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Étale cohomology of arithmetic schemes and zeta values of arithmetic surfaces

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1 Introduction

Let K be an algebraic number field, and let O_K be its integer ring. Let X be a regular scheme which is proper flat over $B := \mathrm{Spec}(O_K)$, and such that $X_K = X \otimes_{O_K} K$ is geometrically connected over K. We fix a prime number p, and assume that

($\star$) *X has good or log smooth reduction at all places v of K dividing p.*

In this paper, we give an approach to the values of the zeta function of X at integers $r \geqq \dim(X)$ using étale cohomology of X with $\mathbb{Q}_p(r)$ and $\mathbb{Z}_p(r)$-coefficients, cf. [KCT], [Mo], [FM], [FS].

1.1 Selmer groups

Let $H^1_f(K, V^i(r))$ be the Selmer group of Bloch-Kato associated with the p-adic Galois representation $V^i(r) := H^i(X_{\overline{K}}, \mathbb{Q}_p(r))$. The first aim of this paper is to relate this group with the étale cohomology group $H^{i+1}(X, \mathbb{Q}_p(r))$, assuming that $r \geqq d := \dim(X)$. Here $H^*(X, \mathbb{Q}_p(r))$ is defined as

$$H^*(X, \mathbb{Q}_p(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geqq 1} H^*(X, \mathfrak{T}_n(r))$$

and $\mathfrak{T}_n(r)$ $(n \geqq 1)$ denotes the complex of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X introduced in [SH] under the assumption that X has good or semi-stable reduction at all places v dividing p (see also [JSS] for the case $r = d$). See §2 below for details on this object under the setting of this paper.

Theorem 1.1 *Assume $r \geqq d$. Then we have*

$$H^{i+1}(X, \mathbb{Q}_p(r)) \cong \begin{cases} \mathbb{Q}_p & ((i,r) = (2d,d)), \\ H^1_f(K, V^i(r)) & (\textit{otherwise}). \end{cases}$$

The key idea of this result is as follows. By a duality result of Jannsen-Saito-Sato [JSS] and the adjunction between $R\pi_{X/B!}(=R\pi_{X/B*})$ and $R\pi^!_{X/B}$, we have

$$R\pi_{X/B*}\mathfrak{T}_n(r)_X \cong R\mathscr{H}om_{B,\mathbb{Z}/p^n\mathbb{Z}}(R\pi_{X/B!}\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B)[2-2d] \qquad (1.1.1)$$

in $D^+(B_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ (see Lemma 3.1 below), where the assumption $r \geqq d$ is crucial and $\mathfrak{T}_n(d-r)_X$ is a constructible sheaf placed in degree 0 by definition. Using this fact, we introduce the following complexes:

$$\mathfrak{H}^{\geqq i}(X, \mathfrak{T}_n(r)) := R\mathscr{H}om_{B,\mathbb{Z}/p^n\mathbb{Z}}(\tau_{\leqq 2d-2-i}R\pi_{X/B!}\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B)[2-2d],$$

$$\mathfrak{H}^{i}(X, \mathfrak{T}_n(d)) := R\mathscr{H}om_{B,\mathbb{Z}/p^n\mathbb{Z}}(R^{2d-2-i}\pi_{X/B!}\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B).$$

By the proper base change theorem for $R\pi_{X/B!}$, we have

$$\mathfrak{H}^i(X, \mathfrak{T}_n(r)) = 0 \quad \text{unless} \quad 0 \leqq i \leqq 2d-2,$$

and the filtration $\{\mathfrak{H}^{\geqq i}(X, \mathfrak{T}_n(r))\}_i$ on the right hand side of (1.1.1) yields a convergent spectral sequence

$$E_2^{a,i} = H^a(B, \mathfrak{H}^i(X, \mathfrak{T}_n(r))) \Longrightarrow H^{a+i}(X, \mathfrak{T}_n(r)).$$

The E_2-terms of this spectral sequence are finite (see Proposition 4.1 below), and we obtain the following spectral sequence of finite-dimensional $\mathbb{Q}_p$-vector spaces:

$$E_2^{a,i} = H^a(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) \Longrightarrow H^{a+i}(X, \mathbb{Q}_p(r)), \qquad (1.1.2)$$

where

$$H^a(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geqq 1} H^a(B, \mathfrak{H}^i(X, \mathfrak{T}_n(r))).$$

Concerning the spectral sequence (1.1.2), we will prove

Theorem 1.2 (§§5–6) *Assume $r \geqq d$. Then the $\mathbb{Q}_p$-vector space $E_2^{a,i}$ is zero, unless $a = 1$ or $(a, i, r) = (3, 2d-2, d)$. Consequently, the spectral sequence* (1.1.2) *degenerates at E_2-terms. Moreover, we have*

$$E_2^{1,i} \cong H^1_f(K, V^i(r))$$

for any i, which is zero unless $0 \leqq i \leqq 2d-2$.

Theorem 1.1 is a direct consequence of this result. An important point of Theorem 1.2 is the vanishing of $E_2^{2,i}$ for any i, which we will prove by computing the cohomology of all local integer rings with $\mathfrak{H}^i(X, \mathbb{Q}_p(r))$-coefficients and by a local-global argument using a Hasse principle of Jannsen [J] p. 337, Theorem 3 (c). As a consequence of the vanishing of $E_2^{2,i}$ (and $E_2^{3,i}$ with $(i, r) \neq (2d-2, d)$), we will obtain the following result (cf. [J] p. 349, Question 2):

Corollary 1.3 (Corollary 6.10 (2)) *Let S be a finite set of places of K including all places which divide $p \cdot \infty$ or where X has bad reduction. Assume $r \geqq d$. Then the restriction map*

$$H^2(G_S, V^i(r)) \longrightarrow \bigoplus_{v \in S} H^2(K_v, V^i(r))$$

is bijective for any $(i, r) \neq (2d-2, d)$, and injective for $(i, r) = (2d-2, d)$. In particular, if $r > d$ or X_K has potentially good reduction at all finite places of K, then

$$H^2(G_S, V^i(r)) = 0 \quad \textit{for any } (i, r) \neq (2d-2, d).$$

1.2 p-Tate-Shafarevich groups $(d = 2)$

We assume that X is an arithmetic surface, i.e., $d = 2$ in what follows. Put $T^i := H^i(X_{\overline{K}}, \mathbb{Z}_p)$. In their paper [BK2] §5, Bloch and Kato introduced a homomorphism

$$\alpha^{i,r} : \frac{H^1(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))}{H^1_f(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))} \longrightarrow \bigoplus_{v \in P} \frac{H^1(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))}{H^1_f(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))}, \tag{1.2.1}$$

where P denotes the set of all places of K, and for each $v \in P$, K_v denotes the local field of K at v; $H^1_f(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))$ (resp. $H^1_f(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))$) denotes the image of $H^1_f(K, V^i(r))$ (resp. $H^1_f(K_v, V^i(r))$). The cokernel $\mathrm{Coker}(\alpha^{i,r})$ is finite and canonically isomorphic to the Pontryagin dual of $H^{2-i}(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(2-r))^{G_K}$, if $i - 2r \leqq -3$. They also proved that $\mathrm{Ker}(\alpha^{i,r}) =: Ш^{(p)}(H^i(X_K)(r))$, the p-Tate-Shafarevich group of the motive $H^i(X_{\overline{K}})(r)$, is finite for the same (i, r). The second main result of this paper compares the maps $\alpha^{i,r}$ with the p-adic Abel-Jacobi mappings

$$\mathrm{aj}^{i,r}_p : H^i_{\mathscr{M}}(X, \mathbb{Z}(r)) \mathbin{\widehat{\otimes}} \mathbb{Z}_p \longrightarrow H^1(B, \mathfrak{H}^{i-1}(X, \mathbb{Z}_p(r)))$$

assuming $r \geqq 2$. Here $H^*_{\mathscr{M}}(X, \mathbb{Z}(r))$ denotes the motivic cohomology of X (see §2.3 below), and for an abelian group M, $M \mathbin{\widehat{\otimes}} \mathbb{Z}_p$ denotes its p-adic completion $\varprojlim_n M/p^n$. We will calculate the above Abel-Jacobi mapping using the Merkur'ev-Suslin theorem [MS] and the Rost-Voevodsky theorem [V1], [V2], which together with Theorem 1.2 will play important roles in the following comparison formula:

Theorem 1.4 (§7) *Assume $r \geqq 2$, and that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$. Assume further that $H^3_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$, the p-primary torsion part of $H^3_{\mathscr{M}}(X, \mathbb{Z}(r))$, is finite. Let S' be the set of the places of K which divide p or where X has bad reduction. Then $\mathrm{aj}^{i,r}_p$ has finite kernel and cokernel for $i = 2, 3$, and we have*

$$\frac{\chi(\alpha^{1,2})}{\chi(\alpha^{0,2})} = \frac{\chi(\mathrm{aj}^{3,2}_p)}{\chi(\mathrm{aj}^{2,2}_p)} \cdot \frac{\#\mathrm{CH}_0(X)\{p\}}{\#\mathrm{Pic}(O_K)\{p\}} \cdot \prod_{v \in S'} \frac{e^{2,1,2}_v \cdot e^{3,0,2}_v}{e^{2,0,2}_v \cdot e^{3,1,2}_v} \qquad (r = 2)$$

$$\frac{\chi(\alpha^{1,r})}{\chi(\alpha^{0,r}) \cdot \chi(\alpha^{2,r})} = \frac{\chi(\mathrm{aj}^{3,r}_p)}{\chi(\mathrm{aj}^{2,r}_p)} \cdot \#H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \cdot \prod_{v \in S'} \frac{e^{2,1,r}_v \cdot e^{3,0,r}_v \cdot e^{3,2,r}_v}{e^{2,0,r}_v \cdot e^{2,2,r}_v \cdot e^{3,1,r}_v} \qquad (r \geqq 3),$$

where we put $\chi(f) := \#\mathrm{Coker}(f)/\#\mathrm{Ker}(f)$ for homomorphisms $f : M \to N$ of abelian groups with finite kernel and cokernel; for each $v \in S'$ and $a = 2, 3$, we put

$$e^{a,i,r}_v := \#H^a(B_v, \mathfrak{H}^i(X, \mathbb{Z}_p(r))), \quad B_v := \text{the completion of } B \text{ at } v.$$

See Corollary 5.6 (2) below for the finiteness of $e^{a,i,r}_v$.

The finiteness of $\mathrm{CH}_0(X)$ is due to Bloch [B1], Kato and Saito [KSa]. By the localization theorem of Levine [Le], $H^i_{\mathscr{M}}(X, \mathbb{Z}(r))$ is zero for any $i > r+2$ (see Lemma 7.1 (1) below). As natural extensions of these facts, we will prove that $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is finite for any $r \geqq 3$, and that $H^i_{\mathscr{M}}(X, \mathbb{Z}(r))$ is uniquely p-divisible for any $i \geqq 5$ and $r \geqq 3$, see Propositions 7.5 and 7.6 below. The formulas in Theorem 1.4 are based on these facts and results.

1.3 Zeta values modulo rational numbers prime to p $\quad (d = 2)$

Assuming a weak version of p-Tamagawa number conjecture (see Conjecture 9.1), we will relate the formulas in Theorem 1.4 with the residue or value at $s = r$ of the zeta function

$$\zeta(X,s) := \prod_{x \in X_0} \frac{1}{1 - q_x^{-s}} \qquad (q_x := \#\kappa(x)),$$

where the product on the right hand side runs through all closed points of X.

Theorem 1.5 (Proposition 9.3) *Assume $r \geqq 2$ and the following conditions*:

(i) $p \geqq r + 2$.

(ii) *For any $v \in B_0$ dividing p, v is absolutely unramified and X has good reduction at v.*

(iii) *A weak p-Tamagawa number conjecture (see Conjecture 9.1 below) holds for the motives $H^i(X_K)(r)$ with $i = 0, 1$ (resp. $i = 0, 1, 2$), if $r = 2$ (resp. $r \geqq 3$).*

Then $H^3_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is finite, and we have

$$\operatorname*{Res}_{s=2} \zeta(X,s) \equiv \operatorname*{Res}_{s=1} \zeta_K(s) \cdot \frac{\chi(\mathrm{aj}_p^{3,2}) \cdot \#\mathrm{CH}_0(X) \cdot R_\Phi^{0,2}}{\chi(\mathrm{aj}_p^{2,2}) \cdot \#\mathrm{Pic}(O_K) \cdot R_\Phi^{1,2}} \mod \mathbb{Z}_{(p)}^\times \qquad (r = 2)$$

$$\zeta(X,r) \equiv \frac{\chi(\mathrm{aj}_p^{3,r}) \cdot \#H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \cdot R_\Phi^{0,r} \cdot R_\Phi^{2,r}}{\chi(\mathrm{aj}_p^{2,r}) \cdot R_\Phi^{1,r}} \mod \mathbb{Z}_{(p)}^\times \qquad (r \geqq 3)$$

where $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at (p). See Conjecture 9.1 below for the definition of the number $R_\Phi^{i,r} \in \mathbb{R}^\times/\mathbb{Z}_{(p)}^\times$, which is a p-adic modification of the Beilinson regulator of the motive $H^i(X_K)(r)$.

This result is deduced from Theorem 1.4 and certain comparison results between the alternate products of local terms that appear in Theorem 1.4 with zeta values of the closed fibers of $X \to B$, see Theorems 8.5 and 8.6 below. The assumptions (i) and (ii) are essential in this comparison at present, while the reductions at the closed points $v \in B_0$ with $v \nmid p$ are arbitrary.

Example 1.6 Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the integer ring of an imaginary quadratic field. Let X be a regular model of E which is proper flat over $\mathbb{Z}$. Let p be a prime number $\geqq 5$ at which X has good reduction and which is regular for E ([So3] 3.3.1). Then we obtain a formula (without assuming any conjectures)

$$\operatorname*{Res}_{s=2} \zeta(X,s) \equiv \frac{\pi^2 \cdot \chi(\mathrm{aj}_p^{3,2}) \cdot \#\mathrm{CH}_0(X)}{6 \cdot \chi(\mathrm{aj}_p^{2,2}) \cdot R_\Phi^{1,2}} \mod \mathbb{Z}_{(p)}^\times$$

from Theorem 1.5 and results of Bloch and Kato [BK2] 6.3 (i), 7.4 (cf. [Ki1], [Ki2]). If we assume that $H^i_{\mathscr{M}}(X, \mathbb{Z}(2))$ is a finitely generated abelian group for $i = 2, 3$, then we have

$$\#H^3_{\mathscr{M}}(X, \mathbb{Z}(2)) < \infty, \qquad \operatorname{rank} H^2_{\mathscr{M}}(X, \mathbb{Z}(2)) = 1$$

by Theorem 1.2 (and Corollary 7.7 (2), (3) below), and obtain a stronger formula

$$\operatorname*{Res}_{s=2} \zeta(X,s) \equiv \frac{\pi^2 \cdot \#\mathrm{Ker}(\mathrm{reg}_{\mathscr{D}}^{2,2}) \cdot \#\mathrm{CH}_0(X)}{3 \cdot R_{\mathscr{M}}^{1,2} \cdot \# H^3_{\mathscr{M}}(X, \mathbb{Z}(2))} \mod \mathbb{Z}[T^{-1}]^{\times}$$

by [BK2] 7.4 and Theorem 9.6 below, where T denotes the set of prime numbers consisting of 2, 3 and the bad prime numbers for X; $\mathrm{reg}_{\mathscr{D}}^{2,2}$ denotes the regulator map to the real Deligne cohomology with $\mathbb{Z}(2)$-coefficients

$$\mathrm{reg}_{\mathscr{D}}^{2,2} : H^2_{\mathscr{M}}(X, \mathbb{Z}(2)) \longrightarrow H^2_{\mathscr{D}}(E_{/\mathbb{R}}, \mathbb{Z}(2)).$$

$R_{\mathscr{M}}^{1,2}$ denotes the volume of $\mathrm{Coker}(\mathrm{reg}_{\mathscr{D}}^{2,2})$ with respect to the same $\mathbb{Z}$-lattice of $H^1_{\mathrm{dR}}(E/\mathbb{Q})$ as used in the definition of $R_{\Phi}^{1,2}$.

Organization of this paper

In §2, we review the definition of the étale complexes $\mathfrak{T}_n(r)$ on $X_{\text{ét}}$ and establish their fundamental properties under the setting of this paper. In §3–§4, we further introduce the étale complexes $\mathfrak{H}^{\geqq i}(X, \mathfrak{T}_n(r))$ and $\mathfrak{H}^i(X, \mathfrak{T}_n(r))$ on $B_{\text{ét}}$ and prove some preliminary results on those new complexes. In §5–§6 we will prove Theorems 1.1 and 1.2. In §7, we will compute p-adic cycle class maps and p-adic Abel-Jacobi mappings assuming $d = 2$, and then prove the formulas in Theorem 1.4. In §8, we will relate the alternate product of local terms in Theorem 1.4 with zeta values of the fibers of $X \to B$. Finally in §9, we will relate the formulas in Theorem 1.4 with zeta values assuming a weak version of p-Tamagawa number conjecture.

1.4 Notation

Throughout this paper, we fix a prime number p, and put $\varLambda_n := \mathbb{Z}/p^n\mathbb{Z}$.

If p is *invertible* on a scheme X, we write $\mu_{p^n} = \mu_{p^n,X}$ ($n \geqq 1$) for the étale sheaf of p^n-th roots of unity on X, and define a $\varLambda_n$-sheaf $\varLambda_n(r) = \varLambda_n(r)_X$ ($r \in \mathbb{Z}$) on $X_{\text{ét}}$ as

$$\varLambda_n(r) := \begin{cases} \mu_{p^n}^{\otimes r} & (r \geqq 1) \\ \varLambda_n & (r = 0) \\ \mathscr{H}om(\varLambda_n(-r), \varLambda_n) & (r < 0). \end{cases} \tag{1.4.1}$$

This notation will be useful mainly in the case that r is negative.

On the other hand, if X is *an $\mathbb{F}_p$-scheme*, then we write $W_n\Omega^r_{X,\log}$ ($r \geqq 0$, $n \geqq 1$) for the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf $W_n\Omega^r_X$ (see [Ill] I (1.12.1)). If $r < 0$, then we define $W_n\Omega^r_{X,\log}$ as the zero sheaf. If X is an equi-dimensional scheme which is of finite type over a field k of characteristic p, then we write $\nu^r_{X,n}$ for the sheaf on $X_{\text{ét}}$ defined as the kernel of Kato's boundary map [KCT]

$$\partial : \bigoplus_{x \in X^0} i_{x*} W_n\Omega^r_{x,\log} \longrightarrow \bigoplus_{x \in X^1} i_{x*} W_n\Omega^{r-1}_{x,\log},$$

where $i_x : x \to X$ denotes the canonical map for any $x \in X$. If X is smooth over k, then we have $\nu^r_{X,n} = W_n\Omega^r_{X,\log}$ by Gros-Suwa [GS].

Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology.

2 Étale coefficients

Let $\mathfrak{O}$ be a Dedekind ring whose fraction field K has characteristic 0, and let p be a prime number. We put

$$B := \mathrm{Spec}(\mathfrak{O}), \qquad B[p^{-1}] := \mathrm{Spec}(\mathfrak{O}[p^{-1}]) \quad \text{and} \quad \Sigma := \mathrm{Spec}\Big(\mathfrak{O}/\sqrt{(p)}\Big).$$

Let X be a regular connected scheme which is separated, flat of finite type over $B = \mathrm{Spec}(\mathfrak{O})$. For a closed point $v \in B$, we put $B_v^{\mathrm{h}} := \mathrm{Spec}(\mathfrak{O}_v^{\mathrm{h}})$ and $Y_v := X \times_B v$, where $\mathfrak{O}_v^{\mathrm{h}}$ denotes the henselization of the local ring $\mathfrak{O}_v = \mathscr{O}_{B,v}$. Throughout this paper, we assume that X satisfies the following condition:

($\star_1$) *For any closed point $v \in B$ of characteristic p, $(Y_v)_{\mathrm{red}}$ has normal crossings on X and the morphism $X \times_B B_v^{\mathrm{h}} \to B_v^{\mathrm{h}}$ is log smooth with respect to the log structure on $X \times_B B_v^{\mathrm{h}}$ associated with $(Y_v)_{\mathrm{red}}$ and that on B_v^{h} associated with v.*

We write $\pi_{X/B} : X \to B$ for the structure morphism, and put $d := \dim(X)$, the absolute dimension of X. Let Y be the disjoint union of Y_v's for all closed points $v \in B$ with $\mathrm{ch}(v) = p$. Let j (resp. ι) be the open immersion $X[p^{-1}] \hookrightarrow X$ (resp. closed immersion $Y \hookrightarrow X$).

In this section, we define a family of complexes of étale sheaves $\{\mathfrak{T}_n(r)\}_{n \geqq 1, r \in \mathbb{Z}}$ on X and check several fundamental properties of them using the main results of [SS], which have been established in [SH] and [Sa2] in the case that X has semi-stable reduction at all $v \in B$ with $\mathrm{ch}(v) = p$. The coefficients $\{\mathfrak{T}_n(r)\}_{n,r}$ play key roles throughout this paper.

2.1 Étale complex $\mathfrak{T}_n(r)$

For $r \geqq 0$, we define $\mathfrak{T}_n(r) = \mathfrak{T}_n(r)_X \in D^b(X_{\text{ét}}, \varLambda_n)$ by the distinguished triangle

$$\iota_*\nu_{Y,n}^{r-1}[-r-1] \xrightarrow{\ g\ } \mathfrak{T}_n(r) \xrightarrow{\ t\ } \tau_{\leqq r} Rj_*\mu_{p^n}^{\otimes r} \xrightarrow{(\star)} \iota_*\nu_{Y,n}^{r-1}[-r]. \tag{2.1.1}$$

See [SH] (3.2.5) and (4.2.1) for the right arrow $(\star)$. By the same arguments as in loc. cit. 4.2.2, $\mathfrak{T}_n(r)$ is concentrated in $[0, r]$ and unique up to a unique isomorphism in $D^b(X_{\text{ét}}, \varLambda_n)$. For $r < 0$, we define $\mathfrak{T}_n(r)$ as

$$\mathfrak{T}_n(r) := j_!\varLambda_n(r).$$

See (1.4.1) for the definition of the (locally constant) sheaf $\varLambda_n(r)$ on $(X[p^{-1}])_{\text{ét}}$.

Lemma 2.1 (1) *If p is invertible in $\mathfrak{O}$, then we have $\mathfrak{T}_n(r) \cong \varLambda_n(r)$ for any $r \in \mathbb{Z}$.*

(2) *Assume that*

($\star_2$) *any residue field of $\mathfrak{O}$ of characteristic p is perfect.*

Then we have $\mathfrak{T}_n(r) \cong Rj_*\mu_{p^n}^{\otimes r}$ for any $r \geqq d+1$.

Proof. (1) is obvious. We prove (2). Without loss of generality, we may assume that $\mathfrak{O}$ is local and strict henselian. Let k be the residue field of $\mathfrak{O}$. Since k is algebraically closed by assumption, we have $\mathrm{cd}_p(K) \leqq 1$ ([Se] Chapter II, §3.3). By this fact and the cohomological dimension of affine varieties [SGA4] X.3.2, we have $\tau_{\leqq r} Rj_*\mu_{p^n}^{\otimes r} \cong Rj_*\mu_{p^n}^{\otimes r}$ for any $r \geqq d$. On the other hand, we have $\nu_{Y,n}^{r-1} = 0$ for any $r \geqq d+1$ again because k is algebraically closed (note that $\dim(Y) = d-1$). The assertion follows from these facts. □

Proposition 2.2 (cf. [SH] 4.2.8) *Let $\mathfrak{O}'$ be another Dedekind ring which is flat over $\mathfrak{O}$, and let X' be a scheme which is regular and flat of finite type over B' and satisfies $(\star_1)$ over B'. Let $f : X' \to X$ be an arbitrary morphism, and let $g : X'[p^{-1}] \to X[p^{-1}]$ be the induced morphism. Then for any $n \geqq 1$ and $r \in \mathbb{Z}$, there exists a unique morphism*

$$f^\sharp : f^*\mathfrak{T}_n(r)_X \longrightarrow \mathfrak{T}_n(r)_{X'} \quad \text{in} \quad D^b(X'_{\text{ét}}, \Lambda_n)$$

that extends the natural isomorphism $g^*\Lambda_n(r)_{X[p^{-1}]} \cong \Lambda_n(r)_{X'[p^{-1}]}$ on $X'[p^{-1}]$.

Proof. The case $r \leqq 0$ is obvious. Assume $r \geqq 1$ and put $U^1\mathscr{O}_X^\times := \mathrm{Ker}\big(\mathscr{O}_X^\times \to \mathscr{O}_{Y_{\mathrm{red}}}^\times\big)$. We define a filtration

$$0 \subset U^1R^rj_*\mu_{p^n}^{\otimes r} \subset FR^rj_*\mu_{p^n}^{\otimes r} \subset R^rj_*\mu_{p^n}^{\otimes r}$$

on the sheaf $R^rj_*\mu_{p^n}^{\otimes r}$ as

$U^1R^rj_*\mu_{p^n}^{\otimes r}$:= the subsheaf generated étale locally by symbols of the form $\{a, b_1, \ldots, b_{r-1}\}$ with $a \in U^1\mathscr{O}_X^\times$ and $b_j \in j_*\mathscr{O}_{X[p^{-1}]}^\times$,

$FR^rj_*\mu_{p^n}^{\otimes r}$:= the subsheaf generated étale locally by $U^1R^rj_*\mu_{p^n}^{\otimes r}$ and the symbols $\{a_1, a_2, \ldots, a_r\}$ with $a_j \in \mathscr{O}_X^\times$.

We have $R^rj_*\mu_{p^n}^{\otimes r}/FR^rj_*\mu_{p^n}^{\otimes r} \cong \iota_*\nu_{Y,n}^{r-1}$ by [SS] 1.1 (see also Remark 2.3 below) and the same arguments as in [SH] 3.4.2, and hence

$$\mathscr{H}^r(\mathfrak{T}_n(r)) \cong FR^rj_*\mu_{p^n}^{\otimes r}. \tag{2.1.2}$$

The assertion follows from this fact and [SH] 2.1.2 (1). □

Remark 2.3 The assumption in [SS] 1.1 that the base field K contains a primitive p-th root of unity can be removed by the following argument due to K. Kato, [KSS]. Without loss of generality, we may assume that $\mathfrak{O}$ is henselian local and that X is an affine scheme of the from

$$X = \mathrm{Spec}(\mathfrak{O}[t_0, t_1, \ldots, t_d]/(t_0^{e_0}t_1^{e_1}\cdots t_c^{e_c} - \pi))$$

for some integers $0 \leqq c \leqq d$ and $e_0, e_1, \ldots, e_c \geqq 1$ and some prime element $\pi \in \mathfrak{O}$. Put $\varpi := \sqrt[p-1]{\pi}$ and $\mathfrak{O}''$:= the valuation ring of $K(\varpi)$. There is a finite flat extension of X

$$X'' = \mathrm{Spec}(\mathfrak{O}''[T_0, T_1, \ldots, T_d]/(T_0^{e_0}T_1^{e_1}\cdots T_c^{e_c} - \varpi))$$

with $T_i := \sqrt[p-1]{t_i}$, which is quasi-log smooth over $\mathfrak{O}''$ and $K(\varpi)$ contains a primitive p-th root of unity. Hence [SS] 1.1 is applicable for X'', and we obtain the same assertion for X by a standard norm argument.

Proposition 2.4 (cf. [SH] 4.3.1) *For any $r \in \mathbb{Z}$ and $m, n \geqq 1$, there exists a canonical distinguished triangle of the following form*:

$$\mathfrak{T}_n(r) \xrightarrow{\underline{p}^m} \mathfrak{T}_{n+m}(r) \xrightarrow{\mathscr{R}^m} \mathfrak{T}_m(r) \xrightarrow{\delta_{m,n}} \mathfrak{T}_n(r)[1] \quad \textit{in} \quad D^b(X_{\text{ét}}).$$

Here $\underline{p}^m$ (resp. $\mathscr{R}^m$) is a unique morphism that extends the natural inclusion $\Lambda_n(r) \hookrightarrow \Lambda_{n+m}(r)$ on $(X[p^{-1}])_{\text{ét}}$ (resp. the natural surjection $\Lambda_{n+m}(r) \twoheadrightarrow \Lambda_m(r)$ on $(X[p^{-1}])_{\text{ét}}$) satisfying

$$\times p^m = \underline{p}^m \circ \mathscr{R}^m : \Lambda_{n+m}(r) \longrightarrow \Lambda_{n+m}(r)$$

The arrow $\delta_{m,n}$ is a canonical morphism which extends the Bockstein morphism $\Lambda_m(r) \to \Lambda_n(r)[1]$ in $D^b((X[p^{-1}])_{\text{ét}})$ associated with the exact sequence $0 \to \Lambda_n(r) \to \Lambda_{n+m}(r) \to \Lambda_m(r) \to 0$.

Proof. On obtains the assertion by repeating the proof of [SH] 4.3.1, using [SS] 1.1 in place of [SH] 3.3.7 (1). □

2.2 Purity and duality

Let Z be an integral closed subscheme of Y', and let $i_Z : Z \hookrightarrow Y$ and $\iota_Z : Z \hookrightarrow X$ be the natural closed immersions. Put $c := \mathrm{codim}_X(Z)$. We define the Gysin morphism for ι_Z as the composite

$$\mathrm{Gys}_{\iota_Z} : \nu_{Z,n}^{r-c}[-r-c] \xrightarrow{\mathrm{Gys}_{i_Z}} Ri_Z^! \nu_{Y,n}^{r-1}[-r-1] \xrightarrow{g} R\iota_Z^! \mathfrak{T}_n(r) \quad \text{in} \quad D^+(Z_{\text{ét}}, \Lambda_n). \qquad (2.2.1)$$

See (2.1.1) for g, and [SH] 2.2.1 for Gys_{i_Z} (see also [Sa1] 2.4.1).

Proposition 2.5 (1) Gys_{ι_Z} *induces an isomorphism $\nu_{Z,n}^{r-c}[-r-c] \cong \tau_{\leqq r+c} R\iota_Z^! \mathfrak{T}_n(r)$ for any $r \in \mathbb{Z}$.*

(2) *Assume further the condition $(\star_2)$ of Lemma 2.1 (2). Then the above* Gys_{ι_Z} *is an isomorphism for any $r \geqq d$.*

Proof. (1) We obtain the assertion by repeating the proof of [SH] 4.4.7, using [SS] 1.1 and 4.5 in place of [SH] 3.3.7. More precisely, our task is to prove that

$$\tau_{\leqq r+c-1} Ri_Z^! (\tau_{\geqq r+1} \iota^* Rj_* \mu_{p^n}^{\otimes r}) = 0,$$

which is reduced, by a standard argument using [SS] 1.1, to showing the semi-purity of Hagihara in our situation:

$$R^q i_Z^! (\iota^* R^m j_* \mu_p^{\otimes r}) = 0 \quad \text{for any } m \text{ and } q \text{ with } q \leqq c-2.$$

This last vanishing is further reduced to the case that K contains a primitive p-th root of unity by the argument in Remark 2.3, and then checked by the arguments in [SH] A.2.9 and the fact that the sheaf $U^1 R^m j_* \mu_p^{\otimes m}$ introduced in the proof of Proposition 2.2 has a finite descending filtration for which each graded quotient is a free $(\mathscr{O}_T)^p$-modules for some irreducible component T of Y, see [SS] 4.5 and the last display in the proof of loc. cit. 4.4.

(2) Under the assumptions, the left arrow in (2.2.1) is an isomorphism by [Sa1] 1.3.2 and 4.3.2. The right arrow in (2.2.1) is an isomorphism as well by the facts that $\tau_{\leqq r}Rj_*\mu_{p^n}^{\otimes r} \cong Rj_*\mu_{p^n}^{\otimes r}$ for any $r \geqq d$ (see the proof of Lemma 2.1 (2)) and that $R\iota^! Rj_* = 0$. □

Corollary 2.6 (cf. [SH] 4.4.9) *For any closed immersion $\iota_Z : Z \hookrightarrow X$ of codimension $\geqq r+1$ and any $q \leqq 2r+1$, we have $R^q\iota_Z^!\mathfrak{T}_n(r) = 0$.*

Proof. One obtains the corollary by the same arguments as in the proof loc. cit. 4.4.9, using Proposition 2.5 (1) in place of loc. cit. 4.4.7. □

Let x and y be points of X such that $y \in \overline{\{x\}}$ and such that $c := \mathrm{codim}_X(y) = \mathrm{codim}_X(x) + 1$. To proceed our preliminaries on the complex $\mathfrak{T}_n(r)$, we introduce the following residue diagram:

$$\begin{array}{ccc} H^{r-c+1}(x, \Lambda_n(r-c+1)) & \xrightarrow{\partial} & H^{r-c}(y, \Lambda_n(r-c)) \\ \downarrow{\scriptstyle \mathrm{Gys}_{\iota_x}} & & \downarrow{\scriptstyle \mathrm{Gys}_{\iota_y}} \\ H_x^{r+c-1}(\mathrm{Spec}(\mathscr{O}_{X,x}), \mathfrak{T}_n(r)) & \xrightarrow{\delta} & H_y^{r+c}(\mathrm{Spec}(\mathscr{O}_{X,y}), \mathfrak{T}_n(r)), \end{array} \tag{2.2.2}$$

where the coefficient $\Lambda_n(s) = \Lambda_n(s)_z$ on a point z denotes the étale complex $W_n\Omega^s_{z,\log}[-s]$ (resp. the étale sheaf defined in (1.4.1)) if $\mathrm{ch}(z) = p$ (resp. $\mathrm{ch}(z) \neq p$). If $\mathrm{ch}(z) \neq p$, then the Gysin map Gys_{ι_z} for $\iota_z : z \hookrightarrow \mathrm{Spec}(\mathscr{O}_{X,z})$ is defined as the cup product with Gabber's cycle class $\mathrm{cl}_X(z) \in H_z^{2c'}(\mathrm{Spec}(\mathscr{O}_{X,z}), \mu_{p^n}^{\otimes c'})$, where $c' := \mathrm{codim}_X(z)$. The arrow ∂ denotes the boundary map of Galois cohomology [KCT], and δ denotes the connecting map of a localization long exact sequence of étale cohomology.

Lemma 2.7 *The diagram* (2.2.2) *is anti-commutative.*

Proof. See [JSS] Theorem 3.1.1 for the case $\mathrm{ch}(y) \neq p$. The case $\mathrm{ch}(x) = \mathrm{ch}(y) = p$ follows from the definition of the Gysin morphism in [SH] 2.2.1. We check the case that $\mathrm{ch}(x) = 0$ and $\mathrm{ch}(y) = p$, using the results in [SH] as follows. Put $Z := \overline{\{y\}}$, the Zariski closure of $\{y\}$ in X. We write $\mathrm{RD}(X, x, y, r)$ for the diagram (2.2.2). Since the problem is étale local on X, we may assume that X is affine and that X is a closed subscheme of an affine space $\mathbb{A}^N_{\mathfrak{O}} =: X'$. Let ξ be the generic point of X and put $c' := \mathrm{codim}_{X'}(X)$. The diagram $\mathrm{RD}(X', \xi, \eta, r+c')$ is anti-commutative for any generic point η of Y by [SH] 6.1.1. Hence there exists a Gysin morphism for $i : X \hookrightarrow X'$

$$\mathrm{Gys}_i : \mathfrak{T}_n(r)[-2c'] \longrightarrow Ri^!\mathfrak{T}_n(r+c') \quad \text{in} \quad D^+(X_{\text{ét}}, \Lambda_n),$$

which induces an isomorphism $\mathfrak{T}_n(r)[-2c'] \cong \tau_{\leqq r+c'}Ri^!\mathfrak{T}_n(r+c')$, by the same arguments as in loc. cit. §6.3. Moreover, one obtains the transitivity assertion in loc. cit. 6.3.3 for the closed immersions $Z \hookrightarrow X \hookrightarrow X'$ by the same arguments as in the proof of loc. cit. 6.3.3, where we have again used the fact that the diagram $\mathrm{RD}(X', \xi, \eta, r+c')$ is anti-commutative for any generic point η of Y. Thus the anti-commutativity of $\mathrm{RD}(X, x, y, r)$ follows from that of $\mathrm{RD}(X', x, y, r+c')$ (loc. cit. 6.1.1) and the purity in Proposition 2.5 (1) for $Z \hookrightarrow X$ and $Z \hookrightarrow X'$. □

The compatibility in Lemma 2.7 plays an important role in the following results:

Proposition 2.8 (1) *Let $\mathfrak{O}'$ be another Dedekind ring which is flat over $\mathfrak{O}$, and let X' be a scheme which is regular and separated flat of finite type over B' and satisfies $(\star_1)$ over B'. Let $f : X' \to X$ be an arbitrary morphism, and let $\psi : X'[p^{-1}] \to X[p^{-1}]$ be the induced morphism. Put $c := \dim(X[p^{-1}]) - \dim(X'[p^{-1}])$. Then for any $n \geqq 1$ and $r \geqq 0$, there exists a unique morphism*

$$\mathrm{tr}_f : Rf_! \mathfrak{T}_n(r-c)_{X'}[-2c] \longrightarrow \mathfrak{T}_n(r)_X \quad \textit{in} \;\; D^+(X_{\text{ét}}, \Lambda_n)$$

that extends the push-forward map $\mathrm{tr}_\psi : R\psi_! \Lambda_n(r-c)[-2c] \to \Lambda_n(r)$ on $(X[p^{-1}])_{\text{ét}}$. We will often write $\mathrm{tr}_{X'/X}$ for tr_f in what follows.

(2) *Assume further the condition $(\star_2)$ of Lemma 2.1 (2). Then the adjunction morphism of $\mathrm{tr}_{X/B} = \mathrm{tr}_{\pi_{X/B}}$ is an isomorphism for any $r \geqq d$:*

$$\mathfrak{T}_n(r)_X[2(d-1)] \cong R\pi^!_{X/B} \mathfrak{T}_n(r+1-d)_B \quad \textit{in} \;\; D^+(X_{\text{ét}}, \Lambda_n).$$

Proof. If f is a locally closed immersion, the assertion (1) follows from Lemma 2.7, see [SH] 6.3.4 (2). One can check (1) in the general case, using [SS] 1.1 and 4.5 and the arguments in [SH] §§7.1–7.2; in the step corresponding to loc. cit. 7.1.2, it is enough to consider locally free $(\mathscr{O}_T)^p$-modules $\mathscr{F}$ for each irreducible component T of Y in place of 'locally free $(\mathscr{O}_Y)^p$-modules $\mathscr{F}$' (and the assumption that k is perfect is unnecessary).

As for the assertion (2) with $r = d$, see loc. cit. 7.3.1, where we have used the absolute purity [FG] and the duality in [JSS] Theorem 4.6.2. The assertion (2) in the case $r > d$ directly follows from the absolute purity, Lemma 2.1 (2) and the base change isomorphism $R\pi^!_{X/B} Rj_{U*} = Rj_* R\pi^!_{X_U/U}$ ([SGA4] XVIII.3.1.12.3), where j_U denotes the open immersion $U := B[p^{-1}] \hookrightarrow B$. □

Corollary 2.9 *Let $\beta : B' \to B$ be a flat morphism such that B' is regular of dimension $\leqq 1$ and such that $X' := X \times_B B'$ satisfies $(\star_1)$ over B'. Let $\alpha : X' \to X$ be the first projection. Then the following diagram commutes in $D^+(B'_{\text{ét}}, \Lambda_n)$ for any $r \geqq d-1$:*

$$\begin{array}{ccc} R\pi_{X'/B'!} \mathfrak{T}_n(r)_{X'}[2(d-1)] & \xrightarrow{\mathrm{tr}_{X'/B'}} & \mathfrak{T}_n(r+1-d)_{B'} \\ \uparrow{\scriptstyle \alpha^*} & & \uparrow{\scriptstyle \beta^*} \\ \beta^* R\pi_{X/B!} \mathfrak{T}_n(r)_X[2(d-1)] & \xrightarrow{\beta^* \mathrm{tr}_{X/B}} & \beta^* \mathfrak{T}_n(r+1-d)_B. \end{array}$$

Proof. The assertion follows from the uniqueness of the trace morphisms for $\mathfrak{T}_n(r)$ and the base change property in [SGA4] XVIII.2.9. □

Corollary 2.10 (1) *Assume that $\mathfrak{O}$ is a strict henselian discrete valuation ring with algebraically closed residue field, and let v be the closed point of B. Then there is a trace map*

$$\mathrm{tr}_{X,Y} : H^{2d}_c(X, \iota_* R\iota^! \mathfrak{T}_n(d)) \xrightarrow{\mathrm{tr}_{X/B}} H^2_v(B, \mathfrak{T}_n(1)) \xleftarrow[\simeq]{\mathrm{Gys}_{\iota_v}} \Lambda_n,$$

where $\iota_v : v \hookrightarrow B$ denotes the closed point of B and the subscript c means the étale cohomology with proper support over B. Moreover, for any constructible Λ_n-sheaf $\mathscr{F}$ on X and any $i \geqq 0$, the induced pairing

$$H^i_c(X, \mathscr{F}) \times \mathrm{Ext}^{2d-i}_{X,\Lambda_n}(\mathscr{F}, \iota_* R\iota^! \mathfrak{T}_n(d)) \longrightarrow \Lambda_n$$

is a non-degenerate pairing of finite Λ_n-modules.

(2) *Assume that $\mathfrak{O}$ is an algebraic integer ring. Then there is a trace map*

$$\mathrm{tr}_X : H^{2d+1}_c(X, \mathfrak{T}_n(d)) \xrightarrow{\mathrm{tr}_{X/B}} H^3_c(B, \mathfrak{T}_n(1)) \xrightarrow[\simeq]{\mathrm{tr}_B} \Lambda_n,$$

where the subscript c means the étale cohomology with compact support (see e.g. [KCT] *§3). Moreover, for any constructible Λ_n-sheaf $\mathscr{F}$ on X and any $i \geqq 0$, the induced pairing*

$$H^i_c(X, \mathscr{F}) \times \mathrm{Ext}^{2d+1-i}_{X,\Lambda_n}(\mathscr{F}, \mathfrak{T}_n(d)) \longrightarrow \Lambda_n$$

is a non-degenerate pairing of finite Λ_n-modules.

Proof. (1) By Proposition 2.8 (2) for $r = d$ and the purity in Proposition 2.5 (2), we have isomorphisms

$$R\iota^! \mathfrak{T}_n(d) \cong R\iota^! R\pi^!_{X/B} \mathfrak{T}_n(1)[-2(d-1)] = R\pi^!_{Y/v} R\iota^!_v \mathfrak{T}_n(1)[-2(d-1)] \cong R\pi^!_{Y/v} \Lambda_n[-2d].$$

The assertion follows from this fact and the isomorphisms compatible with Yoneda pairings

$$H^*_c(X, \mathscr{F}) \cong H^*_c(Y, \iota^* \mathscr{F}), \qquad \mathrm{Ext}^*_{X,\Lambda_n}(\mathscr{F}, \iota_* R\iota^! \mathfrak{T}_n(d)) \cong \mathrm{Ext}^*_{Y,\Lambda_n}(\iota^* \mathscr{F}, R\iota^! \mathfrak{T}_n(d)),$$

where we have used the proper base change theorem to obtain the left isomorphism. See e.g. [KSc] Chapter II, Proposition 2.6.4 for the right isomorphism.

(2) The assertion follows from Proposition 2.8 (2) and [JSS] Proposition 2.4.1 (3), Corollary 2.5.1. □

Remark 2.11 The push-forward morphism tr_f in Proposition 2.8 (1) satisfies the projection formula in [SH] 7.2.4, by the same arguments as in loc. cit. See also the proof of Proposition 2.8 (1) as to how we modified loc. cit. 7.1.2 in our situation.

2.3 Cycle class morphism

To construct a cycle class morphism from Bloch's cycle complex (see (2.3.2) below), we formulate a version of $\mathfrak{T}_n(r)$ with log poles and a purity for this coefficient; see also [Z] for a construction assuming Gersten's conjecture for Bloch's cycle complex. Let D be a reduced normal crossing divisor on X which is flat over B and such that $D \cup Y_{\mathrm{red}}$ also has simple normal crossings on X and such that the pair (X, D) is quasi-log smooth over B in the sense of [SS] 5.2. We define $\mathfrak{T}_n(r)_{(X,D)}$ by the following distinguished triangle analogous to (2.1.1):

$$\iota_* \nu^{r-1}_{(Y,E),n}[-r-1] \xrightarrow{g} \mathfrak{T}_n(r)_{(X,D)} \xrightarrow{t} \tau_{\leqq r} R\psi_* \mu_{p^n}^{\otimes r} \xrightarrow{(\star)} \iota_* \nu^{r-1}_{(Y,E),n}[-r], \tag{2.3.1}$$

where we put $E := Y_{\text{red}} \cap D$ and $\nu^{r-1}_{(Y,E),n} := \phi_* \nu^{r-1}_{Y \smallsetminus E}$ with $\phi : Y \smallsetminus E \hookrightarrow Y$; ψ denotes the open immersion $X \smallsetminus (Y \cup D) \hookrightarrow X$. See also [Sa2] 3.5 and 3.6. When $D = \emptyset$, we have $\mathfrak{T}_n(r)_{(X,\emptyset)} = \mathfrak{T}_n(r)_X$. The following propositions concerning the complex $\mathfrak{T}_n(r)_{(X,D)}$ play fundamental roles in our construction of cycle class maps.

Proposition 2.12 (cf. [Sa2] 6.5) *Let Z be a closed subset of X of codimension $\geqq c$. Then we have*

$$H^q_Z(X, \mathfrak{T}_n(r)_{(X,D)}) \cong \begin{cases} 0 & (q < r+c) \\ H^{r+c}_{Z \smallsetminus D}(X \smallsetminus D, \mathfrak{T}_n(r)) & (q = r+c). \end{cases}$$

In particular, if Z has pure codimension c on X, then we have

$$H^q_Z(X, \mathfrak{T}_n(c)_{(X,D)}) \cong \begin{cases} 0 & (q < 2c) \\ \Lambda_n[Z^0 \smallsetminus D] & (q = 2c), \end{cases}$$

where $\Lambda_n[Z^0 \smallsetminus D]$ means the free Λ_n-module generated over the set $Z^0 \smallsetminus D$.

Proof. One obtains the assertion by repeating the arguments in the proof of loc. cit. 6.5, using [SS] 1.1 and 4.5 (resp. Corollary 2.6 of the previous subsection) in place of [Sa2] 3.3 (resp. [SH] 4.4.9). We do not need to assume the existence of primitive p-th roots of unity in K by the argument in Remark 2.3. □

Proposition 2.13 (cf. [Sa2] 4.3) *Let $E \to X$ be a vector bundle of rank a, and let $f : \mathbb{P} := \mathbb{P}(E \oplus 1) \to X$ be its projective completion. Let $\mathbb{P}' := \mathbb{P}(E)$ the projective bundle associated with E, regarded as the infinite hyperplane section of $\mathbb{P}$. Then the composite morphism*

$$\mathfrak{T}_n(r)_X \longrightarrow Rf_*\mathfrak{T}_n(r)_{\mathbb{P}} \longrightarrow Rf_*\mathfrak{T}_n(r)_{(\mathbb{P},\mathbb{P}')}$$

is an isomorphism in $D^+(X_{\text{ét}}, \Lambda_n)$.

Proof. One can extend the Dold-Thom isomorphism (loc. cit. 4.1) and the distinguished triangle in loc. cit. 3.12 to the situation of this section, by repeating the same arguments as in the proofs of loc. cit. 4.1 and 3.12, using [SS] 1.1 and 4.5 (note also Remark 2.3 of this section). The assertion follows from those facts and Remark 2.11. □

Let Ét$/X$ be the underlying category of X-schemes of the étale site $X_{\text{ét}}$. For a scheme U and $r \geqq 0$, let $z^r(U,*)$ be Bloch's cycle complex [B2]. We define a complex $\mathbb{Z}(r)$ of presheaves on Ét$/X$ by the assignment

$$\mathbb{Z}(r) : U \in \text{Ob(Ét}/X) \longmapsto z^r(U,*)[-2r],$$

which is in fact a complex of sheaves in the Zariski and the étale topologies. We call $\mathbb{Z}(r)$ the *motivic complex* of X of weight r. For a closed subset $C \subset X$ and $U \in$ Ob(Ét$/X$), put $C_U := C \times_X U$ and let $z^r_{C_U}(U,q)$ be the subgroup of $z^r(U,q)$ consisting of the cycles on $U \times \Delta^q$ of codimension r whose support is contained in $C_U \times \Delta^q$ (and which satisfies the face condition). The collection $\{z^r_{C_U}(U,q)\}_{q\geqq 0}$ forms a subcomplex of $z^r(U,*)$, and we define a subcomplex $\mathbb{Z}(r)_{C\subset X} \subset \mathbb{Z}(r)$ by the assignment

$$\mathbb{Z}(r)_{C\subset X} : U \in \mathrm{Ob}(\text{Ét}/X) \;\longmapsto\; z^r_{C_U}(U,*)[-2r].$$

By Propositions 2.12 and 2.13, Lemma 2.7 and the same arguments as in [Sa2] §7 (see also Remark 2.14 below), one obtains a *cycle class morphism*

$$\mathrm{cl}_{C\subset X,\Lambda_n} : \mathbb{Z}(r)_{C\subset X} \otimes \Lambda_n \longrightarrow R\underline{\Gamma}_C(X, \mathfrak{T}_n(r)) \quad \text{in } D(X_{\text{ét}}, \Lambda_n) \tag{2.3.2}$$

for any $r \geqq 0$, which yields the *cycle class map* on hypercohomology groups

$$\mathrm{cl}_{C\subset X,\Lambda_n} : H^*_C(X_{\mathrm{Zar}}, \mathbb{Z}(r) \otimes \Lambda_n) \longrightarrow H^*_C(X, \mathfrak{T}_n(r)).$$

When $C = X$, the group on the left hand side will be denoted by $H^*_{\mathscr{M}}(X, \Lambda_n(r))$, and the map $\mathrm{cl}_{X,\Lambda_n} := \mathrm{cl}_{X\subset X,\Lambda_n}$ will be computed in Lemma 7.1 (3) below under the assumption that $d = 2$.

Remark 2.14 To follow the arguments in [Sa2] §7, we have used the projection formula in [SH] Corollary 7.2.4, which has been extended to our situation in Remark 2.11. We also need to extend the compatibility fact in [SH] Corollary 6.3.3 to our situation, where the push-forward morphism in Proposition 2.8 (1) plays the role of Gys_i of loc. cit. 6.3.3. One can easily check the details by Lemma 2.7 and the proof of loc. cit. 6.3.3.

Let $C_2 \subset C_1$ be closed subsets of X, and let $\phi : X' := X \smallsetminus C_2 \hookrightarrow X$ be the natural open immersion. Put $C' := C_1 \smallsetminus C_2$. Then the squares in $D(X_{\text{ét}}, \Lambda_n)$

$$\begin{array}{ccccc}
\mathbb{Z}(r)_{C_2\subset X} \otimes \Lambda_n & \longrightarrow & \mathbb{Z}(r)_{C_1\subset X} \otimes \Lambda_n & \xrightarrow{\phi^\sharp} & R\phi_*\mathbb{Z}(r)_{C'\subset X'} \otimes \Lambda_n \\
\downarrow{\scriptstyle \mathrm{cl}_{C_2\subset X,\Lambda_n}} & & \downarrow{\scriptstyle \mathrm{cl}_{C_1\subset X,\Lambda_n}} & & \downarrow{\scriptstyle \mathrm{cl}_{C'\subset X',\Lambda_n}} \\
R\underline{\Gamma}_{C_2}(X, \mathfrak{T}_n(r)) & \longrightarrow & R\underline{\Gamma}_{C_1}(X, \mathfrak{T}_n(r)) & \longrightarrow & R\phi_*R\underline{\Gamma}_{C'}(X', \mathfrak{T}_n(r))
\end{array} \tag{2.3.3}$$

are commutative by the construction of cycle class morphisms. From this commutative diagram, one obtains another commutative diagram in $D(X_{\text{ét}}, \Lambda_n)$

$$\begin{array}{ccccc}
R\phi_*\mathbb{Z}(r)_{C'\subset X'} \otimes \Lambda_n & \xrightarrow{\delta} & R\underline{\Gamma}_{C_2}(X, \mathbb{Z}(r)_{C_1\subset X} \otimes \Lambda_n)[1] & \xleftarrow{\gamma} & \mathbb{Z}(r)_{C_2\subset X} \otimes \Lambda_n[1] \\
\downarrow{\scriptstyle \mathrm{cl}_{C'\subset X',\Lambda_n}} & & \downarrow{\scriptstyle \mathrm{cl}_{C_2\subset X,\Lambda_n}} & \swarrow{\scriptstyle \mathrm{cl}_{C_2\subset X,\Lambda_n}} & \\
R\phi_*R\underline{\Gamma}_{C'}(X', \mathfrak{T}_n(r)) & \xrightarrow{\delta} & R\underline{\Gamma}_{C_2}(X, \mathfrak{T}_n(r))[1], & &
\end{array} \tag{2.3.4}$$

where the arrows δ are the connecting morphisms of localization triangles (see [SH] 1.9).

Remark 2.15 The arrow γ of (2.3.4) is NOT an isomorphism, or equivalently, the upper row of (2.3.3) does NOT fit into any localization triangle in $D(X_{\text{ét}}, \Lambda_n)$. If one considers localization triangles in the Zariski topology, then the morphism corresponding to γ of (2.3.4) is an isomorphism by Levine [Le].

2.4 Cospecialization and a residue diagram

In this subsection, we consider a residue map and prove its contravariance, which will be useful in §5.3 below. We suppose that $\pi_{X/B} : X \to B$ is *proper*, and that $\mathfrak{O}$ is a *henselian discrete valuation ring with algebraically closed residue field*. Put $I_K := \mathrm{Gal}(\overline{K}/K)$. By the duality in Corollary 2.10 (1) and the Poincaré duality for $X_{\overline{K}}$, the cospecialization map

$$\mathrm{cosp}_X : H^i(Y, \Lambda_n) \cong H^i(X, \Lambda_n) \longrightarrow H^i(X_{\overline{K}}, \Lambda_n)^{I_K}$$

induces a canonical homomorphism

$$\mathrm{Res}_X : H^{i'}(X_{\overline{K}}, \mu_{p^n}^{\otimes d-1})_{I_K} \longrightarrow H^{i'+2}_Y(X, \mathfrak{T}_n(d)),$$

where we put $i' := 2(d-1) - i$.

Proposition 2.16 *For any $i \geqq 0$, the following diagram is anti-commutative*:

$$\begin{array}{ccc} H^1(I_K, H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes d})) & \xrightarrow{\ \alpha\ } & H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes d-1})_{I_K} \\ \varepsilon \downarrow & & \downarrow \mathrm{Res}_X \\ H^{i+1}(X_K, \mu_{p^n}^{\otimes d}) & \xrightarrow{\ \delta_X\ } & H^{i+2}_Y(X, \mathfrak{T}_n(d)), \end{array}$$

where the left vertical arrow is an edge map of a Hochschild-Serre spectral sequence, and the upper horizontal arrow denotes the composite map

$$\begin{aligned} H^1(I_K, H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes d})) &\longrightarrow H^1(I_K, \mu_{p^n}) \otimes H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes d-1})_{I_K} \\ &\cong\ K^\times \otimes H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes d-1})_{I_K} \xrightarrow{\mathrm{ord}_K} \mathbb{Z} \otimes H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes d-1})_{I_K}. \end{aligned}$$

The bottom horizontal arrow is the connecting map of a localization long exact sequence.

Proof. The following diagram of trace maps and boundary maps are commutative:

$$\begin{array}{ccccc} H^{2d-1}(X_K, \mu_{p^n}^{\otimes d}) & \xrightarrow{\mathrm{tr}_{X/B}} & H^1(I_K, \mu_{p^n}) & \overset{\sim}{=\!=} & K^\times \otimes \Lambda_n \\ \delta_X \downarrow & & \delta_B \downarrow & & \downarrow -\mathrm{ord}_K \\ H^{2d}_Y(X, \mathfrak{T}_n(d)) & \xrightarrow{\mathrm{tr}_{X/B}} & H^2_v(B, \mathfrak{T}_n(1)) & \underset{\sim}{\overset{\mathrm{tr}_{B,v}}{=\!=}} & \Lambda_n, \end{array}$$

where v denotes the closed point of B, and $\mathrm{tr}_{B,v}$ means $\mathrm{tr}_{X,Y}$ for $(X, Y) = (B, v)$ (see Corollary 2.10 (1)). See Lemma 2.7 for the commutativity of the right square. The assertion follows from this commutativity and the following obvious commutative square:

$$\begin{array}{ccc} H^0(I_K, H^{i'}(X_{\overline{K}}, \Lambda_n)) & =\!=\!= & H^{i'}(X_{\overline{K}}, \Lambda_n)^{I_K} \\ \uparrow & & \uparrow \\ H^{i'}(X_K, \Lambda_n) & \longleftarrow & H^{i'}(X, \Lambda_n). \end{array}$$

The details are straight-forward and left to the reader. □

The following consequence of Proposition 2.16 will be useful later. Let $\mathfrak{O}'$ be another strict henselian local ring which is flat over $\mathfrak{O}$ and whose residue field k' is algebraically closed. Let X' be a scheme which is regular, proper flat of finite type over B' and satisfies $(\star_1)$ over B'. Put $L := \mathrm{Frac}(\mathfrak{O}')$, $I_L := \mathrm{Gal}(\overline{L}/L)$ and $Y' := X' \otimes_{\mathfrak{O}'} k'$. Assume that

$$\dim(X') = \dim(X) = d$$

(hence that $\dim(X'_L) = \dim(X_K) = d - 1$). Under this setting we obtain:

Corollary 2.17 *For any morphism $f : X' \to X$ and any $i \geqq 0$, the diagram*

$$\begin{array}{ccc} H^i(X'_{\overline{L}}, \mu_{p^n}^{\otimes d-1})_{I_L} & \xleftarrow{f^\sharp} & H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes d-1})_{I_K} \\ {\scriptstyle \mathrm{Res}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_X} \\ H^{i+2}_{Y'}(X', \mathfrak{T}_n(d)) & \xleftarrow{f^\sharp} & H^{i+2}_Y(X, \mathfrak{T}_n(d)) \end{array}$$

is commutative, that is, the map Res_X *is contravariant in* X.

Proof. In the diagram of Proposition 2.16, the composite map $\delta \circ \epsilon$ is contravariant in X by Proposition 2.2, and the map α is surjective by the fact that $\mathrm{cd}(I_K) = 1$. The corollary follows from these facts and Proposition 2.16. $\square$

3 A filtration on the direct image

Let $\pi_{X/B} : X \to B = \mathrm{Spec}(\mathfrak{O})$ be as in the beginning of §2. In this section, we assume

$(\star_2)$ *any residue field of $\mathfrak{O}$ of characteristic p is perfect.*

Under this assumption, we introduce objects $\mathfrak{H}^*(X, \mathfrak{T}_n(r))$ of $D^+(B_{\text{ét}}, \Lambda_n)$ for $r \geqq d := \dim(X)$, which play central roles throughout this paper. The étale cohomology of B with coefficients in these new objects will be related to the étale cohomology of X with coefficients in $\mathfrak{T}_n(r)$ by the spectral sequence (3.1.6) below.

3.1 Étale complex $\mathfrak{H}^m(X, \mathfrak{T}_n(r))$

Lemma 3.1 *For any $r \geqq d$, we have*

$$R\pi_{X/B*}\mathfrak{T}_n(r) \cong R\mathscr{H}om_{B,\Lambda_n}(R\pi_{X/B!}\mathfrak{T}_n(d - r), \mathfrak{T}_n(1))[2 - 2d] \tag{3.1.1}$$

in $D^+(B_{\text{ét}}, \Lambda_n)$.

Proof. Since $r \geqq d$ by assumption, there exists a canonical isomorphism

$$\mathfrak{T}_n(r) \cong R\mathscr{H}om_{X,\Lambda_n}(\mathfrak{T}_n(d - r), \mathfrak{T}_n(d)) \quad \text{in} \quad D^+(X_{\text{ét}}, \Lambda_n), \tag{3.1.2}$$

which is obvious if $r = d$, and otherwise a consequence of Lemma 2.1 (2) and the adjunction in [SGA4] XVIII.3.1.10 for the open immersion $X[p^{-1}] \hookrightarrow X$. Hence we have

$$\begin{aligned} R\pi_{X/B*}\mathfrak{T}_n(r) &\cong R\pi_{X/B*}R\mathscr{H}om_{X,\Lambda_n}(\mathfrak{T}_n(d-r), R\pi^!_{X/B}\mathfrak{T}_n(1)[2-2d]) \\ &\cong R\mathscr{H}om_{B,\Lambda_n}(R\pi_{X/B!}\mathfrak{T}_n(d-r), \mathfrak{T}_n(1))[2-2d] \end{aligned}$$

in $D^+(B_{\text{ét}}, \Lambda_n)$, by Proposition 2.8 (2) and [SGA4] XVIII.3.1.10 for $\pi_{X/B}$. □

Definition 3.2 For each $m \in \mathbb{Z}$, we define

$$\begin{aligned} \mathfrak{H}^{\leqq m}(X, \mathfrak{T}_n(r)) &:= R\mathscr{H}om_{B,\Lambda_n}(\tau_{\geqq 2(d-1)-m}R\pi_{X/B!}\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B)[2-2d], \\ \mathfrak{H}^m(X, \mathfrak{T}_n(r)) &:= R\mathscr{H}om_{B,\Lambda_n}(R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B), \end{aligned}$$

which are objects of $D^+(B_{\text{ét}}, \Lambda_n)$.

Caution 3.3 $\mathfrak{H}^m(X, \mathfrak{T}_n(r))$ is NOT the sheaf $R^m_{X/B*}\mathfrak{T}_n(r)$, but a complex of sheaves.

By Lemma 3.1 and the proper base change theorem (for $R\pi_{X/B!}$), we have

$$\mathfrak{H}^{\leqq m}(X, \mathfrak{T}_n(r)) \cong \begin{cases} 0 & (m \leqq -1) \\ R\pi_{X/B*}\mathfrak{T}_n(r)_X & (m \geqq 2(d-1)) \end{cases} \tag{3.1.3}$$

$$\mathfrak{H}^m(X, \mathfrak{T}_n(r)) = 0 \qquad \text{unless} \quad 0 \leqq m \leqq 2(d-1). \tag{3.1.4}$$

For any $m \in \mathbb{Z}$, we have a natural distinguished triangle of the form

$$\begin{aligned} \mathfrak{H}^{\leqq m-1}(X, \mathfrak{T}_n(r)) \longrightarrow \mathfrak{H}^{\leqq m}(X, \mathfrak{T}_n(r)) \longrightarrow \mathfrak{H}^m(X, \mathfrak{T}_n(r))[-m] \\ \longrightarrow \mathfrak{H}^{\leqq m-1}(X, \mathfrak{T}_n(r))[1]. \end{aligned} \tag{3.1.5}$$

The data $\{\mathfrak{H}^{\leqq m}(X, \mathfrak{T}_n(r))\}_{m \leqq 2(d-1)}$ form a finite ascending filtration on $\mathfrak{H}^{\leqq 2(d-1)}(X, \mathfrak{T}_n(r)) \cong R\pi_{X/B*}\mathfrak{T}_n(r)_X$, and yield a convergent spectral sequence

$$E_2^{a,b} = \boldsymbol{H}^a(B, \mathfrak{H}^b(X, \mathfrak{T}_n(r))) \Longrightarrow \boldsymbol{H}^{a+b}(X, \mathfrak{T}_n(r)). \tag{3.1.6}$$

To illustrate our complex $\mathfrak{H}^m(X, \mathfrak{T}_n(r))$, we show here the following proposition assuming that $\pi_{X/B}$ is proper. See Proposition 3.6 below for more detailed computations without properness assumption.

Proposition 3.4 *Assume that X is proper over B and that $r \geqq d$.*

(1) *Let $U \subset B[p^{-1}]$ be an open subset for which $\pi_{X_U/U} : X_U = X \times_B U \to U$ is smooth (and proper). Then $\mathfrak{H}^m(X, \mathfrak{T}_n(r))|_U$ is the locally constant constructible sheaf placed in degree 0, associated with $H^m(X_{\overline{K}}, \mu_{p^n}^{\otimes r})$.*

(2) *Assume further that the generic fiber X_K is geometrically connected over K. Then the trace map $\mathrm{tr}_{X/B} : R\pi_{X/B*}\mathfrak{T}_n(r)_X[2(d-1)] \to \mathfrak{T}_n(r+1-d)_B$ induces an isomorphism*

$$\mathfrak{H}^{2(d-1)}(X, \mathfrak{T}_n(r)) \cong \mathfrak{T}_n(r+1-d)_B. \tag{3.1.7}$$

To prove this proposition, we need the following lemma:

Lemma 3.5 *Let Z be a scheme and let $\mathscr{F}$ be a locally constant constructible Λ_n-sheaf on $Z_{\text{ét}}$. Then we have*

$$\mathscr{H}om(\mathscr{F},\Lambda_n)_{\overline{x}} \cong \operatorname{Hom}(\mathscr{F}_{\overline{x}},\Lambda_n) \quad \textit{and} \quad \mathscr{E}xt^q_{Z,\Lambda_n}(\mathscr{F},\Lambda_n)=0 \quad (q \geqq 1).$$

Proof of Lemma 3.5. Since $\mathscr{F}$ is a pseudo-coherent Λ_n-module on $Z_{\text{ét}}$ in the sense of [Mi1] p. 80, we have

$$\mathscr{E}xt^q_{Z,\Lambda_n}(\mathscr{F},\Lambda_n)_{\overline{x}} \cong \operatorname{Ext}^q_{\Lambda_n}(\mathscr{F}_{\overline{x}},\Lambda_n)$$

for any $q \geqq 0$ by loc. cit. II.3.20. The assertions follow from this fact and the fact that Λ_n is an injective Λ_n-module. $\square$

Proof of Proposition 3.4. (1) By definition, we have

$$\mathfrak{H}^m(X,\mathfrak{T}_n(r))|_U = R\mathscr{H}om_{U,\Lambda_n}(R^{2(d-1)-m}\pi_{X_U/U*}\Lambda_n(d-r),\Lambda_n(1)).$$

Since $R^{2(d-1)-m}\pi_{X_U/U*}\Lambda_n(d-r)$ is locally constant and constructible by the proper smooth base change theorem, the object on the right hand side is isomorphic to the sheaf

$$\mathscr{H}om_{U,\Lambda_n}(R^{2(d-1)-m}\pi_{X_U/U*}\Lambda_n(d-r),\Lambda_n(1))$$

placed in degree 0, by Lemma 3.5. Then the assertion follows from the Poincaré duality.

(2) We have

$$\mathfrak{H}^{2(d-1)}(X,\mathfrak{T}_n(r)) = R\mathscr{H}om_{B,\Lambda_n}(\pi_{X/B*}\mathfrak{T}_n(d-r)_X,\mathfrak{T}_n(1)_B)$$

by definition, and $\pi_{X/B*}\mathfrak{T}_n(d-r)_X \cong \mathfrak{T}_n(d-r)_B$ for $r \geqq d$ by the connectedness of the geometric fibers. The assertion follows from this fact and Lemma 2.1 (2) for B. $\square$

3.2 Local computations

We investigate here the local structure of $\mathfrak{H}^m(X,\mathfrak{T}_n(r))$ around the closed points on B *without assuming* that $\pi_{X/B}$ is proper. For a closed point $v \in B$, we often write Y_v (resp. $Y_{\overline{v}}$, $X_{\overline{v}}$) for $X \times_B v$ (resp. $X \times_B \overline{v}$, $X \times_B B^{\text{sh}}_{\overline{v}}$), where $B^{\text{sh}}_{\overline{v}}$ denotes the spectrum of the strict henselization of $\mathfrak{O}_v = \mathscr{O}_{B,v}$ at its maximal ideal.

Proposition 3.6 *Let v be a closed point on B, and let q and m be integers. We write ι_v for the closed immersion $v \hookrightarrow B$ and j_v for the open immersion $B \smallsetminus v \hookrightarrow B$. Assume $r \geqq d$. Then*

(1) *We have $R^q\iota_v^!\mathfrak{H}^m(X,\mathfrak{T}_n(r))=0$ unless $q=2$, and a canonical isomorphism*

$$(R^2\iota_v^!\mathfrak{H}^m(X,\mathfrak{T}_n(r)))_{\overline{v}} \cong H^{m+2}_{Y_{\overline{v}}}(X_{\overline{v}},\mathfrak{T}_n(r)).$$

Moreover, we have $R\iota_v^!\mathfrak{H}^m(X,\mathfrak{T}_n(r))=0$, if $\operatorname{ch}(v)=p$ and $r>d$.

(2) *We have*

$$(R^q j_{v*} j_v^* \mathfrak{H}^m(X, \mathfrak{T}_n(r)))_{\overline{v}} \cong H^q(I_v, H^m(X_{\overline{K}}, \mu_{p^n}^{\otimes r})),$$

where I_v denotes the inertia subgroup of G_K at v. Consequently, we have

$$R^q j_{v*} j_v^* \mathfrak{H}^m(X, \mathfrak{T}_n(r)) = 0$$

unless $q = 0$ or 1, by the fact that $\mathrm{cd}_p(I_v) = 1$ (*see* [Se] II.3.3).

(3) *We have*

$$\mathscr{H}^q(\mathfrak{H}^m(X, \mathfrak{T}_n(r)))_{\overline{v}} \cong \begin{cases} H^m(X_{\overline{K}}, \mu_{p^n}^{\otimes r})^{I_v} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0, 1 \text{ or } 2 \end{cases}$$

and an exact sequence

$$0 \longrightarrow \mathscr{H}^1(\mathfrak{H}^m(X, \mathfrak{T}_n(r)))_{\overline{v}} \longrightarrow H^1(I_v, H^m(X_{\overline{K}}, \mu_{p^n}^{\otimes r})) \xrightarrow{\delta^+} H^{m+2}_{Y_{\overline{v}}}(X_{\overline{v}}, \mathfrak{T}_n(r)) \longrightarrow \mathscr{H}^2(\mathfrak{H}^m(X, \mathfrak{T}_n(r)))_{\overline{v}} \to 0.$$

Here $\mathscr{H}^q(-)$ denotes the q-th cohomology sheaf, and δ^+ denotes the composite map $\delta_X \circ \varepsilon$ in the diagram of Proposition 2.16.

Proof of Proposition 3.6. (1) By the definition of $\mathfrak{H}^m(X, \mathfrak{T}_n(r))$ in Definition 3.2 and the adjunction in [SGA4] XVIII.3.1.12.2, we have

$$\begin{aligned} R\iota_v^! \mathfrak{H}^m(X, \mathfrak{T}_n(r)) &= R\iota_v^! R\mathscr{H}om_{B,\Lambda_n}(R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B) \\ &\cong R\mathscr{H}om_{v,\Lambda_n}(\iota_v^* R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r)_X, R\iota_v^!\mathfrak{T}_n(1)_B) \qquad (3.2.1) \\ &\cong R\mathscr{H}om_{v,\Lambda_n}(R^{2(d-1)-m}\pi_{Y_v/v!}(\iota_{Y_v}^*\mathfrak{T}_n(d-r)_X), \Lambda_n)[-2], \end{aligned}$$

where ι_{Y_v} denotes the closed immersion $Y_v \to X$, and we have used the proper base change theorem for $R\pi_{X/B!}$ and the purity in Proposition 2.5 (2) for $\mathfrak{T}_n(1)_B$ in the last isomorphism. In particular if $\mathrm{ch}(v) = p$ and $r > d$, then $\iota_{Y_v}^*\mathfrak{T}_n(d-r)_X$ is zero and we have $R\iota_v^!\mathfrak{H}^m(X, \mathfrak{T}_n(r)) = 0$, which shows the third assertion of (1). If $\mathrm{ch}(v) \neq p$ or $r = d$, then $R\iota_v^!\mathfrak{H}^m(X, \mathfrak{T}_n(r))$ is acyclic outside of degree 2 by (3.2.1) and Lemma 3.5 for $Z = v$. Moreover, if $r = d$, then we have

$$\begin{aligned} R^2\iota_v^!\mathfrak{H}^m(X, \mathfrak{T}_n(d)) &\overset{(3.2.1)}{\cong} \mathscr{H}om_{v,\Lambda_n}(R^{2(d-1)-m}\pi_{Y_v/v!}\Lambda_n, \Lambda_n) \\ &\cong \mathscr{H}^{m-2(d-1)}(R\mathscr{H}om_{v,\Lambda_n}(R\pi_{Y_v/v!}\Lambda_n, \Lambda_n)) \\ &\cong \mathscr{H}^{m-2(d-1)}(R\pi_{Y_v/v*}R\pi^!_{Y_v/v}\Lambda_n) \end{aligned}$$

again by Lemma 3.5 for $Z = v$ and adjunction, and we have

$$R\pi^!_{Y_v/v}\Lambda_n \cong R\pi^!_{Y_v/v}R\iota_v^!\mathfrak{T}_n(1)_B[2] \cong R\iota_{Y_v}^!\mathfrak{T}_n(d)_X[2d] \qquad (3.2.2)$$

by the purity in Proposition 2.5 (2) for $v \hookrightarrow B$ and Proposition 2.8 (2). Hence we have

$$(R^2\iota_v^!\mathfrak{H}^m(X, \mathfrak{T}_n(d)))_{\overline{v}} \cong H^{m+2}_{Y_{\overline{v}}}(X_{\overline{v}}, \mathfrak{T}_n(d)).$$

The isomorphism in the case that $r > d$ and $\mathrm{ch}(v) \neq p$ is similar and left to the reader.

(2) We may assume that B is local with closed point v, without loss of generality. Put $\eta := B \smallsetminus v$, which is the generic point of B. The sheaf $j_v^* R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r)$ is locally constant on $\eta_{\text{ét}}$, and the object

$$j_v^*\mathfrak{H}^m(X, \mathfrak{T}_n(r)) = R\mathscr{H}om_{\eta,\Lambda_n}(j_v^* R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r), \mu_{p^n})$$

is isomorphic to the sheaf (on $\eta_{\text{ét}}$) associated with $H^m(X_{\overline{K}}, \mu_{p^n}^{\otimes r})$ placed in degree 0 by Lemma 3.5 for $Z = \eta$ and the Poincaré duality. The assertion follows from this fact.

(3) The assertion follows from Proposition 3.6 (1), (2) and the fact that the stalk at $\overline{v}$ of the connecting homomorphism

$$\delta_{B,B\smallsetminus v} : R^1 j_{v*} j_v^* \mathfrak{H}^m(X, \mathfrak{T}_n(r)) \longrightarrow \iota_{v*} R^2 \iota_v^! \mathfrak{H}^m(X, \mathfrak{T}_n(r))$$

agrees with δ^+ up to a sign. □

The following corollary follows from Proposition 3.6 (1) and (3).

Corollary 3.7 (1) *If* $\mathrm{ch}(v) = p$ *and* $r > d$*, then* $\mathfrak{H}^m(X, \mathfrak{T}_n(r)) \cong Rj_{v*}j_v^*\mathfrak{H}^m(X, \mathfrak{T}_n(r))$.

(2) $\mathfrak{H}^m(X, \mathfrak{T}_n(r))$ *is concentrated in* $[0, 2]$*, and* $R\pi_{X/B*}\mathfrak{T}_n(r)$ *is concentrated in* $[0, 2d]$.

3.3 Rigidity

In this subsection, we assume further that $\mathfrak{O}$ is henselian local. Let $\mathfrak{O}'$ be the completion of $\mathfrak{O}$ at its maximal ideal, and put

$$B' := \mathrm{Spec}(\mathfrak{O}) \quad \text{and} \quad X' := X \times_B B'.$$

Let v be the closed point of B', which we identify with the closed point of B. Let Y' be the special fiber of $\pi_{X'/B'} : X' \to B'$, and let Y be the special fiber of $\pi_{X/B} : X \to B$. We have cartesian squares

$$\begin{array}{ccccccc} Y' & \hookrightarrow & X' & \xrightarrow{\pi_{X'/B'}} & B' & \xleftarrow{\iota_v} & v \\ \| & \square & \downarrow \alpha & \square & \downarrow \beta & \square & \| \\ Y & \hookrightarrow & X & \xrightarrow{\pi_{X/B}} & B & \xleftarrow{i_v} & v. \end{array} \tag{3.3.1}$$

We prove here the following preliminary result, where we do *not* assume that $\pi_{X/B}$ is proper:

Proposition 3.8 (rigidity) *For any* $r \geqq d$*, there exist canonical isomorphisms*

$$\psi_1 : R\pi_{X/B*}\mathfrak{T}_n(r)_X \xrightarrow{\simeq} R\beta_* R\pi_{X'/B'*}\mathfrak{T}_n(r)_{X'}$$

$$\psi_2^m : \mathfrak{H}^{\leqq m}(X, \mathfrak{T}_n(r)) \xrightarrow{\simeq} R\beta_*\mathfrak{H}^{\leqq m}(X', \mathfrak{T}_n(r)) \qquad ({}^\forall m \in \mathbb{Z})$$

$$\psi_3^m : \mathfrak{H}^m(X, \mathfrak{T}_n(r)) \xrightarrow{\simeq} R\beta_*\mathfrak{H}^m(X', \mathfrak{T}_n(r)) \qquad ({}^\forall m \in \mathbb{Z})$$

$$\psi_4^m : Ri_v^!\mathfrak{H}^m(X, \mathfrak{T}_n(r)) \xrightarrow{\simeq} R\iota_v^!\mathfrak{H}^m(X', \mathfrak{T}_n(r)) \qquad ({}^\forall m \in \mathbb{Z})$$

in $D^b(B_{\text{ét}}, \Lambda_n)$*, where* $i_v : v \hookrightarrow B$ *and* $\iota_v : v \hookrightarrow B'$ *are canonical closed immersions.*

Corollary 3.9 *We have canonical isomorphisms for any $q, m \in \mathbb{Z}$ and any $r \geqq d$*

$$H^q(X, \mathfrak{T}_n(r)_X) \cong H^q(X', \mathfrak{T}_n(r)_{X'}),$$
$$H^q_Y(X, \mathfrak{T}_n(r)_X) \cong H^q_{Y'}(X', \mathfrak{T}_n(r)_{X'}),$$
$$H^q(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) \cong H^q(B', \mathfrak{H}^m(X', \mathfrak{T}_n(r))),$$
$$H^q_v(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) \cong H^q_v(B', \mathfrak{H}^m(X', \mathfrak{T}_n(r))).$$

Proof of Proposition 3.8. Let Res_X and Res_B be the pull-back morphisms

$$\mathrm{Res}_X : \alpha^*\mathfrak{T}_n(r)_X \longrightarrow \mathfrak{T}_n(r)_{X'} \quad \text{and} \quad \mathrm{Res}_B : \beta^*\mathfrak{T}_n(1)_B \to \mathfrak{T}_n(1)_{B'}.$$

We define ψ_1 as the composite

$$\psi_1 : R\pi_{X/B*}\mathfrak{T}_n(r)_X \longrightarrow R\pi_{X/B*}R\alpha_*\mathfrak{T}_n(r)_{X'} = R\beta_*R\pi_{X'/B'*}\mathfrak{T}_n(r)_{X'},$$

where the first arrow is the adjunction map of Res_X. We define ψ_2^m as the composite

$$\begin{aligned}\psi_2^m : \mathfrak{H}^{\leqq m}(X, \mathfrak{T}_n(r)) &= R\mathscr{H}om_{B,\Lambda_n}(\tau_{\geqq 2(d-1)-m}R\pi_{X/B!}\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B)[2-2d] \\ &\longrightarrow R\beta_*R\mathscr{H}om_{B',\Lambda_n}(\tau_{\geqq 2(d-1)-m}\beta^*R\pi_{X/B!}\mathfrak{T}_n(d-r)_X, \beta^*\mathfrak{T}_n(1)_B)[2-2d] \\ &\longrightarrow R\beta_*R\mathscr{H}om_{B',\Lambda_n}(\tau_{\geqq 2(d-1)-m}R\pi_{X'/B'!}\mathfrak{T}_n(d-r)_{X'}, \mathfrak{T}_n(1)_{B'})[2-2d] \\ &= R\beta_*\mathfrak{H}^{\leqq m}(X', \mathfrak{T}_n(r)),\end{aligned}$$

where the second arrow is induced by Res_B and the isomorphisms

$$\begin{aligned}\beta^*R\pi_{X/B!}\mathfrak{T}_n(d-r)_X &\cong R\pi_{X'/B'!}\alpha^*\mathfrak{T}_n(d-r)_X && \text{(proper base change)} \\ &\cong R\pi_{X'/B'!}\mathfrak{T}_n(d-r)_{X'} && (r \geqq d).\end{aligned}$$

We define ψ_3^m in a similar way. Note that the following diagram is commutative by Corollary 2.9:

$$\begin{array}{ccc} R\pi_{X/B*}\mathfrak{T}_n(r)_X & \xrightarrow[\text{(3.1.3)}]{\simeq} & \mathfrak{H}^{\leqq 2(d-1)}(X, \mathfrak{T}_n(r)) \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2^{2(d-1)}} \\ R\beta_*R\pi_{X'/B'*}\mathfrak{T}_n(r)_{X'} & \xrightarrow[\text{(3.1.3)}]{\simeq} & R\beta_*\mathfrak{H}^{\leqq 2(d-1)}(X', \mathfrak{T}_n(r)). \end{array} \tag{3.3.2}$$

We define ψ_4^m as the composite

$$\psi_4^m : Ri_v^!\mathfrak{H}^m(X, \mathfrak{T}_n(r)) \xrightarrow{\text{base change}} R\iota_v^!\beta^*\mathfrak{H}^m(X, \mathfrak{T}_n(r)) \xrightarrow{\psi_3^m} R\iota_v^!\mathfrak{H}^m(X', \mathfrak{T}_n(r)).$$

See [SGA4] XVIII.3.1.14.2 for the base change morphism. This ψ_4^m is an isomorphism, because both $Ri_v^!\mathfrak{H}^m(X, \mathfrak{T}_n(r))$ and $R\iota_v^!\mathfrak{H}^m(X', \mathfrak{T}_n(r))$ are isomorphic to

$$\begin{cases} R\mathscr{H}om_{v,\Lambda_n}(R^{2(d-1)-m}\pi_{Y_v/v!}\Lambda_n(d-r), \Lambda_n)[-2] & (\text{if } \mathrm{ch}(v) \neq p \text{ or } r = d) \\ 0 & (\text{if } \mathrm{ch}(v) = p \text{ and } r > d) \end{cases}$$

by (3.2.1) and Proposition 3.6 (1). We prove that ψ_1, ψ_2^m and ψ_3^m are isomorphisms. By the triangle (3.1.5) and the commutative diagram (3.3.2), we are reduced to showing that ψ_3^m is an isomorphism for any $m \in \mathbb{Z}$. Put $K' := \mathrm{Frac}(\mathfrak{O}')$, and let us note the following facts:

(i) $H^m(X_{\overline{K}}, \mu_{p^n}^{\otimes r}) \cong H^m(X'_{\overline{K'}}, \mu_{p^n}^{\otimes r})$, see [Mi1] VI.4.3

(ii) $G_K \cong G_{K'}$, see [Mi3] p. 160, (i), (ii)

(iii) ψ_4^m is an isomorphism

By these facts and Proposition 3.6 (2), we see that ψ_3^m is an isomorphism, which completes the proof of Proposition 3.8. □

4 Projective and inductive limits

Let $\pi_{X/B} : X \to B = \mathrm{Spec}(\mathfrak{O})$ be as in §2. We do *not* assume that $\pi_{X/B}$ is proper in this section, but assume that $\mathfrak{O}$ and $K = \mathrm{Frac}(\mathfrak{O})$ satisfy either of the following conditions:

(L) K is a non-archimedean local field of characterictic 0, i.e., a finite field extension of $\mathbb{Q}_\ell$ for some prime number ℓ, and $\mathfrak{O}$ is the valuation ring of K.

(G) K is an algebraic number field, i.e., a finite field extension of $\mathbb{Q}$, and $B = \mathrm{Spec}(\mathfrak{O})$ is an open subset of $\mathrm{Spec}(O_K)$, where O_K denotes the integer ring of K.

The main aims of this section are to prove some standard finiteness results and to construct spectral sequences (4.1.1)–(4.1.3) below, under these assumptions.

Proposition 4.1 *There is a canonical isomorphism*

$$H^q(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) \cong \mathrm{Ext}^q_B(R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r), \mathbb{G}_{\mathrm{m}}) \tag{4.0.1}$$

for any $q, m \geqq 0$, $n \geqq 1$ and $r \geqq d$. Moreover, $H^q(X, \mathfrak{T}_n(r))$ and $H^q(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r)))$ are finite for the same q, m, n and r.

Proof. The isomorphism (4.0.1) follows from the definition of $\mathfrak{H}^m(X, \mathfrak{T}_n(r))$ (see Definition 3.2) and the canonical isomorphism

$$R\mathscr{H}om_B(\Lambda_n, \mathbb{G}_{\mathrm{m}}) \cong \mathfrak{T}_n(1)$$

(a variant of [SH] Proposition 4.5.1). See also [JSS] (2.3.4). The finiteness of the groups in (4.0.1) follows from the finiteness of Ext-groups in the Artin-Verdier duality ([Ma] (2.4)) and the constructibility of $R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r)$. The finiteness of $H^q(X, \mathfrak{T}_n(r))$ follows from the spectral sequence (3.1.6) and that of E_2-terms. □

4.1 Spectral sequences

For $r \geqq d$, we introduce the following groups:

$$H^q(X, \mathbb{Z}_p(r)) := \varprojlim_{n \geqq 1} H^q(X, \mathfrak{T}_n(r)),$$

$$H^q(X,\mathbb{Q}_p(r)) := H^q(X,\mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

$$H^q(X,\mathbb{Q}_p/\mathbb{Z}_p(r)) := \varinjlim_{n\geqq 1} H^q(X,\mathfrak{T}_n(r)),$$

$$H^q(B,\mathfrak{H}^m(X,\mathbb{Z}_p(r))) := \varprojlim_{n\geqq 1} H^q(B,\mathfrak{H}^m(X,\mathfrak{T}_n(r))),$$

$$H^q(B,\mathfrak{H}^m(X,\mathbb{Q}_p(r))) := H^q(B,\mathfrak{H}^m(X,\mathbb{Z}_p(r))) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

$$H^q(B,\mathfrak{H}^m(X,\mathbb{Q}_p/\mathbb{Z}_p(r))) := \varinjlim_{n\geqq 1} H^q(B,\mathfrak{H}^m(X,\mathfrak{T}_n(r))).$$

Here the transition maps in the forth group is defined by the commutative diagram

$$\begin{array}{ccc} H^q(B,\mathfrak{H}^m(X,\mathfrak{T}_{n+1}(r))) & \dashrightarrow & H^q(B,\mathfrak{H}^m(X,\mathfrak{T}_n(r))) \\ {\scriptstyle (4.0.1)}\downarrow\cong & & {\scriptstyle (4.0.1)}\downarrow\cong \\ \mathrm{Ext}^q_B(R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_{n+1}(d-r),\mathbb{G}_{\mathrm{m}}) & \longrightarrow & \mathrm{Ext}^q_B(R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r),\mathbb{G}_{\mathrm{m}}) \end{array}$$

with the bottom arrow induced by $\underline{p} : \mathfrak{T}_n(d-r) \hookrightarrow \mathfrak{T}_{n+1}(d-r)$ of Proposition 2.4. The transition maps in the last group is defined by the commutative diagram

$$\begin{array}{ccc} H^q(B,\mathfrak{H}^m(X,\mathfrak{T}_n(r))) & \dashrightarrow & H^q(B,\mathfrak{H}^m(X,\mathfrak{T}_{n+1}(r))) \\ {\scriptstyle (4.0.1)}\downarrow\cong & & {\scriptstyle (4.0.1)}\downarrow\cong \\ \mathrm{Ext}^q_B(R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_n(d-r),\mathbb{G}_{\mathrm{m}}) & \longrightarrow & \mathrm{Ext}^q_B(R^{2(d-1)-m}\pi_{X/B!}\mathfrak{T}_{n+1}(d-r),\mathbb{G}_{\mathrm{m}}) \end{array}$$

with the bottom arrow induced by $\mathscr{R}^1 : \mathfrak{T}_{n+1}(d-r) \twoheadrightarrow \mathfrak{T}_n(d-r)$ of Proposition 2.4. Taking the projective limit of the spectral sequence (3.1.6) with respect to $n \geqq 1$, we obtain a convergent spectral sequence of $\mathbb{Z}_p$-modules

$$E_2^{a,b} = H^a(B,\mathfrak{H}^b(X,\mathbb{Z}_p(r))) \Longrightarrow H^{a+b}(X,\mathbb{Z}_p(r)). \tag{4.1.1}$$

This spectral sequence yields a spectral sequence of $\mathbb{Q}_p$-vector spaces:

$$E_2^{a,b} = H^a(B,\mathfrak{H}^b(X,\mathbb{Q}_p(r))) \Longrightarrow H^{a+b}(X,\mathbb{Q}_p(r)). \tag{4.1.2}$$

On the other hand, taking the inductive limit of (3.1.6) with respect to $n \geqq 1$, we obtain another convergent spectral sequence of $\mathbb{Z}_p$-modules

$$E_2^{a,b} = H^a(B,\mathfrak{H}^b(X,\mathbb{Q}_p/\mathbb{Z}_p(r))) \Longrightarrow H^{a+b}(X,\mathbb{Q}_p/\mathbb{Z}_p(r)). \tag{4.1.3}$$

4.2 Finite and cofinite generation

The following standard facts will be useful later:

Theorem 4.2 (1) *$H^q(X,\mathbb{Z}_p(r))$ and $H^q(B,\mathfrak{H}^m(X,\mathbb{Z}_p(r)))$ are finitely generated over $\mathbb{Z}_p$ for any $q,m \in \mathbb{Z}$ and any $r \geqq d$.*

(2) *$H^q(X, \mathbb{Q}_p/\mathbb{Z}_p(r))$ and $H^q(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ are cofinitely generated over $\mathbb{Z}_p$ for any $q, m \in \mathbb{Z}$ and any $r \geqq d$.*

(3) *We have* $\mathrm{rank}_{\mathbb{Z}_p} H^q(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))) = \mathrm{corank}_{\mathbb{Z}_p} H^q(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ *for any $q, m \in \mathbb{Z}$ and any $r \geqq d$.*

Proof. The assertions for $H^q(X, \mathbb{Z}_p(r))$ and $H^q(X, \mathbb{Q}_p/\mathbb{Z}_p(r))$ follow from a standard argument using Propositions 4.1 and 2.4. We prove the assertions for $H^q(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r)))$ and $H^q(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ in the case (G); the case (L) is similar and left to the reader.

We first show that $H^q(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r)))$ is finitely generated over $\mathbb{Z}_p$. By the Artin-Verdier duality, it is enough to show that its Pontryagin dual

$$H^{3-q}_c(B, R^{m'}\pi_{X/B!}\mathbb{Q}_p/\mathbb{Z}_p(d-r)) := \varinjlim_{n \geqq 1} H^{3-q}_c(B, R^{m'}\pi_{X/B!}\mathfrak{T}_n(d-r))$$

is cofinitely generated over $\mathbb{Z}_p$, where $m' := 2(d-1) - m$. Let M^s_{Div} to be the maximal p-divisible subsheaf of $M^s := R^s\pi_{X/B!}\mathbb{Q}_p/\mathbb{Z}_p(d-r)$, i.e.,

$$M^s_{\mathrm{Div}} := \mathrm{Im}\Big(\mathscr{H}om_B(\underline{\mathbb{Q}_p}, M^s) \to M^s\Big),$$

where $\underline{\mathbb{Q}_p}$ denotes the constant sheaf on $B_{\text{ét}}$ with values in $\mathbb{Q}_p$. There is an exact sequence

$$0 \longrightarrow {}_{p^n}(M^s_{\mathrm{Div}}) \longrightarrow {}_{p^{n+n'}}(M^s_{\mathrm{Div}}) \longrightarrow {}_{p^{n'}}(M^s_{\mathrm{Div}}) \longrightarrow 0$$

of constructible sheaves for any $n, n' \geqq 1$, and $H^i_c(B, M^s_{\mathrm{Div}})$ is cofinitely generated over $\mathbb{Z}_p$ for any i by a standard argument. On the other hand, the quotient sheaf

$$M^s_{\mathrm{cotor}} := M^s / M^s_{\mathrm{Div}}$$

is the torsion part of $R^{s+1}\pi_{X/B!}\mathbb{Z}_p$, hence constructible ([SGA5] VI.2.2.2), and $H^i_c(B, M^s_{\mathrm{cotor}})$ is finite for any i. Therefore by the long exact sequence

$$\cdots \to H^i_c(B, M^s_{\mathrm{Div}}) \to H^i_c(B, M^s) \to H^i_c(B, M^s_{\mathrm{cotor}}) \to H^{i+1}_c(B, M^s_{\mathrm{Div}}) \to \cdots, \quad (4.2.1)$$

$H^i_c(B, M^s)$ is cofinitely generated over $\mathbb{Z}_p$ for any i and s, and $H^q(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r)))$ is finitely generated over $\mathbb{Z}_p$ for any q and m.

We next show that $H^q(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ is cofinitely generated over $\mathbb{Z}_p$. By similar arguments as before, it is enough to show that the group

$$H^{3-q}_c(B, R^s\pi_{X/B!}\mathbb{Z}_p(d-r)) := \varprojlim_{n \geqq 1} H^{3-q}_c(B, R^s\pi_{X/B!}\mathfrak{T}_n(d-r))$$

is finitely generated over $\mathbb{Z}_p$ for any i and s. Let M^s and M^s_{Div} be as before, and put

$$T^s_n := {}_{p^n}(M^s_{\mathrm{Div}}) \quad (n \geqq 1) \quad \text{and} \quad T^s := (T^s_n)_{n \geqq 1}$$

Note that $T^s := (T^s_n)_{n \geqq 1}$ is a constructible $\mathbb{Z}_p$-sheaf. We further put

$$H^i_c(B, T^s) := \varprojlim_{n \geqq 1} H^i_c(B, T^s_n),$$

which is finitely generated over $\mathbb{Z}_p$ for any i by a standard argument. Noting that there is a short exact sequence of constructible $\mathbb{Z}_p$-sheaves

$$0 \longrightarrow M^{s-1}_{\text{cotor}} \longrightarrow R^s\pi_{X/B!}\mathbb{Z}_p(d-r) \longrightarrow T^s \longrightarrow 0$$

we obtain a long exact sequence

$$\cdots \to H^i_c(B, M^{s-1}_{\text{cotor}}) \to H^i_c(B, R^s\pi_{X/B!}\mathbb{Z}_p(d-r)) \to H^i_c(B, T^s) \to H^{i+1}_c(B, M^{s-1}_{\text{cotor}}) \to \cdots , \qquad (4.2.2)$$

which shows that $H^i_c(B, R^s\pi_{X/B!}\mathbb{Z}_p(d-r))$ is finitely generated over $\mathbb{Z}_p$ for any i.

Finally noting that there is a long exact sequence of $\mathbb{Z}_p$-modules

$$\cdots \to H^i_c(B, T^s) \to H^i_c(B, R^s\pi_{X/B!}\mathbb{Q}_p(d-r)) \to H^i_c(B, M^s_{\text{Div}}) \to H^{i+1}_c(B, T^s) \to \cdots , \qquad (4.2.3)$$

we obtain the equalities

$$\begin{aligned} &\text{rank}_{\mathbb{Z}_p} H^q(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))) \overset{\text{(duality)}}{=} \text{corank}_{\mathbb{Z}_p} H^{3-q}_c(B, M^{m'}) \quad (m' := 2(d-1)-m) \\ &\overset{(4.2.1)}{=} \text{corank}_{\mathbb{Z}_p} H^{3-q}_c(B, M^{m'}_{\text{Div}}) \overset{(4.2.3)}{=} \text{rank}_{\mathbb{Z}_p} H^{3-q}_c(B, T^{m'}) \\ &\overset{(4.2.2)}{=} \text{rank}_{\mathbb{Z}_p} H^{3-q}_c(B, R^{m'}\pi_{X/B!}\mathbb{Z}_p(d-r)) \overset{\text{(duality)}}{=} \text{corank}_{\mathbb{Z}_p} H^q(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))), \end{aligned}$$

which shows the assertion (3). □

5 Comparison with Selmer groups, local case

Let $\pi_{X/B} : X \to B = \text{Spec}(\mathfrak{O})$ be as in §2. In this section, we always assume the following:

- $\pi_{X/B}$ is *proper*, and the generic fiber X_K is *geometrically connected over* K.
- K is a non-archimedean local field of characteristic 0, and $\mathfrak{O}$ is the valuation ring of K, i.e., the case (L) of §4.

Let k be the residue field of $\mathfrak{O}$ and put $\ell := \text{ch}(k)$. We will often write Y (resp. $\overline{Y}$) for $X \otimes_{\mathfrak{O}} k$ (resp. $X \otimes_{\mathfrak{O}} \overline{k}$). A main aim of this section is to compare $H^*(X, \mathbb{Q}_p(r))$ ($r \geqq d$) with Selmer groups H^1_f, whose definition we are going to review briefly.

Let V be a finite-dimensional $\mathbb{Q}_p$-vector space on which the Galois group G_K acts continuously. In their paper [BK2] §3, Bloch and Kato defined the $\mathbb{Q}_p$-subspace $H^1_f(K, V)$ of $H^1(K, V)$ as

$$H^1_f(K, V) := \begin{cases} \text{Ker}(H^1(K, V) \longrightarrow H^1(K^{\text{ur}}, V)) & (\ell \neq p) \\ \text{Ker}(H^1(K, V) \longrightarrow H^1(K, V \otimes_{\mathbb{Q}_p} B_{\text{crys}})) & (\ell = p), \end{cases}$$

where K^{ur} denotes the maximal unramified extension field of K, and B_{crys} denotes the period ring of crystalline representations defined by Fontaine [Fo]. We often write $H^1_{/f}(K,V)$ for the quotient of $H^1(K,V)$ by $H^1_f(K,V)$. Let us recall here the following fundamental theorem of Bloch-Kato [BK2] 3.8, which will be useful in this section:

Lemma 5.1 (Bloch-Kato) *Assume that V is finite-dimensional over $\mathbb{Q}_p$ and has a $\mathbb{Z}_p$-lattice which is preserved under the action of G_K. If $\ell = p$, assume further that V is a de Rham representation. Let $V^*(1)$ be the Kummer dual of V, that is,* $\mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1))$*. Then under the perfect pairing of local Tate duality*

$$H^1(K,V) \times H^1(K,V^*(1)) \to H^2(K,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p,$$

the subspaces $H^1_f(K,V)$ and $H^1_f(K,V^*(1))$ are the exact annihilators of each other.

The following standard fact will be useful later in §§6–8 below.

Lemma 5.2 *Assume that $\ell \neq p$, and that $\pi_{X/B} : X \to B$ is smooth and proper. Then we have $H^a(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) = 0$ for any $a \geqq 2$, $m \geqq 0$, $n \geqq 1$ and $r \geqq d$.*

Proof. Under the assumptions, $\mathfrak{H}^m(X,\mathfrak{T}_n(r))$ is a locally constant sheaf on $B_{\text{ét}}$ placed in degree 0, whose stalk at $\overline{v}$ is $H^m(\overline{Y}, \mu_{p^n}^{\otimes r})$ by Lemma 2.1 (1), Proposition 3.4 (1) and the proper smooth base change theorem. Hence we have

$$H^a(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) \cong H^a(v, H^m(\overline{Y}, \mu_{p^n}^{\otimes r})) = 0$$

for any $a \geqq 2$, as claimed. □

5.1 Comparison results

The main result of this section is the following:

Theorem 5.3 *For any $m \geqq 0$ and $r \geqq d$, we have canonical isomorphisms*

$$H^q(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) \cong \begin{cases} H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) & (q=1) \\ 0 & (\textit{otherwise}) \end{cases}$$

Moreover, if $\ell \neq p$, then we have $H^q(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) = 0$ for any $q, m \geqq 0$ and $r \geqq d$.

Remark 5.4 If $\ell \neq p$, then we have $H^m(X, \mathbb{Q}_p(r)) = 0$ for any $m \in \mathbb{Z}$ and any $r \geqq d$ by the proper base change theorem

$$H^m(X, \mathbb{Q}_p(r)) \cong H^m(Y, \mathbb{Q}_p(r))$$

and a theorem of Deligne [De] 3.3.4 on weights of $H^*(\overline{Y}, \mathbb{Q}_p)$ (note that $\dim(Y) = d-1$). Theorem 5.3 for $\ell \neq p$ refines this fact.

We first state a few consequences of Theorem 5.3. By the theorem and the spectral sequence (4.1.2), we obtain the following corollary:

Corollary 5.5 *The spectral sequence* (4.1.2) *degenerates at E_2, and we have*

$$H^m(X, \mathbb{Q}_p(r)) \cong H^1_f(K, H^{m-1}(X_{\overline{K}}, \mathbb{Q}_p(r)))$$

for any $m \geqq 0$ and any $r \geqq d$.

The following corollary will be useful later:

Corollary 5.6 (1) *There exists a natural map*

$$H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) \longrightarrow H^1(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$$

which fits into a commutative diagram

$$\begin{array}{ccc} H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) & & \\ \vdots & \searrow^{\textit{(natural map)}} & \\ \downarrow & & \\ H^1(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) & \hookrightarrow & H^1(K, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r))). \end{array}$$

See Proposition 3.6 (*1*) *for the injectivity of the bottom arrow.*

(2) *$H^a(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r)))$ and $H^a(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ are finite for any $a \neq 1$, $m \geqq 0$ and $r \geqq d$.*

Proof. The claim (1) immediately follows from Theorem 5.3, and the claim (2) follows from Theorems 5.3 and 4.2. □

We start the proof of Theorem 5.3. A key step is to show Theorem 5.7 below. Fix integers $m \geqq 0$ and $r \geqq d$, and put $V := H^m(X_{\overline{K}}, \mathbb{Q}_p)$ and

$$H^q(B, R^m\pi_{X/B*}\mathbb{Q}_p(d-r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^q(B, R^m\pi_{X/B*}\mathfrak{T}_n(d-r)),$$

Under this notation, we prove the following:

Theorem 5.7 *We have*

$$H^q(B, R^m\pi_{X/B*}\mathbb{Q}_p(d-r)) \cong \begin{cases} V(d-r)^{G_K} & (q=0) \\ H^1_f(K, V(d-r)) & (q=1) \\ 0 & (q \neq 0,1) \end{cases} \tag{5.1.1}$$

and

$$V(r)^{G_K} = 0. \tag{5.1.2}$$

We have $H^q(B, R^m\pi_{X/B*}\mathbb{Q}_p(d-r)) = 0$ if $\ell = p$ and $r > d$, by the definition of $\mathfrak{T}_n(d-r)$ and the proper base change theorem. In this case, the isomorphism (5.1.1) asserts the vanishing of the right hand side. We will prove Theorem 5.7 in §5.2 and §5.3 below.

Proof of "Theorem 5.7 $\Longrightarrow$ Theorem 5.3". Let v be the closed point of B. Put $s := d-r (\leqq 0)$ for simplicity. By the isomorphisms in (5.1.1) and the localization long exact sequence

$$\cdots \longrightarrow H^{q-1}_v(B, R^m\pi_{X/B*}\mathbb{Q}_p(s)) \longrightarrow H^{q-1}(B, R^m\pi_{X/B*}\mathbb{Q}_p(s)) \longrightarrow H^{q-1}(K, V(s)) \longrightarrow H^q_v(B, R^m\pi_{X/B*}\mathbb{Q}_p(s)) \longrightarrow \cdots$$

we have

$$H^q_v(B, R^m\pi_{X/B*}\mathbb{Q}_p(s)) \cong \begin{cases} 0 & (q \neq 2,3) \\ H^1_{/f}(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(s))) & (q=2) \\ H^2(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(s))) & (q=3). \end{cases} \tag{5.1.3}$$

Theorem 5.3 for $q \neq 0$ follows from (5.1.3) with $2(d-1)-m$ in place of m, Lemma 5.1 and the Tate duality for cohomology of B (see [Ma] (2.4)):

$$H^q(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) \times H^{3-q}_v(B, R^{2(d-1)-m}\pi_{X/B*}\mathbb{Q}_p(s)) \longrightarrow H^3_v(B, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p.$$

The assertion for $q = 0$ of Theorem 5.3 is a consequence of the isomorphism

$$H^0(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) \cong V(r)^{G_K}$$

(see Proposition 3.6 (1)) and the vanishing (5.1.2). Finally if $\ell \neq p$, then $H^1_f(K, V(r)) = H^1(k, V(r)^{I_K}) = 0$ again by (5.1.2) and the equality of dimensions

$$\dim_{\mathbb{Q}_p} V(r)^{G_K} = \dim_{\mathbb{Q}_p} H^1(k, V(r)^{I_K}), \tag{5.1.4}$$

which is a consequence of the duality of Galois cohomology of G_k. $\square$

5.2 Proof of Theorem 5.7 (the case $\ell \neq p$)

Let K^{ur} be the maximal unramified extension of K, and let $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{ur}})$ be the inertia group of K. Let $\mathfrak{O}^{\mathrm{ur}}$ be the valuation ring of K^{ur}, and let cosp^m_X be the cospecialization map

$$\mathrm{cosp}^m_X : H^m(\overline{Y}, \mathbb{Q}_p) \cong H^m(X^{\mathrm{ur}}, \mathbb{Q}_p) \longrightarrow H^m(X_{\overline{K}}, \mathbb{Q}_p)^{I_K} \tag{5.2.1}$$

for $m \geqq 0$, where X^{ur} (resp. $\overline{Y}$) denotes $X \otimes_{\mathfrak{O}} \mathfrak{O}^{\mathrm{ur}}$ (resp. $Y \otimes_k \overline{k}$). We first reduce Theorem 5.7 for $\ell \neq p$ to the following proposition:

Proposition 5.8 *Assume that $\ell \neq p$. Let $m \geqq 0$ be an integer, and put $V := H^m(X_{\overline{K}}, \mathbb{Q}_p)$.*

(1) *We have $V(r)^{G_K} = 0$ for any $r \geqq d$.*

(2) *For any $s \leqq 0$ and $q = 0, 1$, the map cosp^m_X induces an isomorphism*

$$H^q(k, H^m(\overline{Y}, \mathbb{Q}_p(s))) \cong H^q(k, V(s)^{I_K}).$$

Proposition 5.8 (1) is the same as (5.1.2) of Theorem 5.7.

Proof of "Proposition 5.8 $\Longrightarrow$ Theorem 5.7". We have

$$H^q(B, R^m\pi_{X/B*}\mathbb{Q}_p(s)) \cong H^q(k, H^m(\overline{Y}, \mathbb{Q}_p(s)))$$

and the last group is zero unless $q = 0$ or 1, because $\mathrm{cd}(G_k) = 1$. The isomorphisms for $q = 0, 1$ of (5.1.1) follow from Proposition 5.8 (1) and the fact that

$$H^1_f(K, V(s)) = H^1(k, V(s)^{I_K})$$

by definition. Thus we obtain Theorem 5.7, admitting Proposition 5.8. □

Proof of Proposition 5.8. If X is smooth over B, then the assertions are clear by the proper smooth base change theorem and Deligne's proof of the Weil conjecture [De] 3.3.9. We are concerned with the case that $\pi_{X/B} : X \to B$ is not smooth, in what follows.

(I) *Strict semi-stable reduction case.* We first prove Proposition 5.8 assuming that X has strict semi-stable reduction. We introduce some notation. Let $\overline{j}$ be the canonical map $X_{\overline{K}} \to X^{\mathrm{ur}} = X \otimes_{\mathfrak{O}} \mathfrak{O}^{\mathrm{ur}}$, and let $\overline{\iota}$ be the closed immersion $\overline{Y} \to X^{\mathrm{ur}}$. By the properness of X/B, we have the following Leray spectral sequence for any $n \geqq 1$:

$$E_2^{a,b} = H^a(\overline{Y}, \overline{\iota}^* R^b\overline{j}_*\Lambda_n) \Longrightarrow H^{a+b}(X_{\overline{K}}, \Lambda_n). \qquad (5.2.2)$$

By a theorem of Rapoport and Zink [RZ] 2.23, there is an exact sequence on $(\overline{Y})_{\text{ét}}$

$$0 \longrightarrow \overline{\iota}^* R^b\overline{j}_*\Lambda_n \longrightarrow u_*^{b+1}\Lambda_n(-b)_{Z^{(b+1)}} \longrightarrow u_*^{b+2}\Lambda_n(-b)_{Z^{(b+2)}} \longrightarrow \cdots \longrightarrow u_*^d\Lambda_n(-b)_{Z^{(d)}} \longrightarrow 0, \qquad (5.2.3)$$

where for each $m > 0$, $Z^{(m)}$ denotes the disjoint union of m-fold intersections distinct irreducible components of $\overline{Y}$ and u^m denotes the canonical (finite) map $Z^{(m)} \to \overline{Y}$; see (1.4.1) for $\Lambda_n(-b)$. Hence the E_2-terms of the spectral sequence of (5.2.2) are finite and we obtain a spectral sequence

$$E_2^{a,b} = H^a(\overline{Y}, \overline{\iota}^* R^b\overline{j}_*\mathbb{Q}_p) \Longrightarrow H^{a+b}(X_{\overline{K}}, \mathbb{Q}_p). \qquad (5.2.4)$$

by taking the projective limit with respect to $n \geqq 1$ and the tensor product with $\mathbb{Q}_p$ over $\mathbb{Z}_p$. Note that the canonical map $E_2^{m,0} = H^m(\overline{Y}, \mathbb{Q}_p) \to E^m = H^m(X_{\overline{K}}, \mathbb{Q}_p)$ agrees with the cospecialization map cosp_X^m of (5.2.1), and that the inertia group I_K acts trivially on the E_2-terms of (5.2.4). We will prove the following:

Lemma 5.9 *In the spectral sequence* (5.2.4)*, we have* $E_2^{a,b} = 0$ *unless* $0 \leqq a \leqq 2(d-b-1)$ *and* $0 \leqq b \leqq d-1$*. Furthermore, for a pair* (a, b) *with* $0 \leqq a \leqq 2(d-b-1)$ *and* $0 \leqq b \leqq d-1$*, the weights of* $E_2^{a,b}$ *are at least* $\max\{2b, 2(a + 2b + 1 - d)\}$ *and at most* $a + 2b$*.*

By this lemma, the kernel and the cokernel of the map cosp_X^m in (5.2.1) have only positive weights and hence we obtain the assertion of Proposition 5.8 (2). Similarly, one can easily derive Proposition 5.8 (1) from this lemma.

Proof of Lemma 5.9. By (5.2.3), the sheaf $\bar{\iota}^* R^b \bar{j}_* \Lambda_n$ (hence $E_2^{a,b}$ of (5.2.4)) is zero unless $0 \leqq b \leqq d-1$. Fix a $b \geqq 0$ in what follows. By the exact sequence (5.2.3), we have a spectral sequence of finite-dimensional G_k-$\mathbb{Q}_p$-vector spaces:

$$'E_1^{s,t} = H^t(Z^{(s+b+1)}, \mathbb{Q}_p(-b)) \Longrightarrow H^{s+t}(\overline{Y}, \bar{\iota}^* R^b \bar{j}_* \mathbb{Q}_p), \tag{5.2.5}$$

Here $'E_1^{s,t}$ is zero unless

$$0 \leqq t \leqq 2(d-s-b-1) \quad \text{and} \quad 0 \leqq s \leqq d-b-1, \tag{5.2.6}$$

because $\dim(Z^{(s+b+1)}) = d-s-b-1$ and $Z^{(s+b+1)} = \emptyset$ if $s+b \geqq d$. Using this spectral sequence, one can easily check that $E_2^{a,b}$ of (5.2.4)) is zero unless $0 \leqq a \leqq 2(d-b-1)$. Moreover, $'E_1^{s,t}$ has weight $t+2b$ by [De] 3.3.9. Therefore one obtains the lemma by computing the span of $t+2b$ under the conditions (5.2.6) and $a = s+t$. □

This completes the proof Proposition 5.8 in the strict semi-stable reduction case.

(II) *General case.* We prove Proposition 5.8 in the general case. By the alteration theorem of de Jong [dJ] 6.5, there exists a proper generically étale morphism $f : X' \to X$ such that X' is regular and flat over B and has strict semi-stable reduction over the normalization B' of B in X'. Let L (resp. F) be the function field of B' (resp. the residue field of the closed point of B'), Y' for the special fiber of $\pi_{X'/B'} : X' \to B'$. Then Proposition 5.8 (2) immediately follows from those for X', proved in Step (I), and the fact that $V = H^m(X_{\overline{K}}, \mathbb{Q}_p)$ is a direct summand of $H^m(X'_{\overline{L}}, \mathbb{Q}_p)$ as G_L-$\mathbb{Q}_p$-vector spaces. To prove Proposition 5.8 (1), we consider the following commutative diagram:

$$\begin{array}{ccccc}
H^q(k, H^m(\overline{Y}, \mathbb{Q}_p(s))) & \xrightarrow{f^\sharp} & H^q(F, H^m(\overline{Y'}, \mathbb{Q}_p(s))) & \xrightarrow{\mathrm{tr}_f} & H^q(k, H^m(\overline{Y}, \mathbb{Q}_p(s))) \\
\Big\downarrow \mathrm{cosp}_X^m & & \Big\downarrow \mathrm{cosp}_{X'}^m & & \Big\downarrow \mathrm{cosp}_X^m \\
H^q(k, V(s)^{I_K}) & \xrightarrow{f^\sharp} & H^q(F, H^m(X'_{\overline{L}}, \mathbb{Q}_p(s))^{I_L}) & \xrightarrow{\mathrm{tr}_f} & H^q(k, V(s)^{I_K}),
\end{array}$$

where the right horizontal arrows are induced by the following homomorphism of étale sheaves on B:

$$\begin{aligned}
\mathrm{tr}_f : \pi_{B'/B*} R^m \pi_{X'/B'*} \Lambda_n(s)_{X'} \cong R^m \pi_{X'/B*} \Lambda_n(s)_{X'} &\overset{(*)}{\cong} R^m \pi_{X'/B*}(Rf^! \Lambda_n(s)_X) \\
= R^m \pi_{X/B*}(Rf_* Rf^! \Lambda_n(s)_X) &\xrightarrow{\text{adjunction}} R^m \pi_{X/B*} \Lambda_n(s)_X
\end{aligned}$$

and we have used the absolute purity [FG] to obtain the isomorphism $(*)$. Since the middle vertical arrow in the above diagram is bijective by Step (I), the assertion of Proposition 5.8 (1) for X follows from the fact that the composite map

$$R^m \pi_{X/B*} \Lambda_n(s)_X \xrightarrow{f^\sharp} \pi_{B'/B*} R^m \pi_{X'/B'*} \Lambda_n(s)_{X'} \xrightarrow{\mathrm{tr}_f} R^m \pi_{X/B*} \Lambda_n(s)_X$$

on $B_{\text{ét}}$ agrees with the multiplication by the extension degree of function fields of $f : X' \to X$. This completes the proof. □

5.3 Proof of Theorem 5.7 (the case $\ell = p$)

By the same arguments as in the proof of "Proposition 5.8 $\Rightarrow$ Theorem 5.7" in §5.2, the assertions of Theorem 5.7 with $\ell = p$ is reduced to the following:

Proposition 5.10 *Assume that $\ell = p$. Let m be an arbitrary integer with $m \geqq 0$, and put $V := H^m(X_{\overline{K}}, \mathbb{Q}_p)$. Then:*

(1) *We have $V(r)^{I_K} = 0$ for any $r \geqq d$.*

(2) *For any $s < 0$, we have $V(s)^{I_K} = 0$. For $s = 0$, the cospecialization map*

$$\mathrm{cosp}_X^m : H^m(\overline{Y}, \mathbb{Q}_p) \longrightarrow H^m(X_{\overline{K}}, \mathbb{Q}_p)^{I_K} = V^{I_K}$$

is bijective.

(3) *For any $s \leqq 0$, we have $H^1(k, V(s)^{I_K}) = H^1_f(K, V(s))$ as subspaces of $H^1(K, V(s))$. In particular, we have $H^1_f(K, V(s)) = 0$ if $s < 0$.*

We will first prove Proposition 5.10 assuming that X has semi-stable reduction, and then prove the log smooth reduction case.

Proof. (I) *Semi-stable reduction case.* Put $D := H^m_{\text{log-crys}}(Y/W(k))$. By the Fontaine-Jannsen conjecture ([HK], [Ts] 0.2), we have a p-adic period isomorphism

$$V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} \cong D \otimes_{W(k)} B_{\mathrm{st}}, \tag{5.3.1}$$

which preserves the Frobenius operator ϕ, the monodromy operator N, the action of G_K, and the Hodge filtration $\mathrm{F}^\bullet_H$ after taking $\otimes_{B_{\mathrm{st}}} B_{\mathrm{dR}}$. By the isomorphism (5.3.1), we have

$$V(r) \cong \left(D \otimes_{W(k)} B_{\mathrm{st}}\right)^{N=0,\, \phi=p^r} \cap \mathrm{F}^r_H \left(D \otimes_{W(k)} B_{\mathrm{dR}}\right)$$

and

$$V(r)^{I_K} \cong (H^m_{\text{log-crys}}(\overline{Y}/W(\overline{k}))_{\mathbb{Q}_p})^{\varphi=p^r} \cap \mathrm{F}^r_H \left(H^m_{\mathrm{dR}}(X_K/K) \otimes_K \widehat{K^{\mathrm{ur}}}\right), \tag{5.3.2}$$

for any $r \in \mathbb{Z}$. Here φ denotes the Frobenius operator acting on $H^m_{\text{log-crys}}(\overline{Y}/W(\overline{k}))$, $\widehat{K^{\mathrm{ur}}}$ denotes the completion of K^{ur}, and we have used the following facts:

(a) $(B_{\mathrm{st}})^{I_K} = \mathrm{Frac}(W(\overline{k}))$ ([Fo] *5.1.2, 5.1.3*), *and* $(B_{\mathrm{dR}})^{I_K} = \widehat{K^{\mathrm{ur}}}$.

(b) $D \otimes_{W(k)} K \cong H^m_{\mathrm{dR}}(X_K/K)$ ([HK]).

Proposition 5.10 (1) and the case $s < 0$ of Proposition 5.10 (2) follow from (5.3.2) and the fact that

$$(H^m_{\text{log-crys}}(\overline{Y}/W(\overline{k}))_{\mathbb{Q}_p})^{\varphi=p^r} = 0 \quad \text{if } r \geqq d \text{ or } r < 0.$$

As for the case $s = 0$ of Proposition 5.10 (2), the map cosp^m is bijective by [W] Theorem 1. To prove Proposition 5.10 (3), it is enough to show the following two claims:

(c) *The restriction map*

$$H^1(K, V \otimes_{\mathbb{Q}_p} B_{\text{crys}}) \longrightarrow H^1(K^{\text{ur}}, V \otimes_{\mathbb{Q}_p} B_{\text{crys}})$$

is injective. Consequently, the image of the inflation map

$$H^1(k, V(s)^{I_K}) \hookrightarrow H^1(K, V(s))$$

is contained in $H^1_f(K, V(s))$ *for any* $s \in \mathbb{Z}$*.*

(d) *We have* $\dim_{\mathbb{Q}_p} H^1(k, V(s)^{I_K}) = \dim_{\mathbb{Q}_p} H^1_f(K, V(s))$ *for any* $s \leqq 0$*.*

Proof of (c). By the exact sequence

$$(0 \to) H^1(k, (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{I_K}) \longrightarrow H^1(K, V \otimes_{\mathbb{Q}_p} B_{\text{crys}}) \longrightarrow H^1(K^{\text{ur}}, V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_k}$$

arising from a Hochschild-Serre spectral sequence, it is enough to show that the first term is zero. We have

$$(V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{I_K} \cong H^m_{\text{log-crys}}(\overline{Y}/W(\overline{k}))^{N=0}_{\mathbb{Q}_p}$$

by the exact sequence ([Fo] 3.2.3)

$$0 \longrightarrow B_{\text{crys}} \longrightarrow B_{\text{st}} \xrightarrow{N} B_{\text{st}} \longrightarrow 0$$

and the period isomorphism (5.3.1). Hence we have

$$H^1(k, (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{I_K}) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geq 1} H^1(k, H^m_{\text{log-crys}}(\overline{Y}/W_n(\overline{k}))^{N=0}).$$

Finally, the group on the right hand side is zero, because $H^m_{\text{log-crys}}(\overline{Y}/W_n(\overline{k}))^{N=0}$ is a finite successive extension of additive G_k-modules.

Proof of (d). Since V is a de Rham representation [Fa], there is an exact sequence of finite-dimensional $\mathbb{Q}_p$-vector spaces ([BK2] Corollary 3.8.4):

$$\begin{aligned} 0 \longrightarrow V(s)^{G_K} \longrightarrow \text{Cris}(V) \oplus \text{DR}(V(s))^0 \\ \longrightarrow \text{Cris}(V) \oplus \text{DR}(V) \longrightarrow H^1_f(K, V(s)) \longrightarrow 0, \end{aligned} \tag{5.3.3}$$

where $\text{Cris}(V)$, $\text{DR}(V(s))^0$ and $\text{DR}(V)$ denote $(V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K}$, $(V(s) \otimes_{\mathbb{Q}_p} B^+_{\text{dR}})^{G_K}$ and $(V \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{G_K}$, respectively, and B^+_{dR} denotes the valuation ring of B_{dR}, see [Fo] 1.5.5. Moreover we have

$$\text{DR}(V) \cong H^m_{\text{dR}}(X_K/K) = \text{F}^s_H H^m_{\text{dR}}(X_K/K) \cong \text{DR}(V(s))^0 \tag{5.3.4}$$

for any $s \leqq 0$. Hence we obtain the claim (d) from the equalities

$$\dim_{\mathbb{Q}_p} H^1_f(K, V(s)) \overset{(5.3.3)}{=} \dim_{\mathbb{Q}_p} V(s)^{G_K} = \dim_{\mathbb{Q}_p} H^1(k, V(s)^{I_K}),$$

where the right equality is similar to the left equality in (5.1.4). This completes the proof of Proposition 5.10 in the semi-stable reduction case.

(II) *Log smooth reduction case.* Let $f : X' \to X$, B and L be as in Step (II) in the proof of Proposition 5.8. The assertions in Proposition 5.10 other than the bijectivity of cosp^m are reduced to the semi-stable reduction case directly by a standard norm argument for $f_K : X'_L \to X_K$. We derive the bijectivity of cosp^m for X from that for X'. Indeed, there exists a homomorphism $\mathrm{tr}_f : \pi_{B'/B*}R^m\pi_{X'/B'*}\Lambda_n \to R^m\pi_{X/B*}\Lambda_n$ on $B_{\text{ét}}$ for each $n \geqq 1$ given by the following left commutative square, which is by definition the Pontryagin dual of the right commutative square ($m' := 2d' - m$):

$$\begin{array}{ccc} H^m(\overline{Y'}, \Lambda_n) & \longrightarrow & H^m(\overline{Y}, \Lambda_n) \\ \downarrow{\scriptstyle \mathrm{cosp}^m_{X'}} & & \downarrow{\scriptstyle \mathrm{cosp}^m_X} \\ H^m(X'_{\overline{L}}, \Lambda_n)^{I_L} & \longrightarrow & H^m(X_{\overline{K}}, \Lambda_n)^{I_K} \end{array} \qquad \begin{array}{ccc} H^{m'+2}_{\overline{Y'}}((X')^{\mathrm{ur}}, \mathfrak{T}_n(d)) & \xleftarrow{f^\sharp} & H^{m'+2}_{\overline{Y}}(X^{\mathrm{ur}}, \mathfrak{T}_n(d)) \\ \uparrow{\scriptstyle \mathrm{Res}_{X'}} & & \uparrow{\scriptstyle \mathrm{Res}_X} \\ H^{m'}(X'_{\overline{L}}, \mu_{p^n}^{\otimes d-1})_{I_L} & \xleftarrow{f^\sharp} & H^{m'}(X_{\overline{K}}, \mu_{p^n}^{\otimes d-1})_{I_K} \end{array}$$

where we put $(X')^{\mathrm{ur}} := X' \times_{B'} (B')^{\mathrm{ur}}$ and $X^{\mathrm{ur}} := X \times_B B^{\mathrm{ur}}$, and the right square is the commutative diagram in Corollary 2.17. Thus we see that cosp^m_X is bijective by a similar norm argument as in Step (II) in the proof of Proposition 5.8. This completes the proof of Proposition 5.10 and Theorem 5.7. □

By Proposition 5.10 (1) and [BK2] Corollary 3.8.4 for $V(r) = H^m(X_{\overline{K}}, \mathbb{Q}_p(r))$, we obtain the following corollary:

Corollary 5.11 *The exponential map of Bloch-Kato induces an isomorphism*

$$\exp : H^m_{\mathrm{dR}}(X_K/K) \xrightarrow{\simeq} H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))$$

for any $m \geqq 0$ and $r \geqq d$.

6 Comparison with Selmer groups, global case

Let $\pi_{X/B} : X \to B = \mathrm{Spec}(\mathfrak{O})$ be as in §2. In the rest of this paper, we always assume:

- $\pi_{X/B}$ is *proper*, and the generic fiber X_K is *geometrically connected over* K.
- K is an algebraic number field, and $\mathfrak{O}$ is the integer ring of K, i.e., the case (G) of §4.

In this section, we compare $H^*(X, \mathbb{Q}_p(r))$ ($r \geqq d$) with Selmer groups H^1_f, using the results of the previous section. For a place v of K, we write K_v for the completion of K at v. For a finite place v of K (i.e., a closed point of B), we put $B_v := \mathrm{Spec}(O_v)$, where O_v denotes the valuation ring of K_v. We often write X_v for $X \times_B B_v$.

We first review the definition of H^1_f briefly. Let V be a G_K-$\mathbb{Q}_p$-vector space satisfying the following two conditions:

◦ *V is finite-dimensional over $\mathbb{Q}_p$, and the action of G_K on V is continuous.*

- *There exists a finite set S of places of K such that V is unramified at v, i.e., the inertia group I_v acts trivially on V, for any $v \notin S$.*

The Selmer group $H^1_f(K,V) = H^1_{f,B}(K,V)$ is defined as the kernel of the restriction map

$$\mathrm{Res} : H^1(G_S, V) \longrightarrow \bigoplus_{v\in S} H^1_{/f}(K_v, V).$$

Here S denotes a finite set of places of K which contains all ramified places of V and all places dividing p or ∞, and G_S denotes the Galois group $\mathrm{Gal}(K_S/K)$; K_S denotes the maximal S-ramified extension of K (i.e., the maximal Galois extension of K which is unramified at every finite place of K outside of S). See §5 for the definition of $H^1_f(K_v,V)$ and $H^1_{/f}(K_v,V)$. One can easily check that $H^1_f(K,V)$ is independent of the choice of S.

6.1 Fast computations

Proposition 6.1 *Let r be an integer with $r \geqq d$.*

(1) *$H^q(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r)))$ is finite in each of the following cases*:

(i) $m < 0$ (ii) $m > 2(d-1)$ (iii) $q \leqq 0$ (iv) $q > 3$

(v) $q = 3$ *and* $0 \leqq m \leqq 2d-3$ (vi) $q = 3$, $m = 2(d-1)$ *and* $r > d$.

Consequently, the spectral sequence (4.1.1) *degenerates at E_2-terms up to finite p-primary torsion.*

(2) *For any $m \geqq 0$, we have*

$$H^1(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) \cong H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))).$$

Proof of Proposition 6.1. (1) We put

$$H^{q,m,r} := H^q(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))).$$

The cases (i) and (ii) are clear by the definition of $\mathfrak{H}^m(X, \mathfrak{T}_n(r))$ (see Definition 3.2). The case (iii) with $q < 0$ follows from the fact that $\mathfrak{H}^m(X, \mathfrak{T}_n(r))$ is concentrated in degrees $\geqq 0$ (see Proposition 3.6 (3)). When $q = 0$, the restriction map

$$H^{0,m,r} \longrightarrow H^m(X_{\overline{K}}, \mathbb{Z}_p(r))^{G_K}$$

is injective by Proposition 3.6 (1) and the last group is finite by [De] 3.3.9. Hence $H^{0,m,r}$ is finite. The case (iv) follows from the Artin-Verdier duality [Ma] (2.4). Indeed, we have

$$H^q(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) \cong \mathrm{Ext}^q_B(R^{2(d-1)-m}\pi_{X/B*}\mathfrak{T}_n(d-r), \mathbb{G}_{\mathrm{m}})$$

by (4.0.1), and its dual

$$H^{3-q}_c(B, R^{2(d-1)-m}\pi_{X/B*}\mathfrak{T}_n(d-r))$$

is finite 2-torsion for any $n \geqq 1$ and $q > 3$. Finally we prove the cases (v) and (vi). Fix a dense open subset $U \subset B[p^{-1}]$ such that $X_U \to U$ is smooth (and proper). Let j be the open immersion $U \hookrightarrow B$, and for each $v \in B$ let $\iota_v : v \hookrightarrow B$ be the canonical map. There is an exact sequence

$$H^3(B, j_!j^*\mathfrak{H}^m(X, \mathbb{Z}_p(r))) \longrightarrow H^{3,m,r} \longrightarrow \bigoplus_{v \in B \smallsetminus U} H^3(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))),$$

where we identified $H^3(v, \iota_v^*\mathfrak{H}^m(X, \mathbb{Z}_p(r)))$ with $H^3(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r)))$ for each $v \in B \smallsetminus U$ by Corollary 3.9. The first term in this sequence is finite unless $(m, r) = (2(d-1), d)$ by the Artin-Verdier duality and a weight argument which is similar as for the case $q = 0$. The last term is finite as well by Corollary 5.6 (2). Thus $H^{3,m,r}$ is finite in the cases (v) and (vi), which completes the proof of Proposition 6.1 (1).

(2) We put $V^m := H^m(X_{\overline{K}}, \mathbb{Q}_p)$, for simplicity. Let S be a finite set of places of K containing all places dividing p or ∞, and all finite places where X has bad reduction. To prove Proposition 6.1 (2), it is enough to check the following:

Lemma 6.2 *There is an exact sequence of $\mathbb{Q}_p$-vector spaces*

$$0 \longrightarrow H^{1,m,r} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow H^1(G_S, V^m(r)) \xrightarrow{\text{Res}} \bigoplus_{v \in S} H^1_{/f}(K_v, V^m(r)).$$

where $H^1_{/f}(K_v, V^m(r))$ means zero for the places $v|\infty$.

Proof. Let P_∞ be the set of all infinite places of K, and consider the localization long exact sequence of cohomology groups for each $n \geqq 1$

$$\cdots \to H^q(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) \to H^q(G_S, H^m(X_{\overline{K}}, \mu_{p^n}^{\otimes r})) \to \bigoplus_{v \in S \smallsetminus P_\infty} H^{q+1}_v(B_v, \mathfrak{H}^m(X_v, \mathfrak{T}_n(r)))$$
$$\to H^{q+1}(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) \to \cdots,$$

where we have used the fact that $\mathfrak{H}^m(X, \mathfrak{T}_n(r))|_{B \smallsetminus S}$ is a locally constant sheaf on $B \smallsetminus S$ associated with the G_S-module $H^m(X_{\overline{K}}, \mu_{p^n}^{\otimes r})$ (see Proposition 3.4 (1)). We have also used the isomorphisms

$$H^*_v(B, \mathfrak{H}^m(X, \mathfrak{T}_n(r))) \cong H^*_v(B_v, \mathfrak{H}^m(X_v, \mathfrak{T}_n(r))) \qquad (v \in S \smallsetminus P_\infty)$$

obtained from étale excision and the rigidity of Corollary 3.9. The groups in this long exact sequence are finite by Proposition 4.1. Therefore we obtain the following long exact sequence by taking the projective limit with respect to $n \geqq 1$ and then $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$:

$$\cdots \longrightarrow \bigoplus_{v \in S} H^q_v(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p(r))) \longrightarrow H^{q,m,r} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow H^q(G_S, V^m(r))$$
$$\longrightarrow \bigoplus_{v \in S} H^{q+1}_v(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p(r))) \longrightarrow \cdots.$$

Moreover we have

$$H^q_v(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p(r))) \cong \begin{cases} 0 & (q = 1) \\ H^1_{/f}(K_v, V^m(r)) & (q = 2) \\ H^2(K_v, V^m(r)) & (q = 3) \end{cases}$$

by Theorem 5.3; the case $q = 3$ will be useful later in the proof of Corollary 6.10 (2) below. The assertion follows from these facts. $\square$

This completes the proof of Proposition 6.1. $\square$

Corollary 6.3 *For any $r \geqq d$, the spectral sequence* (4.1.2) *degenerates at E_2, and we have*

$$H^m(X, \mathbb{Q}_p(r)) \cong \begin{cases} H^1_f(K, H^{m-1}(X_{\overline{K}}, \mathbb{Q}_p(r))) & \\ \qquad \oplus H^2(B, \mathfrak{H}^{m-2}(X, \mathbb{Q}_p(r))) & (1 \leqq m \leqq 2d-1) \\ \mathbb{Q}_p & ((m, r) = (2d+1, d)) \\ 0 & (\textit{otherwise}). \end{cases}$$

We will prove that $H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) = 0$ for any (m, r) with $r \geqq d$, in Theorem 6.6 below. The following corollary is a global analogue of Corollary 5.6 (1), which will be useful later.

Corollary 6.4 *For any $r \geqq d$, there exists a natural map*

$$H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) \longrightarrow H^1(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$$

which fits into a commutative diagram

$$\begin{array}{ccc} H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) & & \\ \vdots & \searrow^{(\textit{natural map})} & \\ \downarrow & & \\ H^1(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) & \hookrightarrow & H^1(K, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r))). \end{array}$$

See Proposition 3.6 (1) for the injectivity of the bottom arrow.

Remark 6.5 For any $s \leqq 0$, one can easily check the following canonical isomorphism by (5.1.1), (5.1.3) and similar arguments as for the proof of Proposition 6.1:

$$H^1(B, R^m\pi_{X/B*}\mathbb{Q}_p(s)) \cong H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(s))).$$

6.2 A global finiteness of étale cohomology

In this subsection, we prove the following vanishing and finiteness result:

Theorem 6.6 *For any $m \geqq 0$ and $r \geqq d$, we have*

$$H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) = 0,$$

and the groups $H^2(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r)))$ and $H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ are finite.

As a direct consequence of this theorem and Corollary 6.3, we obtain:

Corollary 6.7 *For any $m \geqq 0$ and $r \geqq d$ with $(m, r) \neq (2d+1, d)$, we have*

$$H^m(X, \mathbb{Q}_p(r)) \cong H^1_f(K, H^{m-1}(X_{\overline{K}}, \mathbb{Q}_p(r))).$$

On the other hand, Theorem 6.6 and Remark 6.5 imply the following vanishing result by the Artin-Verdier duality:

Corollary 6.8 *For any $m \geqq 0$ and $s \leqq 0$, we have $H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(s))) = 0$.*

Proof of Theorem 6.6. By Theorem 4.2, it is enough to show that $H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ is finite. When $(m, r) = (2(d-1), d)$, we have

$$H^2(B, \mathfrak{H}^{2(d-1)}(X, \mathbb{Q}_p/\mathbb{Z}_p(d))) \overset{(3.1.7)}{\cong} H^2(B, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \mathrm{Br}(O_K)\{p\},$$

by the finiteness of $\mathrm{Pic}(O_K)$, and moreover $\mathrm{Br}(O_K)$ is finite 2-torsion by the classical Hasse principle for Brauer groups. Thus we obtain the finiteness in question.

In what follows, we assume $(m, d) \neq (2(d-1), d)$ and consider the following commutative diagram with exact rows, where the coefficients $\mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))$ in the upper row and $\mathfrak{H}^m(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))$ in the lower row are omitted:

$$\begin{array}{ccccccccc}
H^1(K) & \longrightarrow & \bigoplus_{v \in B_0} H^2_v(B) & \longrightarrow & H^2(B) & \longrightarrow & H^2(K) & \longrightarrow & \bigoplus_{v \in B_0} H^3_v(B) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\
\bigoplus_{v \in B_0} H^1_{/f}(K_v) & \xrightarrow{(\star)} & \bigoplus_{v \in B_0} H^2_v(B_v) & \longrightarrow & \bigoplus_{v \in B_0} H^2(B_v) & \longrightarrow & \bigoplus_{v \in B_0} H^2(K_v) & \longrightarrow & \bigoplus_{v \in B_0} H^3_v(B_v)
\end{array}$$

Here B_0 denotes the set of the closed points on B, and the both rows are obtained from localization sequences of étale cohomology; we put

$$H^1_{/f}(K_v) := \mathrm{Coker}\big(H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) \to H^1(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r)))\big)$$

for each $v \in B_0$ (note also Proposition 3.4 (1)), and used Corollary 5.6 (1) to verify the existence of the bottom left arrow $(\star)$. The arrows δ are bijective by étale excision and the rigidity (Corollary 3.9). The arrow γ has finite kernel and cokernel by the Hasse principle of Jannsen [J] p. 337, Theorem 3 (c). The map α has finite cokernel by [BK2] Proposition 5.14 (ii). Hence β is bijective up to finite groups. Finally, $H^2(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ is finite for all $v \in B_0$ by Corollary 5.6 (2), and zero for any $v \in (B[p^{-1}])_0$ at which X has good reduction by Lemma 5.2. Thus $H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ is finite. □

Remark 6.9 (1) By Theorem 6.6 for $r = d = 2$ and $m = 1$ and Lemma 7.1 (3) below, Bloch's conjecture ([B1] Remark 1.24) for a projective smooth curve C over K is reduced to a variant of Bass' conjecture (cf. [Ba]) that $H^3_{\mathscr{M}}(X, \mathbb{Z}(2))$ is finitely generated for an arbitrarily taken proper regular model X/B of C.

(2) Corollary 6.8 removes an assumption of a result of Morin [Mo] Theorem 1.5.

The following corollary of Theorem 6.6 follows from a similar argument as for the proof of Lemma 6.2 (see also [J] p. 349, Question 2):

Corollary 6.10 *Assume $r \geqq d$, and let S be as in the proof of Theorem 6.1 (2). Then*:

(1) *We have*

$$\frac{H^1(G_S, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))}{H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))} \cong \bigoplus_{v \in S} \frac{H^1(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))}{H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))}$$

for any m. Moreover, both hand sides are zero for $r > d$.

(2) *The restriction map*

$$H^2(G_S, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) \longrightarrow \bigoplus_{v \in S} H^2(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))$$

is bijective for any $(m, r) \neq (2(d-1), d)$ and injective for $(m, r) = (2(d-1), d)$. In particular, if $r > d$ or X_K has potentially good reduction at all finite places of K, then

$$H^2(G_S, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) = 0 \quad \text{for any } (m, r) \neq (2(d-1), d).$$

7 p-adic Abel-Jacobi mappings $(d = 2)$

The setting remains as in §6. From this section on, we assume further that $d = 2$.

7.1 Cycle class maps

See §2.3 for the definition of the motivic complex $\mathbb{Z}(r)$ on $(\text{Ét}/X)_{\text{Zar}}$. We regard $\mathbb{Z}(r)$ as a complex on X_{Zar} by restriction of topology. We define the motivic cohomology of X as

$$H^m_{\mathscr{M}}(X, \mathbb{Z}(r)) := H^m_{\text{Zar}}(X, \mathbb{Z}(r)),$$

and define the motivic cohomology with $\Lambda_n (= \mathbb{Z}/p^n\mathbb{Z})$-coefficients as

$$H^m_{\mathscr{M}}(X, \Lambda_n(r)) := H^m_{\text{Zar}}(X, \mathbb{Z}(r) \otimes \Lambda_n) \qquad (n \geqq 1).$$

Lemma 7.1 (1) *We have*

$$H^m_{\mathscr{M}}(X, \mathbb{Z}(r)) \cong \begin{cases} H^m_{\mathscr{M}}(K(X), \mathbb{Z}(2)) & (m \leqq 1,\ r = 2) \\ 0 & (m > r + 2) \end{cases}$$

where $K(X)$ denotes the function field of X.

(2) *$H^m_{\mathscr{M}}(X, \mathbb{Z}(2))$ is isomorphic to the cohomology at deree $m-2$ of the Gersten complex of Milnor K-groups*

$$K_2^M(K(X)) \longrightarrow \bigoplus_{x \in X^1} \kappa(x)^\times \longrightarrow \bigoplus_{x \in X^2} \mathbb{Z}$$
$$\text{(deg 0)} \qquad\qquad \text{(deg 1)} \qquad\qquad \text{(deg 2)}$$

for any $m \geqq 2$. In particular, we have $H^4_{\mathscr{M}}(X, \mathbb{Z}(2)) \cong \mathrm{CH}_0(X)$, the Chow group of 0-cycles modulo rational equivalence.

(3) *Assume that $r \geqq 2$, and that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$. Then the cycle class map (see §2.3)*

$$\mathrm{cl}^{m,r}_{\Lambda_n} : H^m_{\mathscr{M}}(X, \Lambda_n(r)) \longrightarrow H^m(X, \mathfrak{T}_n(r))$$

is bijective for any $m \in \mathbb{Z}$ and $n \geqq 1$. Consequently, there exists a short exact sequence

$$0 \longrightarrow H^m_{\mathscr{M}}(X, \mathbb{Z}(r))/p^n \longrightarrow H^m(X, \mathfrak{T}_n(r)) \longrightarrow {}_{p^n}H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r)) \longrightarrow 0$$

for the same m and n.

Proof. There exists a coniveau spectral sequence

$$E_1^{a,m} = \bigoplus_{x \in X^a} H^{m-a}_{\mathscr{M}}(x, \mathbb{Z}(r-a)) \Longrightarrow H^{a+m}_{\mathscr{M}}(X, \mathbb{Z}(r)) \qquad (7.1.1)$$

by [Ge] Proposition 2.1, whose $E_1^{a,m}$-term is zero in each of the following cases for the reason of the dimension of cycles and the codimension of points:

$\circ\ m > r \qquad \circ\ a < 0 \qquad \circ\ a > 2 \qquad \circ\ m < a = r \qquad \circ\ m \leqq a = r-1$

See [B2] Theorem 6.1 for the vanishing in the last case. The assertions (1) and (2) follow from these facts and the Nesterenko-Suslin-Totaro theorem

$$H^q_{\mathscr{M}}(\mathrm{Spec}(F), \mathbb{Z}(q)) \cong K^M_q(F)$$

for any field F and any $q \geqq 0$, see [NS], [To].

To prove the assertion (3), we consider a coniveau spectral sequence analogous to (7.1.1)

$$E_1^{a,m} = \bigoplus_{x \in X^a} H^{m-a}_{\mathscr{M}}(x, \Lambda_n(r-a)) \Longrightarrow H^{a+m}_{\mathscr{M}}(X, \Lambda_n(r)), \qquad (7.1.2)$$

whose $E_1^{a,m}$-terms are zero in each of the following cases:

$\circ\ m > r \qquad \circ\ a < 0 \qquad \circ\ a > 2$

On the other hand, since $r \geqq 2$, there is a coniveau spectral sequence of étale cohomology (see [JSS] (5.10.1))

$$E_1^{a,m} = \bigoplus_{x \in X^a} H^{m-a}(x, \Lambda_n(r-a)) \Longrightarrow H^{a+m}(X, \mathfrak{T}_n(r)), \qquad (7.1.3)$$

where the coefficients $\Lambda_n(s)$ ($s \in \mathbb{Z}$) on the points are those in (2.2.2). The $E_1^{a,m}$-terms of (7.1.3) are zero in each of the following cases:

$\circ\ m > 3 \qquad \circ\ m < a \qquad \circ\ a < 0 \qquad \circ\ a > 2.$

Here we have used the well-known fact that the $\mathrm{cd}_p(\kappa(x)) = 3 - a$ with for any $a \geqq 0$ and $x \in X^a$ (see e.g., [T] Theorem 3.1, [Se] Chapter II, §4.2 Proposition 11). There is a map of spectral sequences from (7.1.2) to (7.1.3) induced by cycle class maps of motivic cohomology groups by the commutative diagrams (2.3.3) and (2.3.4) in §2.3. The cycle class map

$$H^{m-a}_{\mathscr{M}}(x, \Lambda_n(r-a)) \longrightarrow H^{m-a}(x, \Lambda_n(r-a))$$

is bijective for any $a \geqq 0$, any point $x \in X^a$ and any $m \leqq r$ by Rost-Voevodsky [V1], [V2] Theorem 6.16 and Geisser-Levine [GL2] Theorem 7.5 (resp. Bloch-Gabber-Kato [BK1] Theorem 2.1 and Geisser-Levine [GL1] Theorem 1.1), when $\mathrm{ch}(x) \neq p$ (resp. when $\mathrm{ch}(x) = p$). If $r \geqq 3$, then the map $\mathrm{cl}^{m,r}_{\Lambda_n}$ in question is bijective by these facts. As for the case $r = 2$, it remains to check that the $E^{a,3}_\infty$-term of (7.1.3) is zero for any $a = 0, 1, 2$, which is a consequence of Kato's Hasse principle [KCT] p. 145, Corollary. □

Remark 7.2 If we assume the Beilinson-Soulé vanishing conjecture ([So2] p. 501, Conjecture) for points of X, then we would have

$$H^m_{\mathscr{M}}(X, \mathbb{Z}(r)) \cong \begin{cases} H^1_{\mathscr{M}}(K(X), \mathbb{Z}(r)) & (m = 1) \\ 0 & (m \leqq 0) \end{cases}$$

up to small torsion for any $r \geqq 2$, by the same arguments as in the proof of Lemma 7.1 (1).

7.2 p-adic Abel-Jacobi mappings and finiteness results

Let r be an integer with $r \geqq 2$. We define a p-adic cycle class map

$$\mathrm{cl}^{m,r}_p : H^m_{\mathscr{M}}(X, \mathbb{Z}(r)) \mathbin{\widehat{\otimes}} \mathbb{Z}_p \longrightarrow H^m(X, \mathbb{Z}_p(r))$$

as the projective limit with respect to $n \geqq 1$ of the cycle class map

$$\mathrm{cl}^{m,r}_{/p^n} : H^m_{\mathscr{M}}(X, \mathbb{Z}(r))/p^n \longrightarrow H^m_{\mathscr{M}}(X, \Lambda_n(r)) \xrightarrow[\simeq]{\mathrm{cl}^m_{\Lambda_n}} H^m(X, \mathfrak{T}_n(r)).$$

See Lemma 7.1 (3) for the isomorphism $\mathrm{cl}^{m,r}_{\Lambda_n}$. Since $X_{\overline{K}}$ is a curve, $H^m(X_{\overline{K}}, \mathbb{Z}_p(r))$ is torsion-free, and

$$H^0(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))) \subset H^m(X_{\overline{K}}, \mathbb{Z}_p(r))^{G_K} = 0 \tag{7.2.1}$$

by Proposition 3.6 (1) and for the reason of weights. We define a p-adic Abel-Jacobi mapping

$$\mathrm{aj}^{m,r}_p : H^m_{\mathscr{M}}(X, \mathbb{Z}(r)) \mathbin{\widehat{\otimes}} \mathbb{Z}_p \longrightarrow H^1(B, \mathfrak{H}^{m-1}(X, \mathbb{Z}_p(r)))$$

as the map induced by $\mathrm{cl}^{m,r}_p$ and an edge map of the spectral sequence (4.1.1):

$$E^{a,b}_2 = H^a(B, \mathfrak{H}^b(X, \mathbb{Z}_p(r))) \Longrightarrow H^{a+b}(X, \mathbb{Z}_p(r)). \tag{7.2.2}$$

We first observe the following abstract nonsense:

Proposition 7.3 *Let m and r be integers with $r \geqq 2$, and assume that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$. Then the following five conditions are equivalent to one another*:

(i) $\mathrm{aj}_p^{m,r}$ *has finite cokernel.*

(ii) $\mathrm{cl}_p^{m,r}$ *has finite cokernel.*

(iii) $\mathrm{cl}_p^{m,r}$ *is surjective.*

(iv) $H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ *is finite.*

(v) $H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r))_{p\text{-Div}}$ *is uniquely p-divisible.*

Moreover if $m \leqq 1$, these conditions are equivalent to

(i′) $\mathrm{aj}_p^{m,r}$ *is surjective.*

Proof. The E_2-term $E_2^{a,m}$ of (7.2.2) is finite for any $a \geqq 2$ by Theorems 6.1 (1) and 6.6, which shows (iii)⇒(i). The assertion (i)⇒(ii) is a consequnece of the following fact (a), and the assertion (ii)⇒(iii) is a consequnece of the following (b), where T_p denotes the p-Tate module:

(a) *The canonical map*

$$H^m(X, \mathbb{Z}_p(r)) \longrightarrow H^1(B, \mathfrak{H}^{m-1}(X, \mathbb{Z}_p(r)))$$

has finite kernel by Theorem 6.6.

(b) *By taking the projective limit with respect to $n \geqq 1$ of the short exact sequence of Lemma 7.1 (3), we see that* $\mathrm{Coker}(\mathrm{cl}_p^{m,r}) \cong T_p(H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r)))$, *where the latter group is torsion-free.*

We next prove (iii)⇔(iv). Indeed, by taking the inductive limit with respect to $n \geqq 1$ of the short exact sequence of Lemma 7.1 (3), we get an exact sequence

$$0 \to H^m_{\mathscr{M}}(X, \mathbb{Z}(r)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)) \to H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \to 0, \quad (7.2.3)$$

which imply that $H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is cofinitely generated over $\mathbb{Z}_p$, see Theorem 4.2 (2). Hence

$$\text{(iii)} \overset{\text{(b)}}{\iff} T_p(H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r))) = 0 \iff \text{(iv)}.$$

The assertion (iv)⇒(v) is obvious, and the assertion (v)⇒(iv) also follows from the fact that $H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is cofinitely generated over $\mathbb{Z}_p$. Finally, if $m \leqq 1$, the canonical map in (a) is bijective by (7.2.1), which shows that (iii) is equivalent to (i′). □

The following lemma will be useful in what follows.

Lemma 7.4 *Assume that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$. Then*:

(1) $\mathrm{cl}_p^{m,r}$ *is injective for any $m \in \mathbb{Z}$ and $r \geqq 2$.*

(2) *We have* $H^5(X, \mathfrak{T}_n(2)) \cong \Lambda_n$ *for any* $n \geqq 1$*, and* $H^m(X, \mathfrak{T}_n(r)) = 0$ *for any* $m \geqq 5$*,* $r \geqq 2$ *and* $n \geqq 1$ *with* $(m, r) \neq (5, 2)$*.*

Proof. The assertion (1) follows from Lemma 7.1 (3). The assertions in (5) follow from the duality (see Corollary 2.10 (2), (3.1.2))

$$H^m(X, \mathfrak{T}_n(r)) \cong H^{5-m}(X, \mathfrak{T}_n(2-r))^*.$$

The details are straight-forward and left to the reader. □

The following result gives an extension of the vanishing assertion in Lemma 7.1 (1):

Proposition 7.5 *Assume that* $p \geqq 3$ *or* $B(\mathbb{R}) = \emptyset$*. Then*

$$H^m_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}, \quad H^m_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p \quad \textit{and} \quad H^m(X, \mathbb{Z}_p(r))$$

are zero for any $m \geqq 5$ *and* $r \geqq 3$*. In particular,* $H^m_{\mathscr{M}}(X, \mathbb{Z}(r))$ *is uniquely* p*-divisible for the same* m *and* r*.*

Proof. We have $H^m(X, \mathbb{Z}_p(r)) = 0$ by Lemma 7.4 (2), so $H^m_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p = 0$ by Lemma 7.4 (1). To show that $H^m_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} = 0$, we use the surjectivity of the boundary map

$$H^{m-1}(X, \mathbb{Q}_p/\mathbb{Z}_p(r)) \twoheadrightarrow H^m_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$$

of (7.2.3). By Lemma 7.4 (2), we have $H^{m-1}(X, \mathbb{Q}_p/\mathbb{Z}_p(r)) = 0$ for any $m \geqq 6$, which implies that $H^m_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is zero for any $m \geqq 6$ by (7.2.3). As for the case $m = 5$, we have $H^4(X, \mathbb{Q}_p/\mathbb{Z}_p(r)) = 0$. Indeed, it is finite by Corollary 6.7, and p-divisible by the exact sequence obtained from Proposition 2.4

$$\cdots \longrightarrow H^4(X, \mathbb{Q}_p/\mathbb{Z}_p(r)) \xrightarrow{\times p} H^4(X, \mathbb{Q}_p/\mathbb{Z}_p(r)) \longrightarrow H^5(X, \mathfrak{T}_1(r)) \longrightarrow \cdots$$

and Lemma 7.4 (2). Thus $H^5_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is zero. □

Proposition 7.6 *Assume that* $p \geqq 3$ *or* $B(\mathbb{R}) = \emptyset$*. Then for any* $r \geqq 3$*, we have*

$$H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \cong H^4_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p \xrightarrow[\mathrm{cl}_p^{4,r}]{\simeq} H^4(X, \mathbb{Z}_p(r)),$$

which are all finite.

Proof. The cycle class map $\mathrm{cl}_p^{4,r}$ is injective by Lemma 7.4 (1), and surjective by Proposition 7.3 (iv)⇒(iii) and the vanishing of $H^5_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ in Proposition 7.5. The finiteness of $H^4(X, \mathbb{Z}_p(r))$ follows from Corollary 6.7.

We next prove that $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is finite. By Proposition 7.3 (i)⇒(iv), it is enough to check that the map

$$\mathrm{aj}_p^{3,r} : H^3_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p \longrightarrow H^1(B, \mathfrak{H}^2(X, \mathbb{Z}_p(r))) \cong H^1(B[p^{-1}], \mathbb{Z}_p(r-1))$$

has finite cokernel, where the last isomorphism follows from Proposition 3.4 (2) and Lemma 2.1 (2) for B. The finiteness of Coker($\mathrm{aj}_p^{3,r}$) follows from [Ka] Theorem 5.3 and a standard norm argument (see also [So1] Theorem 1 for the case $p \geqq 3$). Thus $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is finite.

Finally, the natural map $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \to H^4_{\mathscr{M}}(X, \mathbb{Z}(r)) \mathbin{\widehat{\otimes}} \mathbb{Z}_p$ is injective by the finiteness of $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$. To show the surjectivity of this map, consider the following commutative triangle:

$$\begin{array}{ccc} H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(r)) & \xrightarrow{\ \delta\ } & H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \\ & \searrow^{\delta'} & \downarrow {\scriptstyle \mathrm{cl}_p^{4,r}|_{\mathrm{tors}}} \\ & & H^4(X, \mathbb{Z}_p(r)), \end{array}$$

where the arrow δ denotes the boundary map of (7.2.3), and the arrow δ' denotes the boundary map of the long exact sequence obtained from Proposition 2.4

$$\cdots \to H^3(X, \mathbb{Q}_p(r)) \to H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(r)) \xrightarrow{\ \delta'\ } H^4(X, \mathbb{Z}_p(r)) \to H^4(X, \mathbb{Q}_p(r)) \to \cdots.$$

The arrow $\mathrm{cl}_p^{4,r}|_{\mathrm{tors}}$ means the restriction of $\mathrm{cl}_p^{4,r}$ to $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$. Since δ' is surjective by the finiteness of $H^4(X, \mathbb{Z}_p(r))$, $\mathrm{cl}_p^{4,r}|_{\mathrm{tors}}$ is surjective as well, which completes the proof. □

The following corollary is a summary of known facts and our results on $\mathrm{cl}_p^{m,r}$ and $\mathrm{aj}_p^{m,r}$:

Corollary 7.7 *Let r be an integer $\geqq 2$, and assume that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$. Then*:

(0) *$H^m_{\mathscr{M}}(X, \mathbb{Z}(r))$ is uniquely p-divisible for any $m \leqq 0$ and any $m \geqq 5$, and zero for any $m > r + 2$.*

(1) *$\mathrm{cl}_p^{1,r}$ and $\mathrm{aj}_p^{1,r}$ are injective.*

(2) *$\mathrm{cl}_p^{2,r}$ is injective, and $\mathrm{aj}_p^{2,r}$ has finite kernel.*

(3) *$\mathrm{cl}_p^{3,r}$ is bijective, and $\mathrm{aj}_p^{3,r}$ has finite kernel and cokernel.*

(4) *$\mathrm{cl}_p^{4,r}$ is bijective, and $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is finite. Moreover, $\mathrm{cl}_p^{4,r}$ induces an isomorphism $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \cong H^4(X, \mathbb{Z}_p(r))$, and $\mathrm{aj}_p^{4,r}$ is zero.*

Proof. The assertion (0) for $m \leqq 0$ follows from Lemmas 7.1 (3) (for $m < 0$) and 7.4 (1) (for $m = 0$) and the vanishing of $H^m(X, \mathfrak{T}_n(r))$ for $m < 0$ and $H^0(X, \mathbb{Z}_p(r))$. See Lemma 7.1 (1) and Proposition 7.5 for the other claims in (0). The injectivity of $\mathrm{cl}_p^{m,r}$ in (1)–(4) is the same as Lemma 7.4 (1), and the finiteness of Ker($\mathrm{aj}_p^{m,r}$) in (2)–(4) follows from (a) in the proof of Proposition 7.3. The injectivity of $\mathrm{aj}_p^{1,r}$ in (1) is that of $\mathrm{cl}_p^{1,r}$. By Proposition 7.3, the surjectivity of $\mathrm{cl}_p^{m,r}$ and the finiteness of Coker($\mathrm{aj}_p^{m,r}$) are both equivalent to the finiteness of $H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$. This last finiteness for the case $(m, r) = (3, 2)$ (resp. the case $(m, r) = (4, 2)$, the case $m \geqq 4$ and $r \geqq 3$) is a consequence of Lemma 7.1 (2) and the finiteness of $H^4_{\mathscr{M}}(X, \mathbb{Z}(2)) \cong \mathrm{CH}_0(X)$ due to Bloch [B1], Kato-Saito [KSa] (resp. Lemma 7.1 (1), Propositions 7.5 and 7.6). Finally, $\mathrm{aj}_p^{4,r}$ is zero for any $r \geqq 2$, because $H^1(B, \mathfrak{H}^3(X, \mathbb{Z}_p(r))) = 0$ by (3.1.4). □

7.3 p-Tate-Shafarevich groups

Let r be an integer with $r \geqq 2$. We put

$$H^1_{/f}(K, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r))) := \frac{H^1(K, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{\text{Image of } H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))}.$$

Let P (resp. P_∞) be the set of all places of K (resp. all infinite places of K). We often identify a finite place of K with a closed point of B. For each $v \in P$, we put

$$H^1_{/f}(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r))) := \frac{H^1(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{\text{Image of } H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))}$$

where we defined $H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) := 0$ for any $v \in P_\infty$. This group for $v \in B_0$ has been (defined and) used in the proof of Theorem 6.6. For $m \geqq 0$ and $r \geqq 2$ with $(m, r) \neq (2, 2)$, the natural map

$$\alpha^{m,r} : H^1_{/f}(K, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r))) \longrightarrow \bigoplus_{v \in P} H^1_{/f}(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r))) \qquad (7.3.1)$$

has finite kernel and cokernel, and we have

$$\text{Coker}(\alpha^{m,r}) \cong (H^{2-m}(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(2-r))^{G_K})^* \qquad (7.3.2)$$

by [BK2] Proposition 5.14 (i), (ii). The p-Tate-Shafarevich group of the motive $H^m(X_K)(r)$ is defined as $\text{Ker}(\alpha^{m,r})$ and often denoted by $Ш^{(p)}(H^m(X_K)(r))$. We fix a finite subset $S' \subset B_0$ containing all points of characteristic p and all points where X has bad reduction.

Theorem 7.8 *Assume that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$, and assume further that $H^3_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is finite. For each $v \in S'$ and $a = 2, 3$, we put*

$$e_v^{a,m,r} := \# H^a(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))),$$

which is finite by Corollary 5.6 (2). Then we have

$$\frac{\chi(\alpha^{1,2})}{\chi(\alpha^{0,2})} = \frac{\chi(\mathrm{aj}_p^{3,2})}{\chi(\mathrm{aj}_p^{2,2})} \cdot \frac{\#\mathrm{CH}_0(X)\{p\}}{\#\mathrm{Pic}(O_K)\{p\}} \cdot \prod_{v \in S'} \frac{e_v^{2,1,2} \cdot e_v^{3,0,2}}{e_v^{2,0,2} \cdot e_v^{3,1,2}} \qquad (r = 2)$$

$$\frac{\chi(\alpha^{1,r})}{\chi(\alpha^{0,r}) \cdot \chi(\alpha^{2,r})} = \frac{\chi(\mathrm{aj}_p^{3,r})}{\chi(\mathrm{aj}_p^{2,r})} \cdot \# H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \cdot \prod_{v \in S'} \frac{e_v^{2,1,r} \cdot e_v^{3,0,r} \cdot e_v^{3,2,r}}{e_v^{2,0,r} \cdot e_v^{2,2,r} \cdot e_v^{3,1,r}} \qquad (r \geqq 3),$$

where we put $\chi(f) := \#\mathrm{Coker}(f)/\#\mathrm{Ker}(f)$ for homomorphisms $f : M \to N$ of abelian groups with finite kernel and cokernel.

See Proposition 7.6 for the finiteness of $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$. The alternate products of local terms $e_v^{a,m,r}$ will be computed in §8 below. To prove Theorem 7.8, we first prove Lemma 7.9 below as a preparation, which relies on the assumption that $d = 2$. We put

$$H^1_{/f}(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) := \frac{H^1(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{\text{Image of } H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))}$$

using Corollary 6.4. For each $v \in B_0$, we put

$$H^1_{/f}(B_v, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) := \frac{H^1(B_v, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{\text{Image of } H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))}$$

cf. Corollary 5.6 (1).

Lemma 7.9 *There are canonical isomorphisms of finite p-groups*

$$H^1_{/f}(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \cong H^2(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))), \tag{7.3.3}$$
$$H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \cong H^3(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))), \tag{7.3.4}$$

for any $m \geqq 0$ and $r \geqq 2$. Similarly, there are canonical isomorphisms of finite p-groups

$$H^1_{/f}(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) \cong H^2(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))), \tag{7.3.5}$$
$$H^2(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) \cong H^3(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))). \tag{7.3.6}$$

for any $m \geqq 0$, $r \geqq 2$ and $v \in B_0$. Moreover, the groups in (7.3.5) *and* (7.3.6) *are zero for any $v \in B_0 \smallsetminus S'$.*

Proof. We prove only (7.3.3) and (7.3.4), and omit the proof of (7.3.5) and (7.3.6). We start with the following short exact sequence on $X_{\text{ét}}$, which is a simple case of Proposition 2.4:

$$0 \longrightarrow \mathfrak{T}_{n'}(2-r) \longrightarrow \mathfrak{T}_{n'+n}(2-r) \longrightarrow \mathfrak{T}_n(2-r) \longrightarrow 0.$$

Since the fibers of $\pi_{X/B} : X \to B$ are proper curves, the associated long exact sequence of higher direct image sheaves breaks up into short exact sequneces on $B_{\text{ét}}$

$$0 \to R^{2-m}\pi_{X/B*}\mathfrak{T}_{n'}(2-r) \to R^{2-m}\pi_{X/B*}\mathfrak{T}_{n'+n}(2-r) \to R^{2-m}\pi_{X/B*}\mathfrak{T}_n(2-r) \to 0$$

for $m = 0, 1, 2$, which yield distinguished triangles in $D(B_{\text{ét}})$

$$\mathfrak{H}^m(X, \mathfrak{T}_n(r)) \longrightarrow \mathfrak{H}^m(X, \mathfrak{T}_{n+n'}(r)) \longrightarrow \mathfrak{H}^m(X, \mathfrak{T}_{n'}(r)) \longrightarrow \mathfrak{H}^m(X, \mathfrak{T}_n(r))[1].$$

One obtains the following long exact sequence by the finiteness in Proposition 4.1 and a standard argument:

$$\cdots \to H^a(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) \to H^a(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \to H^{a+1}(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))) \\ \to H^{a+1}(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r))) \to \cdots$$

Now (7.3.4) follows from the finiteness of $H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ (Theorem 6.6) and the vanishing of $H^3(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r)))$ (Theorem 6.1 (1)). Similarly, (7.3.3) follows from Theorems 4.2 (2) and 6.1 (2) and the vanishing of $H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p(r)))$.Finally, the groups on the right hand side of (7.3.5) and (7.3.6) are zero for any $v \in B_0 \smallsetminus S'$ by Lemma 5.2. □

Proof of Theorem 7.8. The map $\mathrm{cl}_p^{m,r}$ is bijective for $m = 2$ by the finiteness assumption on $H^3_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ (see Proposition 7.3 (iv)⇒(iii)), and bijective for $m = 3, 4$ by Corollary 7.7 (3), (4). In particular for $m = 2, 3$, the map $\mathrm{aj}_p^{m,r}$ is identified with the canonical map

$$H^m(X, \mathbb{Z}_p(r)) \longrightarrow H^1(B, \mathfrak{H}^{m-1}(X, \mathbb{Z}_p(r))).$$

We put $e^{a,m,r} := \# H^a(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r)))$ for each $a \geqq 2$, $m \geqq 0$ and $r \geqq 2$ with $(a, m, r) \neq (3, 2, 2)$, which is finite by Theorems 6.1 (1) and 6.6. One can easily derive an equality

$$\frac{\chi(\mathrm{aj}_p^{3,r})}{\chi(\mathrm{aj}_p^{2,r})} = \frac{e^{2,0,r} \cdot e^{2,2,r} \cdot e^{3,1,r}}{e^{2,1,r} \cdot e^{3,0,r} \cdot \# H^4(X, \mathbb{Z}_p(r))}$$

for any $r \geqq 2$, from the spectral sequence (7.2.2) and the vanishing (7.2.1). Therefore by Corollary 7.7 (4) and the isomorphisms

$$\begin{aligned} H^2(B, \mathfrak{H}^2(X, \mathbb{Z}_p(2))) &\overset{(3.1.7)}{\cong} H^2(B, \mathbb{Z}_p(1)) \cong \mathrm{Pic}(O_K) \otimes \mathbb{Z}_p \cong \mathrm{Pic}(O_K)\{p\}, \\ H^3(B, \mathfrak{H}^2(X, \mathbb{Z}_p(r))) &\cong H^3(B, \mathbb{Z}_p(r-1)) \cong H^3(B[p^{-1}], \mathbb{Z}_p(r-1)) = 0 \qquad (r \geqq 3) \end{aligned}$$

we are reduced to showing that

$$\chi(\alpha^{m,r}) = \frac{e^{3,m,r}}{e^{2,m,r}} \times \prod_{v \in S'} \frac{e_v^{2,m,r}}{e_v^{3,m,r}} \qquad \text{for } {}^{\forall}(m, r) \neq (2, 2), \ \ r \geqq 2. \qquad (7.3.7)$$

To prove (7.3.7), we use the same notation as in the proof of Theorem 6.6, and consider the following commutative diagram with exact rows for $(m, r) \neq (2, 2)$ with $r \geqq 2$, where the coefficients $\mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))$ in the upper row and $\mathfrak{H}^m(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))$ in the lower row are omitted:

$$\begin{array}{ccccccc} H^1_{/f}(B) & \hookrightarrow & H^1_{/f}(K) & \longrightarrow & \bigoplus_{v \in B_0} H^2_v(B) & \xrightarrow{(*)} & \\ \downarrow & & \downarrow \alpha^{m,r} & & \delta \downarrow \wr & & \\ \bigoplus_{v \in B_0} H^1_{/f}(B_v) & \hookrightarrow & \bigoplus_{v \in B_0} H^1_{/f}(K_v) & \longrightarrow & \bigoplus_{v \in B_0} H^2_v(B_v) & \xrightarrow{(**)} & \end{array}$$

$$\begin{array}{ccccccc} \xrightarrow{(*)} & H^2(B) & \longrightarrow & H^2(K) & \longrightarrow & \bigoplus_{v \in B_0} H^3_v(B) \\ & \beta \downarrow & & \gamma \downarrow \wr & & \delta \downarrow \wr \\ \xrightarrow{(**)} & \bigoplus_{v \in B_0} H^2(B_v) & \longrightarrow & \bigoplus_{v \in B_0} H^2(K_v) & \longrightarrow & \bigoplus_{v \in B_0} H^3_v(B_v) \end{array}$$

In this diagram, the arrows δ are bijective as explained in the proof of Theorem 6.6. The arrow γ is bijective by the Hasse principle of Jannsen ([J] p. 337, Theorem 3 (d)) and the fact that $H^m(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r))$ is divisible. From the above commutative diagram, we obtain a six-term exact sequence

$$0 \to \mathrm{Ker}(\alpha^{m,r}) \to H^1_{/f}(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \to \bigoplus_{v \in B_0} H^1_{/f}(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r)))$$
$$\to \mathrm{Coker}(\alpha^{m,r}) \to H^2(B, \mathfrak{H}^m(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \to \bigoplus_{v \in B_0} H^2(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) \to 0.$$

By Lemma 7.9, this sequence yields an exact sequence of the following from:

$$0 \longrightarrow \mathrm{Ker}(\alpha^{m,r}) \longrightarrow H^2(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))) \longrightarrow \bigoplus_{v \in S'} H^2(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r)))$$
$$\longrightarrow \mathrm{Coker}(\alpha^{m,r}) \longrightarrow H^3(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))) \longrightarrow \bigoplus_{v \in S'} H^3(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))) \longrightarrow 0,$$

which implies the formula (7.3.7). □

8 Local terms and zeta values $(d = 2)$

In this section, we compute the local terms $e_v^{2,m,r}$ and $e_v^{3,m,r}$ that appear in Theorem 7.8. The results in §§8.1–8.2 below were obtained in discussions with Takao Yamazaki.

The setting and the notation remain as in §7. In particular, we assume $d = 2$. We further fix the following notation. For a finite place v of K, we write k_v (resp. Y_v, $Y_{\overline{v}}$) for the residue field at v (resp. $X \otimes_{O_K} k_v$, $X \otimes_{O_K} \overline{k_v}$), and X_v (resp. $X_{\overline{v}}$) for $X \otimes_{O_K} O_v$ (resp. $X \otimes_{O_K} O_{\overline{v}}^{\mathrm{sh}}$), where O_v (resp. $O_{\overline{v}}^{\mathrm{sh}}$) denotes the completion of O_K at v (resp. the strict henselization of O_v at its maximal ideal). We put $q_v := \# k_v$.

8.1 Comparison with local points

We first show the following lemma, which refines the case of $q = 1$ of Theorem 5.3 under the assumption that $d = 2$:

Lemma 8.1 *We have*

$$H^1(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))) = H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Z}_p(r)))$$

as subgroups of $H^1(K_v, H^m(X_{\overline{K}}, \mathbb{Z}_p(r)))$, *for any finite place* v *of* K, $m \geqq 0$ *and* $r \geqq 2$.

Proof. Consider a commutative diagram

$$\begin{array}{ccccc} H^1(K_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))) & & \xrightarrow{\quad d \quad} & & H^2_v(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))) \\ \| & & & & \downarrow b \\ H^1(K_v, H^m(X_{\overline{K}}, \mathbb{Z}_p(r))) & \xrightarrow{a} & H^1_{/f}(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) & \xhookrightarrow{d'} & H^2_v(B_v, \mathfrak{H}^m(X_v, \mathbb{Q}_p(r))), \end{array}$$

where the arrows d and d' are connecting maps of localization sequences of cohomology of B_v, and the existence and the injectivity of d' is a consequence of Theorem 5.3 for $q = 1$. The arrow a is a natural map, and we have

$$\mathrm{Ker}(a) = H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Z}_p(r))).$$

On the other hand, since $H^1_v(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))) = 0$ by Proposition 3.6 (1), we have

$$\mathrm{Ker}(d) = H^1(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))).$$

Thus it remains to check that the arrow b is injective, which follows from the facts that

$$H^2_v(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))) = 0 \qquad \text{if } v|p \text{ and } r \geqq 3 \qquad \text{(Corollary 3.7 (1))}$$

and that otherwise

$$H^2_v(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r))) \cong H^1(k_v, H^{2-m}(Y_{\overline{v}}, \mathbb{Q}_p/\mathbb{Z}_p(2-r)))^* \qquad \text{([Ma] (2.4))}$$

is torsion-free because $\dim(Y_v) = 1$ and $\mathrm{cd}(k_v) = 1$. □

The following corollary follows from Proposition 3.6 (1), Lemma 8.1 and a similar argument as in the proof of Lemma 6.2:

Corollary 8.2 *We have*

$$H^1(B, \mathfrak{H}^m(X, \mathbb{Z}_p(r))) = H^1_f(K, H^m(X_{\overline{K}}, \mathbb{Z}_p(r)))$$

as subgroups of $H^1(K, H^m(X_{\overline{K}}, \mathbb{Z}_p(r)))$*, for any* $m \geqq 0$ *and* $r \geqq 2$.

8.2 Comparison with zeta values of the fibers (the case $v \nmid p$)

In this subsection, we always assume that $v \nmid p$ and $r \geqq 2$. Note that $H^a(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r)))$ is finite for any (a, m, r) by Theorems 4.2 (1) and 5.3, and zero unless $a = 0, 1, 2, 3$ and $m = 0, 1, 2$. We put

$$e_v^{a,m,r} := \# H^a(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r)))$$

for each (a, m, r). Note that $\zeta(Y_v, r)$ is a non-zero rational number, since $\dim(Y_v) = 1$. Let $|\ |_p$ be the p-adic absolute value on $\mathbb{Q}_p$ such that $|p|_p = p^{-1}$.

Lemma 8.3 *We have*

$$|\zeta(Y_v, r)|_p^{-1} = \prod_{(a,m)} (e_v^{a,m,r})^{(-1)^{a+m}},$$

where (a, m) *on the right hand side runs through all pairs with* $0 \leqq a \leqq 3$ *and* $0 \leqq m \leqq 2$.

Proof. Let G_v be the absolute Galois group of k_v, and let T_p be a free $\mathbb{Z}_p$-module of finite rank on which G_v acts continuously and $\mathbb{Z}_p$-linearly. It is well-known that

$$\# H^1(k_v, T_p) = \left| \det_{\mathbb{Q}_p}(1 - \varphi_v^{-1} \,|\, T_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \right|_p^{-1}, \qquad (8.2.1)$$

where $\varphi_v \in G_v$ denotes the arithmetic Frobenius element. Now let Fr_v be the geometric Frobenius operator acting on $H^i(Y_{\overline{v}}, \mathbb{Q}_p)$. We have $\varphi_v = q_v^r \cdot \mathrm{Fr}_v^{-1}$ on $H^i(Y_{\overline{v}}, \mathbb{Q}_p(r))$, and

$$\begin{aligned}
|\zeta(Y_v, r)|_p^{-1} &= \prod_{i \geqq 0} \left|\det_{\mathbb{Q}_p}(1 - q_v^{-r} \cdot \mathrm{Fr}_v \,|\, H^i(Y_{\overline{v}}, \mathbb{Q}_p))\right|_p^{(-1)^i} && \text{(trace formula [G], §2)} \\
&= \prod_{i \geqq 0} (\# H^1(k_v, H^i(Y_{\overline{v}}, \mathbb{Z}_p(r))))^{(-1)^{i+1}} && \text{(by (8.2.1))} \\
&\overset{(\star)}{=} \prod_{i \geqq 0} (\# H^i(Y_v, \mathbb{Z}_p(r)))^{(-1)^i} && \text{(see below)} \\
&= \prod_{i \geqq 0} (\# H^i(X_v, \mathbb{Z}_p(r)))^{(-1)^i} && \text{(proper base change)} \\
&= \prod_{(a,m)} (e_v^{a,m,r})^{(-1)^{a+m}} && \text{(spectral sequence (4.1.1))}
\end{aligned}$$

as claimed, where the equality $(\star)$ follows from the fact that $H^i(Y_{\overline{v}}, \mathbb{Z}_p(r))^{G_v} = 0$ for any $i \geqq 0$ (because $\dim(Y_v) = 1$ and $r \geqq 2$). □

As a consequence of Lemma 8.1 and (8.2.1) we obtain:

Corollary 8.4 *Assume that X_v is smooth over O_v. Then we have*

$$\# H^1_f(K_v, H^1(X_{\overline{K}}, \mathbb{Z}_p(r))) = \left|\det_{\mathbb{Q}_p}(1 - q_v^{-r} \cdot \mathrm{Fr}_v \,|\, H^1(Y_{\overline{v}}, \mathbb{Q}_p))\right|_p^{-1}.$$

The following theorem extends Corollary 8.4 to the general $v \nmid p$ case (see also Lemma 5.2):

Theorem 8.5 *We have $e_v^{a,2,r} = 1$ for $a = 2, 3$, and*

$$\frac{\# H^1_f(K_v, H^1(X_{\overline{K}}, \mathbb{Z}_p(r)))}{\left|\zeta(Y_v, r)(1 - q_v^{1-r})(1 - q_v^{-r})\right|_p^{-1}} = \frac{e_v^{2,1,r} \cdot e_v^{3,0,r}}{e_v^{2,0,r} \cdot e_v^{3,1,r}}.$$

Proof. We first show that $e_v^{a,2,r} = 1$ for $a = 2, 3$. Indeed, we have

$$H^a(B_v, \mathfrak{H}^2(X_v, \mathbb{Z}_p(r))) \overset{(3.1.7)}{\cong} H^a(B_v, \mathbb{Z}_p(r-1)) \cong H^a(v, \mathbb{Z}_p(r-1)) = 0$$

for any $a \geqq 2$. To prove the second assertion, we note the following facts:

(a) $e_v^{0,m,r} = 1$ *for any $m \geqq 0$, by Proposition 3.6 (1), Theorem 5.3 and the fact that $H^m(X_{\overline{K}}, \mathbb{Z}_p)$ is torsion-free.*

(b) $e_v^{1,m,r} = \# H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Z}_p(r)))$ *by Lemma 8.1.*

(c) $e_v^{1,0,r} = |1 - q_v^{-r}|_p^{-1}$ *and* $e_v^{1,2,r} = |1 - q_v^{1-r}|_p^{-1}$, *by (b) and* [BK2] *Theorem 4.1 (i).*

Combining these facts with Lemma 8.3, we have

$$\begin{aligned}
&\left|\zeta(Y_v,r)(1-q_v^{1-r})(1-q_v^{-r})\right|_p^{-1}\\
&= \left|(1-q_v^{1-r})(1-q_v^{-r})\right|_p^{-1}\cdot\frac{e_v^{1,1,r}\cdot e_v^{2,2,r}}{e_v^{1,0,r}\cdot e_v^{1,2,r}\cdot e_v^{3,2,r}}\cdot\frac{e_v^{2,0,r}\cdot e_v^{3,1,r}}{e_v^{2,1,r}\cdot e_v^{3,0,r}} \qquad \text{(Lemma 8.3 and (a))}\\
&= \#H_f^1(K_v,H^1(X_{\overline{K}},\mathbb{Z}_p(r)))\cdot\frac{e_v^{2,0,r}\cdot e_v^{3,1,r}}{e_v^{2,1,r}\cdot e_v^{3,0,r}}, \qquad \text{((b), (c), } e_v^{2,2,r}=e_v^{3,2,r}=1)
\end{aligned}$$

which shows the assertion. □

8.3 Comparison with zeta values of the fibers (the case $v|p$)

Let v be a finite place of K dividing p. For each $m = 0, 1, 2$, we fix a Haar measure μ_v^m on $H^m_{\mathrm{dR}}(X_{K_v}/K_v)$ such that

$$\mu_v^m(H^m_{\mathrm{dR}}(X_v/O_v)) = 1.$$

Via the exponential isomorphism of Corollary 5.11:

$$\exp : H^m_{\mathrm{dR}}(X_{K_v}/K_v) \xrightarrow{\simeq} H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) \qquad (r \geqq 2),$$

we regard μ_v^m as a Haar measure on $H^1_f(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))$. Let $K_0 = K_{v,0}$ be the fraction field of the Witt ring $W := W(k_v)$, and let σ be the Frobenius automorphism of K_0. Let $|\ |_p$ be the p-adic absolute value on $\mathbb{Q}$ such that $|p|_p = p^{-1}$. We prove here a p-adic counterpart of Theorem 8.5 under some assumptions.

Theorem 8.6 *Assume that $p - 2 \geqq r \geqq 2$, $K_v/\mathbb{Q}_p$ is unramified, and that X_v is smooth over O_v. Then we have $e_v^{a,2,r} = 1$ for $a = 2, 3$, and*

$$\frac{\mu_v^1(H^1_f(K_v, H^1(X_{\overline{K}}, \mathbb{Z}_p(r))))}{\left|\zeta(Y_v, r)(1 - q_v^{1-r})(1 - q_v^{-r})\right|_p^{-1}} = \frac{e_v^{2,1,r}\cdot e_v^{3,0,r}}{e_v^{2,0,r}\cdot e_v^{3,1,r}},$$

where we put $e_v^{a,m,r} := \#H^a(B_v, \mathfrak{H}^m(X_v, \mathbb{Z}_p(r)))$ for $a \neq 1$.

To prove this theorem, we need Lemma 8.7 below, which is a p-adic analogue of Lemma 8.3. For a homomorphism $\phi : M \to N$ of locally compact $\mathbb{Z}_p$-modules with finite kernel and with open image, and for a Haar measure ν on N, we define a Haar measure μ' on M by

$$\mu'(Z) := \sum_{i=1}^{r} \nu(\phi(Z_i))$$

for any Borel subset $Z \subset M$, where $Z = Z_1 \amalg Z_2 \amalg \cdots \amalg Z_r$ is a partition of Z by Borel subsets $Z_1, Z_2, \ldots, Z_r$ with each $\phi|_{Z_i}$ injective. We call μ' the *measure induced by* ν and often denote it by ν.

Lemma 8.7 *Under the same assumptions as in Theorem 8.6, we have*

$$|\zeta(Y_v, r)|_p^{-1} = \prod_{(a,m)} (e_v^{a,m,r})^{(-1)^{a+m}}$$

where (a, m) *on the right hand side runs through all pairs with* $0 \leqq a \leqq 3$ *and* $0 \leqq m \leqq 2$; *we put*

$$e_v^{1,m,r} := \mu_v^m(\boldsymbol{H}^1(B_v, \mathfrak{H}^m(X, \mathbb{Z}_p(r))))$$

with μ_v^m *the measure induced by* μ_v^m *on* $\boldsymbol{H}_f^1(K_v, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))$.

Proof. We first note that $e_v^{0,m,r} = 1$ for any $m \geqq 0$, by Proposition 3.6 (1), Theorem 5.3 and the fact that $H^m(X_{\overline{K}}, \mathbb{Z}_p)$ is torsion-free. Hence there exists an edge map induced by the spectral sequence (4.1.1)

$$\boldsymbol{H}^{m+1}(X_v, \mathbb{Z}_p(r)) \longrightarrow \boldsymbol{H}^1(B_v, \mathfrak{H}^m(X, \mathbb{Z}_p(r))),$$

which has finite kernel and cokernel by Theorem 5.3. We have the Haar measure μ_v^m on $\boldsymbol{H}^{m+1}(X_v, \mathbb{Z}_p(r))$ induced by that on $\boldsymbol{H}^1(B_v, \mathfrak{H}^m(X, \mathbb{Z}_p(r)))$, and

$$\prod_{i \geqq 0} \mu_v^{i-1}(\boldsymbol{H}^i(X_v, \mathbb{Z}_p(r)))^{(-1)^i} = \prod_{(a,m)} (e_v^{a,m,r})^{(-1)^{a+m}}$$

by the spectral sequence (4.1.1). It remains to show that

$$|\zeta(Y_v, r)|_p^{-1} = \prod_{i \geqq 0} \mu_v^{i-1}(\boldsymbol{H}^i(X_v, \mathbb{Z}_p(r)))^{(-1)^i}. \tag{8.3.1}$$

By the assumption on O_v, it is isomorphic to $W := W(k_v)$, the ring of Witt vectors in k_v. For each $n \geqq 1$, we put $X_n := X_v \otimes_W W_n$, and let $\mathcal{S}_n(r)_{X_v}$ be the syntomic complex associated with the smooth scheme X_v over $W = O_v$. Let $p(r)\Omega^{\bullet}_{X_n/W_n}$ (resp. $p(r)\Omega^{\bullet}_{X_v/W}$) be the subcomplex

$$p^r \cdot \mathscr{O}_{X_n} \xrightarrow{d} p^{r-1} \cdot \Omega^1_{X_n/W_n} \qquad \big(\text{resp. } p^r \cdot \mathscr{O}_{X_v} \xrightarrow{d} p^{r-1} \cdot \Omega^1_{X_v/W}\big)$$

of the de Rham complex $\Omega^{\bullet}_{X_n/W_n}$ (resp. $\Omega^{\bullet}_{X_v/W}$). We note the following facts:

(a) *There exists an isomorphism*

$$\big(p(r)\Omega^{\bullet}_{X_n/W_n}\big)_n[-1] \cong (\mathcal{S}_n(r)_{X_v})_n$$

for any r *with* $2 \leqq r < p$ *in the derived category of complexes of pro-sheaves on* $(Y_v)_{\text{ét}}$, *by* [BEK] *Theorem 5.4.*

(b) *The Euler characteristic*

$$\begin{aligned}\chi(X_v, \Omega^{\bullet}_{X_v/W}/p(r)\Omega^{\bullet}_{X_v/W}) &:= \prod_{i \geqq 0} (\#H^i(X_v, \Omega^{\bullet}_{X_v/W}/p(r)\Omega^{\bullet}_{X_v/W}))^{(-1)^i} \\ &= \prod_{(a,b)} (\#H^a(Y_v, \Omega^b_{Y_v/k_v}))^{(-1)^{a+b}(r-m)}\end{aligned}$$

agrees with $|\zeta(Y_v, r)|_p^{-1}$ ([Mi2] *Theorem 0.1*).

(c) *We have* $\mathcal{S}_n(r)_{X_v} \cong i^*\mathfrak{T}_n(r)$ *in* $D(Y_v, \Lambda_n)$ *for any* r *with* $r < p-1$ *and any* $n \geqq 1$ ([Ku] *p. 275, Theorem), where* i *denotes the closed immersion* $Y_v \hookrightarrow X_v$.

By these facts, we have

$$\begin{aligned}
|\zeta(Y_v, r)|_p^{-1} &= \chi(X_v, \Omega^\bullet_{X_v/W}/p(r)\Omega^\bullet_{X_v/W}) && \text{(by (b))}\\
&= \prod_{i \geqq 0} \frac{\mu_v^i(H^i_{\mathrm{dR}}(X_v/W))^{(-1)^i}}{\mu_v^i(H^i(X_v, p(r)\Omega^\bullet_{X_v/W}))^{(-1)^i}}\\
&= \prod_{i \geqq 0} \mu_v^i(H^i(X_v, p(r)\Omega^\bullet_{X_v/W}))^{(-1)^{i+1}} && (\mu_v^i(H^i_{\mathrm{dR}}(X_v/W)) = 1)\\
&= \prod_{i \geqq 0} \mu_v^i(H^{i+1}(X_v, \mathbb{Z}_p(r)))^{(-1)^{i+1}} && \text{(by (a), (c)).}
\end{aligned}$$

Thus we obtain (8.3.1) and Lemma 8.7. □

Proof of Theorem 8.6. We first show that $e_v^{a,2,r} = 1$ for any $a \geqq 2$. Indeed, we have

$$H^a(B_v, \mathfrak{H}^2(X_v, \mathbb{Z}_p(r))) \overset{(3.1.7)}{\cong} H^a(B_v, \mathbb{Z}_p(r-1)).$$

If $r = 2$, then the last group is zero for any $a \geqq 2$ because $H^a(B_v, \mathbb{G}_{\mathrm{m}}) = 0$ for any $a \geqq 1$. On the other hand, if $r \geqq 3$, then by the Tate duality, we have

$$H^a(B_v, \mathbb{Z}_p(r-1)) \cong H^a(K_v, \mathbb{Z}_p(r-1)) \cong H^{2-a}(K_v, \mathbb{Q}_p/\mathbb{Z}_p(2-r))^*,$$

which is zero for any $a \geqq 2$ by the assumption that K_v is unramified over $\mathbb{Q}_p$. Noting that

(a$^+$) $e_v^{1,0,r} = |1 - q_v^{-r}|_p^{-1}$ *and* $e_v^{1,2,r} = |1 - q_v^{1-r}|_p^{-1}$ *by* [BK2] *Theorem 4.1 (iii) for* $V = \mathbb{Q}_p(r)$ *and* $\mathbb{Q}_p(r-1)$*, and again by the assumption that* K_v *is unramified over* $\mathbb{Q}_p$*,*

one obtains the second assertion from the same computations as in Theorem 8.5. □

9 Global points and zeta values $(d = 2)$

The setting and the notation remain as in §7 (in particular, $d = 2$). In this section, we relate the formula in Theorem 7.8 with zeta values assuming Conjecture 9.1 below for the motives $H^m(X_K)(r)$ with $m = 0, 1, 2$, a weak version of p-Tamagawa number conjecture of Bloch-Kato [BK2] §5. Let S' be a finite set of closed points of B containing all points of characteristic p, and all points where X has bad reduction. For $m = 0, 1, 2$ and $r \geqq 2$ with $(m, r) \neq (2, 2)$, we put

$$L_{S'}(H^m(X_K), r) := \prod_{v \in B_0 \smallsetminus S'} \det(1 - q_v^{-r} \cdot \mathrm{Fr}_v \,|\, H^m(X_{\overline{K}}, \mathbb{Q}_p))^{-1} = \prod_{v \in B_0 \smallsetminus S'} (\# A_p^{m,r}(K_v))^{-1}.$$

This infinite product on the right hand side converges, because $m - 2r \leqq -3$. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal (p).

9.1 p-Tamagawa number conjecture

Conjecture 9.1 (Bloch-Kato) *For any $m = 0, 1, 2$ and $r \geqq 2$ with $(m, r) \neq (2, 2)$, there exists a finite-dimensional $\mathbb{Q}$-subspace $\Phi^{m,r} = \Phi_p^{m,r}$ of the $\mathbb{Q}$-space*

$$H_{\mathscr{M}}^{m+1}(X_K, \mathbb{Q}(r))_{\mathbb{Z}} := \mathrm{Im}\big(H_{\mathscr{M}}^{m+1}(X, \mathbb{Q}(r)) \to H_{\mathscr{M}}^{m+1}(X_K, \mathbb{Q}(r))\big)$$

satisfying the following conditions (i) *and* (ii):

(i) *The p-adic Abel-Jacobi map*

$$H_{\mathscr{M}}^{m+1}(X_K, \mathbb{Q}(r)) \longrightarrow H^1(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))$$

induces an isomorphism $\Phi^{m,r} \otimes \mathbb{Q}_p \cong H_f^1(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r)))$, and Beilinson's regulator map to the real Deligne cohomology

$$H_{\mathscr{M}}^{m+1}(X_K, \mathbb{Q}(r)) \longrightarrow H_{\mathscr{D}}^{m+1}(X_{/\mathbb{R}}, \mathbb{R}(r))$$

induces an isomorphism $\Phi^{m,r} \otimes \mathbb{R} \cong H_{\mathscr{D}}^{m+1}(X_{/\mathbb{R}}, \mathbb{R}(r))$.

(ii) *We define $A_p^{m,r}(K)$, the group of p-global points as the pull-back of $\Phi^{m,r}$ under the natural map*

$$H_f^1(K, H^m(X_{\overline{K}}, \mathbb{Z}_p(r))) \longrightarrow H_f^1(K, H^m(X_{\overline{K}}, \mathbb{Q}_p(r))) \cong \Phi^{m,r} \otimes \mathbb{Q}_p,$$

which is a finitely generated $\mathbb{Z}_{(p)}$-module. We further fix an O_K-lattice L^m of the de Rham cohomology $H_{\mathrm{dR}}^m(X_K/K)$, and define a number $R_\Phi^{m,r} \in \mathbb{R}^\times/\mathbb{Z}_{(p)}^\times$ to be the volume of the space

$$H_{\mathscr{D}}^{m+1}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))/\textit{Image of } A_p^{m,r}(K)$$

with respect to L^m. See Remark 9.2 (1) *below for an explicit description of the real Deligne cohomology $H_{\mathscr{D}}^{m+1}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))$. On the other hand, for each $v \in B_0$ we put*

$$A_p^{m,r}(K_v) := H_f^1(K_v, H^m(X_{\overline{K}}, \mathbb{Z}_p(r))),$$

which we call the group of p-local points at v. Then we have

$$L_{S'}(H^m(X_K), r) \equiv \chi(\alpha^{m,r})^{-1} \cdot R_\Phi^{m,r} \cdot \prod_{v \in S'} \mu_v^m(A_p^{m,r}(K_v)) \mod \mathbb{Z}_{(p)}^\times, \qquad (9.1.1)$$

where μ_v^m for $v \nmid p$ means the cardinarity, and μ_v^m for $v | p$ denotes the Haar measure on $A_p^{m,r}(K_v)$ constructed from that on $H_{\mathrm{dR}}^m(X_{K_v}/K_v)$ such that $\mu_v^m(L^m \otimes_{O_K} O_v) = 1$; see (7.3.1) *for the map $\alpha^{m,r}$.*

Remark 9.2 (1) The map $A_p^{m,r}(K) \to H_{\mathscr{D}}^{m+1}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))$ induced by the regulator map is injective, by the condition (i) for $\Phi^{m,r}$ and [BK2] Lemma 5.10. Here

$$H_{\mathscr{D}}^{m+1}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r)) = \left(\frac{H_{\mathrm{dR}}^m(X/\mathbb{Z}) \otimes \mathbb{C}}{H_{\mathrm{sing}}^m(X \otimes_{\mathbb{Z}} \mathbb{C}, (2\pi i)^r \cdot \mathbb{Z}_{(p)})} \right)^+$$

for any $m = 0, 1, 2$ and $r \geqq 2$, by definition.

(2) The product on the right hand side of (9.1.1) is independent of the choice of L^m.

(3) Conjecture 9.1 for $m = 0$ (resp. $m = 2$) implies that

$$\zeta_K(r) \equiv \chi(\alpha^{0,r})^{-1} \cdot R_\Phi^{0,r} \qquad \text{(resp. } \zeta_K(r-1) \equiv \chi(\alpha^{0,r-1})^{-1} \cdot R_\Phi^{0,r-1}\text{)}$$

modulo $\mathbb{Z}_{(p)}^\times$ if $r \geqq 2$ (resp. $r \geqq 3$) and p is unramified in K. Here we have used the fact (c) in the proof of Theorem 8.5 for all $v \not| p$ belonging to S', and the fact (a$^+$) in the proof of Theorem 8.6. See also [FM] §5.8.3.

(4) We have $R_\Phi^{m,r} = 1$ for any $m \geqq 3$, because $H_{\mathscr{D}}^{m+1}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))$ is zero for such m's.

(5) If $(m, r) = (2, 2)$, there exists a $\mathbb{Q}$-subspace $\Phi^{2,2}$ of $H_{\mathscr{M}}^3(X_K, \mathbb{Q}(2))_\mathbb{Z}$ which is isomorphic to $H_{\mathscr{M}}^1(B, \mathbb{Q}(1))$ under the push-forward map

$$H_{\mathscr{M}}^3(X_K, \mathbb{Q}(2)) \longrightarrow H_{\mathscr{M}}^1(\mathrm{Spec}(K), \mathbb{Q}(1)) \cong K^\times \otimes \mathbb{Q}.$$

Indeed, by a standard norm argument, the push-forward map

$$H_{\mathscr{M}}^3(X, \mathbb{Q}(2)) \longrightarrow H_{\mathscr{M}}^1(B, \mathbb{Q}(1)) \cong O_K^\times \otimes \mathbb{Q}$$

is surjective, and there is a $\mathbb{Q}$-subspace $\widetilde{\Phi}^{2,2} \subset H_{\mathscr{M}}^3(X, \mathbb{Q}(2))$ which maps bijectively onto $H_{\mathscr{M}}^1(B, \mathbb{Q}(1))$. One can define a desired space $\Phi^{2,2}$ by

$$\Phi^{2,2} := \mathrm{Im}\big(\widetilde{\Phi}^{2,2} \to H_{\mathscr{M}}^3(X_K, \mathbb{Q}(2))\big).$$

By this construction of $\Phi^{2,2}$, we have

$$\Phi^{2,2} \otimes \mathbb{Q}_p \cong H_f^1(K, H^2(X_{\overline{K}}, \mathbb{Q}_p(2))) \ (= H_f^1(K, \mathbb{Q}_p(1))).$$

See also Lemma 7.7 (3). For $(m, r) = (2, 2)$, we will use the classical class number formula instead of (9.2.1), later in Theorem 9.6 below.

Proposition 9.3 *Let r be an integer, and let p be a prime number. Assume all the following conditions*:

(i) $p - 2 \geqq r \geqq 2$.

(ii) *For any $v \in B_0$ dividing p, v is absolutely unramified and X has good reduction at v.*

(iii) *Conjecture 9.1 holds for $m = 0, 1$ (resp. $m = 0, 1, 2$), if $r = 2$ (resp. $r \geqq 3$).*

Then the equivalent conditions (i)–(v) *of Proposition 7.3 are satisfied for $m = 1, 2$ (resp. $m = 1, 2, 3$), if $r = 2$ (resp. if $r \geqq 3$). Moreover, we have*

$$\operatorname*{Res}_{s=2} \zeta(X, s) \equiv \operatorname*{Res}_{s=1} \zeta_K(s) \cdot \frac{\chi(\mathrm{aj}_p^{3,2}) \cdot \#\mathrm{CH}_0(X) \cdot R_\Phi^{0,2}}{\chi(\mathrm{aj}_p^{2,2}) \cdot \#\mathrm{Pic}(O_K) \cdot R_\Phi^{1,2}} \mod \mathbb{Z}_{(p)}^\times \qquad (r = 2)$$

$$\zeta(X, r) \equiv \frac{\chi(\mathrm{aj}_p^{3,r}) \cdot \#H_{\mathscr{M}}^4(X, \mathbb{Z}(r))\{p\} \cdot R_\Phi^{0,r} \cdot R_\Phi^{2,r}}{\chi(\mathrm{aj}_p^{2,r}) \cdot R_\Phi^{1,r}} \mod \mathbb{Z}_{(p)}^\times \qquad (r \geqq 3)$$

Proof. The first assertion is obvious. For any $r \geqq 2$, we have

$$\begin{aligned}
\lim_{s\to r} \frac{\zeta(X,s)}{\zeta_K(s)\zeta_K(s-1)} &= \frac{1}{L_{S'}(H^1(X_K),r)} \cdot \prod_{v\in S'} \frac{\zeta(Y_v,r)}{(1-q_v^{-r})^{-1}(1-q_v^{1-r})^{-1}} \\
&\equiv \frac{\chi(\alpha^{1,r})}{R_\Phi^{1,r}} \cdot \prod_{v\in S'} \frac{1}{\mu_v^m(A_p^{1,r}(K_v))} \cdot \prod_{v\in S'} \frac{e_v^{2,0,r}\cdot e_v^{3,1,r}\cdot \mu_v^1(A_p^{1,r}(K_v))}{e_v^{2,1,r}\cdot e_v^{3,0,r}} \mod \mathbb{Z}_{(p)}^\times \\
&= \frac{\chi(\alpha^{1,r})}{R_\Phi^{1,r}} \cdot \prod_{v\in S'} \frac{e_v^{2,0,r}\cdot e_v^{3,1,r}}{e_v^{2,1,r}\cdot e_v^{3,0,r}}
\end{aligned}$$

by the assumptions (i)–(iii) for $m = 1$ and Theorems 8.5 and 8.6 (see Remark 9.2 (2)). Hence for $r = 2$, we have

$$\begin{aligned}
\operatorname*{Res}_{s=2} \zeta(X,s) &\equiv \operatorname*{Res}_{s=1} \zeta_K(s) \cdot \frac{R_\Phi^{0,2}\cdot\chi(\alpha^{1,2})}{\chi(\alpha^{0,2})\cdot R_\Phi^{1,2}} \cdot \prod_{v\in S'} \frac{e_v^{2,0,2}\cdot e_v^{3,1,2}}{e_v^{2,1,2}\cdot e_v^{3,0,2}} \mod \mathbb{Z}_{(p)}^\times \\
&= \operatorname*{Res}_{s=1} \zeta_K(s) \cdot \frac{\chi(\mathrm{aj}_p^{3,2})\cdot \#\mathrm{CH}_0(X)\cdot R_\Phi^{0,2}}{\chi(\mathrm{aj}_p^{2,2})\cdot \#\mathrm{Pic}(O_K)\cdot R_\Phi^{1,2}}
\end{aligned}$$

by the assumption (iii) for $m = 0$ and Theorem 7.8. See also Remark 9.2 (3). Similarly for any $r \geqq 3$, we have

$$\begin{aligned}
\zeta(X,r) &\equiv \frac{R_\Phi^{0,r}\cdot\chi(\alpha^{1,r})\cdot R_\Phi^{2,r}}{\chi(\alpha^{0,r})\cdot R_\Phi^{1,r}\cdot\chi(\alpha^{2,r})} \cdot \prod_{v\in S'} \frac{e_v^{2,0,r}\cdot e_v^{3,1,r}}{e_v^{2,1,r}\cdot e_v^{3,0,r}} \mod \mathbb{Z}_{(p)}^\times \\
&= \frac{\chi(\mathrm{aj}_p^{3,r})\cdot \#H^4_{\mathscr{M}}(X,\mathbb{Z}(r))\{p\}\cdot R_\Phi^{0,r}\cdot R_\Phi^{2,r}}{\chi(\mathrm{aj}_p^{2,r})\cdot R_\Phi^{1,r}}
\end{aligned}$$

as claimed. □

9.2 Zeta value formula without étale cohomology

Let p be an arbitrary prime number. Assuming Conjecture 9.1 for p, we define a number $R_{\mathscr{M}}^{m,r} = R_{\mathscr{M},p}^{m,r} \in \mathbb{R}^\times/\mathbb{Z}_{(p)}^\times$ ($m \geqq 0$, $r \geqq 2$) as follows. We first take the inverse image $\widetilde{A}_p^{m,r}$ of $A_p^{m,r}(K)$ under the composite map

$$H_{\mathscr{M}}^{m+1}(X,\mathbb{Z}(r))\otimes\mathbb{Z}_{(p)} \to H_{\mathscr{M}}^{m+1}(X_K,\mathbb{Z}(r))\otimes\mathbb{Z}_{(p)} \to H^1(K, H^m(X_{\overline{K}},\mathbb{Z}_p(r))),$$

where for $(m,r) = (2,2)$, $A_p^{2,2}(K)$ is considered with respect to $\Phi^{2,2}$ constructed in Remark 9.2 (5). Since $A_p^{m,r}(K)$ is finitely generated over $\mathbb{Z}_{(p)}$, the canonical map $\widetilde{A}_p^{m,r} \to A_p^{m,r}(K)$ factors through a homomorphism

$$c^{m,r} : \overline{A}_p^{m,r} := \widetilde{A}_p^{m,r}/(\widetilde{A}_p^{m,r})_{\mathrm{Div}} \longrightarrow A_p^{m,r}(K).$$

This map fits into a commutative diagram

$$\begin{array}{ccccc}
\overline{A}_p^{m,r}\otimes\mathbb{Z}_p & \xrightarrow{\gamma^{m,r}} & H_{\mathscr{M}}^{m+1}(X,\mathbb{Z}(r))\mathbin{\widehat{\otimes}}\mathbb{Z}_p & \xrightarrow{\mathrm{aj}_p^{m+1,r}} & H^1(B,\mathfrak{H}^m(X,\mathbb{Z}_p(r))) \\
\downarrow{\scriptstyle c^{m,r}\otimes\mathrm{id}} & & & & \| \\
A_p^{m,r}(K)\otimes\mathbb{Z}_p & & \overset{\sim}{=\!=\!=} & & H^1_f(K, H^m(X_{\overline{K}},\mathbb{Z}_p(r))),
\end{array} \qquad (9.2.1)$$

where $\gamma^{m,r}$ denotes the natural map. See Corollary 8.2 for the right vertical equality.

Lemma 9.4 *Assume that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$, and that Conjecture 9.1 holds. Then $\gamma^{m,r}$ and $c^{m,r}$ have finite cokernel.*

Proof. $\mathrm{Coker}(c^{m,r})$ is finite, because it is finitely generated over $\mathbb{Z}_{(p)}$ and torsion by the definition of $\widetilde{A}^{m,r}_p$. The map $\gamma^{m,r}$ has finite cokernel as well, because $c^{m,r} \otimes \mathrm{id}_{\mathbb{Z}_p}$ has finite cokernel and $\mathrm{aj}^{m+1,r}_p$ has finite kernel by Lemma 7.7 (1) and (2). $\square$

By the finiteness of $\mathrm{Coker}(c^{m,r})$, we define $R^{m,r}_{\mathscr{M}} \in \mathbb{R}^\times/\mathbb{Z}^\times_{(p)}$ to be the volume of the space

$$\begin{cases} H^{m+1}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))/\text{Image of } \overline{A}^{m,r}_p & (\text{for } (m,r) \neq (2,2)) \\ \widetilde{H}^3_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(2))/\text{Image of } \overline{A}^{2,2}_p & (\text{for } (m,r) = (2,2)) \end{cases}$$

with respect to L^m that we fixed in Conjecture 9.1, where $\widetilde{H}^3_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(2))$ denotes the kernel of the canonical trace map

$$\mathrm{tr} : \widetilde{H}^3_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(2)) \longrightarrow \mathbb{R}.$$

We have $R^{m,r}_{\mathscr{M}} = 1$ for any $m \geqq 3$ by definition.

Proposition 9.5 $\gamma^{m,r}$ of (9.2.1) is bijective for $(m,r) = (3,2)$.

Proof. The assertion follows immediately from the facts that $H^4_{\mathscr{M}}(X, \mathbb{Z}(2)) \cong \mathrm{CH}_0(X)$ is finite (Lemma 7.7 (4)) and that $A^{3,2}_p(K) = 0$. $\square$

Theorem 9.6 *Under the same assumptions as in Proposition 9.3, assume further that*

(iv) $\gamma^{m,r}$ *of* (9.2.1) *is bijective for any* $m = 0, 1, 2, 3$.

Then $c^{m,r}$ has finite kernel for any $m = 0, 1, 2, 3$, and we have

$$\zeta^*(X, r) \equiv \prod_{m=0}^{3} \left(\frac{R^{m,r}_{\mathscr{M}}}{\mathrm{Ker}(c^{m,r})} \right)^{(-1)^m} \mod \mathbb{Z}^\times_{(p)},$$

where $\zeta^*(X, r)$ denotes $\operatorname*{Res}_{s=2} \zeta(X, s)$ (resp. $\zeta(X, r)$) if $r = 2$ (resp. $r \geqq 3$).

Remark 9.7 A stronger version of Conjecture 9.1 asserts that

(h1) *The $\mathbb{Q}$-vector space $\Phi^{m,r}$ agrees with $H^{m+1}_{\mathscr{M}}(X_K, \mathbb{Q}(r))_{\mathbb{Z}}$.*

The above condition (iv) holds true, under this stronger hypothesis and a variant of Bass' conjecture (cf. [Ba]) that

(h2) $H^{m+1}_{\mathscr{M}}(X, \mathbb{Z}(r))$ *is finitely generated for* $m = 0, 1, 2, 3$.

Under the hypotheses (h1) and (h2), $\mathrm{Ker}(c^{m,r})$ agrees with the p-primary torsion part of the kernel of the regulator map

$$\mathrm{reg}_{\mathscr{D}}^{m+1,r} : H_{\mathscr{M}}^{m+1}(X, \mathbb{Z}(r)) \longrightarrow H_{\mathscr{D}}^{m+1}(X_{/\mathbb{R}}, \mathbb{Z}(r))$$

by Remark 9.2 (1), and $R_{\mathscr{M}}^{m,r}$ is exactly the volume of its cokernel (modulo $\mathbb{Z}_{(p)}^{\times}$).

Proof of Theorem 9.6. The map $\mathrm{aj}_p^{m+1,r}$ has finite kernel for any $m \geqq 0$ (see Lemma 7.7 (1), (2)). By this fact and the assumption (iv), $c^{m,r} \otimes \mathrm{id}$ in the diagram (9.2.1) has finite kernel for any $m \geqq 0$. Thus $c^{m,r}$ has finite kernel, because $\mathbb{Z}_p$ is faithfully flat over $\mathbb{Z}_{(p)}$.

We rewrite the number on the right hand side in the formulas in Proposition 9.3. By the classical class number formula, we have

$$\operatorname*{Res}_{s=1} \zeta_K(s) = \mathrm{vol}(\mathrm{Coker}(\varrho)) \cdot \#\mathrm{Pic}(O_K),$$

where $\varrho = \varrho_K$ denotes the regulator map to (the reduced part of) the integral Deligne cohomology

$$\varrho : O_K^{\times} \longrightarrow \widetilde{H}_{\mathscr{D}}^1(B_{/\mathbb{R}}, \mathbb{Z}(1)) := \mathrm{Ker}\big(\mathrm{tr} : H_{\mathscr{D}}^1(B_{/\mathbb{R}}, \mathbb{Z}(1)) \to \mathbb{R}\big)$$

and the volume of $\mathrm{Coker}(\varrho)$ has been taken with respect to $O_K \subset K = H_{\mathrm{dR}}^0(\mathrm{Spec}(K)/K)$ (note that ϱ is injective). To prove the formula in Theorem 9.6, it is enough to check

$$\frac{R_{\mathscr{M}}^{m,r}}{\mathrm{Ker}(c^{m,r})} = \begin{cases} R_{\Phi}^{0,r} & (m=0) \\ \chi(\mathrm{aj}_p^{2,r}) \cdot R_{\Phi}^{1,r} & (m=1) \\ \chi(\mathrm{aj}_p^{3,2}) \cdot \mathrm{vol}(\mathrm{Coker}(\varrho)) & ((m,r)=(2,2)) \\ \chi(\mathrm{aj}_p^{3,r}) \cdot R_{\Phi}^{2,r} & (m=2,\ r \geqq 3) \\ (\#\mathrm{CH}_0(X)\{p\})^{-1} & ((m,r)=(3,2)) \\ (\#H_{\mathscr{M}}^4(X,\mathbb{Z}(r))\{p\})^{-1} & (m=3,\ r\geqq 3) \end{cases} \tag{9.2.2}$$

We have

$$\mathrm{Ker}(\mathrm{aj}_p^{m,r}) = \mathrm{Ker}(c^{m,r}) \quad \text{and} \quad \mathrm{Coker}(\mathrm{aj}_p^{m,r}) \cong \mathrm{Coker}(c^{m,r}) \tag{9.2.3}$$

for any (m,r) by the diagram (9.2.1) and the hypothesis (iv). See also Remark 9.5 for the case $(m,r) = (3,2)$. This fact implies (9.2.2) for $m = 0,1,2$ with $(m,r) \neq (2,2)$. See also Proposition 7.3 and Lemma 7.7 (1) for the fact that $\chi(\mathrm{aj}_p^{1,r}) = \#\mathrm{Ker}(c^{1,r}) = 1$. The formula (9.2.2) for $m = 3$ follows from (9.2.3) and the fact that $R_{\mathscr{M}}^{m,r} = 1$ for $m \geqq 3$. Finally, noting that $\gamma^{2,2}$ is bijective by assumption, consider the diagram (9.2.1) for $(m,r) = (2,2)$:

$$\begin{array}{ccccc} \overline{A}_p^{2,2} \otimes \mathbb{Z}_p & \xrightarrow{c^{2,2}\otimes \mathrm{id}} & A_p^{2,2}(K) \otimes \mathbb{Z}_p & \overset{\sim}{=\!=\!=} & O_K^{\times} \otimes \mathbb{Z}_p \\ \downarrow{\scriptstyle \mathrm{aj}_p^{3,2}} & & \| \wr & & \| \wr \\ H^1(B, \mathfrak{H}^2(X, \mathbb{Z}_p(2))) & =\!=\!= & H_f^1(K, H^2(X_{\overline{K}}, \mathbb{Z}_p(2))) & \overset{\sim}{=\!=\!=} & H_f^1(K, \mathbb{Z}_p(1)), \end{array}$$

which shows (9.2.2) for $(m,r) = (2,2)$. This completes the proof. □

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