CHUO MATH NO.128(2019)

ON SYNTOMIC COMPLEX WITH MODULUS FOR SEMI-STABLE REDUCTION CASE

by
KENTO YAMAMOTO

DEPARTMENT OF MATHEMATICS
CHUO UNIVERSITY
BUNKYOKU TOKYO JAPAN

Sept. 24 , 2019

ON SYNTOMIC COMPLEX WITH MODULUS FOR SEMI-STABLE REDUCTION CASE

KENTO YAMAMOTO

Abstract. In this paper, we define syntomic complex for modulus pair (X, D), where X is regular semi-stable family and D is an effective Cartier divisor on X. We compute its cohomology sheaves.

Contents

1. Introduction

In their paper [KSY], Bruno Kahn, Shuji Saito and Takao Yamazaki construct and study a triangulated category of motives with modulus MDM_{gm}^{eff} over a field k that extends Voevodsky's category DM_{gm}^{eff} with non $\mathbb{A}^1$-homotopy invariant property. While the Voevodsky's category DM_{gm}^{eff} is constructed from smooth k-varieties, the category of motives with modulus MDM_{gm}^{eff} is constructed from proper modulus pairs (X, D), that is, pairs of a proper k-variety X and an effective divisor D on X such that $X - |D|$ is smooth.

Date: September 24, 2019.
2010 *Mathematics Subject Classification.* 14F30.
Key words and phrases. syntomic complex.

Let K be a p-adic field, and let $\mathscr{O}_K$ be its valuation ring with k the residue field. Let X be a regular semistable family over $\mathscr{O}_K$ and put $Y := X \otimes_{\mathscr{O}_K} k$. Let D be an effective Cartier divisor which is flat over $\mathscr{O}_K$ and such that $D_{red} \cup Y$ has normal crossings on X. The first aim of this paper is to define the syntomic complex $\mathscr{S}_n(r)_{X|D}$ with modulus for such pairs (X, D) for $n \geq 1$ and $0 \leq r \leq p-1$, which is a generalization of Tsuji's syntomic complex $\mathcal{S}_n(r)_{(X,M_X)}$ (cf. [Ka1], [Ka2], [Ku], [Tsu1], [Tsu2], [Tsu3] etc.). More explicitly, we have $\mathscr{S}_n(r)_{X|\emptyset} = \mathcal{S}_n(r)_{(X,M_X)}$. In [Tsu1], [Tsu2] and [Tsu3], Tsuji constructed the symbol map

$$\mathrm{Symb}_X : (M_{X_n}^{gp})^{\otimes r} \longrightarrow \mathcal{H}^r(\mathcal{S}_n(r)_{(X,M_X)})$$

and proved the surjectivity of this map. The second aim of this paper is to construct a symbol map for $\mathscr{S}_n(r)_{X|D}$ and to investigate the surjectivity of the symbol map for $\mathscr{S}_n(r)_{X|D}$. We will prove the following main result:

Theorem 1.1. (Theorem 3.4) *Let $n \geq 1$ be an integer. If $0 \leq r \leq p-2$, the cokernel of the symbol map*

$$\mathrm{Symb}_{X|D} : (1 + I_{D_{n+1}})^{\times} \otimes (M_{X_{n+1}}^{gp})^{\otimes r-1} \longrightarrow \mathcal{H}^r(\mathscr{S}_n(r)_{X|D})$$

is Mittag-Leffler zero with respect to the multiplicities of the prime components of D. Here $X_n := X \otimes \mathbb{Z}/p^n\mathbb{Z}$, $D_n := D \otimes \mathbb{Z}/p^n\mathbb{Z}$ and $I_{D_{n+1}} (\subset \mathscr{O}_{X_{n+1}})$ is the definition ideal of D_{n+1}; M_X denotes the log structure on X associated with $D_{red} \cup Y$, and M_{X_n} is the inverse image of M_X onto X_n.

In fact, the cokernel of $\mathrm{Symb}_{X|D}$ is non-zero unless D is zero or reduced, and deeply depends on the multiplicities of the prime components of D. Nevertheless our main result asserts that those cokernels are Mittag-Leffler zero as a projective system. A key fact to understand this phenomenon is a Cartier inverse isomorphism in a modulus situation (see Lemma 3.2 below). From this key lemma, we will obtain an explicit description of the cokernel of the symbol map in a sufficiently local situation.

As an application of the subject of this paper, we will consider a p-adic étale Tate twists for a modulus pair (X, D) in a forthcoming paper [Y], which is a generalization of Sato's p-adic étale Tate twists ([Sat]). We will show that our new object is a "dual" of the usual p-adic étale Tate twists of $X - D$.

Notation and conventions.

(i) Throughout this paper, p denotes a prime number and K denotes a henselian discrete valuation field of characteristic 0 whose residue field k is a perfect field of characteristic p. We write $\mathscr{O}_K$ for the integer ring of K, and π denotes a prime element of $\mathscr{O}_K$.

(ii) For a scheme X, we put $X_n := X \otimes \mathbb{Z}/p^n\mathbb{Z}$.

(iii) Let X be a pure-dimensional scheme which is flat of finite type over $\mathrm{Spec}(\mathscr{O}_K)$. We call X be a *regular semistable family* over $\mathrm{Spec}(\mathscr{O}_K)$, if it is regular and evrywhere étale locally isomorphic to

$$\mathrm{Spec}\big(\mathscr{O}_K[T_0, \ldots, T_d]/(T_0 \cdots T_a - \pi)\big),$$

for some a such that $0 \leq a \leq d := \dim(X/\mathscr{O}_K)$.

2. SYNTOMIC COMPLEX WITH MODULUS

In this section, we will define syntomic complex with modulus $\mathscr{S}_n(r)_{X|D}$ for $0 \leq q \leq p-1$.

Setting:

- Let X be a *regular semistable family* over $\mathrm{Spec}(\mathscr{O}_K)$. We denote $Y := X \otimes_{\mathscr{O}_K} k$ and $X_K := X \otimes_{\mathscr{O}_K} K$. Let $D \subset X$ be an effective Cartier divisor on X which is flat over $\mathrm{Spec}(\mathscr{O}_K)$ and $Y \cup D_{red}$ has normal crossings on X.
- Let M_X be a logarithmic structure on X associated with $D_{red} \cup Y$. Let M_D be a logarithmic structure on D defined as the restriction of M_X. For $n \geq 1$, we write M_{X_n} for the inverse image of log structure of M_X onto X_n. Let (Y, M_Y) be the reduction of mod π of (X, M_X).

To define the syntomic complex with modulus in a sufficiently local situation, we assume the existence of the following data:

Assumption 2.1.

- *There exist an exact closed immersions* $\beta_n : (X_n, M_{X_n}) \hookrightarrow (Z_n, M_{Z_n})$ *and* $\beta_{n,D} : (D_n, M_{D_n}) \hookrightarrow (\mathscr{D}_n, M_{\mathscr{D}_n})$ *of log schemes for* $n \geq 1$ *such that* (Z_n, M_{Z_n}) *and* $(\mathscr{D}_n, M_{\mathscr{D}_n})$ *are smooth over* $W := W(k)$*, and such that the following diagram is commutative:*

$$\begin{array}{ccc} X_n & \xrightarrow{\beta_n} & Z_n \\ \downarrow & & \downarrow \\ D_n & \xrightarrow{\beta_{D,n}} & \mathscr{D}_n \end{array}$$

- *There exist a compatible system of lifting frobenius endomorphisms* $\{F_{Z_n} : (Z_n, M_{Z_n}) \to (Z_n, M_{Z_n})\}$ *and* $\{F_{\mathscr{D}_n} : (\mathscr{D}_n, M_{\mathscr{D}_n}) \to (\mathscr{D}_n, M_{\mathscr{D}_n})\}$ *for each* $n \in \mathbb{N}$.

Let $(\mathscr{E}_n, M_{\mathscr{E}_n})$ be the PD-envelope of (X_n, M_{X_n}) in (Z_n, M_{Z_n}) which is compatible with the canonical PD-structure on the ideal $(p)\subset \mathbb{Z}/p^n\mathbb{Z}$. Let $(\mathscr{E}_{n,D}, M_{\mathscr{E}_{n,D}})$ be the PD-envelope of (D_n, M_{D_n}) in $(\mathscr{D}_n, M_{\mathscr{D}_n})$. By the assumption the flatness of D, we have $\mathscr{E}_{n,D} \cong \mathscr{E}_n \otimes_{Z_n} \mathscr{D}_n$. The morphism F_{Z_n} induces a lifting of Frobenius $F_{\mathscr{E}_n}$ of $(\mathscr{E}_n, M_{\mathscr{E}_n})$. For $i \geq 1$, let $J^{[i]}_{\mathscr{E}_n} \subset \mathscr{O}_{\mathscr{E}_n}$ be the i-th devided power of the ideal $J_{\mathscr{E}_n} := \mathrm{Ker}\Big(\mathscr{O}_{\mathscr{E}_n} \to \mathscr{O}_{X_n}\Big)$. For $i \leq 0$, we put $J^{[i]}_{\mathscr{E}_n} := \mathscr{O}_{\mathscr{E}_n}$. We put

$$\omega^q_{Z_n} := \Omega^q_{Z_n/W_n}\big(\log M_{Z_n}\big), \quad \omega^q_{Z_n|\mathscr{D}_n} := \omega^q_{Z_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n) \quad (q \geq 0)$$

which are locally free $\mathscr{O}_{Z_n}$-modules.

Let us recall that the syntomic complex defined as follows:

$$\mathcal{S}_n(q)_{(X_n,M_{X_n}),(Z_n,M_{Z_n})} := \mathrm{Cone}\big(1 - p^{-q}\varphi : \omega^{\cdot}_{Z_n} \otimes_{\mathscr{O}_{Z_n}} J^{[r-\cdot]}_{\mathscr{E}_n} \to \omega^{\cdot}_{Z_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{\mathscr{E}}\big)[-1]$$

for $0 \leq q \leq p-1$ (cf. [Tsu1], [Tsu2], [Tsu3]).

Definition 2.2. (*syntomic complex with modulus, sufficiently local case*)
We assume $q \leq p-1$. We define

$$\mathscr{S}_n(q)^{loc}_{X|D,(Z_n,M_{Z_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})} := \mathrm{Cone}\big(\mathcal{S}_n(q)_{(X_n,M_{X_n}),(Z_n,M_{Z_n})} \longrightarrow \mathcal{S}_n(q)_{(D_n,M_{D_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})}\big)[-1]$$

*under the **Assumption** 2.1.*

Lemma 2.3. *The syntomic complex with modulus $\mathscr{S}_n(q)^{loc}_{X|D,(Z_n,M_{Z_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})}$ is independent of the choice of (Z_n, M_{Z_n}) and $(\mathscr{D}_n, M_{\mathscr{D}_n})$.*

Proof. If we choose another $(Z'_n, M_{Z'_n})$ and $(\mathscr{D}'_n, M_{\mathscr{D}'_n})$, we consider the following commutative diagrams:

$$\begin{array}{ccc} (X_n, M_{X_n}) & \xhookrightarrow{\beta'_n} & (Z'_n, M_{Z'_n}) \\ \downarrow{\scriptstyle id} & & \downarrow \\ (X_n, M_{X_n}) & \xhookrightarrow{\beta_n} & (Z_n, M_{Z_n}), \end{array} \qquad \begin{array}{ccc} (D_n, M_{D_n}) & \xhookrightarrow{\beta'_{D,n}} & (\mathscr{D}'_n, M_{\mathscr{D}'_n}) \\ \downarrow{\scriptstyle id} & & \downarrow \\ (D_n, M_{D_n}) & \xhookrightarrow{\beta_{D,n}} & (\mathscr{D}_n, M_{\mathscr{D}_n}), \end{array}$$

where β'_n, β_n, $\beta'_{D,n}$ and $\beta_{D,n}$ are exact closed immersions. Let $(\mathscr{E}_{X,n}, M_{\mathscr{E}_{X,n}})$, $(\mathscr{E}'_{X,n}, M_{\mathscr{E}'_{X,n}})$ (resp. $(\mathscr{E}_{D,n}, M_{\mathscr{E}_{D,n}})$, $(\mathscr{E}'_{D,n}, M_{\mathscr{E}'_{D,n}})$) denote the PD-envelopes of β_n and β'_n (resp. $\beta_{D,n}$ and $\beta'_{D,n}$). From [Tsu3], Cororally 1.11, we have quasi-isomorphisms

$$\mathcal{S}_n(q)_{(X_n,M_{X_n}),(Z_n,M_{Z_n})} \overset{\mathrm{qis}}{\cong} \mathcal{S}_n(q)_{(X_n,M_{X_n}),(Z'_n,M_{Z'_n})},$$

$$\mathcal{S}_n(q)_{(D_n,M_{D_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})} \overset{\mathrm{qis}}{\cong} \mathcal{S}_n(q)_{(D_n,M_{D_n}),(\mathscr{D}'_n,M_{\mathscr{D}'_n})}.$$

Thus we have a quasi-isomorphism

$$\mathscr{S}_n(q)^{loc}_{X|D,(Z_n,M_{Z_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})} \overset{\text{qis}}{\cong} \mathscr{S}_n(q)^{loc}_{X|D,(Z'_n,M_{Z'_n}),(\mathscr{D}'_n,M_{\mathscr{D}'_n})}.$$

This completes the proof. □

In the general case, we define $\mathscr{S}_n(q)_{X|D} \in D^+\big(X_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z}\big)$ by gluing the local complexes: We choose a hyper-covering $X^\bullet$ of X(resp. $D^\bullet$ of D) in the étale topology and a closed immersions $\beta_n^\bullet : (X_n^\bullet, M_{X_n^\bullet}) \longrightarrow (Z_n^\bullet, M_{Z_n^\bullet})$ (resp. $\beta_{n,D^\bullet} : (D_n^\bullet, M_{D_n^\bullet}) \longrightarrow (\mathscr{D}_n^\bullet, M_{\mathscr{D}_n^\bullet})$), with the property that, for each integer $\mu \geq 0$, ${\beta_n}^\mu$(resp. ${\beta_{n,D}}^\mu$) is an immersion of log schemes and (Z^μ, M_{Z^μ}) (resp $(\mathscr{D}^\mu, M_{\mathscr{D}^\mu})$) is a smooth log scheme over W, in such a way that there exists a compatible system of liftings of frobenius $\{F_{Z^\bullet{}_n} : (Z_n^\bullet, M_{Z_n^\bullet}) \to (Z_n^\bullet, M_{Z_n^\bullet})\}$ (resp. $\{F_{\mathscr{D}^\bullet{}_n} : (\mathscr{D}_n^\bullet, M_{\mathscr{D}_n^\bullet}) \to (\mathscr{D}_n^\bullet, M_{\mathscr{D}_n^\bullet})\}$).

Definition 2.4. (*syntomic complex with modulus, the general case; cf.* [Tsu3]*, p. 540*) *We define the syntomic complex with modulus $\mathscr{S}_n(q)_{X|D}$ to be the object*

$$R\theta_*\big(\mathscr{S}_n(q)^{loc}_{X^\bullet|D^\bullet,(Z_n^\bullet,M_{Z_n^\bullet}),(\mathscr{D}_n^\bullet,M_{\mathscr{D}_n^\bullet})}\big)$$

of $D^+(X^\bullet_{1,\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$, where θ denotes the canonical morphism of topoi $(X_1^\bullet)\tilde{}_{\acute{e}t} \longrightarrow X_{1,\acute{e}t}$.

If we choose another $X'^\bullet$, $\beta_n^\bullet : (X_n'^\bullet, M_{X_n'^\bullet}) \hookrightarrow (Z_n'^\bullet, M_{Z_n'^\bullet})$, $\beta^\bullet_{n,D} : (D_n'^\bullet, M_{D_n'^\bullet}) \hookrightarrow (\mathscr{D}'^\bullet{}_n, M_{\mathscr{D}'^\bullet{}_n})$, $\{F_{Z_n}\}_n$ and $\{F_{\mathscr{D}_n}\}_n$, then by taking the fiber products

$$X''^\bullet := X^\bullet \times_X X'^\bullet,\ (Z_n''^\bullet, M_{Z_n''^\bullet}) := (Z_n^\bullet, M_{Z_n^\bullet}) \times_{\mathbb{Z}/p^n\mathbb{Z}} (Z_n'^\bullet, M_{Z_n'^\bullet})$$

$$D''^\bullet := D^\bullet \times_D D'^\bullet,\ (\mathscr{D}^{''\bullet}{}_n, M_{\mathscr{D}''^\bullet{}_n}) := (\mathscr{D}^\bullet{}_n, M_{\mathscr{D}^\bullet{}_n}) \times_{\mathbb{Z}/p^n\mathbb{Z}} (\mathscr{D}^{'\bullet}{}_n, M_{\mathscr{D}'^\bullet{}_n}),$$

$$F_{Z_n''^\bullet} := F_{Z_n^\bullet} \times F_{Z_n'^\bullet},\ F_{\mathscr{D}''^\bullet{}_n} := F_{\mathscr{D}^\bullet{}_n} \times F_{\mathscr{D}'^\bullet{}_n}$$

and using [Tsu3], Cororally1.11, we obtain canonical quasi-isomorphisms

$$\mathrm{pr}^{-1}\mathcal{S}_n(r)_{(X_n^\bullet,M_{X_n^\bullet}),(Z_n^\bullet,M_{Z_n^\bullet})} \longrightarrow \mathcal{S}_n(r)_{(X''^\bullet_n,M_{X''^\bullet_n}),(Z''^\bullet_n,M_{Z''^\bullet_n})},$$

$$\mathrm{pr}'^{-1}\mathcal{S}_n(r)_{(X'^\bullet_n,M_{X'^\bullet_n}),(Z'^\bullet_n,M_{Z'^\bullet_n})} \longrightarrow \mathcal{S}_n(r)_{(X''^\bullet_n,M_{X''^\bullet_n}),(Z''^\bullet_n,M_{Z''^\bullet_n})},$$

$$\mathrm{pr}^{-1}\mathcal{S}_n(r)_{(D_n^\bullet,M_{D_n^\bullet}),(\mathscr{D}_n^\bullet,M_{\mathscr{D}_n^\bullet})} \longrightarrow \mathcal{S}_n(r)_{(D''^\bullet_n,M_{D''^\bullet_n}),(\mathscr{D}''^\bullet_n,M_{\mathscr{D}'^\bullet_n})},$$

$$\mathrm{pr}'^{-1}\mathcal{S}_n(r)_{(D'^\bullet_n,M_{D'^\bullet_n}),(\mathscr{D}'^\bullet_n,M_{\mathscr{D}'^\bullet_n})} \longrightarrow \mathcal{S}_n(r)_{(D''^\bullet_n,M_{D''^\bullet_n}),(\mathscr{D}''^\bullet_n,M_{\mathscr{D}''^\bullet_n})}$$

and a canonical quasi-isomorphisms

$$\begin{aligned} R\theta_*\Big(\mathcal{S}_n(r)_{(X_n^\bullet,M_{X_n^\bullet}),(Z_n^\bullet,M_{Z_n^\bullet})}\Big) &\overset{\cong}{\to} R\theta''_*\Big(\mathcal{S}_n(r)_{(X''^\bullet_n,M_{X''^\bullet_n}),(Z''^\bullet_n,M_{Z''^\bullet_n})}\Big) \\ &\overset{\cong}{\leftarrow} R\theta'_*\Big(\mathcal{S}_n(r)_{(X'^\bullet_n,M_{X'^\bullet_n}),(Z'^\bullet_n,M_{Z'^\bullet_n})}\Big), \end{aligned}$$

$$R\theta_*\Big(\mathcal{S}_n(r)_{(D_n^\bullet, M_{D_n^\bullet}),(\mathscr{D}_n^\bullet, M_{\mathscr{D}_n^\bullet})}\Big) \overset{\cong}{\to} R\theta''_*\Big(\mathcal{S}_n(r)_{(D''^\bullet_n, M_{D''^\bullet_n}),(\mathscr{D}''^\bullet_n, M_{\mathscr{D}''^\bullet_n})}\Big) \overset{\cong}{\leftarrow} R\theta'_*\Big(\mathcal{S}_n(r)_{(D'^\bullet_n, M_{D'^\bullet_n}),(\mathscr{D}'^\bullet_n, M_{\mathscr{D}'^\bullet_n})}\Big),$$

where pr, pr′, θ', θ'' denote the canonical morphism of topoi

$$(X''^\bullet_1)\tilde{}_{\acute{e}t} \longrightarrow (X^\bullet_1)\tilde{}_{\acute{e}t},\ (X''^\bullet_1)\tilde{}_{\acute{e}t} \longrightarrow (X'^\bullet_1)\tilde{}_{\acute{e}t},$$

$$(X'^\bullet_1)\tilde{}_{\acute{e}t} \longrightarrow (X_1{}')\tilde{}_{\acute{e}t},\ (X''^\bullet_1)\tilde{}_{\acute{e}t} \longrightarrow (X_1{}'')\tilde{}_{\acute{e}t}.$$

Hence we obtain

$$R\theta_*\Big(\mathscr{S}_n(q)^{loc}_{X^\bullet|D^\bullet,(Z_n^\bullet, M_{Z_n^\bullet}),(\mathscr{D}_n^\bullet, M_{\mathscr{D}_n^\bullet})}\Big) \overset{\cong}{\to} R\theta''_*\Big(\mathscr{S}_n(q)^{loc}_{X''^\bullet|D''^\bullet,(Z''^\bullet_n, M_{Z''^\bullet_n}),(\mathscr{D}''^\bullet_n, M_{\mathscr{D}''^\bullet_n})}\Big) \overset{\cong}{\leftarrow} R\theta'_*\Big(\mathscr{S}_n(q)^{loc}_{X'^\bullet|D'^\bullet,(Z'^\bullet_n, M_{Z'^\bullet_n}),(\mathscr{D}'^\bullet_n, M_{\mathscr{D}'^\bullet_n})}\Big).$$

This quasi-isomorphism satisfies the transitivity, and then $\mathscr{S}_n(q)_{X|D}$ is independent of the choice of hyper coverings $X^\bullet$ and $D^\bullet$ up to a canonical isomorphism.

Lemma 2.5. *Let $n \geq 1$ be an integer. We have a distinguished triangle*

$$\mathscr{S}_n(q)_{X|D} \longrightarrow \mathscr{S}_{n+1}(q)_{X|D} \longrightarrow \mathscr{S}_1(q)_{X|D} \longrightarrow \mathscr{S}_n(q)_{X|D}[1].$$

Proof. The assertion follows form the following two distinct distinguished triangles

$$\mathcal{S}_n(q)_{(X,M_X)} \longrightarrow \mathcal{S}_{n+1}(q)_{(X,M_X)} \longrightarrow \mathcal{S}_1(q)_{(X,M_X)} \longrightarrow \mathcal{S}_n(q)_{(X,M_X)}[1]$$

and

$$\mathcal{S}_n(q)_{(D,M_D)} \longrightarrow \mathcal{S}_{n+1}(q)_{(D,M_D)} \longrightarrow \mathcal{S}_1(q)_{(D,M_D)} \longrightarrow \mathcal{S}_n(q)_{(D,M_D)}[1].$$

The details are straight-forward and left to the reader. □

In what follows, we assume the following assumption:

Assumption 2.6.

- *There exist an exact closed immersions $\beta_n : (X_n, M_{X_n}) \hookrightarrow (Z_n, M_{Z_n})$ and $\beta_{n,D} : (D_n, M_{D_n}) \hookrightarrow (\mathscr{D}_n, M_{\mathscr{D}_n})$ of log schemes for $n \geq 1$ such that (Z_n, M_{Z_n}) and $(\mathscr{D}_n, M_{\mathscr{D}_n})$ are smooth over $W := W(k)$, and such that the following diagram is cartesian :*

$$\begin{array}{ccc} X_n & \xrightarrow{\beta_n} & Z_n \\ \uparrow & \square & \uparrow \\ D_n & \xrightarrow{\beta_{D,n}} & \mathscr{D}_n \end{array}$$

- *There exist a compatible system of lifting frobenius endomorphisms* $\{F_{Z_n} : (Z_n, M_{Z_n}) \to (Z_n, M_{Z_n})\}$ *and* $\{F_{\mathscr{D}_n} : (\mathscr{D}_n, M_{\mathscr{D}_n}) \to (Z_n, M_{Z_n})\}$ *for each* $n \in \mathbb{N}$.
- *an effective Cartier divisor* $\mathscr{D}_n \subset Z_n$ *such that* $\beta_n^* \mathscr{D}_n = D_n$ *and* F_{Z_n} *which induces a morphism* $\mathscr{D}_n \longrightarrow \mathscr{D}_n$.

We denote $\varphi : J^{[r]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n) \longrightarrow J^{[r]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n)$; $a \otimes b \mapsto \varphi(a) \otimes \varphi(b)$, where the homomorphism φ induced by $F_{\mathscr{E}_n}$. We will define the Frobenius morphism "devided by p^r" $p^{-r}\varphi$ (or φ_r) : $J^{[r-\cdot]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n)) \to \mathscr{O}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n)$ in the following:

We have

$$\varphi(J^{[r]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n)) \subset p^r(\mathscr{O}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n)),$$

(cf. [Ka1], I, Lemma 1.3 (1)). On the other hand, $J^{[r]}_{\mathscr{E}_n}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and

$$\big(J^{[r]}_{\mathscr{E}_{n+1}} \otimes \mathscr{O}_{Z_{n+1}}(-\mathscr{D}_{n+1})\big) \otimes \mathbb{Z}/p^n\mathbb{Z} \cong J^{[r]}_{\mathscr{E}_n} \otimes \mathscr{O}_{Z_n}(-\mathscr{D}_n)$$

for every $n \geq 1$ and $r \geq 0$. Hence, for $0 \leq r \leq p-1$, there exists a unique homomorphism

$$\varphi_r : J^{[r]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n) \to \mathscr{O}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n)$$

which makes the following diagram commute:

$$\begin{array}{ccc} J^{[r]}_{\mathscr{E}_{n+r}} \otimes_{\mathscr{O}_{Z_{n+r}}} \mathscr{O}_{Z_{n+r}}(-\mathscr{D}_{n+r}) & \xrightarrow{\varphi} & \mathscr{O}_{\mathscr{E}_{n+r}} \otimes_{\mathscr{O}_{Z_{n+r}}} \mathscr{O}_{Z_{n+r}}(-\mathscr{D}_{n+r}) \\ \downarrow & & \downarrow{\scriptstyle p^r} \\ J^{[r]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n) & \xrightarrow{\varphi_r} & \mathscr{O}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n). \end{array}$$

From the fact that

$$d\varphi(\omega^1_{Z_n/W_n}) \subset p \cdot \omega^1_{Z_n/W_n} \quad (n \in \mathbb{N},\ n > 0),$$

we can define a frobenius "divided by p"

$$\frac{d\varphi}{p} : \omega^1_{Z_n/W_n} \longrightarrow \omega^1_{Z_n/W_n}.$$

Definition 2.7. (*another syntomic complex with modulus, sufficiently local case*)
We assume $r \leq p-1$. *We define*

$$s_n(r)_{X|D} := \mathrm{Cone}\big(1 - \varphi_r : J^{[r-\cdot]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \dot{\omega}_{Z_n|\mathscr{D}_n} \to \mathscr{O}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \dot{\omega}_{Z_n|\mathscr{D}_n}\big)[-1],$$

where $\varphi_r = \varphi_{r-q} \otimes \wedge^q \frac{d\varphi}{p}$ *in degree* q.

Lemma 2.8. *Under* ***Assumption*** *2.6,* $s_n(q)_{X|D}$ *and* $\mathscr{S}_n(q)^{loc}_{X|D,(Z_n,M_{Z_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})}$ *are quasi-isomorphic.*

Proof. By the definition of $\mathscr{S}_n(q)^{loc}_{X|D}$, we have a quasi-isomorphism

$$\mathrm{Ker}\big(\mathcal{S}_n(q)_{(X_n,M_{X_n}),(Z_n,M_{Z_n})} \longrightarrow \mathcal{S}_n(q)_{(D_n,M_{D_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})}\big) \overset{\mathrm{qis}}{\cong} \mathscr{S}_n(q)^{loc}_{X|D,(Z_n,M_{Z_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})}.$$

We will show that the isomorphism of complexes

$$\mathrm{Ker}\big(\mathcal{S}_n(q)_{(X_n,M_{X_n}),(Z_n,M_{Z_n})} \longrightarrow \mathcal{S}_n(q)_{(D_n,M_{D_n}),(\mathscr{D}_n,M_{\mathscr{D}_n})}\big) \cong s_n(q)_{X|D}.$$

It sufficies to show that

$$J^{[q-\cdot]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \omega^{\cdot}_{Z_n|\mathscr{D}_n} \cong \mathrm{Ker}\Big(J^{[q-\cdot]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \omega^{\cdot}_{Z_n} \longrightarrow J^{[q-\cdot]}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \omega^{\cdot}_{Z_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{\mathscr{D}_n}\Big).$$

The surjectivity of this morphism is trivial from short exact sequence

$$0 \to \mathscr{O}_{Z_n}(-\mathscr{D}_n) \to \mathscr{O}_{Z_n} \to \mathscr{O}_{\mathscr{D}_n} \to 0.$$

We will prove that the injectivity of this morphism. It suffices to show that $\mathscr{O}_{\mathscr{E}_n} \xrightarrow{\times f} \mathscr{O}_{\mathscr{E}_n}$ is injective, where f is the definition equation of $\mathscr{D}_n$. The problem is reduced to the case

$$X_n = \mathrm{Spec}(\mathscr{O}_K/p^n[t_1,\dots,t_d]/(t_i\cdots t_d - \pi)),$$

$$Z_n = \mathrm{Spec}(W_n[T_0,T_1,\dots,T_d,T_\infty]),$$

$$D_n = \{t_i^{m_i}\cdots t_{i+j}^{m_{i+j}} = 0\},$$

$$\mathscr{D}_n = \{T_i^{m_i}\cdots T_{i+j}^{m_{i+j}} = T_\infty\},$$

$$\psi : W_n[T_0,T_1,\dots,T_d,T_\infty] \to \mathscr{O}_K/p^n[t_1,\dots,t_d]/(t_i\cdots t_d - \pi);$$

$$T_0 \mapsto \pi,\ T_i \mapsto t_i\ (1 \le i \le d),\ T_\infty \mapsto 0.$$

In this case, $f = T_i^{m_i}\cdots T_{i+j}^{m_{i+j}} - T_\infty$, and the kernel of the ring homomorphism ψ is

$$J := (T_0 - \pi,\ T_\infty,\ T_1\cdots T_d - T_0).$$

We put $g_1 := T_0 - \pi$, $g_2 := T_\infty$ and $g_3 := T_1\cdots T_d - T_0$. The affine ring $\mathcal{A}_n$ of $\mathscr{E}_n$ is generated by $\Delta := \{g_1^{[m_1]}\cdot g_2^{[m_2]}\cdot g_3^{[m_3]} \mid \sum_{i=1}^3 m_i = m, m \in \mathbb{N}\}$ as a $W_n[T_0,T_1,\dots,T_d,T_\infty]$-module. Then any element of $\mathcal{A}_n$ can be written as $\Sigma_{i,\, n\ge 1,\, x\in\Delta}\, a_i x$, where $a_i \in W_n[T_0,T_1,\dots,T_d,T_\infty]$. The generators g_1,g_2,g_3 of J is a linearly independent on $W_n[T_0,T_1,\dots,T_d,T_\infty]$. Thus Δ is a basis for $\mathcal{A}_n$ as a $W_n[T_0,T_1,\dots,T_d,T_\infty]$-module ([Ber], p.31, 1.4.2 and Corollarie 2.3.2 (ii)). Since the polynomial f is non-zero divisor on $W_n[T_0,T_1,\dots,T_d,T_\infty]$, f is non-zero divisor on $\mathcal{A}_n$. This completes the proof. □

In what follows, we will use the complex $s_n(q)_{X|D}$ when we compute the cohomology sheaf of the syntomic complex with modulus in sufficiently local situation. By definition, $s_n(q)_{X|D}$ is concentrated in $[0,q]$. Note that $s_n(q)_{X|\emptyset} = \mathcal{S}_n(q)_{(X,M_X)}$, the syntomic complex defined in [Tsu2], [Tsu3].

Lemma 2.9. *For $q, q' \geq 0$, there is a morphism in $D^-(Y_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$:*

$$s_n(q)_{X|D} \otimes^{\mathbb{L}} \mathcal{S}_n(q')_{(X_n, M_{X_n}),(Z_n, M_{Z_n})} \longrightarrow s_n(q+q')_{X|D}. \tag{2.1}$$

Proof. Proof is straightforward from the product structure of $\mathcal{S}_n(q)_{(X,M_X)}$ (cf. [Tsu2], [Tsu3]). □

Let us define a symbol map

$$(1 + I_{D_{n+1}})^{\times} \otimes (M^{gp}_{X_{n+1}})^{\otimes q-1} \longrightarrow \mathcal{H}^q(\mathscr{S}_n(q)_{X|D}) \tag{2.2}$$

for $q \geq 0$. Here $I_{D_{n+1}} \subset \mathscr{O}_{X_{n+1}}$ is the definition ideal of D_{n+1} and

$$(1 + I_{D_{n+1}})^{\times} := (1 + I_{D_{n+1}}) \cap \mathscr{O}^{\times}_{X_{n+1}}.$$

We construct a symbol map in the local situation in the following. By taking $R\theta_*$, we immediately obtain its global case.

Recall that (X_n, M_{X_n}) denotes the reduction of $\mod p^n$ of (X, M_X). Let C_{n+1} be the complex

$$\big(1 + J_{\mathscr{E}_{n+1}}\big) \cap \big(1 + \mathscr{O}_{Z_{n+1}}(-\mathscr{D}_{n+1})\big)^{\times} \longrightarrow \big(1 + \mathscr{O}_{Z_{n+1}}(-\mathscr{D}_{n+1})\big)^{\times} \tag{2.3}$$

$$\deg.\, 0 \qquad\qquad\qquad\qquad \deg.\, 1$$

We define the morphism of complexes $C_{n+1} \longrightarrow s_n(1)_{X|D}$ by

$$(1 + J_{\mathscr{E}_{n+1}}) \cap (1 + \mathscr{O}_{Z_{n+1}}(-\mathscr{D}_{n+1}))^{\times} \longrightarrow (s_n(1)_{X|D})^0 = J_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n); \tag{2.4}$$

$$a \mapsto \log(a) \mod p^n$$

at degree 0 and

$$\big(1 + \mathscr{O}_{Z_{n+1}}(-\mathscr{D}_{n+1})\big)^{\times} \longrightarrow (s_n(1)_{X|D})^1 = \big(\mathscr{O}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \omega^1_{Z_n|\mathscr{D}_n}\big) \oplus \big(\mathscr{O}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{\mathscr{Z}_n}(-\mathscr{D}_n)\big); \tag{2.5}$$

$$b \mapsto \Big(d\log b \mod p^n, p^{-1}\log(b^p \varphi_{\mathscr{E}_{n+1}}(b)^{-1})\Big),$$

at degree 1, where $\varphi_{\mathscr{E}_n} : \mathscr{O}_{\mathscr{E}_n} \to \mathscr{O}_{\mathscr{E}_n}$ denotes the Frobenius operator induced by $\{F_{Z_n}\}$ and we have used the fact that $\log(b^p \phi_{\mathscr{E}_{n+1}}(b)^{-1})$ is contained in

$$p(\mathscr{O}_{\mathscr{E}_{n+1}} \otimes_{\mathscr{O}_{Z_{n+1}}} \mathscr{O}_{Z_{n+1}}(-\mathscr{D}_{n+1})) \overset{\cong}{\leftarrow} \mathscr{O}_{\mathscr{E}_n} \otimes_{\mathscr{O}_{Z_n}} \mathscr{O}_{Z_n}(-\mathscr{D}_n),$$

since $b^p \varphi_{\mathscr{E}_{n+1}}(b)^{-1} \in 1 + p(\mathscr{O}_{\mathscr{E}_{n+1}} \otimes_{\mathscr{O}_{Z_{n+1}}} \mathscr{O}_{Z_{n+1}}(-\mathscr{D}_{n+1}))$. Taking $\mathcal{H}^1$, we obtain

$$\mathrm{Symb}_{X|D} : \big(1 + I_{D_{n+1}}\big)^{\times} = \mathcal{H}^1(C_{n+1}) \longrightarrow \mathcal{H}^1(s_n(1)_{X|D}) \cong \mathcal{H}^1(\mathscr{S}_n(1)^{loc}_{X|D}). \tag{2.6}$$

We obtain the symbol map (2.2) as following composite maps:

$$(1 + I_{D_{n+1}})^{\times} \otimes (M^{gp}_{X_{n+1}})^{\otimes q-1} \xrightarrow{\mathrm{Symb}_{X|D} \otimes \mathrm{Symb}_X} \mathcal{H}^1(s_n(1)_{X|D}) \otimes \mathcal{H}^{q-1}(\mathcal{S}_n(q-1)_{(X,M_X)}) \tag{2.7}$$

$$\longrightarrow \mathcal{H}^q(s_n(q)_{X|D}) \cong \mathcal{H}^q(\mathscr{S}_n(q)^{loc}_{X|D}).$$

Here $\mathrm{Symb}_X : (M_{X_{n+1}}^{gp})^{\otimes q-1} \to \mathcal{H}^{q-1}(\mathcal{S}_n(q-1)_{(X,M_X)})$ is symbol map defined by [Tsu2] §2. The second morphism is product structure $s_n(1)_{X|D} \otimes^{\mathbb{L}} \mathcal{S}_n(q-1)_{(X,M_X)} \to s_n(q)_{X|D}$.

3. Main Rsults

In this and the next section, for $0 \leq q \leq p-2$, we calculate the cohomology sheaf

$$\mathcal{H}^q(s_1(q)_{X|D}) \quad (0 \leq q \leq p-2). \tag{3.1}$$

We first define tow filtrations on the sheaf $\mathcal{H}^q(s_1(q)_{X|D})$using symbols and state our main results on the associated graded pieces.

Definition 3.1. *We define the filtrations $U^{\cdot}$ and $V^{\cdot}$ on $(1+I_{D_2})^{\times} \otimes (M_{X_2}^{gp})^{\otimes(q-1)}$ $(q \geq 1)$ by*

$$U^0((1+I_{D_2})^{\times}) := (1+I_{D_2})^{\times}, \quad V^0((1+I_{D_2})^{\times}) := (1+\pi I_{D_2})^{\times} \cdot < \pi >, \tag{3.2}$$

$$U^m((1+I_{D_2})^{\times}) := (1+\pi^m I_{D_2})^{\times}, \quad V^m((1+I_{D_2})^{\times}) := U^{m+1}((1+I_{D_2})^{\times}) \ (m \geq 1), \tag{3.3}$$

if $q = 1$, and

$$U^m\Big((1+I_{D_2})^{\times} \otimes (M_{X_2}^{gp})^{\otimes(q-1)}\Big) := (\text{the image of } U^m((1+I_{D_2})^{\times})) \otimes (M_{X_2}^{gp})^{\otimes(q-1)}, \tag{3.4}$$

$$\begin{aligned} V^m\Big((1+I_{D_2})^{\times} \otimes (M_{X_2}^{gp})^{\otimes(q-1)}\Big) &:= (\text{the image of } U^m((1+I_{D_2})^{\times})) \otimes (M_{X_2}^{gp})^{\otimes(q-2)} \otimes < \pi > \\ &+ U^{m+1}\Big((1+I_{D_2})^{\times} \otimes (M_{X_2}^{gp})^{\otimes(q-1)}\Big) \end{aligned} \tag{3.5}$$

if $q \geq 2$. Here $(1+\pi^m I_{D_2})^{\times} := (1+\pi^m I_{D_2}) \cap \mathscr{O}_{X_2}^{\times}$ for $m \geq 0$.

We define the filtration $U^{\cdot}$ and $V^{\cdot}$ on $\mathcal{H}^q(s_1(q)_{X|D})(q \geq 0)$ to be the images of these filtrations under the symbol map 2.2. Put

$$\mathrm{gr}_0^m \mathcal{H}^q(s_1(q)_{X|D}) := U^m\mathcal{H}^q(s_1(q)_{X|D})/V^m\mathcal{H}^q(s_1(q)_{X|D}), \tag{3.6}$$

$$\mathrm{gr}_1^m \mathcal{H}^q(s_1(q)_{X|D}) := V^m\mathcal{H}^q(s_1(q)_{X|D})/U^{m+1}\mathcal{H}^q(s_1(q)_{X|D}). \tag{3.7}$$

To describe these graded pieces, we introduce some differential sheaves on Y. We define

$$\omega_{Y|D}^q := \omega_Y^q \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(-D_s), \tag{3.8}$$

where $s := \mathrm{Spec}(k)$, $\omega_Y^q := \Omega_{Y/s}^q\big(\log(M_Y/N_s)\big)$, $D_s := D \otimes_{\mathscr{O}_K} k$ and (s, N_s) denotes the log point over s. We define the subsheaves $Z_{Y|D}^q$ and $B_{Y|D}^q$ of $\omega_{Y|D}^q$ by

$$Z_{Y|D}^q := \mathrm{Ker}(d^q : \omega_{Y|D}^q \to \omega_{Y|D}^{q+1}), \tag{3.9}$$

$$B_{Y|D}^q := \mathrm{Im}(d^{q-1} : \omega_{Y|D}^{q-1} \to \omega_{Y|D}^q). \tag{3.10}$$

Let $\omega^q_{Y|D,\log}$ be the subsheaf of abelian groups of ω^q_Y generated by local sections of the form

$$d\log(x) \wedge d\log(a_1) \wedge \cdots \wedge d\log(a_{q-1}),$$

where $x \in \Big(1 + \mathscr{O}_Y\big(-D_s\big)\Big)^{\times}$ and $a_1, \ldots, a_{q-1} \in M_Y$.

If $D = \sum_{\lambda\in\Lambda} m_\lambda D_\lambda$, we denote $D' := \sum_{\lambda\in\Lambda} m'_\lambda D_\lambda$. Here $m'_\lambda := \min\{l \in \mathbb{N} \mid p \cdot l \geq m_\lambda\}$. We put $D_s := \sum_{\lambda\in\Lambda} m_\lambda (D_s)_\lambda$. We define a map $d : \omega^q_Y \otimes \mathscr{O}_Y(-D_s) \to \omega^{q+1}_Y \otimes \mathscr{O}_Y(-D_s)$ by the local assignment

$$\omega \otimes \prod_{\lambda\in\Lambda} \pi_\lambda^{m_\lambda} \mapsto \big(d\omega + \sum_{\lambda\in\Lambda} m_\lambda \cdot d\log(\pi_\lambda) \wedge \omega\big) \otimes \prod_{\lambda\in\Lambda} \pi_\lambda^{m_\lambda} \quad (\omega \in \omega^q_Y),$$

where $\pi_\lambda \in \mathscr{O}_Y$ denotes a local uniformizer of $(D_s)_\lambda$, for each $\lambda \in \Lambda$. Using this d, we regard $(\omega^{\cdot}_Y \otimes \mathscr{O}_Y(-D_s), d)$ as a complex.

We have the following Lemma:

Lemma 3.2. (cf. [SS], Theorem 3.2) *For each integer $q \geq 0$, there exists an isomorphism*

$$C^{-1} : \omega^q_{Y|D'_s} \xrightarrow{\cong} \mathcal{H}^q(\omega^{\cdot}_{Y|D_s}) \tag{3.11}$$

$$a\ \mathrm{dlog}(b_1) \wedge \mathrm{dlog}(b_2) \wedge ... \wedge \mathrm{dlog}(b_q) \mapsto \text{the class of } a^p\ \mathrm{dlog}(b_1) \wedge \mathrm{dlog}(b_2) \wedge ... \wedge \mathrm{dlog}(b_q), \tag{3.12}$$

where $a \in \mathscr{O}_Y(-D'_s)$ and $b_1, ..., b_q \in M_Y$.

Proof. We use a similar argument as in [SS], Theorem 3.2. If p divides m_λ for any $\lambda \in \Lambda$, then the map $d : \omega^q_Y \otimes \mathscr{O}_Y(-D_s) \to \omega^{q+1}_Y \otimes \mathscr{O}_Y(-D_s)$ sends

$$\omega \otimes \prod_{\lambda\in\Lambda} \pi_\lambda^{m_\lambda} \mapsto d\omega \otimes \prod_{\lambda\in\Lambda} \pi_\lambda^{m_\lambda}.$$

Thus we have the isomorphism

$$C^{-1} : \omega^q_Y \otimes \mathscr{O}_Y(-D'_s) \xrightarrow{\cong} \mathcal{H}^q\big(\omega^{\cdot}_Y \otimes \mathscr{O}_Y(-D_s)\big)$$

from [SS], Theorem 2.3 (2).

We next show the general case. We see that the natural inclusion

$$\omega^{\cdot}_Y \otimes \mathscr{O}_Y(-p \cdot D'_s) \hookrightarrow \omega^{\cdot}_Y \otimes \mathscr{O}_Y(-D_s)$$

is a quasi-isomorphism. We define $\omega^{\cdot}_{\mathbf{m}} := \omega^{\cdot}_Y \otimes \mathscr{O}_Y(-D_s)$, where $\mathbf{m} = (m_\lambda)_{\lambda\in\Lambda}$. We can consider a filtration

$$\omega^{\cdot}_{p\cdot\mathbf{m}'} = \omega^{\cdot}_{\mathbf{m}_t} \subset \cdots \subset \omega^{\cdot}_{\mathbf{m}_1} \subset \omega^{\cdot}_{\mathbf{m}_0} = \omega^{\cdot}_{\mathbf{m}}$$

such that

$$\sum_{\lambda\in\Lambda} m_{\lambda,i+1} - \sum_{\lambda\in\Lambda} m_{\lambda,i} = 1 \quad for \quad 0 \leq i < t,$$

where $\mathbf{m}_i := (m_{\lambda,i})_\lambda$ and $\mathbf{m}'_i := (m'_{\lambda,i})_\lambda$.

The graded pieces of the above filtration are of the form $\frac{\omega^q_{\mathbf{m}_i}}{\omega^q_{\mathbf{m}_{i+1}}}$ such that $(m_{\lambda,i}, p) = 1$, for all $\lambda \in \Lambda$. The sheaf $\frac{\omega^q_{\mathbf{m}_i}}{\omega^q_{\mathbf{m}_{i+1}}}$ are acyclic from a similar argument as in [SS], Lemma 3.4.

Using above facts, we obtain the isomorphism

$$C^{-1} : \omega^q_Y \otimes \mathscr{O}_Y(-D'_s) \overset{\cong}{\to} \mathcal{H}^q\big(\omega^{\cdot}_Y \otimes \mathscr{O}_Y(-p \cdot D'_s)\big) \overset{\cong}{\to} \mathcal{H}^q\big(\omega^{\cdot}_Y \otimes \mathscr{O}_Y(-D_s)\big).$$

This completes the proof. □

For each integer $q \geq 0$, we have the following morphism which restricts a morphism (3.11) to $\omega^q_{Y|D}$:

$$C^{-1} : \omega^q_{Y|D} \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{Y|D}) \tag{3.13}$$

$$a\, \mathrm{dlog}(b_1) \wedge \mathrm{dlog}(b_2) \wedge ... \wedge \mathrm{dlog}(b_q) \mapsto \text{the class of } a^p\, \mathrm{dlog}(b_1) \wedge \mathrm{dlog}(b_2) \wedge ... \wedge \mathrm{dlog}(b_q), \tag{3.14}$$

where $a \in \mathscr{O}_Y(-D_s)$, and $b_1, ..., b_q \in M_Y$.

Lemma 3.3. (cf. [JSZ], Theorem 1.2.1, Proposition 1.2.3) *We assume that the notation and the assumption be as above. Then, for each integer $q \geq 0$, we have the following exact sequence.*

$$0 \longrightarrow \omega^q_{Y|D,\log} \longrightarrow Z^q_{Y|D} \overset{1-C^{-1}}{\longrightarrow} \mathcal{H}^q(\omega^{\cdot}_{Y|D}) \longrightarrow 0. \tag{3.15}$$

Proof. This claim is trivial except the exactness in the middle term (The surjectivity of $1 - C^{-1}$ is reduced to that of $1 - C^{-1} : \omega^q_{Y|D} \to \omega^q_{Y|D}/B^q_{Y|D}$). One can check the exactness at the middle term in the same way as that of [JSZ] Theorem 1.2.1. □

We have the following main results.

Theorem 3.4. *Let $n \geq 1$ be an integer. If $0 \leq r \leq p-2$, the cokernel of the symbol map*

$$\mathrm{Symb}_{X|D} : (1 + I_{D_{n+1}})^\times \otimes (M^{gp}_{X_{n+1}})^{\otimes r-1} \longrightarrow \mathcal{H}^r(\mathscr{S}_n(r)_{X|D})$$

is Mittag-Leffler zero with respect to the multiplicities of the prime components of D.

Theorem 3.5. *We assume that $p \geq 3$. Let e be the absolute ramification index of K. Then the sheaf $\mathcal{H}^q\big(s_1(q)_{X|D}\big)$ has the folllowing structure:*

(1) For $m = 0$, we have short exact sequences:

$$0 \longrightarrow \frac{\mathcal{R}}{\mathcal{R} \cap \mathrm{gr}^0_1 \mathcal{H}^q\big(s_1(q)_{X|D}\big)} \longrightarrow \mathrm{gr}^0_0 \mathcal{H}^q\big(s_1(q)_{X|D}\big) \longrightarrow \omega^q_{Y|D,\log} \longrightarrow 0,$$
$$\{x, a_1, \ldots, a_{q-1}\} \mapsto d\log \overline{x} \wedge d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_q}$$

Here $x \in (1+I_{D_2})^{\times}$, $a_1, \dots, a_{q-1} \in M_{X_2}^{gp}$ *and* $y \in \mathscr{O}_{X_2}(-D_2)$. *We denote* $\overline{x}$ (resp. $\overline{a_i}$) *the image of* x (resp. a_i) *in* M_Y^{gp}, *and we denote* $\overline{y}$ *the image of* y *in* $\mathscr{O}_Y(-D_s)$.

$$0 \longrightarrow \mathcal{R} \cap \mathrm{gr}_1^0 \mathcal{H}^q\big(s_1(q)_{X|D}\big) \longrightarrow \mathrm{gr}_1^0 \mathcal{H}^q\big(s_1(q)_{X|D}\big) \longrightarrow \omega_{Y|D,\log}^{q-1} \longrightarrow 0,$$

$$\{x, a_1, \dots, a_{q-2}, \pi\} \mapsto d\log \overline{x} \wedge d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_{q-2}}$$

where

$$\mathcal{R} := \mathrm{Ker}\Big(\mathrm{gr}_U^0 \mathcal{H}^q(S_D^{\cdot}) \to \mathrm{Ker}\big(Z^q\big(\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big) \xrightarrow{1-\varphi\otimes\wedge^q d\varphi/p} \mathcal{H}^q\big(\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big)\big)\Big).$$

(2) If $0 < m < pe/(p-1)$ *and* $p \nmid m$, *then we have*

$$\mathrm{gr}_0^m \mathcal{H}^q\big(s_1(q)_{X|D}\big) \cong \frac{\omega_{Y|D}^{q-1}}{B_{Y|D}^{q-1}}$$

$$\{1+\pi^m y, a_1, \dots, a_{q-1}\} \mapsto \overline{y} d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_q}$$

$$\mathrm{gr}_1^m \mathcal{H}^q\big(s_1(q)_{X|D}\big) \cong \frac{\omega_{Y|D}^{q-2}}{Z_{Y|D}^{q-2}}$$

$$\{1+\pi^m y, a_1, \dots, a_{q-2}, \pi\} \mapsto \overline{y} d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_{q-2}}$$

(3) If $0 < m < pe/(p-1)$ *and* $p|m$, *then we have short exact sequences*

$$0 \longrightarrow \frac{\mathcal{L}^m}{\mathcal{L}^m \cap \mathcal{H}^q\big(s_1(q)_{X|D}\big)} \longrightarrow \mathrm{gr}_0^m \mathcal{H}^q\big(s_1(q)_{X|D}\big) \longrightarrow \frac{\omega_{Y|D}^{q-1}}{Z_{Y|D}^{q-1}} \to 0,$$

$$\{1+\pi^m y, a_1, \dots, a_{q-1}\} \mapsto \overline{y} d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_q}$$

$$0 \longrightarrow \mathcal{L}^m \cap \mathcal{H}^q\big(s_1(q)_{X|D}\big) \longrightarrow \mathrm{gr}_1^m \mathcal{H}^q\big(s_1(q)_{X|D}\big) \longrightarrow \frac{\omega_{Y|D}^{q-2}}{Z_{Y|D}^{q-2}} \longrightarrow 0,$$

$$\{1+\pi^m y, a_1, \dots, a_{q-2}, \pi\} \mapsto \overline{y} d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_{q-2}}$$

where $\mathcal{L}^m$ *is an certain subsheaf of* $\mathrm{gr}_U^m \mathcal{H}^q\big(s_1(q)_{X|D}\big)$ *which is given more explicitly in a sufficientlly local situation (see Lemma 4.10 below).*

(4) If $m \geq pe/(p-1)$, *then* $U^m \mathcal{H}^q\big(s_1(q)_{X|D}\big) = 0$.

4. Proof of Main Results

4.1. **Proof of Theorem 3.4.** From Lemma 2.5 and induction on n, it suffices to show the claim in the case $n = 1$. By Lemma 4.11 and Lemma 4.12 below, the cokernel of the morphism

$$\mathrm{gr}_U^m\big((1+I_{D_2})^{\times} \otimes (M_{X_2}^{gp})^{\otimes q-1}\big) \longrightarrow \mathrm{gr}_U^m \mathcal{H}^q(s_1(q)_{X|D})$$

will be Mittag-Leffler zero with respect to the multiplicities of the prime components of D.. Then we will obtain that $\mathrm{Coker}(\mathrm{Symb}_{X|D})$ is Mittag-Leffler zero by the finiteness of the filtration $\{U^m\}_{m\in\mathbb{N}}$ in Theorem 3.5 (4).

4.2. **Proof of Theorem 3.5.** By Lemma 4.2 and Lemma 4.10 below, we will obtain (1), (2) and (3). From Lemma 4.5 (3) below, we will obtain $\mathrm{gr}^m_U\mathcal{H}^q(S^{\cdot}_D) = 0$ for $pe/(p-1) \leq m < pe$. Since $U^{pe}\mathcal{H}^q(S^{\cdot}_D) = 0$ by Lemma 4.4 and Corollary 4.6 below, this implies (4). Thus we will obtain Theorem 3.5 from Lemmas 4.2, 4.4, 4.5, 4.10, and Corollary 4.6 below.

In the rest of this section we prove the lemmas that have been mentioned in the above proof of Theorem 3.4 and Theorem 3.5. We will work with the following local situation.

4.3. **Local computation.** We denote (S, N) the scheme $\mathrm{Spec}(\mathscr{O}_K)$ with log sturcture N defined by the closed point. Let (V, M_V) be the scheme $\mathrm{Spec}(W[T])$ with the log structure defined by the divisor $\{T = 0\}$, and let $i_V : (S, N) \to (V, M_V)$ be the exact closed immersion defined by $T \mapsto \pi$. We assume that there exists a factorization $(Z, M_Z) \to (V, M_V) \to \mathrm{Spec}(W)$ such that $(Z, M_Z) \to (V, M_V)$ is smooth and compatible with the liftings of frobenii, and that the following diagram is cartesian (the left cartesian diagram is mentioned in **Assumption** 2.6):

$$\begin{array}{ccccc} (D, M_D) & \longrightarrow & (X, M_X) & \longrightarrow & (S, N) \\ \downarrow{\scriptstyle \beta_D} & \square & \downarrow{\scriptstyle \beta} & \square & \downarrow{\scriptstyle i_V} \\ (\mathscr{D}, M_{\mathscr{D}}) & \longrightarrow & (Z, M_Z) & \longrightarrow & (V, M_V). \end{array}$$

Lemma 4.1. *Let n be a non-negative integer.*

(1) From the reduction $\bmod T$ *of the short exact sequence*

$$0 \longrightarrow \omega^{q-1}_{Z_1/V_1} \otimes_{\mathscr{O}_{Z_1}} \mathscr{O}_{Z_1}(-\mathscr{D}_1) \xrightarrow{\wedge \mathrm{dlog} T} \omega^q_{Z_1|\mathscr{D}_1} \longrightarrow \omega^q_{Z_1/V_1} \otimes_{\mathscr{O}_{Z_1}} \mathscr{O}_{Z_1}(-\mathscr{D}_1) \longrightarrow 0, \tag{4.1}$$

and the $\mathscr{O}_{Z_1}$-linear isomorphism

$$\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^q_{Z_1|\mathscr{D}_1} \xrightarrow{\cong} \left(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\right) \otimes \omega^q_{Z_1|\mathscr{D}_1} \tag{4.2}$$

induced by the multiplication by T^m on $\omega^q_{Z_1|\mathscr{D}_1}$ for each integer $q \geq 0$, we obtain a short exact sequence of complexes:

$$0 \longrightarrow \omega^{\cdot}_{Y|D}[-1] \longrightarrow \left(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\right) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1} \longrightarrow \omega^{\cdot}_{Y|D} \longrightarrow 0. \tag{4.3}$$

(∗) *Furthermore, for each integer $q \geq 0$, the connecting homomorphism $\mathcal{H}^q(\omega^{\cdot}_{Y|D}) \to \mathcal{H}^q(\omega^{\cdot}_{Y|D})$ of the long exact sequence associated to* (4.3) *is the multiplication by $(-1)^q m$.*

(2) If $p \nmid m$, $\mathcal{H}^q(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}) = 0$.

(3) If $p|m$, there is an isomorphism:

$$T^m \cdot \Big(\varphi \otimes \wedge^q \frac{d\varphi}{p}\Big) : \mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^q_{Z_1|\mathscr{D}'_1} \xrightarrow{\cong} \mathcal{H}^q\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big). \tag{4.4}$$

Proof. The assertions 4.1, 4.2 and 4.3 are easy follows from Lemma 2.4.2, [Tsu2] and (2) follows from (∗). We prove (*) and (3). There is a commutative diagram of complexes with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega^{q-1}_{Y|D'}[-q] & \xrightarrow{\wedge d\log(T)} & \mathscr{O}_Y \otimes \omega^q_{Z_1|\mathscr{D}'}[-q] & \longrightarrow & \omega^q_{Y|D'}[-q] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \wedge^{q-1}\frac{d\varphi}{p}} & & \downarrow{\scriptstyle T^m\cdot\left(\varphi\otimes\wedge^q\frac{d\varphi}{p}\right)} & & \downarrow{\scriptstyle \wedge^q\frac{d\varphi}{p}} & & \\
0 & \longrightarrow & \omega^{\cdot}_{Y|D}[-1] & \xrightarrow{T^m\cdot(\wedge d\log(T))} & \left(\frac{T^m\mathscr{O}_{Z_1}}{T^{m+1}\mathscr{O}_{Z_1}}\right) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1} & \longrightarrow & \omega^{\cdot}_{Y|D} & \longrightarrow & 0
\end{array}$$

and taking cohomology, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega^{q-1}_{Y|D'} & \xrightarrow{\wedge d\log(T)} & \mathscr{O}_Y \otimes \omega^q_{Z_1|\mathscr{D}'} & \longrightarrow & \omega^q_{Y|D'} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle C^{-1}} & & \downarrow{\scriptstyle T^m\cdot\left(\varphi\otimes\wedge^q\frac{d\varphi}{p}\right)} & & \downarrow{\scriptstyle C^{-1}} & & \downarrow \\
\mathcal{H}^{q-1}(\omega^{\cdot}_{Y|D}) & \longrightarrow & \mathcal{H}^{q-1}(\omega^{\cdot}_{Y|D}) & \xrightarrow{T^m\cdot(\wedge d\log(T))} & \mathcal{H}^q\Big(\big(\frac{T^m\mathscr{O}_{Z_1}}{T^{m+1}\mathscr{O}_{Z_1}}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) & \longrightarrow & \mathcal{H}^q(\omega^{\cdot}_{Y|D}) & \longrightarrow & \mathcal{H}^q(\omega^{\cdot}_{Y|D}).
\end{array}$$

The commutativity of the above two diagrams follows from Lemma 7.1.4 in [Tsu2] and the characterization of Cartier isomorphism. Then we have (3) from Lemma 3.2. The claim (∗) is follows from a similar arguments as in the proof of Lemma 7.4.3 (2) [Tsu1]. □

Lemma 4.2. *Let m be a non-negative integer.*

(1) If $p \nmid m$, there is a short exact sequence

$$0 \longrightarrow \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}} \longrightarrow B^q\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \longrightarrow \frac{\omega^{q-1}_{Y|D}}{B^{q-1}_{Y|D}} \longrightarrow 0 \tag{4.5}$$

which is characterized by the following properties. For $x \in \mathscr{O}_Y(-\mathscr{D}_1)$ and $a_1, \ldots, a_{q-1} \in M^{gp}_{Z_1}$, the image of

$$d\big(T^m x \otimes d\log a_1 \wedge \cdots \wedge d\log a_{q-1}\big) \in B^q\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}}\Big) \tag{4.6}$$

in $\frac{\omega^{q-1}_{Y|D}}{B^{q-1}_{Y|D}}$ *is* $xd\log\overline{a_1}\wedge\cdots\wedge d\log\overline{a_{q-1}}$, *and*

$$d\big(T^m x\otimes d\log a_1\wedge\cdots\wedge d\log a_{q-2}\wedge d\log T\big)\in B^q\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big)\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \tag{4.7}$$

is the image of $xd\log\overline{a_1}\wedge\cdots\wedge d\log\overline{a_{q-2}}\in\frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}}$, *where* $\overline{a_i}$ *denote the images of* a_i *in* M^{gp}_Y.

(2) If $p|m$, *there is a short exact sequence*

$$0\longrightarrow\frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}}\longrightarrow B^q\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big)\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)\longrightarrow\frac{\omega^{q-1}_{Y|D}}{Z^{q-1}_{Y|D}}\longrightarrow 0 \tag{4.8}$$

which is characterized in the same way as (1).

(3) The homomorphism

$$1-\varphi\otimes\wedge^q\frac{d\varphi}{p}:Z^q\big(\mathscr{O}_Y\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\big)\to\mathcal{H}^q\big(\mathscr{O}_Y\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\big) \tag{4.9}$$

is surjective. Its kernel $\mathcal{K}$ *is the subsheaf of abelian groups of* $:Z^q\big(\mathscr{O}_Y\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\big)$ *generated by local sections of the form*

$$1\otimes d\log(a_1)\wedge d\log(a_2)\wedge\cdots\wedge d\log(a_q),\quad(a_1\in 1+\mathscr{O}_{Z_1}(-\mathscr{D}_1),a_2\ldots,a_q\in M^{gp}_{Z_1}.) \tag{4.10}$$

and there is a short exact sequence

$$0\to\omega^{q-1}_{Y|D,\log}\to\mathcal{K}\to\omega^q_{Y|D,\log}\to 0 \tag{4.11}$$

which is characterized by the following properties:

For $a_1\in 1+\mathscr{O}_{Z_1}(-\mathscr{D}_1),a_2,\ldots,a_q\in M^{gp}_{Z_1}$, *the image of*

$$1\otimes d\log(a_1)\wedge d\log(a_2)\wedge\cdots\wedge d\log(a_q)\in\mathcal{K} \tag{4.12}$$

in $\omega^q_{Y|D,\log}$ *is* $d\log(\overline{a_1})\wedge d\log(\overline{a_2})\wedge\cdots\wedge d\log(\overline{a_q})$, *and*

$$1\otimes d\log(a_1)\wedge d\log(a_2)\wedge\cdots\wedge d\log(a_{q-1})\wedge d\log T\in\mathcal{K} \tag{4.13}$$

is the image of $d\log(\overline{a_1})\wedge d\log(\overline{a_2})\wedge\cdots\wedge d\log(\overline{a_{q-1}})\in\omega^{q-1}_{Y|D,\log}$, *where* $\overline{a_i}$ *denote the images of* a_i *in* M^{gp}_Y.

Proof. If $p\not|m$, $Z^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big)\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)=B^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big)\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)$ by Lemma 4.1 (2). Then we have from 4.3 the following exact sequence:

$$0\to Z^{q-2}_{Y|D}\to Z^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big)\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)\to B^{q-1}_{Y|D}\to 0. \tag{4.14}$$

If $p|m$,the homomorphism

$$\mathcal{H}^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big)\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)\to\mathcal{H}^{q-1}(\omega^{\cdot}_{Y|D}) \tag{4.15}$$

is surjective by Lemma 4.1 (1). Hence, the homomorphism

$$Z^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big)\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)\to Z^{q-1}(\omega^{\cdot}_{Y|D}) \tag{4.16}$$

is surjective and 4.3 induces a short exact sequence:

$$0\to Z^{q-2}_{Y|D}\to Z^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big)\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)\to Z^{q-1}_{Y|D}\to 0. \tag{4.17}$$

(1) and (2) follows from these two short exact sequences and 4.3. (3) follows from the latter exact sequence with $m=0$, Lemma 4.1 (1) and Lemma 3.3. □

Let $A^{\cdot}$(resp. $B^{\cdot}$) be the subcomplex of $J^{[q-\cdot]}_{\mathscr{E}_1}\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}$ (resp. $\mathscr{O}_{\mathscr{E}_1}\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}$) which coincide with $J^{[q-\cdot]}_{\mathscr{E}_1}\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}$ (resp. $\mathscr{O}_{\mathscr{E}_1}\otimes\omega^{\cdot}_{Z_1|\mathscr{D}_1}$) in degree $q-1$, q, and $q+1$ (resp. degree $q-2$, $q-1$, and q), and is 0 in other degree. The inclusion map (resp. $\varphi_1\otimes\wedge^{q-1}d\varphi/p$) $J^{[q-\cdot]}_{\mathscr{E}_1}\otimes\omega^{q-1}_{Z_1|\mathscr{D}_1}\to\mathscr{O}_{\mathscr{E}_1}\otimes\omega^{q-1}_{Z_1|\mathscr{D}_1}$ and the identity map (resp.$\varphi\otimes\wedge^q d\varphi/p$) $\mathscr{O}_{\mathscr{E}_1}\otimes\omega^{q}_{Z_1|\mathscr{D}_1}\to\mathscr{O}_{\mathscr{E}_1}\otimes\omega^{q}_{Z_1|\mathscr{D}_1}$ give a morphism of complexes 1 (resp. φ_q):$A^{\cdot}\to B^{\cdot}$. We put $S_D{}^{\cdot}$ the mapping fiber of the morphism $1-\varphi_q: A^{\cdot}\to B^{\cdot}$. Then we have $\mathcal{H}^q(s_1(q)_{X|D})=\mathcal{H}^q(S_D{}^{\cdot})$.

We define the descending filtration $\tilde{U}^m$ $(0\le m\le pe)$ on $A^{\cdot}$(resp. $B^{\cdot}$) as follows:

$$\begin{aligned}\cdots\to 0\to\big(T^{m'}\mathscr{O}_{\mathscr{E}_1}+J^{[p]}_{\mathscr{E}_1}\big)\otimes\omega^{q-1}_{Z_1|\mathscr{D}_1}&\to\big(T^{m}\mathscr{O}_{\mathscr{E}_1}+J^{[p]}_{\mathscr{E}_1}\big)\otimes\omega^{q}_{Z_1|\mathscr{D}_1}\\&\to\big(T^{m}\mathscr{O}_{\mathscr{E}_1}+J^{[p]}_{\mathscr{E}_1}\big)\otimes\omega^{q+1}_{Z_1|\mathscr{D}_1}\to 0\to\cdots\end{aligned} \tag{4.18}$$

$$\begin{aligned}\Big(\text{resp.}\cdots\to 0\to\big(T^{m}\mathscr{O}_{\mathscr{E}_1}+J^{[p]}_{\mathscr{E}_1}\big)\otimes\omega^{q-2}_{Z_1|\mathscr{D}_1}&\to\big(T^{m}\mathscr{O}_{\mathscr{E}_1}+J^{[p]}_{\mathscr{E}_1}\big)\otimes\omega^{q-1}_{Z_1|\mathscr{D}_1}\\&\to\big(T^{m}\mathscr{O}_{\mathscr{E}_1}+J^{[p]}_{\mathscr{E}_1}\big)\otimes\omega^{q}_{Z_1|\mathscr{D}_1}\to 0\to\cdots\Big),\end{aligned} \tag{4.19}$$

where m' denotes the smallest integer which is $\ge\max(e+m/p,m)$. The morphism $1: A^{\cdot}\to B^{\cdot}$ is compatible with the filtrations $\tilde{U}^{\cdot}$. By the assumption $p\ge 3$, we have $\varphi_1(J^{[p]}_{\mathscr{E}_1})=0$. Then the morphism $\varphi_q: A^{\cdot}\to B^{\cdot}$ is also compatible with the filtrations $\tilde{U}^{\cdot}$.

We define the filtration $\tilde{U}^m(0\le m\le pe)$ on $S_D{}^{\cdot}$ to be the mapping fiber of $1-\varphi_q:\tilde{U}^mA^{\cdot}\to\tilde{U}^mB^{\cdot}$ and define the filtration $\tilde{U}^m$ on $\mathcal{H}^q(S_D{}^{\cdot})$ to be the image of $\mathcal{H}^q(\tilde{U}^mS_D{}^{\cdot})$. We will show that $\tilde{U}^m\mathcal{H}^q(S_D{}^{\cdot})=\mathcal{H}^q(\tilde{U}^mS_D{}^{\cdot})$ $(0\le m\le pe)$.

Next we calculate the image of $(1+I_{D_2})^{\times}\otimes(M^{gp}_{X_2})^{\otimes(q-1)}$ under the symbol map 2.2.

Lemma 4.3. *For $x\in\big(1+\mathscr{O}_{Z_2}(-\mathscr{D}_2)\big)^{\times}$, $a_1,\dots,a_{q-1}\in M^{gp}_{\mathscr{E}_2}$, the image of $\overline{x}\otimes\overline{a_1}\otimes\cdots\otimes\overline{a_{q-1}}$ in $\mathcal{H}^q(S_D{}^{\cdot})$ under the symbol map 2.2, is the class of the cocycle*

$$\begin{aligned}\Big(d\log x\wedge d\log a_1\wedge\cdots\wedge d\log a_{q-1},\ p^{-1}\log(x^p\varphi_{\mathscr{E}_2}(x)^{-1})\cdot d\varphi/p(d\log a_1)\wedge\cdots\wedge d\varphi/p(d\log a_{q-1})\\ +\Sigma_{i=1}^{q-1}(-1)^{i-1}p^{-1}\log(a_i^p\varphi_{\mathscr{E}_2}(a_i)^{-1})\cdot d\log x\wedge d\log a_1\wedge\cdots\\ \wedge d\log a_{i-1}\wedge\frac{d\varphi}{p}(d\log a_{i+1})\wedge\cdots\wedge\frac{d\varphi}{p}(d\log a_{q-1})\Big)\in(\mathscr{O}_{\mathscr{E}_1}\otimes\omega^{q}_{Z_1|\mathscr{D}_1})\oplus(\mathscr{O}_{\mathscr{E}_1}\otimes\omega^{q-1}_{Z_1|\mathscr{D}_1}),\end{aligned} \tag{4.20}$$

where $\overline{x}$ denote the image of x in $(1+I_{D_2})^{\times}$ and $\overline{a_i}$ denote the images of a_i in $M^{gp}_{X_2}$.

Proof. This is a straightforward calculation by (2.2). □

Lemma 4.4. *For $0 \leq m \leq pe$, we have $U^m\mathcal{H}^q(S_D{}^{\cdot}) \subset \mathcal{H}^q(\tilde{U}^m S_D{}^{\cdot})$.*

Proof. We use a similar argument as in [Tsu2], Lemma 2.5.2. By Lemma 4.3, $Z^q(\tilde{U}^m S_D{}^{\cdot}) = Z^q(S_D{}^{\cdot}) \cap \tilde{U}^m S_D{}^{\cdot}$, and the definition of $\tilde{U}^m S_D{}^{\cdot}$, it suffices to prove that:

$$d\log(1+T^m x) \in \left(T^m\mathscr{O}_{\mathscr{E}_1} + J_{\mathscr{E}_1}^{[p]}\right) \otimes \omega^1_{Z_1|\mathscr{D}_1} \tag{4.21}$$

$$p^{-1}\log\left((1+T^m x)^p \varphi_{\mathscr{E}_2}(1+T^m x)^{-1}\right) \in \left(T^m\mathscr{O}_{\mathscr{E}_1} + J_{\mathscr{E}_1}^{[p]}\right) \otimes_{\mathscr{O}_{Z_1}} \mathscr{O}_{Z_1}(-\mathscr{D}_1) \tag{4.22}$$

for $x \in \mathscr{O}_{\mathscr{E}_2} \otimes_{\mathscr{O}_{Z_2}} \mathscr{O}_{Z_2}(-\mathscr{D}_2)$ and $1 \leq m \leq pe$. We denote $T \in \Gamma(\mathscr{E}_2, \mathscr{O}_{\mathscr{E}_2})$ a lifting of $\pi \in \Gamma(X_2, \mathscr{O}_{X_2})$. The former is trivial. We will prove the latter. We have $\varphi_{\mathscr{E}_2}(x) = x^p + py$ for some $y \in \mathscr{O}_{\mathscr{E}_2} \otimes_{\mathscr{O}_{Z_2}} \mathscr{O}_{Z_2}(-\mathscr{D}_2)$. Put $z = y(1+T^{pm}x^p)^{-1}$. Then we have $\varphi_{\mathscr{E}_2}(1+T^m x) = (1+T^{pm}x^p)(1+pT^{pm}z)$. On the other hand, $(1+T^m x)^p = 1+T^{pm}x^p+pT^m w$ for some $w \in \mathscr{O}_{\mathscr{E}_2} \otimes_{\mathscr{O}_{Z_2}} \mathscr{O}_{Z_2}(-\mathscr{D}_2)$. Hence we obtain

$$(1+T^m x)^p \varphi_{\mathscr{E}_2}(1+T^m x)^{-1} = \left(1+pT^m w(1+T^{pm}x^p)^{-1}\right)(1+pT^{pm}z)^{-1}. \tag{4.23}$$

This completes the proof. □

Next we calculate $\mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot})$ for $0 \leq m < pe$. By definition, we have a long exact sequence:

$$\begin{aligned} 0 \to Z^{q-2}(\mathrm{gr}^m_{\tilde{U}} B^{\cdot}) \to \mathcal{H}^{q-1}(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}) \to Z^{q-1}(\mathrm{gr}^m_{\tilde{U}} A^{\cdot}) \xrightarrow{1-\varphi_q} \mathcal{H}^{q-1}(\mathrm{gr}^m_{\tilde{U}} B^{\cdot}) \\ \to \mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}) \to \mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} A^{\cdot}) \xrightarrow{1-\varphi_q} \frac{\mathrm{gr}^m_{\tilde{U}} B^q}{B^q(\mathrm{gr}^m_{\tilde{U}} B^{\cdot})} \end{aligned} \tag{4.24}$$

Since $m \geq e + m/p$ (resp. $m \leq e + m/p$)$\Longleftrightarrow m \geq pe/(p-1)$ (resp. $m \leq pe/(p-1)$), we have the following:

$$Z^{q-1}(\mathrm{gr}^m_{\tilde{U}} A^{\cdot}) = \begin{cases} 0, & (0 \leq m < pe/(p-1), p \not| m), \\ \left(T^{e+m/p}\mathscr{O}_{Z_1}/T^{e+m/p+1}\mathscr{O}_{Z_1}\right) \otimes \omega^{q-1}_{Z_1|\mathscr{D}_1}, & (0 \leq m < pe/(p-1), p|m), \\ Z^{q-1}\Big(\left(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\right) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) & (pe/(p-1) \leq m < pe), \end{cases} \tag{4.25}$$

$$\mathcal{H}^{q-1}(\mathrm{gr}^m_{\tilde{U}} B^{\cdot}) = \mathcal{H}^{q-1}\left(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\right), \tag{4.26}$$

$$\mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} A^{\cdot}) = \begin{cases} Z^q\Big(\left(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\right) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) & (0 \leq m < pe/(p-1)), \\ \mathcal{H}^q\Big(\left(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\right) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) & (pe/(p-1) \leq m < pe), \end{cases} \tag{4.27}$$

$$\frac{\mathrm{gr}^m_{\tilde{U}} B^q}{B^q(\mathrm{gr}^m_{\tilde{U}} B^{\cdot})} = \frac{\left(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\right) \otimes \omega^q_{Z_1|\mathscr{D}_1}}{B^q\Big(\left(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\right) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)}. \tag{4.28}$$

Lemma 4.5. *Let m be an integer such that $0 \leq m < pe$. Then :*

(1) If $m = 0$, we have an exact sequence

$$0 \longrightarrow \frac{\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{q-1}_{Z_1|\mathscr{D}'_1}}{\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{q-1}_{Z_1|\mathscr{D}_1}} \longrightarrow \mathcal{H}^q(\mathrm{gr}^0_{\tilde{U}} S_D{}^{\cdot}) \tag{4.29}$$
$$\longrightarrow \mathrm{Ker}\Big(Z^q\big(\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big) \xrightarrow{1-\varphi\otimes\wedge^q \frac{d\varphi}{p}} \mathcal{H}^q\big(\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big)\Big) \longrightarrow 0.$$

(2) We have an isomorphism

$$Z^{q-2}\big(\mathrm{gr}^m_{\tilde{U}} B^{\cdot}\big) \xrightarrow{\cong} \mathcal{H}^{q-1}(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}). \tag{4.30}$$

(3) If $0 < m < pe/(p-1)$ and $p|m$, we have a surjective morphism,

$$\mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}) \twoheadrightarrow B^q\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big), \tag{4.31}$$

and it is isomorhism if $0 < m < pe/(p-1)$ and $p \not| m$.

(4) If $pe/(p-1) \leq m < pe$,

$$\mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}) = 0. \tag{4.32}$$

Proof. We describe the homomorphism $Z^{q-1}(\mathrm{gr}^m_{\tilde{U}} A^{\cdot}) \xrightarrow{1-\varphi_q} \mathcal{H}^{q-1}(\mathrm{gr}^m_{\tilde{U}} B^{\cdot})$ as follows:

(i)
$$0 \to \mathcal{H}^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big),\ (0 \leq m < pe/(p-1), p \not| m),$$

(ii)
$$\big(T^{e+m/p}\mathscr{O}_{Z_1}/T^{e+m/p+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{q-1}_{Z_1|\mathscr{D}_1} \xrightarrow{\varphi_1\otimes\wedge^{q-1}d\varphi/p} \mathcal{H}^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big),$$
$$(0 \leq m < pe/(p-1), p|m),$$

(iii)
$$Z^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \xrightarrow{1-\varphi_1\otimes\wedge^{q-1}d\varphi/p} \mathcal{H}^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big),$$
$$(m = pe/(p-1)),$$

(iv)
$$Z^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \xrightarrow{1} \mathcal{H}^{q-1}\Big(\big(T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big),$$
$$(pe/(p-1) < m < pe).$$

The first homomorphism (i) is isomorphism by Lemma 4.1 (2). The second homomorphism (ii) is injective by Lemma 4.1 (3). The third homomorphism (iii) is surjective by Lemma 4.2 (3). It is trivial that (iv) are surjective. Hence we have

$$\mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}) \xrightarrow{\cong} \mathrm{Ker}\Big(\mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} A^{\cdot}) \xrightarrow{1-\varphi_q} \frac{\mathrm{gr}^m_{\tilde{U}} B^q}{B^q(\mathrm{gr}^m_{\tilde{U}} B^{\cdot})}\Big) \tag{4.33}$$

except for $0 \leq m < pe/(p-1), p|m$ case. (1) and (2) follows from the injectivity of the homomorphism of (ii). □

Lemma 4.6. *We have* $\mathcal{H}^q(\tilde{U}^m S_D{}^{\cdot}) = 0$ *for* $pe/(p-1) < m \leq pe$

Proof. If $pe/(p-1) < m \leq pe$, we have

$$\tilde{U}^m A^{q-1} = \tilde{U}^m B^{q-1} = \big(T^m \mathscr{O}_{\mathscr{E}_1} + J^{[p]}_{\mathscr{E}_1}\big) \otimes \omega^q_{Z_1|\mathscr{D}_1} \tag{4.34}$$

$$\Big(\text{resp. } \tilde{U}^m A^q = \tilde{U}^m B^q = \big(T^m \mathscr{O}_{\mathscr{E}_1} + J^{[p]}_{\mathscr{E}_1}\big) \otimes \omega^q_{Z_1|\mathscr{D}_1}\Big) \tag{4.35}$$

and $\varphi_1 \otimes \wedge^{q-1} d\varphi/p$(resp. $\varphi \otimes \wedge^q d\phi/p$) are nilpotent on them since $pe/(p-1) < m$ implies $m > e + m/p$ and $\varphi_1(J^{[p]}_{\mathscr{E}_1}) = 0$. Then $1 - \varphi_q : \tilde{U}^m A^{\cdot} \to \tilde{U}^m B^{\cdot}$ are bijective in degree $q-1$ and degree q. □

Corollaly 4.7. *We have* $\tilde{U}^m \mathcal{H}^q(S_D{}^{\cdot}) = 0$ *for* $pe/(p-1) < m \leq pe$.

Lemma 4.8. *The homomorphism* $\mathcal{H}^q(\tilde{U}^{m+1} S_D{}^{\cdot}) \to \tilde{U}^m \mathcal{H}^q(S_D{}^{\cdot})$ *are injective for* $0 \leq m < pe$.

Proof. By Lemma 4.6, we may assume that $m + 1 \leq pe/(p-1)$. It is enough to show that

$$\mathcal{H}^{q-1}(\tilde{U}^m S_D{}^{\cdot}) \to \mathcal{H}^{q-1}(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}) \tag{4.36}$$

is surjective. From the argument before Lemma 4.5 (i), we obtain an isomorphism

$$Z^{q-2}\big(\mathrm{gr}^m_{\tilde{U}} B^{\cdot}\big) = Z^{q-2}\Big(\big(T^m \mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \xrightarrow{\cong} \mathcal{H}^{q-1}(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}). \tag{4.37}$$

Then it suffices to prove that the natural homomorphism

$$\begin{aligned} Z^{q-2}\big(\tilde{U}^m B^{\cdot}\big) &= Z^{q-2}\Big(\big(T^m \mathscr{O}_{Z_1} + J^{[p]}_{\mathscr{E}_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \\ \to Z^{q-2}\big(\mathrm{gr}^m_{\tilde{U}} B^{\cdot}\big) &= Z^{q-2}\Big(\big(T^m \mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \end{aligned} \tag{4.38}$$

is surjective or equivalently that the homomorphism

$$\mathcal{H}^{q-2}\Big(\big(T^m \mathscr{O}_{Z_1} + J^{[p]}_{\mathscr{E}_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \to \mathcal{H}^{q-2}\Big(\big(T^m \mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \tag{4.39}$$

is surjective. When $p \not| m$, this is obvious by Lemma 4.1 (2). In the case of $p|m$, this follows from the following commutative diagram in which the lower horizontal arrow is surjective and the right vertical arrow is an isomorphism by Lemma 4.1 (3).

$$\begin{array}{ccc} \mathcal{H}^{q-2}\Big(\big(T^m \mathscr{O}_{Z_1} + J^{[p]}_{\mathscr{E}_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) & \longrightarrow & \mathcal{H}^{q-2}\Big(\big(T^m \mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \\ \Big\uparrow{\scriptstyle T^m \cdot \varphi \otimes \wedge^{q-2} \frac{d\varphi}{p}} & & \Big\uparrow{\scriptstyle T^m \cdot \varphi \otimes \wedge^{q-2} \frac{d\varphi}{p}} \\ \omega^{q-2}_{Z_1|\mathscr{D}'_1} & \longrightarrow & \mathscr{O}_Y \otimes \omega^{q-2}_{Z_1|\mathscr{D}'_1}. \end{array}$$

□

Corollaly 4.9. *$\mathcal{H}^q(\tilde{U}^m S_D{}^{\cdot}) = \tilde{U}^m \mathcal{H}^q(S_D{}^{\cdot})$ for $0 \leq m \leq pe$.*

From lemma 4.4 and 4.9, we have homomorphisms

$$\alpha_{m,D} : \mathrm{gr}^m_U \mathcal{H}^q(S_D{}^{\cdot}) \to \mathrm{gr}^m_{\tilde{U}} \mathcal{H}^q(S_D{}^{\cdot}) \tag{4.40}$$

and injective homomorphisms

$$\beta_{m,D} : \mathrm{gr}^m_{\tilde{U}} \mathcal{H}^q(S_D{}^{\cdot}) \to \mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}) \tag{4.41}$$

for $0 \leq m < pe$.

Lemma 4.10. *Let m be a non-negative integer. Let $x \in (1 + I_{\mathscr{D}_2})^{\times}$, let $a_1, \dots a_{q-1} \in M^{gp}_{Z_2}$ and let $y \in \mathscr{O}_{Z_2}(-\mathscr{D}_2)$. Let $\overline{x}$ denote the image of x in $(1 + I_{D_2})^{\times}$, let $\overline{a_i}$ denote the image of a_i in $M^{gp}_{X_2}$ and let $\overline{y}$ denote the image of y in $\mathscr{O}_{X_2}(-D_2)$. Then we have:*

(1) If $m = 0$, the image of

$$\overline{x} \otimes \overline{a_1} \otimes \dots \overline{a_{q-1}} \in (1 + I_{D_2})^{\times} \otimes (M^{gp}_{X_2})^{\otimes (q-1)}$$

under the composite

$$\begin{aligned}(1 + I_{D_2})^{\times} \otimes (M^{gp}_{X_2})^{\otimes (q-1)} \to \mathrm{gr}^0_U \mathcal{H}^q(S_D{}^{\cdot}) \xrightarrow{\alpha_{0,D}} \mathrm{gr}^0_{\tilde{U}} \mathcal{H}^q(S_D{}^{\cdot}) \\ \xrightarrow{\beta_{0,D}} \mathcal{H}^q(\mathrm{gr}^0_{\tilde{U}} S_D{}^{\cdot}) \twoheadrightarrow \mathrm{Ker}\Big(Z^q\big(\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big) \xrightarrow{1-\varphi \otimes \wedge^q d\varphi/p} \mathcal{H}^q\big(\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big)\Big)\end{aligned} \tag{4.42}$$

is $1 \otimes d\log x \wedge d\log a_1 \wedge \dots \wedge d\log a_{q-1}$.

(2) If $0 < m < pe/(p-1)$ and $p \nmid m$, the image of

$$(1 + \pi^m \overline{y}) \otimes \overline{a_1} \otimes \dots \otimes \overline{a_{q-1}} \in U^m\Big((1 + I_{D_2})^{\times} \otimes (M^{gp}_{X_2})^{\otimes (q-1)}\Big)$$

under the composite

$$\begin{aligned}U^m\Big((1 + I_{D_2})^{\times} \otimes (M^{gp}_{X_2})^{\otimes (q-1)}\Big) \to \mathrm{gr}^m_U \mathcal{H}^q(S_D{}^{\cdot}) \xrightarrow{\beta_{m,D}\alpha_{m,D}} \mathcal{H}^q(\mathrm{gr}^m_{\tilde{U}} S_D{}^{\cdot}) \\ \xrightarrow[Lemma\ 4.5\ (3)]{\cong} B^q\Big(\big(T^m \mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)\end{aligned} \tag{4.43}$$

is $d\big(T^m y \otimes d\log a_1 \wedge \dots \wedge d\log a_{q-1}\big)$.

We put

$$\mathcal{L}^m := \mathrm{Ker}\Big(\mathrm{gr}^m_U \mathcal{H}^q(S^{\cdot}_D) \twoheadrightarrow B^q\Big(\big(T^m \mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}\big) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)\Big).$$

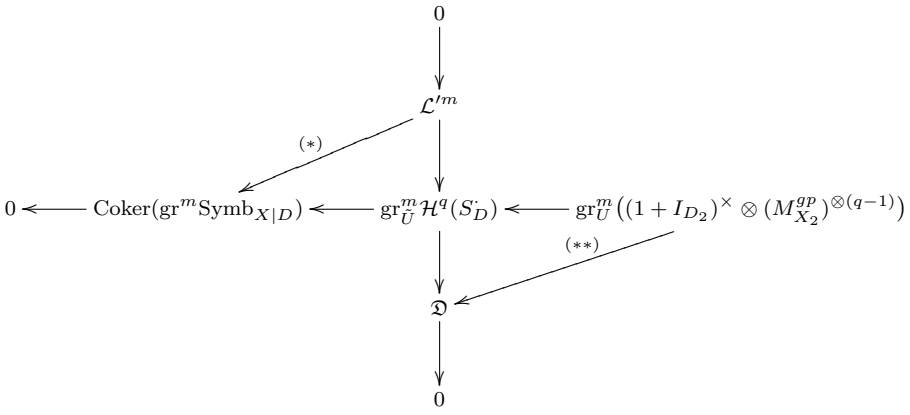
Proof. We obtain the lemma from Lemma 4.5. Note that $T \in \Gamma(Z_2, \mathscr{O}_{Z_2})$ is alifting of $\pi \in \Gamma(X_2, \mathscr{O}_{X_2})$. □

Lemma 4.11. *The cokernel of the morphism*

$$\mathrm{gr}^m\mathrm{Symb}_{X|D} : \mathrm{gr}^m_U\big((1+I_{D_2})^\times \otimes (M^{gp}_{X_2})^{\otimes(q-1)}\big) \longrightarrow \mathrm{gr}^m_{\tilde{U}}\mathcal{H}^q(S^{\cdot}_D)$$

is Mittag-Leffler zero with respect to the multiplicities of the prime components of D.

Proof. We have the following diagram:



where the vertical and horizontal sequences is exact, $\mathcal{L}'^m := \mathcal{L}^m \cap \mathrm{gr}^m_{\tilde{U}}\mathcal{H}^q(S^{\cdot}_D)$ and $\mathfrak{D}$ is certain differenital sheaves which is explicitly written in Lemma 4.5. The morphism $(**)$ is surjective, then $(*)$ is also. Here $\mathcal{L}'^m$ is Mittag-Leffler zero with respect to the multiplicities of the prime components of D. Hence $\mathrm{Coker}(\mathrm{gr}^m\mathrm{Symb}_{X|D})$ is also. □

Lemma 4.12. *The kernel and the cokernel of the morphism*

$$\alpha_{m,D} : \mathrm{gr}^m_U\mathcal{H}^q(S^{\cdot}_D) \to \mathrm{gr}^m_{\tilde{U}}\mathcal{H}^q(S^{\cdot}_D)$$

are Mittag-Leffler zero with respect to the multiplicities of the prime components of D.

Proof. We consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U^{m+1}\mathcal{H}^q(S^{\cdot}_D) & \longrightarrow & U^m\mathcal{H}^q(S^{\cdot}_D) & \longrightarrow & \mathrm{gr}^m_U\mathcal{H}^q(S^{\cdot}_D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{U}^{m+1}\mathcal{H}^q(S^{\cdot}_D) & \longrightarrow & \tilde{U}^m\mathcal{H}^q(S^{\cdot}_D) & \longrightarrow & \mathrm{gr}^m_{\tilde{U}}\mathcal{H}^q(S^{\cdot}_D) & \longrightarrow & 0
\end{array}$$

The left and central vertical morphism is injective by Lemma 4.4. If $m > pe/(p-1)$, the claim is trivial. We assume that $0 \leq m \leq pe/(p-1)$. If $m = pe/(p-1)$, the right vertical morphism is injective by Corollaly 4.7 and the cokernel of $\alpha_{m,D}$ is Mittag-Leffler zero from Lemma 4.11. We can easily to show by induction on m. □

5. Calculation of $\mathcal{H}^q(s_1(r)_{X|D})$ for $0 \leq r < q \leq p-2$

In this section, for $0 \leq q < r \leq p-2$, we will calculate the cohomology sheaf $\mathcal{H}^q(s_1(r)_{X|D})$ by a similar computations as in Appendix [Tsu3]. The setting remains as in §4.3.

We define a descending filtration on $\tilde{\mathfrak{U}}^m, m \in \mathbb{N}$ on $s_1(r)_{X|D}$ for an integer $0 \leq r \leq p-2$ as follows: we define the filtration $\tilde{\mathfrak{U}}^m, m \in \mathbb{N}$ on $\mathscr{O}_{\mathscr{E}_1}$ (resp.$J^{[r]}_{\mathscr{E}_1}$ $(r \leq p-2)$) by

$$T^m \mathscr{O}_{\mathscr{E}_1} + J^{[p]}_{\mathscr{E}_1} \quad (\text{resp. } T^{\max\{er+\lceil \frac{m}{p} \rceil, m\}} \mathscr{O}_{\mathscr{E}_1} + J^{[p]}_{\mathscr{E}_1}).$$

Here $\lceil x \rceil$ for $x \in \mathbb{R}$ denotes the smallest integer $\geq x$. We can easy to see that the morphism $1, \varphi_r : J^{[r-\cdot]}_{\mathscr{E}_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1} \to \mathscr{O}_{\mathscr{E}_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}$ are compatible with $\tilde{\mathfrak{U}}^m$. We define the filtration $\tilde{\mathfrak{U}}^m$ on $s_1(r)_{X|D}$ to be the mapping fiber of $1 - \varphi_q : \tilde{\mathfrak{U}}^m(J^{[r-\cdot]}_{\mathscr{E}_1}) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1} \to \tilde{\mathfrak{U}}^m(\mathscr{O}_{\mathscr{E}_1}) \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}$.

Lemma 5.1. *Let m be a non-negative integer. For $a \in k^*$, the homomorphism*

$$1 - a^p \cdot \varphi \otimes \wedge^q \frac{d\varphi}{p} : Z^q\big(\mathscr{O}_Y \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big) \to \mathcal{H}^q\big(\mathscr{O}_Y \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big) \tag{5.1}$$

is surjective. Its kernel $\mathcal{K}$ is the subsheaf of abelian groups of $: Z^q\big(\mathscr{O}_Y \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big)$ generated by local sections of the form

(5.2)

$$x \otimes d\log(a_1) \wedge d\log(a_2) \wedge \cdots \wedge d\log(a_q), \quad (x \in \mathrm{Ker}(1 - a^p\varphi : \mathscr{O}_Y \to \mathscr{O}_Y), a_1 \in 1 + \mathscr{O}_{Z_1}(-\mathscr{D}_1), a_2 \ldots, a_q \in M^{gp}_{Z_1}).$$

and there is a short exact sequence

(5.3)

$$0 \to \mathrm{Ker}\big(1 - a^p C^{-1} : Z^{q-1}_{Y|D} \longrightarrow \mathcal{H}^{q-1}(\omega^{\cdot}_{Y|D})\big) \to \mathcal{K} \to \mathrm{Ker}\big(1 - a^p C^{-1} : Z^q_{Y|D} \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{Y|D})\big) \to 0$$

which is characterized by the following properties:

For $a_1 \in 1 + \mathscr{O}_{Z_1}(-\mathscr{D}_1), a_2, \ldots, a_q \in M^{gp}_{Z_1}$ and $x \in \mathrm{Ker}(1 - a^p\varphi : \mathscr{O}_Y \to \mathscr{O}_Y)$, the image of

$$x \otimes d\log(a_1) \wedge d\log(a_2) \wedge \cdots \wedge d\log(a_q) \in \mathcal{K} \tag{5.4}$$

in the right term is $d\log(\overline{a_1}) \wedge d\log(\overline{a_2}) \wedge \cdots \wedge d\log(\overline{a_q})$, and

$$x \otimes d\log(a_1) \wedge d\log(a_2) \wedge \cdots \wedge d\log(a_{q-1}) \wedge d\log T \in \mathcal{K} \tag{5.5}$$

is the image of $d\log(\overline{a_1}) \wedge d\log(\overline{a_2}) \wedge \cdots \wedge d\log(\overline{a_{q-1}})$ in the left term, where $\overline{a_i}$ denote the images of a_i in M^{gp}_Y.

Proof. We can prove this in the same way as Lemma 4.2 (3), [Tsu3], Lemma A8. □

Lemma 5.2. *(cf.* [Tsu1], Lemma 4.5*) Let q and r be integers such that $0 \leq q \leq r \leq p-2$. We have the following description of the kernel of*

$$\mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(J^{[r-\cdot]}_{\mathscr{E}_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1})\big) \longrightarrow \mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(\mathscr{O}_{\mathscr{E}_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1})\big) \quad \cdots(\bigstar).$$

(1) If $m < ep(r-q)/(p-1)$ or $m \geq ep(r-q+1)/(p-1)$, then $(\bigstar)$ is isomorphism.

(2) If $m = ep(r-q)/(p-1)$, then the kernel of $(\bigstar)$ is isomorphic to the kernel of

$$1 - a_0^{p(r-q)} \cdot C^{-1} : Z^q\big((\mathscr{O}_{Z_1}/T\mathscr{O}_{Z_1}) \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big) \longrightarrow \mathcal{H}^q\big((\mathscr{O}_{Z_1}/T\mathscr{O}_{Z_1}) \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big),$$

where $a_0 := N_{\mathscr{O}_{\hat{K}/W}}(-\pi) \cdot p^{-1} \mod p \in k^*$.

(3) Suppose $ep(r-q)/(p-1) < m < ep(r-q+1)/(p-1)$, then the kernel of $(\bigstar)$ is isomorphic to $B^q\big((\frac{T^m\mathscr{O}_{Z_1}}{T^{m+1}\mathscr{O}_{Z_1}} \otimes \omega_{Z_1|\mathscr{D}_1})\big)$. If $p \nmid m$, $(\bigstar)$ is surjective. If $p|m$, $(\bigstar)$ is not surjective.

Proof. We note that

$$\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(\mathscr{O}_{\mathscr{E}_1} \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1}) \xleftarrow{\cong} (T^m\mathscr{O}_{Z_1}/T^{m+1}\mathscr{O}_{Z_1}) \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1},$$

and

$$m \lessgtr e(r-q) + m/p \leftrightarrow m \lessgtr ep(r-q)/(p-1).$$

- If $m \geq ep(r-q+1)/(p-1)$, $\tilde{\mathfrak{U}}^m J^{[r-\cdot]}_{\mathscr{E}_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}$ coincides with $\tilde{\mathfrak{U}}^m\mathscr{O}_{\mathscr{E}_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}$ in degree $\geq q-1$ and

  $$\varphi_{r-q}\Big(\tilde{\mathfrak{U}}^m J^{[r-q]}_{\mathscr{E}_1}\Big) \subset \tilde{\mathfrak{U}}^{m+1}\mathscr{O}_{\mathscr{E}_1}.$$

  Hence the morphism $(\bigstar)$ is the identity.

- If $ep(r-q)/(p-1) \leq m \leq ep(r-q+1)/(p-1)$, $\tilde{\mathfrak{U}}^m J^{[r-\cdot]}_{\mathscr{E}_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}$ coincides with $\tilde{\mathfrak{U}}^m\mathscr{O}_{\mathscr{E}_1} \otimes \omega^{\cdot}_{Z_1|\mathscr{D}_1}$ in degree $\geq q$ and

  $$d^{q-1}\Big(\tilde{\mathfrak{U}}^m J^{[r-q+1]}_{\mathscr{E}_1} \otimes \omega^{q-1}_{Z_1|\mathscr{D}_1}\Big) \subset \tilde{\mathfrak{U}}^m J^{[r-q]}_{\mathscr{E}_1} \otimes \omega^{q}_{Z_1|\mathscr{D}_1}.$$

- If $m > ep(r-q)/(p-1)$, we have

  $$\varphi_{r-q}\Big(\tilde{\mathfrak{U}}^m J^{[r-q]}_{\mathscr{E}_1}\Big) \subset \tilde{\mathfrak{U}}^{m+1}\mathscr{O}_{\mathscr{E}_1},$$

  then we have (3).

If $m = ep(r-q)/(p-1)$, we have (2) from Lemma 4.1 and Lemma 5.1 and

$$\varphi_{r-q}\big(T^{e(r-q)}\big) = \big(a_0^p + a_1^p T^p + \cdots + a_{e-1}^p T^{(e-1)p} + (T^e)^{[p]}\big)^{r-q},$$

where $T^e + p(a_{e-1}T^{e-1} + \cdots + a_1 T + a_0)$ $(a_i \in W)$ denotes the Eisenstein polynomial of π over W. If $m < ep(r-q)/(p-1)$ and $p \nmid m$, both sides of the homomorphism $(\bigstar)$ vanish by Lemma 4.1 (2).

If $m < ep(r-q)/(p-1)$ and $p|m$, the claim (1) follows from Lemma 5.1 and the above description of $\varphi_{r-q}\big(T^{e(r-q)}\big)$. □

If K contains a primitive p-th root of unity, then we have $a_0 \in (k^*)^{p-1}$(See [Tsu3], the proof of Proposition A17). Choose a $(p-1)$-th root $b_0 \in k$ of a_0. Then, by Lemma 3.2, for integers $q \geq 0, \theta \geq 0$, we have

$$(*1) \quad \omega^q_{Y|D,\log} \xrightarrow{\cong} \mathrm{Ker}\big(1 - a_0^{p\theta} C^{-1} : Z^q_{Y|D} \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{Y|D})\big), \quad \omega \mapsto b_0^{-p\theta} \cdot \omega.$$

Proposition 5.3. *Let the notation and assumption be as above. Let q and r be an integers such that $0 \leq q \leq r \leq p-2$. Then, for every integer $m \geq 0$, we have the structure of $\mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(s_1(r)_{X|D})\big)$ as follows:*

(1) If $m < ep(r-q)/(p-1)$ or $m \geq ep(r-q+1)/(p-1)$, then

$$\mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(s_1(r)_{X|D})\big) = 0.$$

(2) If $m = ep(r-q)/(p-1)$, then there exists an exact sequence

$$0 \longrightarrow \omega^{q-1}_{Y|D,\log} \longrightarrow \frac{\mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(s_1(r)_{X|D})\big)}{\mathfrak{K}^q} \longrightarrow \omega^q_{Y|D,\log} \longrightarrow 0,$$

where $\mathfrak{K}^q := \frac{\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{q-1}_{Z_1|\mathscr{D}'_1}}{\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{q-1}_{Z_1|\mathscr{D}_1}}$.

(3) Suppose $ep(r-q)/(p-1) < m < ep(r-q+1)/(p-1)$. Then

(a) If $p \nmid m$, there exists an exact sequence

$$0 \longrightarrow \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}} \longrightarrow \mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(s_1(r)_{X|D})\big) \longrightarrow \frac{\omega^{q-1}_{Y|D}}{B^{q-1}_{Y|D}} \longrightarrow 0.$$

(b) If $p|m$, there exists an exact sequence

$$0 \longrightarrow \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}} \longrightarrow \frac{\mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(s_1(r)_{X|D})\big)}{\mathfrak{K}^q} \longrightarrow \frac{\omega^{q-1}_{Y|D}}{Z^{q-1}_{Y|D}} \longrightarrow 0.$$

Proof. We immediately obtain this Proposition from Lemma 4.2, Lemma 5.1 and Lemma 5.2. □

For integers $0 \leq q \leq r < p-2$, we define the filtration $\tilde{\mathfrak{U}}^m$ on $\mathcal{H}^q(s_1(r)_{X|D})$ to be the image of $\mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(s_1(r)_{X|D})\big)$. By the same argument as in proposition A6 in [Tsu3], we have

$$(*2) \quad \mathrm{gr}^m_{\tilde{\mathfrak{U}}}\big(\mathcal{H}^q(s_1(r)_{X|D})\big) \cong \mathcal{H}^q\big(\mathrm{gr}^m_{\tilde{\mathfrak{U}}}(s_1(r)_{X|D})\big),$$

$$(*3) \quad \tilde{\mathfrak{U}}^{ep}\mathcal{H}^q(s_1(r)_{X|D} = 0.$$

For $x \in \tilde{\mathfrak{U}}^m\mathcal{H}^q(s_1(r)_{X|D})$ and $x' \in \tilde{\mathfrak{U}}^{m'}\mathcal{H}^{q'}(s_1(r')_{X|D})$, where $m, m', q, q' \geq 0$ and $0 \leq r, r', r+r' \leq p-2$, the product $x \cdot x'$ is contained in $\tilde{\mathfrak{U}}^{m+m'}\mathcal{H}^{q+q'}(s_1(r+r')_{X|D})$. By Proposition 5.3 and $(*2)$, for

each integer $0 \leq r \leq p-2$, we have an isomorphism

$$(*4) \quad \mathcal{H}^0(s_1(r)_{X|D}) \xleftarrow{\cong} \tilde{\mathfrak{U}}^{epr/(p-1)}\mathcal{H}^0(s_1(r)_{X|D}) \xrightarrow{\cong} \mathrm{gr}_{\tilde{\mathfrak{U}}}^{epr/(p-1)}\mathcal{H}^0(s_1(r)_{X|D}) \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z}.$$

Here the last isomorphism, we use $(*1)$ with $q=0, \theta = r$.

Definition 5.4. *We define a filtrations on $\mathcal{H}^q\big(s_1(r)_{X|D}\big)$ as follows:*

$$\mathcal{U}^m\mathcal{H}^q\big(s_1(r)_{X|D}\big) := \textit{the image of } \mathbb{Z}/p\mathbb{Z} \otimes U^m\mathcal{H}^q\big(s_1(q)_{X|D}\big) \textit{ under the product morphism}$$

$$\mathbb{Z}/p\mathbb{Z} \otimes \mathcal{H}^q\big(s_1(q)_{X|D}\big) \longrightarrow \mathcal{H}^q\big(s_1(r)_{X|D}\big),$$

$$\mathcal{V}^m\mathcal{H}^q\big(s_1(r)_{X|D}\big) := \textit{the image of } \mathbb{Z}/p\mathbb{Z} \otimes V^m\mathcal{H}^q\big(s_1(q)_{X|D}\big) \textit{ under the product morphism}$$

$$\mathbb{Z}/p\mathbb{Z} \otimes \mathcal{H}^q\big(s_1(q)_{X|D}\big) \longrightarrow \mathcal{H}^q\big(s_1(r)_{X|D}\big).$$

As in Lemma 4.4, we see that the image of $U^m((1+I_{D_2})^\times \otimes (M_{X_2}^{gp})^{\otimes(q-1)})$ $(m \in \mathbb{N})$ under the symbol map is contained in $\tilde{\mathfrak{U}}^m\mathcal{H}^q(s_1(r)_{X|D})$. Hence we have a homomorphism

$$\mathrm{gr}_{\mathcal{U}}^m\big(\mathcal{H}^q(s_1(r)_{X|D})\big) \longrightarrow \mathrm{gr}_{\tilde{\mathfrak{U}}}^{ep(r-q)/(p-1)+m}\big(\mathcal{H}^q(s_1(r)_{X|D})\big)$$

by using $(*4)$. Put

$$\mathrm{gr}_0^m\mathcal{H}^q(s_1(r)_{X|D}) := \mathcal{U}^m\mathcal{H}^q(s_1(r)_{X|D})/\mathcal{V}^m\mathcal{H}^q(s_1(r)_{X|D}), \tag{5.6}$$

$$\mathrm{gr}_1^m\mathcal{H}^q(s_1(r)_{X|D}) := \mathcal{V}^m\mathcal{H}^q(s_1(r)_{X|D})/\mathcal{U}^{m+1}\mathcal{H}^q(s_1(r)_{X|D}). \tag{5.7}$$

Proposition 5.5. *Let m be a non-negative integer. Let $x \in (1+I_{\mathscr{D}_2})^\times$, let $a_1, \dots a_{q-1} \in M_{Z_2}^{gp}$ and let $y \in \mathscr{O}_{Z_2}(-\mathscr{D}_2)$. Let $\overline{x}$ denote the image of x in $(1+I_{D_2})^\times$, let $\overline{a_i}$ denote the image of a_i in $M_{X_2}^{gp}$ and let $\overline{y}$ denote the image of y in $\mathscr{O}_{X_2}(-D_2)$. Then we have:*

(1) If $m=0$, the image of

$$\overline{x} \otimes \overline{a_1} \otimes \dots \overline{a_{q-1}} \in (1+I_{D_2})^\times \otimes (M_{X_2}^{gp})^{\otimes(q-1)}$$

under the composite

$$\begin{aligned}(1+I_{D_2})^\times \otimes (M_{X_2}^{gp})^{\otimes(q-1)} &\longrightarrow \mathrm{gr}_{\mathcal{U}}^0\big(\mathcal{H}^q(s_1(r)_{X|D})\big) \\ &\longrightarrow \mathrm{gr}_{\tilde{\mathfrak{U}}}^{ep(r-q)/(p-1)}\big(\mathcal{H}^q(s_1(r)_{X|D})\big) \\ &\twoheadrightarrow \mathrm{Ker}\Big(Z^q\Big(\left(\frac{\mathscr{O}_{Z_1}}{T\mathscr{O}_{Z_1}}\right) \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big) \xrightarrow{1-a_0^{p(r-q)}\cdot C^{-1}} \mathcal{H}^q\Big(\left(\frac{\mathscr{O}_{Z_1}}{T\mathscr{O}_{Z_1}}\right) \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\Big)\Big)\end{aligned}$$

is $b_0^{-p(r-q)} d\log(x) \wedge d\log(a_1) \wedge \cdots \wedge d\log(a_{q-1})$.

By Proposition 5.3, $(*1)$ and $(*2)$, we get an exact sequence :

$$0 \longrightarrow \omega_{Y|D,\log}^{q-1} \longrightarrow \frac{\mathrm{gr}_{\tilde{\mathfrak{U}}}^{ep(r-q)/(p-1)}\mathcal{H}^q\big(s_1(r)_{X|D}\big)}{\mathfrak{K}^q} \longrightarrow \omega_{Y|D,\log}^{q} \longrightarrow 0,$$

(2) Suppose $1 \leq m < ep/(p-1)$. *If* $p \nmid m$, *the image of*

$$1 + \pi^m \overline{y} \otimes \overline{a_1} \otimes \dots \overline{a_{q-1}} \in U^m\Big((1+I_{D_2})^\times \otimes (M_{X_2}^{gp})^{\otimes(q-1)}\Big)$$

under the composite

$$U^m\Big((1+I_{D_2})^\times \otimes (M_{X_2}^{gp})^{\otimes(q-1)}\Big) \longrightarrow \mathrm{gr}^m_{\mathcal{U}}\big(\mathcal{H}^q(s_1(r)_{X|D})\big)$$
$$\longrightarrow \mathrm{gr}^{ep(r-q)/(p-1)+m}_{\tilde{\mathfrak{U}}}(\mathcal{H}^q(s_1(r)_{X|D})) \xrightarrow{\cong} B^q\left(\frac{T^{ep(r-q)/(p-1)+m}\mathscr{O}_{Z_1}}{T^{ep(r-q)/(p-1)+m+1}\mathscr{O}_{Z_1}} \otimes \omega_{Z_1|\mathscr{D}_1}\right)$$

is $d\Big(T^{ep(r-q)/(p-1)+m} b_0^{-p(r-q)} y \cdot d\log(x) \wedge d\log(a_1) \wedge \cdots \wedge d\log(a_{q-1})\Big)$.

If $p \nmid m$ *(resp.* $p|m$*), by Proposition 5.3 and* $(*2)$*, we get an exact sequence:*

$$0 \longrightarrow \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}} \longrightarrow \mathrm{gr}^{ep(r-q)/(p-1)+m}_{\tilde{\mathfrak{U}}}\mathcal{H}^q\big(s_1(r)_{X|D}\big) \longrightarrow \frac{\omega^{q-1}_{Y|D}}{B^{q-1}_{Y|D}} \longrightarrow 0$$

$$(\ \textit{resp.}\quad 0 \longrightarrow \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}} \longrightarrow \frac{\mathrm{gr}^{ep(r-q)/(p-1)+m}_{\tilde{\mathfrak{U}}}\mathcal{H}^q\big(s_1(r)_{X|D}\big)}{\mathfrak{K}^q} \longrightarrow \frac{\omega^{q-1}_{Y|D}}{Z^{q-1}_{Y|D}} \longrightarrow 0 \quad).$$

Proof. We put $m_0 := ep(r-q)/(p-1)$ and denote by c the image of $1 \in \mathbb{Z}/p\mathbb{Z}$ in

$$\tilde{\mathfrak{U}}^{m_0}\mathcal{H}^0(s_1(r-q)_{X|D}) = \big(\tilde{\mathfrak{U}}^{m_0} J^{[r-q]}_{\mathscr{E}_1} \otimes \mathscr{O}_{Z_1}(-\mathscr{D}_1)\big)^{\varphi_{r-q}=1,\nabla=0}$$

under $(*4)$. Then we have $c \equiv T^{m_0} b_0^{-p(r-q)}(\mod \tilde{\mathfrak{U}}^{m_0+1} J^{[r-q]}_{\mathscr{E}_1} \otimes \mathscr{O}_{Z_1}(-\mathscr{D}_1))$. By using this fact, Lemma 4.2, Lemma 4.3, Lemma 5.1 and Lemma 5.2, we obtain this Proposition. □

Corollaly 5.6. *If* K *contains a primitive* p*-th roots of unity, for any integer* q *and* r *such that* $0 \leq q \leq r \leq p-2$, *the homomorphism*

$$\mathcal{H}^0(s_1(r-q)_{X|D}) \otimes \mathcal{H}^q(s_1(q)_{X|D}) \longrightarrow \mathcal{H}^q(s_1(r)_{X|D})$$

induced by the product structure is an isomorphism.

Proof. Applying Proposition 5.5 to $\mathcal{H}^q(s_1(r)_{X|D})$ and $\mathcal{H}^q(s_1(q)_{X|D})$ and using $(*4)$, we can verify that

$$\mathcal{H}^0(s_1(r-q)_{X|D}) \otimes \mathcal{H}^q(s_1(q)_{X|D}) \longrightarrow \mathcal{H}^q(s_1(r)_{X|D})$$

induces an isomorphism

$$\mathcal{H}^0(s_1(r-q)_{X|D}) \otimes \mathrm{gr}^m_{\tilde{\mathfrak{U}}}\mathcal{H}^q(s_1(q)_{X|D}) \xrightarrow{\cong} \mathrm{gr}^{ep(r-q)/(p-1)+m}_{\tilde{\mathfrak{U}}}\mathcal{H}^q(s_1(r)_{X|D})$$

for every non-negative integer m. □

Corollaly 5.7. *Let* e *be the absolute ramification index of* K. *Then the sheaf* $\mathcal{H}^q\big(s_1(r)_{X|D}\big)$ *has the folllowing structure:*

(1) For $m = 0$, we have short exact sequences:

$$0 \longrightarrow \frac{\mathfrak{R}}{\mathfrak{R} \cap \mathrm{gr}_1^0 \mathcal{H}^q\big(s_1(r)_{X|D}\big)} \longrightarrow \mathrm{gr}_0^0 \mathcal{H}^q\big(s_1(r)_{X|D}\big) \longrightarrow \omega^q_{Y|D,\log} \longrightarrow 0,$$
$$\{x, a_1, \ldots, a_{q-1}\} \mapsto d\log \overline{x} \wedge d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_q}$$

Here $x \in (1 + I_{D_2})^{\times}$, $a_1, \ldots, a_{q-1} \in M_{X_2}^{gp}$ and $y \in \mathscr{O}_{X_2}(-D_2)$. We denote $\overline{x}$ (resp. $\overline{a_i}$) *the image of x* (resp. a_i) *in M_Y^{gp}, and we denote $\overline{y}$ the image of y in $\mathscr{O}_Y(-D_s)$.*

$$0 \longrightarrow \mathfrak{R} \cap \mathrm{gr}_1^0 \mathcal{H}^q\big(s_1(r)_{X|D}\big) \longrightarrow \mathrm{gr}_1^0 \mathcal{H}^q\big(s_1(r)_{X|D}\big) \longrightarrow \omega^{q-1}_{Y|D,\log} \longrightarrow 0,$$
$$\{x, a_1, \ldots, a_{q-2}, \pi\} \mapsto d\log \overline{x} \wedge d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_{q-2}}$$

where

$$\mathfrak{R} := \mathrm{Ker}\Big(\mathrm{gr}^0_{\mathcal{U}}\big(\mathcal{H}^q(s_1(r)_{X|D})\big) \to \mathrm{Ker}\big(Z^q\big(\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big) \xrightarrow{1-\varphi\otimes\wedge^q d\varphi/p} \mathcal{H}^q\big(\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{\cdot}_{Z_1|\mathscr{D}_1}\big)\big)\Big).$$

(2) If $0 < m < pe/(p-1)$ and $p \nmid m$, then we have

$$\mathrm{gr}_0^m \mathcal{H}^q\big(s_1(r)_{X|D}\big) \cong \frac{\omega^{q-1}_{Y|D}}{B^{q-1}_{Y|D}}$$
$$\{1 + \pi^m y, a_1, \ldots, a_{q-1}\} \mapsto b_0^{-p(r-q)} \overline{y} d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_q}$$
$$\mathrm{gr}_1^m \mathcal{H}^q\big(s_1(r)_{X|D}\big) \cong \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}}$$
$$\{1 + \pi^m y, a_1, \ldots, a_{q-2}, \pi\} \mapsto b_0^{-p(r-q)} \overline{y} d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_{q-2}}$$

(3) If $0 < m < pe/(p-1)$ and $p|m$, then we have short exact sequences

$$0 \longrightarrow \frac{\mathfrak{L}}{\mathfrak{L} \cap \mathcal{H}^q\big(s_1(r)_{X|D}\big)} \longrightarrow \mathrm{gr}_0^m \mathcal{H}^q\big(s_1(r)_{X|D}\big) \longrightarrow \frac{\omega^{q-1}_{Y|D}}{Z^{q-1}_{Y|D}} \to 0,$$
$$\{1+\pi^m y, a_1, \ldots, a_{q-1}\} \mapsto b_0^{-p(r-q)} \overline{y} d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_q}$$

$$0 \longrightarrow \mathfrak{L} \cap \mathcal{H}^q\big(s_1(r)_{X|D}\big) \longrightarrow \mathrm{gr}_1^m \mathcal{H}^q\big(s_1(r)_{X|D}\big) \longrightarrow \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}} \longrightarrow 0,$$
$$\{1+\pi^m y, a_1, \ldots, a_{q-2}, \pi\} \mapsto b_0^{-p(r-q)} \overline{y} d\log \overline{a_1} \wedge \cdots \wedge d\log \overline{a_{q-2}}$$

where $\mathfrak{L} := \mathrm{Ker}\Big(\mathrm{gr}^m_{\mathcal{U}}\big(\mathcal{H}^q(s_1(r)_{X|D})\big) \longrightarrow B^q\Big(\frac{T^{ep(r-q)/(p-1)+m}\mathscr{O}_{Z_1}}{T^{ep(r-q)/(p-1)+m+1}\mathscr{O}_{Z_1}} \otimes \omega_{Z_1|\mathscr{D}_1}\Big)\Big)$.

(4) If $m \geq pe/(p-1)$, $\mathcal{U}^m \mathcal{H}^q\big(s_1(r)_{X|D}\big) = 0$.

Proof. By Proposition 5.5, we obtain (1), (2) and (3). Since $U^{pe}\mathcal{H}^q(s_1(q)_{X|D})\big) = 0$ by Lemma 4.4 and Corollary 4.6 , this implies (4). This completes the proof of this Proposition. □

Next we do not assume that K contains a primitive p-th root of unity. Let $\mathscr{O}_{K'}$ be a totally ramified extension of $\mathscr{O}_K$ of degree d. We denote (S', N') the scheme Spec$\mathscr{O}_{K'}$ with the log structure defined by the closed point. Assume that there exists a prime π' of $\mathscr{O}_{K'}$ such that $\pi'^d = \pi$. We choose such a prime π'. Let $(V', M_{V'})$ be the scheme $\mathrm{Spec}(W[\mathbb{N}]) = \mathrm{Spec}(W[T'])$ endowed with the log structure associated to the inclusion $\mathbb{N} \hookrightarrow W[\mathbb{N}]$. We define the exact closed immersion $i_{V'_n} : (S'_n, M_{S'_n}) \hookrightarrow (V', M_{V'})$ in the same way as i_{V_n}, by using π' (see the argument before Lemma 4.1). We have a cartesian diagram:

$$\begin{array}{ccc} (S'_n, M_{S'_n}) & \longrightarrow & (V'_n, M_{V'_n}) \\ \downarrow & \square & \downarrow (\clubsuit) \\ (S_n, N_n) & \longrightarrow & (V_n, M_{V_n}), \end{array}$$

where the morphism ($\clubsuit$) is defined by the multiplication by d on $\mathbb{N}$. We define $(X', M_{X'}) := (X, M_X) \times_{(S,N)} (S', N')$, $\tilde{D} := D \times_S S'$, and denote $\tilde{\mathscr{D}}_n$, $(Z'_n, M_{Z'_n})$ and $\{F_{Z'_n}\}$ the base changes of $\mathscr{D}_n$, (Z_n, M_{Z_n}) and $\{F_{Z_n}\}$ under the morphism ($\clubsuit$) above. Then one can apply the above arguments to $\mathscr{O}_{K'}$, π', $(X', M_{X'})$, $(Z'_n, M_{Z'_n})$ and $\{F_{Z'_n}\}$. We denote by $'$ the corresponding things. Since $(Y', M_{Y'}) := (Y, M_Y) \times_{(s,M_s)} (s', M'_s)$, $s = s'$ and $Y' = Y$, then we have $\omega^{\cdot}_{Y|D_s} \xrightarrow{\cong} \omega^{\cdot}_{Y'|\tilde{D}_s}$. Thus we obtain the following relations of the filtrations $\tilde{\mathfrak{U}}^m$ on $\mathcal{H}^q(s_1(r)_{X|D})$ and $\mathcal{H}^q(s_1(r)'_{X|D})$ from Proposition 5.3 and $(*2)$:

Lemma 5.8. *(cf.* [Tsu3]*, Lemma A18) Let r and q be are integers such that $0 \leq q \leq r \leq p-2$. Then there exists a canonical morphism*

$$\mathcal{H}^q(s_1(r)_{X|D}) \longrightarrow \mathcal{H}^q(s_1(r)'_{X|D})$$

sends $\tilde{\mathfrak{U}}^m$ into $\tilde{\mathfrak{U}}^{dm}$ for $m \in \mathbb{N}$. If $ep(r-q)/(p-1) \leq m < ep(r-q+1)/(p-1)$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{W}_1 & \longrightarrow & \frac{\mathrm{gr}^m_{\tilde{\mathfrak{U}}} \mathcal{H}^q\left(s_1(r)_{X|D}\right)}{\mathfrak{K}} & \longrightarrow & \mathfrak{W}_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d \cdot id_{\mathfrak{W}}} & & \downarrow & & \downarrow{\scriptstyle pr.} & & \\ 0 & \longrightarrow & \mathfrak{W}_1 & \longrightarrow & \frac{\mathrm{gr}^{dm}_{\tilde{\mathfrak{U}}} \mathcal{H}^q\left(s_1(r)'_{X|D}\right)}{\mathfrak{K}'} & \longrightarrow & \mathfrak{W}'_2 & \longrightarrow & 0, \end{array}$$

where

$$\mathfrak{W}_1 := \mathrm{Ker}\big(1 - a_0^{p(r-q)} C^{-1} : Z^{q-1}_{Y|D} \longrightarrow \mathcal{H}^{q-1}(\omega^{\cdot}_{Y|D})\big) \quad (resp.\ \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}},\ resp.\ \frac{\omega^{q-2}_{Y|D}}{Z^{q-2}_{Y|D}}),$$

$$\mathfrak{W}_2 := \mathrm{Ker}\big(1 - a_0^{p(r-q)}C^{-1} : Z^q_{Y|D} \longrightarrow \mathcal{H}^q(\omega^{\cdot}_{Y|D})\big) \quad (resp.\ \frac{\omega^{q-1}_{Y|D}}{Z^{q-1}_{Y|D}},\ resp.\ \frac{\omega^{q-1}_{Y|D}}{Z^{q-1}_{Y|D}}),$$

if

$$m = ep(r-q)/(p-1) \quad (resp.\ m > ep(r-q)/(p-1),\ p \not| m,\ resp.\ m > ep(r-q)/(p-1),\ p|m),$$

$$\mathfrak{W}'_2 := \frac{\omega^{q-1}_{Y|D}}{Z^{q-1}_{Y|D}} \quad if\ m > ep(r-q)/(p-1),\ p|dm,$$

and $\mathfrak{W}'_2 = \mathfrak{W}_2$ *otherwise,*

$$\mathfrak{K} := \mathfrak{K}^q \quad (resp.\ 0,\ resp.\ \mathfrak{K}^q)$$

$$if\ m = ep(r-q)/(p-1) \quad (resp.\ m > ep(r-q)/(p-1),\ p \not| m,\ resp.\ m > ep(r-q)/(p-1),\ p|m),$$

$$\mathfrak{K}' := \mathfrak{K}'^q \quad (resp.\ 0,\ resp.\ \mathfrak{K}'^q)$$

$$if\ m = ep(r-q)/(p-1) \quad (resp.\ m > ep(r-q)/(p-1),\ p \not| dm,\ resp.\ m > ep(r-q)/(p-1),\ p|dm).$$

Here $\mathfrak{K}^q := \frac{\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{q-1}_{Z_1|\mathscr{D}'_1}}{\mathscr{O}_Y \otimes_{\mathscr{O}_{Z_1}} \omega^{q-1}_{Z_1|\mathscr{D}_1}}$, $\mathfrak{K}'^q := \frac{\mathscr{O}_Y \otimes_{\mathscr{O}_{Z'_1}} \omega^{q-1}_{Z'_1|\mathscr{D}'_1}}{\mathscr{O}_Y \otimes_{\mathscr{O}_{Z'_1}} \omega^{q-1}_{Z'_1|\mathscr{D}_1}}$. *We denote pr. the canoical projection or the identity. If* K' *is tamely ramified filed over* K*, we have an isomorphism:*

$$\frac{\mathrm{gr}^m_{\tilde{\mathfrak{U}}}\mathcal{H}^q\big(s_1(r)_{X|D}\big)}{\mathfrak{K}} \xrightarrow{\cong} \frac{\mathrm{gr}^{dm}_{\tilde{\mathfrak{U}}}\mathcal{H}^q\big(s_1(r)'_{X|D}\big)}{\mathfrak{K}'},$$

Proof. The first claim is trivial by $T = T'^d$. The second claim follows from $d\log T = d' \cdot d\log T'$. □

Corollaly 5.9. *If* $0 \leq m \leq ep/(p-1)$ *and* K' *is tamely ramified filed over* K*, the kenel and the cokernel of*

$$\mathrm{gr}^m_{\tilde{\mathfrak{U}}}\mathcal{H}^q\big(s_1(r)_{X|D}\big) \longrightarrow \mathrm{gr}^{dm}_{\tilde{\mathfrak{U}}}\mathcal{H}^q\big(s_1(r)'_{X|D}\big)$$

are Mittag-Leffler zero with respect to the multiplicities of the prime components of D*.*

Proof. We consider a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{K} & \longrightarrow & \mathrm{gr}^m_{\tilde{\mathfrak{U}}}\mathcal{H}^q\big(s_1(r)_{X|D}\big) & \longrightarrow & \frac{\mathrm{gr}^m_{\tilde{\mathfrak{U}}}\mathcal{H}^q\left(s_1(r)_{X|D}\right)}{\mathfrak{K}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{K}' & \longrightarrow & \mathrm{gr}^{dm}_{\tilde{\mathfrak{U}}}\mathcal{H}^q\big(s_1(r)'_{X|D}\big) & \longrightarrow & \frac{\mathrm{gr}^{dm}_{\tilde{\mathfrak{U}}}\mathcal{H}^q\left(s_1(r)'_{X|D}\right)}{\mathfrak{K}'} & \longrightarrow & 0.
\end{array}$$

From Lemma 5.8, the right vertical arrow is isomorphism. The kernel and the cokernel of the left vertical arrow are Mittag-Leffler zero. Thus we obtain the claim. □

By the same arguments as in Lemma 4.11 and Lemma 4.12, we have the following Lemma:

Lemma 5.10. *The kernel and the cokernel of the morphism*

$$\mathrm{gr}^m_{\mathcal{U}}\mathcal{H}^q\big(s_1(r)_{X|D}\big) \to \mathrm{gr}^{m+ep(r-q)/(p-1)}_{\tilde{\mathcal{U}}}\mathcal{H}^q\big(s_1(r)_{X|D}\big)$$

are Mittag-Leffler zero with respect to the multiplicities of the prime components of D.

6. Acknowledgements

I am thankful to Professor Kanetomo Sato for his great help and discussion. This problem was suggested to me by him.

References

[Ber] Berthelot., U., Cohomologie cristalline des schemas de caract éristiqué $p > 0$. Lecture Notes in Mathematics, Vol. 407, SpringerVerlag, Berlin (1974)

[JSZ] Jannsen., U., Saito., S., Zhao.,Y.: Duality for relative logarithmic de Rham-Witt sheaves and wildly ramified class field theory over finite fields. Compositio Math. 154, 1306–1331 (2018)

[Ka1] Kato, K.: On p-adic vanishing cycles (application of ideas of Fontaine-Messing). Advan. Stud. Pure Math., t. 10, 207–251 (1987)

[Ka2] Kato, K.: Semi-stable reduction and p-adic étale cohomology. Périodes p-adiques. Séminaire de Bures. 1988, Astérisque, t. 223, 269–293 (1994)

[KSY] Kahn., B., Saito., S., and Yamazaki., T.: Motives of modulus. arXiv preprint http://arxiv.org/abs/1511.07124 (2015).

[Ku] Kurihara, M.: A note on p-adic étale cohomology. Proc. Japan Academy 63, 275–278 (1987) Trans. AMS. 370, 987–1043 (2018)

[Sat] Sato, K.: p-adic étale Tate twists and arithmetic duality(with an appendix by Hagihara, K.). Ann. Sci. Ećole. Norm. Sup. (4) 40, 519–588 (2007)

[SS] Saito, S., Sato, K.: On p-adic vanishing cycles of log smooth families. arXiv:1807.11128

[Tsu1] Tsuji, T.: Syntomic complexes and p-adic vanishing cycles. J. reine angew. Math., t.472, 69-138 (1996)

[Tsu2] Tsuji, T.: p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. Invent. Math., t.137, 233–411 (1999)

[Tsu3] Tsuji, T.: On p-adic nearby cycles of log smooth families.Bull. Soc. Math. France. 128, 529–575 (2000)

[Y] Yamamoto, K.: Duality for p-adic étale Tate twists with modulus, in preparation.

Department of Mathematics,Chuo University 1-13-27 Kasuga, Bunkyo-Ku, Tokyo 112-8551, Japan
Email address: k-yamamo@gug.math.chuo-u.ac.jp