$ and the vector field $W = \tilde{L}$ generates the Cauchy characteristic $\mathcal{W}$.

Proposition 4.6 For the Lorentz prolongation $\mathcal{D}$ of the magnetic extension of a hyperbolic surface Σ, the Cauchy characteristic $\mathcal{W}$ is regarded as an S^1-family of the Anosov strong stable foliations associated with the geodesic flow of Σ.

In particular, it is of genuine-parabolic type. The invariant sub-line bundle in $\mathcal{E}/\mathcal{W}$ is generated by $\tilde{Y}$, the S^1-family of geodesic flows.

Proof. We verify the second statement. The commutation relation $[Y(\theta), L(\theta)] = L(\theta)$ on each $S^1(T\Sigma) \times \{\theta\}$ implies that the plane fied spanned by $Y(\theta)$ and $L(\theta)$ is integrable and in fact it is nothing but the Anosov stable foliation $\mathcal{F}^s(\theta)$ of the geodesic flow generated by $Y(\theta)$. We have seen $\mathcal{W} = \langle W \rangle = \langle \tilde{L} \rangle$, $\mathcal{D} = \langle \tilde{L}, \Theta \rangle$, and $\mathcal{E} = \langle \tilde{L}, \Theta, \tilde{Y} \rangle$.

The action of $W = \tilde{L}$ on $\mathcal{E}/\mathcal{W}$ is easily computed as

$$[\tilde{L}, \Theta] = -\tilde{Y}, \quad [\tilde{L}, \tilde{Y}] = \tilde{L} \equiv 0 \text{ in } \mathcal{E}/\mathcal{W}.$$

The first one is a general phnomenon, while the second one is characteristic in this case. Along W, θ inclines towards $\pm\tilde{Y}$ but it never reaches. This proves the $\langle \tilde{Y} \rangle$ is an invariant sub-line bundle of $\mathcal{E}/\mathcal{W}$ with which $\mathcal{D}/\mathcal{W} = \langle \Theta \rangle$ never coincides. Q.E.D. □

Remark 4.7 The 2-dimensional foliation $\tilde{\mathcal{F}} = \{\mathcal{F}^s(\theta)\}_\theta$ on $S^1(T\Sigma) \times S^1$ naturally extends to two different 3-dimensional foliations $\mathcal{G}_0 = \{S^1(T\Sigma) \times \{\theta\}$ and $\mathcal{G}_1$, whose intersection is exactly $\tilde{\mathcal{F}}$. Because $\mathcal{F}^s(\theta) = \exp(\theta Z)^* \mathcal{F}^s = \exp(-\theta Z)_* \mathcal{F}^s$, the leaves of G_1 is the trace of a leaf of $\mathcal{F}^s$ by the S^1 action generated by $-\tilde{Z} + \Theta$. Indeed, apparently we have $[-\tilde{Z} + \Theta, \tilde{Y}] = [-\tilde{Z} + \Theta, \tilde{L}] = 0$.

Example 4.8 (Lorentzian extension of flat torus) If the surface Σ is a flat torus, namely the case of $\kappa \equiv 0$, the magnetic extension and the product extension coincide to each other. In this case also the Engel structure by Lorentz prolongation is of genuine-parabolic type.

Now we proceed to more general case. The goal of this subsection is the following result.

Theorem 4.9 Let (Σ, dh) is a compact (or complete) Riemannian surface with constant curvature κ. The Engel structure by Lorentz prolongation of the magnetic extension of (Σ, dh) is of

(1) elliptic type iff $\kappa > 0$ or $\kappa < -1$

(2) genuine-parabolic type iff $\kappa = 0$ or -1

(3) genuine-hyperbolic type iff $-1 < \kappa < 0$.

The sign of the quadratic function $\kappa(\kappa + 1)$ controlls this phenomemon.

Apart from Engel structures, as a problem of magnetic Lorentzian extensions of Riemaniann surface in general, the following results are of certain interest. Also it is fundamental to understand the above theorem.

Proposition 4.10 Let (Σ, dh) be any Riemannian surface (the curvature κ can vary) and $(V = S^1(T\Sigma), dg)$ is its magnetic Lorentzian extension.

1) Any null-geodesic $\Gamma(t)$ is of constant speed if projected down to Σ.

2) Let $\gamma(t)$ be the projected image of a null-geodesic $\Gamma(t)$ as in 1). Then γ is a curve with geodesic curvature $-\kappa(\gamma(t))$.

One consequence of the above proposition and its proof in Engel structure is the following.

Corollary 4.11 If the curvature κ is constant, the natural lift of any null-geodesic stays in a single $S^1(T\Sigma) \times \{\theta\}$.

Proof of Proposition 4.10. Let (Σ, dh) be any Riemannian surface and κ its (Gaussian) curvature. The unique Levi-Civita connection of the magnetic Lorentzian extension $(V = S^1(T\Sigma), dg)$ is denoted by ∇.

Though the statements are described on V, it is easier to prove them on $M = NC(TV)$. Therefore on $M = V \times S^1$, we extend ∇ as $\nabla \times \Theta$ and by abuseof notation ∇ denotes it again. Also we pull back dg to M and $\langle\, , \rangle$ denotes both dg on V and the pulled-back on M as in the proof of Proposition 4.6.

Any point $(v, l) \in NC(TV)$ provides an initial condition for a null-geodesic $\Gamma(t)$ as $\Gamma(0) = v$, $\dot{\Gamma}(0) = X(\theta(0)) + Z \in l$ which admits a unique solution $\Gamma(t)$. As it is already explained in the second paragraph of Subsection 3.3, the unique existence for the initial condition implies $NC(TV)$ admits a line field $\mathcal{W}$, which is nothing but the Cauchy characteristic of the Engel structure, whose integral curves are the natural lifts of null-geodesics. A priori, we do not have natural choice of vector field spanning $\mathcal{W}$. Therefore we consider the problem locally. Take a point $(v_*, l_*) \in M$ and a local transversal $T \cong \operatorname{int} D^3$ to $\mathcal{W}$ which contains (v_*, l_*). On T the initial conditions for the geodesics is given as $\Gamma(0) = v \in T$, $\dot{\Gamma}(0) = X(\theta(0)) + Z \in \mathcal{W}_v$. Then, for a small $\varepsilon > 0$ $U = \{(\Gamma(t), \dot{\Gamma}(t))\,;\, (\Gamma(0), \dot{\Gamma}(0)) \in T,\, |t| < \varepsilon\}$ is an open set of M which contains (v_*, l_*) and is diffeomorphic to $T \times (-\varepsilon, \varepsilon)$ and the natural lifts of geodesics with such initial conditions define a local flow on U, whose velocity field is denoted by W. On U, W takes the following form.

$$W = r(\tilde{X} + \tilde{Z}) + f\Theta$$

Here r is a positive smooth function on U satisfying $r|_T \equiv 1$ and f is a smooth function.

The geodesic equation $\nabla_{\dot{\Gamma}}\dot{\Gamma} = 0$ for a null-geodesic $\Gamma(t)$ on V lifts to M and W satisfies $\nabla_W W = 0$ on U. Therefore it is equivalent to $\langle \nabla_W W,\, F\rangle$ for each member F of a framing of $\pi^* V$ on U, *e.g.*, for $\tilde{X}$, $\tilde{Y}$, and $\tilde{L} = \tilde{X} + \tilde{Z}$. For 1) $F = \tilde{X}$ safices. We have

$$\begin{aligned} 0 &= \langle \nabla_W W,\, \tilde{X}\rangle = W\langle W,\, \tilde{X}\rangle - \langle W,\, \nabla_W \tilde{X}\rangle \\ &= W \cdot r - \langle W,\, [W, \tilde{X}]\rangle - \langle W,\, \nabla_{\tilde{X}} W\rangle \\ &= W \cdot r - \langle W,\, [W, \tilde{X}]\rangle - \frac{1}{2}\tilde{X}\langle W,\, W\rangle = W \cdot r - \langle W,\, [W, \tilde{X}]\rangle \end{aligned}$$

while

$$\begin{aligned} [W, \tilde{X}] &= [r\tilde{X} + r\tilde{Z} + f\Theta,\, \tilde{X}] = [r\tilde{X},\, \tilde{X}] + [r\tilde{Z},\, \tilde{X}] + [fTheta,\, \tilde{X}] \\ &= -(\tilde{X} \cdot r)\tilde{X} + (r + f)\tilde{Y} - (\tilde{X} \cdot r)\tilde{Z} + (\tilde{X} \cdot f)\Theta \end{aligned}$$

implies $\langle W,\, [W, \tilde{X}]\rangle = 0$, so that we can conlude $W \cdot r = 0$, namely, we can take $r \equiv 1$ on U and eventually on M. This completes the proof of 1).

Let us proceed to prove 2), which is done by a similar computation for $F = \tilde{Y}$. Form 1) we can assume that globally on M

$$W = \tilde{X} + \tilde{Z} + f\Theta$$

generates the null-geodesic flow.

Assertion 4.12 $\quad f = -(1 + \kappa)$.

Proof of Assertion. $\quad$ We have

$$\begin{aligned} 0 &= \langle \nabla_W W,\, \tilde{Y}\rangle = W\langle W,\, \tilde{Y}\rangle - \langle W,\, \nabla_W \tilde{Y}\rangle \\ &= -\langle W,\, [W, \tilde{Y}]\rangle - \langle W,\, \nabla_{\tilde{Y}} W\rangle = -\langle W,\, [W, \tilde{Y}]\rangle \end{aligned}$$

and

$$\begin{aligned} [W, \tilde{Y}] = [\tilde{X} + \tilde{Z} + f\Theta,\, \tilde{Y}] &= [\tilde{X},\, \tilde{Y}] + [\tilde{Z},\, \tilde{Y}] - f\tilde{X} - (\tilde{Y} \cdot f)\Theta \\ &= \kappa\tilde{Z} - (1 + f)\tilde{X} - (\tilde{Y} \cdot f)\Theta. \end{aligned}$$

Therefore we obtain

$$
\begin{aligned}
0 = -\langle W, [W, \tilde{Y}] \rangle &= -\langle \tilde{X} + \tilde{Z} + f\Theta, \kappa\tilde{Z} - (1+f)\tilde{X} - (\tilde{Y} \cdot f)\Theta \rangle \\
&= \kappa + 1 + f\,.
\end{aligned}
$$

Q.E.D. □Assertion 4.12.

Now the statement 2) follows from the following Proposition.
Q.E.D. □Prpopsition 4.10.

Proposition 4.13 Let $\gamma(t)$ be a curve on Σ with unit speed, $\Gamma(t)$ be a null-curve lift of γ to $V = S^1(T\Sigma)$, and $\tilde{\Gamma}(t)$ be the natural lift of Γ to $M = NC(TV)$, namely,

- $p \circ \Gamma = \gamma, \quad \pi \circ \tilde{\Gamma} = \Gamma$

where p and π denote the projections $V \to \Sigma$ and $M \to V$ respectively,

- $\|\dot{\gamma}(t)\| \equiv 1, \;\; \dot{\Gamma}(t) = \text{a horizontal lift of } \dot{\gamma}(t) + Z, \;\; \tilde{\Gamma}(t) = \langle \dot{\Gamma}(t) \rangle.$

We present the velocity of $\tilde{\Gamma}$ is presented as $\dot{\tilde{\Gamma}}(t) = \dot{\Gamma}(t) + \varphi(t)\Theta$, respecting the product structure $M = V \times S^1$. Then the geodesic curvature of $\gamma(t)$ on Σ is equal to $\varphi(t) + 1$. In the integral form, if $\tilde{\Gamma}(t) = (\Gamma(t), \Phi(t))$, then the geodesic curvature is $\dot{\Phi}(t) + 1$.

Remark 4.14 1) Proposition 4.10, in particular the statement 1) implies somehow the Magnetic Lorentzian extension remembers the Riemannian metric of Σ. If we look at the global symmetry of the magnetic Lorentzian extension, , unless the surface is very special type, *e.g.*, the global isometry group is the standard S^1-action in the fibre direction of the projection $p : S^1(T\Sigma) \to \Sigma$.

But even as a local geometry, we can find a reminiscence of the surface as in the next assertion, which we will partly use in the proof of the above proposition.
2) It is seen from the proof that the correction term $+1$ in the above proposition is the $+Z$ which makes the null-lift going up in Z-direction.

In the magnetic extension, if we take the sum of the metrics of the horizontal and vertical spaces $H_{(\sigma,v)}$ and $V_{(\sigma,v)}$, both with the positive sign, we obtain the standard Riemannian metric on $S^1(T\Sigma)$, *i.e.*, the magnetic Riemannian extension of Σ.

Assertion 4.15 1) Any horizontal lift of a geodesic on Σ is again a geodesic on $S^1(T\Sigma)$ with magnetic Riemannian extension.
2) Consequently the horizontal lift of any geodesic on Σ is also a geodesic for the magnetic Lorentzian extension.
3) The same applies to the fibre circles, *i.e.*, the Z-curves.

Proof of Assertion 4.15. 1) is easily understood because the horizontal lift is apparently locally minimizing the length. Then 2) is concluded from the fact that the infinitesimal deformation of the given horizontal lift and the derivation of the energy is computed by decomposing the deformation in the horizontal and the vertical directions. Q.E.D. □Assertion 4.15.

Proof of Proposition 4.13. First let us verify the proposition in the case of a geodesic γ on Σ of unit speed. Let $\zeta(\theta)$ denote the standard S^1-action in the fibre direction of $S^1(T\Sigma)$. Let $\Gamma_h(t)$ denote a horizontal lift of $\gamma(t)$. Its velocity $\dot{\Gamma}_h(t)$ is described as $X(\theta(t)_{\Gamma_h(t)})$. Then it is clear that $\theta_{\Gamma_h} = \theta(t)$ is constant in t. Any null-lift $\Gamma(t)$ of $\gamma(t)$ is then given as $\Gamma(t) = \zeta(t-c)(\Gamma_h(t))$ for some constant c. Then clearly the velocity satisfies $\dot{\Gamma}(t) = X(\theta_\Gamma - t + c)_{\Gamma(t)}$. Its natural lift to $M = NC(TV) = V \times S^1$ is $(\Gamma_n(t), \Phi(t) = \theta_\Gamma - t + c)$ and thus $\dot{\Phi}(t) \equiv -1$ for any geodesic $\gamma(t)$ on Σ.

Now we consider the general case. The geodesic curvature $\kappa_g(t)$ of $\gamma(t)$ is defined to be $\nabla^\Sigma_{\dot{\gamma}(t)}\dot{\gamma}(t)$ or numerically to be $dh(\nabla^\Sigma_{\dot{\gamma}(t)}\dot{\gamma}(t), \nu(t))$. Here, $\nu(t)$ denotes the unit normal vector field along $\gamma(t)$ such that $\dot{\gamma}(t)$ and $\dot{\nu}(t)$ form an oriented orthonormal basis of $T_\gamma(t)\Sigma$ with the Riemannian metric dh, and its Levi-Civita connection ∇^Σ.

Let $\Gamma_h(t)$ and $\Gamma_n(t)$ be horizontal and null- lifts of $\gamma(t)$ respectively. Present the velocity of $\Gamma_h(t)$ and of $\Gamma_n(t)$ as $\dot{\Gamma}_h(t) = X(\theta_h(t))_{\Gamma_h(t)}$. $\dot{\Gamma}_n(t) = X(\theta_n(t))_{\Gamma_n(t)}$. Then for some constant c, we have $\Gamma_n(t) = \zeta(t-c)\Gamma_h(t)$ and thus $\theta_n(t) = \theta_h(t) - t + c$. Therefore it is enough to show that

$$\dot{\theta}_h(t) = \kappa_g(t).$$

We prove the above equality at $t = t_0$. The following computations and arguments are done locally. Take the vector field $X(\theta_h(t_0))$ and consider an orbit $\Omega(t)$ with $\Omega(t_0) = \Gamma_h(t_0)$, so that $\dot{\Omega}(t_0) = \dot{\Gamma}_h(t_0)$. Take a local triviality of the S^1-bundle $S^1(T\Sigma) \to \Sigma$ so that we have a product neighborhood $U \times S^1$ of $\Gamma_h(t_0)$, $\gamma(t_0) \in U \subset \Sigma$, $\Gamma_h(t_0) = (\gamma(t_0), \theta_0)$. Also we can assume that all orbits of $X(\theta_h(t_0))$ in this product neighborhoos is horizontal with respect to this product structure. Then we also take a product connection $\nabla^P = \nabla^\Sigma \times (d\theta \otimes \frac{d}{d\theta})$ of $T(U \times S^1)$ with respect to this product structure. Then it is clear from the definition that

$$\langle \nabla^P_{\dot{\Gamma}_h(t)}\dot{\Gamma}_h(t), Y(\theta(t))\rangle = \kappa_g(t)$$

holds.

Now we calim that even if we replace ∇^P in this equation with the Levi-Civita conection ∇ of the magnetic Lorentzian extension, the following argument shows that the equation still holds. (Instead of Lorentzian extension, we can use the magnetic Riemannian extension and its Livi-Civita connection. Even then exactly the same argument holds.)

From the construction we have $\nabla^P_{\dot{\Omega}(t)}\dot{\Omega}(t) = \nabla_{\dot{\Omega}(t)}\dot{\Omega}(t) = 0$. Also it is clear that both ∇^P and ∇ are torsion free as affine connection. Therefore for vector fiedls A and B with $B_Q = 0$ at a point Q, $(\nabla^P_A B)_Q = (\nabla_A B)_Q = [A, B]_Q$. If necessary we extend the vector field $\dot{\Gamma}_h(t)$ along $\Gamma_h(t)$ as a genuine vector field around $\Gamma_h(t_0)$, this applies to the vector fields $A = X(\theta(t_0))$, $B = \dot{\Gamma}_h(t) - X(\theta(t_0))$, and the point $Q = \Gamma_h(t_0)$. Therefore we obtain

$$\nabla^P_{X(\theta(t_0))_{\Gamma_h(t_0)}} \dot{\Gamma}_h(t) = \nabla_{X(\theta(t_0))_{\Gamma_h(t_0)}} \dot{\Gamma}_h(t) = \nabla_{\dot{\Gamma}_h(t)} \dot{\Gamma}_h(t)\Big|_{t=t_0}.$$

From the presentation $\dot{\Gamma}_h(t) = X(\theta_h(t))_{\Gamma_h(t)}$ and the above computations, we see

$$\begin{aligned}
\dot{\theta}_h(t_0) &= \frac{d}{dt}\Big|_{t=t_0} \left\langle \dot{\Gamma}_h(t), Y(\theta_h(t_0))_{\Gamma_h(t)} \right\rangle \\
&= \left\langle \nabla_{\dot{\Gamma}_h(t_0)} \dot{\Gamma}_h(t), Y(\theta_h(t_0))_{\Gamma_h(t_0)} \right\rangle + \left\langle \dot{\Gamma}_h(t_0), \nabla_{\dot{\Gamma}_h(t_0)} Y(\theta_h(t_0))_{\Gamma_h(t)} \right\rangle \\
&= \kappa_g(t_0) + \left\langle X(\theta_h(t_0)), \nabla_{X(\theta_h(t_0))} Y(\theta_h(t_0)) \right\rangle = \kappa_g(t_0).
\end{aligned}$$

Q.E.D. □Propostion 4.13.

Proof of Theorem 4.9. We are ready to compute the linear holonomy of W inside $\mathcal{E}/\mathcal{W}$. We start the computation without assuming that κ is constant. With respect to the framing Θ and $\tilde{Y}$ of $\mathcal{E}/\mathcal{W}$, the infinitesimal action of W is computed as

$$[W, \Theta] = [\tilde{X} + \tilde{Z} - (\kappa+1)\Theta,\, \Theta] = -\tilde{Y}$$

and

$$\begin{aligned}
[W, \tilde{Y}] &= [\tilde{X} + \tilde{Z} - (\kappa+1)\Theta,\, \tilde{Y}] = \kappa(\tilde{Z}\tilde{X}) + (\tilde{Y}\cdot\kappa)\Theta \\
&\equiv \kappa(\kappa+1)\Theta + (\tilde{Y}\cdot\kappa)\Theta = \{\kappa(\kappa+1) + (\tilde{Y}\cdot\kappa)\}\Theta \pmod{\mathcal{W}}.
\end{aligned}$$

This implies, modulo $\mathcal{W}$

$$\frac{d}{dt}\Big|_{t=0} \exp(tW)_*(\Theta,\, \tilde{Y}) = (\Theta,\, \tilde{Y}) \begin{pmatrix} 0 & -\{\kappa(\kappa+1) + (\tilde{Y}\cdot\kappa)\} \\ 1 & 0 \end{pmatrix}.$$

Finally we assume hereafter the curvature κ to be constant. Then the above matrix reduces to $\begin{pmatrix} 0 & -\kappa(\kappa+1) \\ 1 & 0 \end{pmatrix}$, from which it is almost clear that the theorem holds. However we take a slightly closer look at what happens in each cases.

Parabolic case "$\kappa = 0, -1$" : If $\kappa \equiv 0$ or -1, we have $\exp(tW)_* = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Therefore in $\mathcal{E}/\mathcal{W}$ along $\mathcal{W}$, $\tilde{Y}$ is invariant, without expanding nor contracting, and $\Theta \equiv \mathcal{D}/\mathcal{W}$ inclines to $\tilde{Y}$. There is no other invariant line field than $\langle \tilde{Y} \rangle$. They are of genuine-parabolic type.

Elliptic case "$\kappa < -1$" or "$\kappa > 0$" : If $\kappa < -1$ or $\kappa > 0$, as $\kappa(\kappa+1) > 0$, we take $K\Theta$ and $\tilde{Y}$ as a global framing in place of Θ and $\tilde{Y}$, where $K = \sqrt{\kappa(\kappa+1)}$. Then the matrix for the holonomy becomes $\exp(tW)_* =$

$\begin{pmatrix} \cos Kt & -\sin Kt \\ \sin Kt & \cos Kt \end{pmatrix}$, from which we see clearly that the system is of elliptic type.

Hyperbolic case "$-1 < \kappa < 0$" : If $-1 < \kappa < 0$, as $\kappa(\kappa+1) < 0$, we take $K = \sqrt{-\kappa(\kappa+1)}$ and employ the same change of global framing. Then Then the matrix for the holonomy becomes $\exp(tW)_* = \begin{pmatrix} \cosh Kt & \sinh Kt \\ \sinh Kt & \cosh Kt \end{pmatrix}$. There fore we have two invariant subline bundles, $l^u = \langle K\Theta + \tilde{Y} \rangle$ which is expanding, and $l^s = \langle K\Theta - \tilde{Y} \rangle$ which is contracting.

$\mathcal{D}/\mathcal{W} = \langle \Theta \rangle$ is expanding and coming closer to l^u as $t \to \infty$ while expanding and coming closer to l^s as $t \to -\infty$ without passing through l^u nor l^s. The system is of genuine-hyperbolic type.

Q.E.D. □Theorem 4.9.

Remark 4.16 On the magnetic extension of surfaces of constant curvature κ, considering the symmetry, we see that the null-geodesics should project down to the surface to be curves with constant curvature. Let us look at the case of negative curvature. In order to compare the cases of different curvatures, we take homotethic transofromations (conformal transformation by multiplying constant) so that all surfaces are of constant curvature -1 and then look at projected curves. If the original surface is of constant curvarure κ, then multiplying $(-\kappa)^{-1/2}$ to the metric ($(-\kappa)^{-1}$ to the Riemannian metric tensor dh) provides a surface of constant curvature -1. Then curves of constant geodesic curvature $-\kappa$ is transformed into those of $(-\kappa)^{1/2}$.

$\kappa \equiv -1$ is the critical case. The curves of constant geodesic curvarture 1 is nothing but the horocycles. If $\kappa < -1$, then the correponding curves on the universal covering $\mathbb{H}$ after the conformal transformation are those of constant geodesic curvature $(-\kappa)^{1/2} > 1$, which are closed circles, while in the case of $-1 < \kappa < 0$ they are of geodesic curvature $0 < (-\kappa)^{1/2} < 1$ and are not compact. They have an intermediate character between geodesics and horocycles.

In the spherical case $\kappa > 0$, always those curves are small circles on S^2, *e.g.*, for $\kappa \equiv 1$ then a circle which passes through the north pole and touches the equator is one of those.

These observation explains Theorem 4.9 from a slightly different point of view.

4.3 Lorentz prolongation-III : Lorentzian surfaces*

The product Lorentzian extension of a hyperbolic surface (or a surface of negative curvature in general) gives rise to one of the bi-Engel structure corresponding to the negative one of the bi-contact structure associated with the geodesic Anosov flow. The seek for a natural construction of

Engel structure of the partner of the above mentioned one motivated the examples presented in this subsection.

Starting with a Lorentzian surface, we can perform the magnetic extension in order to obtain a Lorentzian 3-manifold and thus its Lorentz prolongation, while what we obtain so far still seems a bit mysterious. It is an interesting problem to analyze them. This is left to readers as a problem.

In this subsection we take the product of a Lorentzian surface (Σ, dh) with $(S^1, -d\theta^2)$, so that we obtain a Lorentzian 3-manifold (V, dg). We do not assume Σ to be compact, while for the sake of simplicity we assume the orientations not only of Σ but also of the positive and negative directions. It means the following. At each point $\sigma \in \Sigma$ the set $S^c = S^c(T_\sigma\Sigma) = \{u \in T_\sigma\Sigma | dg(u,u) = c\}$ is a hyperbola if $c \neq 0$ or a pair of straight lines crossing at the origin if $C = 0$. The orientation of the positive or negative directions implies the choice of a connected component $S_0^{\pm 1}$ of $S^{\pm 1}$. We also assume that the surface is oriented and the pair $\langle u_+, u_- \rangle$ ($u_\pm \in S_0^{\pm 1}$) forms an oriented basis.

In this situation the structural group $O(1;1)$ reduces to the connected component $SO(1;1)_0$ of the identity and the set of oriented Lorentzian orthonormal frame $\cup_{\sigma\in\Sigma}\{\langle u_+, u_- \rangle \,|\, u_\pm \in S_0^{\pm 1}\}$ is naturally identified with the associated principal $SO(1;1)_0$-bundle over Σ. If we take an oriented Lorentzian orthonormal basis $u_\pm$ at a point $\sigma \in \Sigma$ as above, then the other (negative) unit vectors in $S_0^{\pm 1}$ are indicated as $\cosh tu_+ + \sinh tu_-$ or as $\sinh tu_+ + \cosh tu_-$.

Like in the Riemannian case, using the Levi-Civita connection, we have the tautological horizontal vector fields X and Y corresponding to u_+ and u_- on the unit tangent bundle $S^1(T\Sigma)$. The vertical vector field which generates the action of $\mathbb{R} = \{t\}$ in the above sense is denoted by $Z = \frac{\partial}{\partial t}$. Then we have the structural equations

$$[Z, X] = Y\,, \quad [Z, Y] = X\,, \quad [X, Y] = \kappa Z$$

where κ denotes the Lorentzian curvature of (Σ, dh). The positive geodesic flow is generated by X, the negative one by Y.

The three dimensional Lorentzian metric dg has signature $(1,2)$, namely, one positive dimension and two negative dimensions. At each point $v \in V$ the set $NC(T_vV)$ of null-lines in the tangent space T_vV is again a circle. On the other hand, in this construction the circle contains two special points ℓ_+ and ℓ_- which are the fixed points of the involution $\iota : v = (\sigma, \theta) \mapsto (\sigma, -\theta)$. These two lines are those who already exist in $T_\sigma\Sigma$ as null-lines $\ell_+ = \langle u_+ + u_- \rangle$ and $\ell_- = \langle u_+ - u_- \rangle$. The complement of these two points consists of two open arcs which are exchanged by the involution. So we take the only one open arc in this construction. This open arc $NC^+ = NC^+(T_\sigma\Sigma)$ is also regarded as the real line $\mathbb{R} = \{t\}$ by the correspondence $t \leftrightarrow \langle \cosh tu_+ + \sinh tu_- + \Theta \rangle$ after fixing an oriented Lorentzian ON basis $u_\pm$. Here again Θ denotes $\dfrac{\partial}{\partial\theta}$. Therefore like in 4.1, our 4-manifold $M = NC^+(TV)$ is naturally identified with $S^1(T\Sigma) \times S^1$ by $((\sigma, \theta), u_+ + \Theta) \leftrightarrow ((\sigma, u_+), \theta)$. The natural lift of the null-geodesics are generated by $W = X + \Theta$ and the Engel structure is given by $\mathcal{D} = \langle W, Z \rangle$. and we obtain the even contact structure $\mathcal{E} = \langle W, Z, Y \rangle$. As Θ commutes with X, Y, and Z, this is also verified by the commutation relation.

Now we focus our attentions to a specific model.

Example 4.17 (de Sitter space) Let us take the de Sitter space $dS_2 = \{(s_1, s_2, s_3) \in \mathbb{R}^3 \mid s_1^2 + s_2^2 - s_3^2 = 1\}$ as Σ. which is also identified with $PSL(2;\mathbb{R})/\{\exp tl \mid t \in \mathbb{R}\}$. Here we are following the notations in Example 4.4. More precisely, $\mathbb{R}^3$ and its standard basis correspond to the Lie algebra $psl(2, \mathbb{R})$ and their basis h, l, k and the $(2,1)$-type metric on $\mathbb{R}^3$ to the adjoint invariant metric (*i.e.*, the Killing form) $\mathrm{tr}(ad(\cdot) \circ ad(\cdot)) = 2\mathrm{tr}(\cdot \times \cdot)$. Then as is well-known, the principal $\mathrm{SO}(1,1)_0$-bundle which coincides with the unit tangent bundle $S^1(TdS_2)$ over dS_2 is identified with the principal $\mathbb{R}$-bundle $PSL(2;\mathbb{R}) \to PSL(2;\mathbb{R})/\{\exp tl \mid t \in \mathbb{R}\} = dS_2$. On the total space $S^1(TdS_2) = PSL(2;\mathbb{R})$ the canonical vector fields X, Y, and Z are now nothing but the left invariant vector fields h, k, and l. From the commutation relation, we see that $\kappa \equiv -1$.

If we take a closed hyperbolic surface $\Gamma \setminus \mathbb{H}^2$ with $\pi_1(\Gamma \setminus \mathbb{H}^2) \cong \Gamma \subset PSL(2;\mathbb{R})$, Γ acts as orientation preserving isometry on $\Gamma \setminus \mathbb{H}^2$, the whole construction is invariant with respect to this action, namely, to the left translation of Γ to $PSL(2;\mathbb{R}) \times S^1$. The action on the second factor is trivial. Thus we obtained the 4-manifold $M' = S^1(T\Gamma \setminus \mathbb{H}^2) \times S^1$ and an Engel structure $\mathcal{D}' = \langle h + \Theta, l \rangle$, the Cauchy characteristic $\mathcal{W}' = \langle W = h + \Theta \rangle$, and the even contact structure $\mathcal{E}' = \langle h + \Theta, l, k \rangle$.

As mentioned in Remark 4.3, M', $\mathcal{W}'$, and $\mathcal{E}'$ are exactly the same as in Subsection 4.1 for a hyperbolic surface $\Gamma \setminus \mathbb{H}^2$ and the Engel structure obtained here is one of the bi-Engel structure ([KV]) which corresponds to the contact structure ξ_+ explained in Remark 4.3.

Problem 4.18 1) We do not know when we start with the magnetic extension of a Lorentzian surface and adopt the above construction with NC^+ whether if there exists some good discrete group action which yields a compact quotient.
2) If we could have a nice compactification in the above problem, it should be interesting to look at the dynamics and look for the parabolic ones.
3) Also, even in the above example, we have not yet understood the meaning of or the geometry corresponding to the compactification of the subspace NC^+ of null-lines by $\{\ell_+, \ell_-\}$.

Looking both on the product extension and the magnetic extension concerning these problems might be of some interest.

4.4 Pre-quantum prolongation*

Our aim in this subsection is to give examples of pre-quantum prolongations and look for more Engel structures of hyperbolic or possibly parabolic type. Basically we follow the notations in Subsection 1.4.

Example 4.19 (Nil-Solv hybrid) Consider a solvable 3-manifold V^3 which fibers over the circle S^1 with fibre T^2 and a hyperbolic monodromy $\varphi \in SL(2;\mathbb{R})$ ($\operatorname{tr}\varphi > 2$). The standard area form of area 1 is invariant under the monodromy φ, we can take a closed 2-form ω whose restriction to any fibre is the area form, which represents the cohomology class $\alpha \in H^2(V;\mathbb{Z}) \cong \mathbb{Z}$ corresponding to $1 \in \mathbb{Z}$. We take the the volume form $d\mathrm{vol} = d\theta \wedge \alpha/2\pi$. Then the suspension flow $\underline{W}$ is a lift of $2\pi\dfrac{d}{d\theta}$ and is the Poincaré dual to α with respect to $d$vol, *i.e.*, $\alpha = \iota_{\underline{W}} d\mathrm{vol}$. $\underline{W}$ is one of the standard suspension Anosov flows. Now we take the bi-contact structure $\xi_\pm$ associated with this Anosov flow ([KV], [Mi]). $\underline{W}$ is Legendrian for both of $\xi_\pm$. Performing the pre-quantum prolongation process, we obtain a bi-Engel structure $\mathcal{D}_\pm$ whose even contact structure is the horizontal space defined by the connection 1-form whose curvature form is exactly ω. The resulting 4-manifold W is the circle bundle over V, whose restriction to each fibre $T^2 \subset V$ is a nilpotent 3-manifold $Nil^3(1)$ of euler class 1.

The vector field W which spans the Cauchy characteristic $\mathcal{W}$ is a lift of $\underline{W}$ and preserves the length of the fibre circle. Therefore the dynamics is nothing almost the same as that of the Anosov flow W.

This is one way to construct a bi-Engel structure introduced in [KV]. We can also start from the 4-dimensional Lie group G which is the central extension of the 3-dimensional solvable Lie group $Solv^3$. Namely, G is obtained by

$$0 \to \mathbb{R}^2 \to Solv^3 \to \mathbb{R} \to 0\,, \qquad 0 \to \mathbb{R} \to G \to Solv^3 \to 0$$

where the first exact sequence is a semi-direct product by the action of $\varphi^t \in \mathbb{R}$ on $\mathbb{R}^2$ and the second one is the unique (up to constant) non-trivial central extension. Then we can find a co-compact lattice in G.

This Lie group is also presented as a non-central extension of the 3-dimensional Heisenberg Lie group $H_\mathbb{R}$ by an automorphism $\tilde{\varphi}$ which preserves the integral lattice $H_\mathbb{Z}$ where

$$H_\mathbb{R} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \supset H_\mathbb{Z} = \{ \ * \ \mid x, y, z \in \mathbb{Z}\}$$

and $\tilde{\varphi}$ acts as $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \varphi \begin{pmatrix} x \\ y \end{pmatrix}$.

Example 4.20 (Geodesic flow of Riemannian surface) Let (Σ, dg) be a Riemannian surface with no-where zero curvature κ. Consider the unit tangent bundle $p : V = S^1(T\Sigma) \to \Sigma$, the geodesic flow on V generated by the horizontal canonical vector field X, and the Levi-Civita connection whose connection 1-form is denoted by β. $\xi = \ker\beta$ is the horizontal plane field to which X is tangent. If the curvature κ is everywhere positive [*resp.* negative] ξ is a negative [*resp.* positive] contact structure, because $d\beta = -\kappa p^* d\mathrm{area}_\Sigma$. Because X is the Reeb vector field of the canonical contact 1-form (*i.e.*, of the associated Liouville 1-form) λ, X is an exact vector field in the sense of asymptotic cycle. We adopt X as W for the pre-quantum prolongation. The invariant volume is $\lambda \wedge d\lambda$ and the closed 2-form to which the pre-quantization is performed is $\omega = d\lambda$, so that the corresponding cohomology class α can be any of the torsion part $\mathrm{Tor}(H^2(V;\mathbb{Z}))$.

The resulting manifold M is the circle bundle over V whose euler class is α. The action of the Cauchy characteristic W on $\mathcal{E}/\mathcal{W}$ is almost same as the dynamics of the geodesic flow on V. Therefore if the curvature κ is everywhere positive, the Engel structure is of elliptic type, and if κ is everywhere negative, it is of hyperbolic type. Especially if we take the cohomology class $\alpha = 0 = [d\lambda]$, on the resulting 4-manifold $M = V \times S^1$, with respect to this product structure we can take the connection form to be $d\theta + \pi^*\lambda$. Therefore the Cauchy characteristic $\mathcal{W}$ is generated by the vector field $W = X - \dfrac{\partial}{\partial\theta}$. Therefore the Engel structure we obtained coincides with one in Proposition 4.1 which is obtained by the suspension of the time 2π map of the geodesic flow.

For a closed oriented surface $\Sigma = \Sigma_g$ of genus $g \neq 1$, we have non-unique candidates for the resulting manifold W on which the Engel structure because $\mathrm{Tor}(H^2(V;\mathbb{Z})) \cong \mathbb{Z}/|2g-2|$.

For example, if we assume that $\kappa > 0$, Σ is topologically S^2, V is diffeomorphic to $\mathbb{R}P^3$, and $H^2(V;\mathbb{Z}) \cong \mathbb{Z}/2$. If we take $\alpha = 0$, $M \cong V \times S^1$ and $\pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}/2$. On the other hand, if we take $\alpha \neq 0 \in \mathbb{Z}/2$, the fiber bundle structure $S^1 \to M \to \mathbb{R}P^3$ and the euler class imply that $\pi_1(M) \cong \mathbb{Z}$ (the π_1's of the fibration is an extension : $0 \to \mathbb{Z} \overset{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}/2 \to 0$). In fact M is diffeomorphic to $U(2)$ and even to $S^3 \times S^1$.

Unfortunately, even the manifold is different, the Engel structure is not so new. The natural projection $M \to S^1$ to the circle of the half length $S^1/\mathbb{Z}/2$ tells that M is the mapping torus of the antipodal map τ of S^3, which is isotopic to the identity of S^3. Therefore M is diffeomorphic to the product. The antipodal map τ is the deck transformation of the universal covering $S^3 \to \mathbb{R}P^3$ which pulls the geodesic flow ϕ_t back to S^3 as the flow $\tilde{\phi}_t$ and the Liouville contact structure $\xi_0 = \ker\lambda$ to $\tilde{\xi}_0$ as well. Therefore τ and $\tilde{\phi}_t$ commute to each other. Our Engel structure is also obtained by the suspension construction given in Subsection 1.5 with respect to $V = S^3$, $\tilde{\xi}_0$, and $\varphi = \tau \circ \tilde{\phi}_{2\pi}$.

Like in the case of $\kappa > 0$, for a surface of $g > 1$ and $\kappa < 0$, consider finite converings in the fibre direction of $V \to \Sigma_g$, we see that more or less similar situations appear.

Example 4.21 (Propellor constructions-I) This class of examples are a generalization of the Nil-Solv hybrid example. For the suspension construction, we need a contact structure on 3-manifold which gives rise to the even contact structure, while in the pre-quantum prolongation, the non-vanishing closed 2-form replaces the roll of the non-integrability which assures the bracket generation $[\mathcal{E},\mathcal{E}] = TM$.

Let V be the mapping cylinder of a linear automorphism $\varphi \in SL(2;\mathbb{R})$ of T^2. It fibers over the circle and is also presented as the quotient of $\tilde{V} = T^2 \times S^1 \ni (x,y,t)$ $(x,y,\in \mathbb{R}/\mathbb{Z})$ by the identification $(x,y,t+1) \sim (x',y',t)$ with $\begin{pmatrix} x' \\ y' \end{pmatrix} = \varphi \begin{pmatrix} x \\ y \end{pmatrix}$. $\frac{\partial}{\partial t}$ on $\tilde{V}$ descends to $\underline{W}$ on V.

The propellor construction of contact structure is to choose a linear foliation $\tilde{l}_t = \left\langle a(t)\dfrac{\partial}{\partial x} + b(t)\dfrac{\partial}{\partial y} \right\rangle$ on each $T^2 \times \{t\}$ in such a way that l_t rotates in positive or negative direction when $t \in \mathbb{R}$ increases and satisfies $l_{t+1} = \varphi_* l_t$. Then $\tilde{\xi} = l \oplus \left\langle \dfrac{\partial}{\partial t} \right\rangle$ is a contact structure on $\tilde{V}$ which descends to ξ on V.

The fiberwise area form $\omega = dx \wedge dy$, the standard volume form $d\mathrm{vol} = dx \wedge dy \wedge dt$ are also well-defined on V and satisfies $\omega = \iota_{\underline{W}} d\mathrm{vol}$ which is a closed 2-form. The cohomology class $\alpha = [\omega]$ is the Poincaré dual to the suspension circle and is integral. Therefore the pre-quantum prolongation can be performed to obtain an Engel structure on M which fibers over V with euler class α.

According to φ being elliptic (including the case φ is the identity), parabolic, or hyperbolic, the Engel structure is of elliptic, (genuine or trans-) parabolic, or (genuine or trans-) hyperbolic type.

Example 4.22 (Propellor constructions-II) In the above propellor construction, we choose a non-singular vector field $\underline{\widetilde{W}} = a(t)\dfrac{\partial}{\partial x} + b(t)\dfrac{\partial}{\partial y}$ in such a way that not only $\tilde{l}$ is invariant under the monodromy but also the vector field itself is invariant, namely it satisfies $\varphi \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} a(t+1) \\ b(t+1) \end{pmatrix}$, so that $\underline{\widetilde{W}}$ descends to $\underline{W}$ on V. $\underline{W}$ is Legendrian and preserves $d\mathrm{vol}$.

In hyperbolic case or in elliptic case except for the identity, because through the projection $H_1(V;\mathbb{Z}) \cong H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$, $\omega = \iota_d \mathrm{vol}$ is always exact. In parabolic case, if the eigenvalue of φ is equal to -1, then again we have $H_1(V;\mathbb{Z}) \cong H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ and $\alpha = 0$, while if it is equal to 1, we have rank 1 choice for α, which depends on the choice of $\underline{W}$. In the case where φ is the identity, we have choice of rank 2. Multiplying appropriate non-vanishing smooth function on S^1 to $\underline{W}$ achieves the choice.

It is not hard to see that in any case the resulting Engel structure is of parabolic type, while it is not surprizing because on the cyclic covering of M which covers the cyclic covering $\tilde{V} \to V$, all of them look alike.

5 Problems and discussions

To close this note we collect problems concerning the topics discussed in this note. Some of them have already been mentioned before and some other may be accompanied with short discussions. Also some are concrete and others are rather vague.

We have not discussed the symmetry of Engel manifolds. One reason is that for an Engel structure on a closed 4-manifold M, in general the symmetry is expected to be very small. First of all, if a diffeomorphism $\Phi : M \to M$ preserves the Engel structure $\mathcal{D}$, necessarily it also preserves $\mathcal{W}$. In generic cases it implies Φ preserves each orbit of $\mathcal{W}$. Then inside each $\mathcal{W}$-orbit the position of $\mathcal{D}/\mathcal{W}$ in $\mathcal{E}/\mathcal{W}$ almost determines the point of the orbit.

The Cartan prolongation is an exceptional case in this sense. If we start from a contact 3-manifold (V, ξ) and obtain its Cartan prolongation $(M, \mathcal{D})$, the symmetry of $(M, \mathcal{D})$ is exactly that of (V, ξ), *i.e.*, naturally we have $Diff^{\infty}(M, \mathcal{D}) \cong Diff^{\infty}(V, \xi)$.

If a closed Lorentzain 3-manifold (V, dg) admits an isometry, it automatically lifts to a symmetry of its Lorentz prolongation.

There are many other closed Engel manifolds for which we have continuous symmetries.

Problem 5.1 Classify all the closed Engel manifolds with continuous symmetry.

In particular, study the case where in the support of continuous symmetry the $\mathcal{W}$-orbits are not closed.

On the other hand, the 2-jet space $J^2(1,1)$ and the standard Engel structure $\mathcal{D}_0$ with the standard Engel-Darboux coordinates admits a big symmetry group, which is naturally isomorphic to the group of contactomorphisms of the standard contact structure $(\mathbb{R}^3, \xi_0)$. Here we follow the notations in [EF] in Subsection 1.1. In side this group we can find a 4-dimensional 3-step nilpotent Lie group G_E which acts transitively and freely on $\mathbb{R}^4 = J^2(1,1)$.

The vector fields W, X, Z, and Y on $\mathbb{R}^4$ generate a 4-dimensional nilpotent Lie algebra $\mathfrak{g}_E$ and thus the corresponding nilpotent Lie group G_E. In fact they satisfy the commutation relations $[W, X] = Z$, $[Z, X] = Y$, and others are trivial. These relations exactly corresponde to the flag generation $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset T\mathbb{R}^4$. However, apparently among these vector fields W and Z are out of symmetries of $\mathcal{D}_0$. If we identify $\mathbb{R}^4 = J^2(1,1)$ with the Lie group G_E and $\mathfrak{g}_E$ is the set of left invariant vector fields on G_E, the full flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ is understood as left invariant fields on G_E. In this formulation the action, *e.g.*, $\exp(tW)$ generated by W is a right translation on G_E and thus necessarily preserve left invariant fields. Relying on the left action which preserves the Engel structure the element $\Phi_{dcba} \in G_E$ is indicated as $\Phi_{dcba} : (x, y, z, w) \mapsto (x + d,\ y + \frac{1}{2}ax^2 + bx + c,\ z + ax + b,\ w + a)$ where $\Phi_{dcba}(0,0,0,0) = (d, c, b, a)$ for $(d, c, b, a) \in \mathbb{R}^4$.

The association of the Lie algebra $\mathcal{G}_E$ or the Lie group G_E to an Engel structure seems to depends on the choice of local Engel-Darboux chart. However, even at any point and its neighbourhood, the Lie algebra structure seems to survive as the generation of the flag. Therefore $\mathfrak{g}_E$ msut be very fundamental.

Problem 5.2 An Engel structure may have continuous symmetry like in the case of Lorentz prolongation from a Lorentzian 3-manifold with isometry group of positive dimension. Formulate relations between such global symmetry to $\mathfrak{g}_E$ or if possible to G_E.

Problem 5.3 In which sense dess the h-principle for Engel structures hold? Describe the path-components of the set of Engel structures.

Problem 5.4 (See 2) in Remark 1.7, [Ch], and [SY].) Formulate the Wünschmann invariant for a line field $\mathcal{L} \subset \mathcal{D}$ to be the fibre of the Lorentz prolongation in a local sense in a manner which is suitable for the following purpose. For a given Engel structure study the existence or an obstruction to the existence of a line field $\mathcal{L} \subset \mathcal{D}$ with vanishing Wünschmann invariant, or the deformability to such one in a 1-parameter family of Engel structures of a given one.

Problem 5.5 Which kind of informations on Lorentzian manifold can we deduce from their Lorentz prolongation? How about the same question for the above generalized case, Engel structures equipped with line fields $\mathcal{L} \subset \mathcal{D}$ with vanishing Wünschmann invariant?

Problem 5.6 Inaba's accessible set should be reconsidered from the coordinate-free point of view.

It might be possible to distribte the germ of the accessible set along a $\mathcal{W}$-curve of each point as a kind of field on the Engel manifold. Formulate this notion and relate it to the global and dynamical structure of Engel manifolds.

Consider the Minkowski space $\mathbb{R}^{2,1} = \{(r, s, t)\}$ with $dg = dr^2 + ds^2 - dt^2$ and its Lorentz prolongation. We can arrange an Engel-Darboux coordinate in such a way that $r = \frac{1}{\sqrt{2}}(w - y)$, $s = z$, $t = \frac{1}{\sqrt{2}}(w + y)$, and the positive ineteriot A_+ of Inaba's accessible set from the origin coincides with the interior of the causal set of the origin in $\mathbb{R}^{2,1}$. In this case, the fibre direction of the Lorentz prolongation is exactly $\langle X \rangle$ in [EF]. The function $z^2 - 2wy = r^2 + s^2 - t^2$ is a 1st integral of the vector field X.

Problem 5.7 Study the relationship between this fact and the problems 5.3 - 5.5 above.

Problem 5.8 (cf. Subsection 3.4.) Give a proof of Theorem 1.3 relying on the causality and the Bryant-Hsu rigidity.

Problem 5.9 Develop the study of (the group of) contactomorphisms of 3-dimensional contact structures in order not only for the construction but also for classification problem of Engel structures.

Problem 5.10 Study the Engel structure of Lorentz prolongation from one more family of Lorentzian extension of Riemannian surface; $(\Sigma, dh) \times (S^1, -f(\theta)(d\theta)^2)$ for a positive non-constant function f. S^1 can be repalced with any 1-manifold. From the point of view of Engel structures, it is equivalent to study $(\Sigma, f(\theta)^{-1}dh) \times (S^1, -(d\theta)^2)$.

Problem 5.11 In the construction of compact Lorentzian 3-manifold from product extension of Lorentzian surfaces in Subsection 4.3, we did not take the whole of the null circle bundle but a portion of it corresponding to an open interval in null circle. After taking a quotient by a discrete group action, the boundary disappeared. Which kind of informations do the boundary or the compactification carry?

Problem 5.12 Study the magnetic extensions of Lorentzian surfaces and associated Engel structures.

Problem 5.13 By definition, paraboic/hyperbolic Engel structures require only two dimensional foliations to which the Cauchy characteristic $\mathcal{W}$ is tangent, while in any example in this note it is raised to 3-dimensional ones! Does there exist examples which can not be raised, or is it a natural consequence?

How about the Engel structures coming from Eliashberg-Thurston's example based on laminations [ET]?

Problem 5.14 (Discussions on Example 4.20 and Proposition 4.1.) What is the limit of Example 4.20 with $\alpha = 0 \in H^2(V;\mathbb{Z})$ when κ tends to 0? In the family of isomorphic examples in Proposition 4.1, $\kappa = 0$ does not matter and we have an Engel structure.

For a Riemannian surface, in Subsection 4.1 first we performed a (special) relativistic procedure (taking the product with $(S^1, -d\theta^2)$) and then we took a phase space for lightlike motions and obtained an Engel structure.

On the other hand, in Example 4.20 with $\alpha = 0 \in H^2(V;\mathbb{Z})$, we first took the phase space for motions of fixed kinetic energy on the surface and then we perform the pre-quantization, then again we obtained the same Engel structure.

Does there any Physical significance of the coincidence of the results of these procedures? (Taking phase spaces conjugates the special relativistic procedure and quantum one.) If we look into these examples, the coincidence is natural and nothing is mysterious...

If there is some meaning, then does there exists further implication of the non-uniqueness of the case where $H^2(V;\mathbb{Z})$ has torsions?

We have also the third construction for the same Engel structure, namely the suspension by the geodesic flow (at certain time) (the method of Subsection 1.5) of the Liouville contact structure. This construction sounds more purely mathematical, but in this case we need to put a plane field $\mathcal{D}$ somehow by hand.

Problem 5.15 Provide the study of Engel structures with singularities with interesting examples and a framework or guiding principle.

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