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Global well-posedness of the Kirchhoff equation

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Abstract. The aim of this paper is to prove the global existence of solutions for the Kirchhoff equation without any smallness condition on data both in Sobolev spaces and in Gevrey ones. The approach to the construction of global solutions is to obtain absolute integrability of time-derivative of the coefficient of the principal term. The key of the proof is a uniform energy estimate in a suitable Sobolev space for global in time analytic solutions. This estimate yields the boundedness of solutions in Sobolev norm at the life span. The global existence of low regular solutions is also proved.

1. Introduction

We consider the problem for the Kirchhoff-type equation of the form

$$\begin{cases} \partial_t^2 u - \varphi\left(\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = 0, & t>0, \quad x\in\Omega, \\ u(0,x)=u_0(x), \quad \partial_t u(0,x) = u_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is the whole space $\mathbb{R}^n$ $(n \geq 1)$, or an open set with a smooth boundary $\partial\Omega$ and $u(t,x)$ satisfies the Dirichlet boundary condition

$$u|_{[0,\infty)\times\partial\Omega} = 0.$$

We assume that $\varphi(\rho)$ is a C^1 real function on $\mathbb{R}$ satisfying

$$\varphi(\rho) \geq \nu_0 \quad \text{for all } \rho \geq 0 \quad (\nu_0 > 0). \tag{1.2}$$

Equation (1.1) has been previously considered for various positive functions $\varphi(\rho)$. In the case when

$$n = 1, \quad \varphi(\rho) = a + b\rho \quad (a, b > 0),$$

Equation (1.1) was proposed by Kirchhoff in 1876 to describe the transversal motions of the elastic string (see [10]). Several authors have investigated the global existence for these equations with real analytic data. In 1940, Bernstein first studied the global existence for analytic data in one space dimension (see [5]). After him, in 1975, Pohozaev extended Bernstein's result to several space dimensions (see [15]). Later, global solvability in real analytic class was studied by D'Ancona and Spagnolo under the assumption that

$$\varphi \text{ is continuous on } [0,\infty),$$

$$\varphi(\rho) \geq 0 \quad (\rho \geq 0)$$

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(see [6], and also Arosio and Spagnolo [3]). Kajitani and Yamaguti obtained the same result under a more general principal term (see [8]). It is natural to ask whether (1.1) admits a unique global solution with data in wider function spaces, say, quasi-analytic class or Sobolev spaces. The global solvability for quasi-analytic data was studied by Nishihara (see [14]), and a variant of his class in [14] was discussed by Manfrin (see [11], and also Ghisi and Gobbino [7]). Recently, the large time existence of solutions was proved in Gevrey spaces (see [13]). As is well known, the results on global existence in Sobolev spaces $H^{3/2}$, or H^2 with small data are well established (see [12], and the references therein). As to the existence of periodic solutions, there is a result of Baldi (see [4]).

The Kirchhoff equation has a first integral (see Lemma 3.1 below). Nevertheless, it has been a long-standing open problem whether or not, one can prove the existence of time global solutions in Sobolev spaces or Gevrey ones without smallness condition on data. Moreover, the existence of local solutions in low regular Sobolev spaces, say, $H^\sigma \times H^{\sigma-1}$, $\sigma \in [1, 3/2)$, is still not known. The main point of the proof of global existence of high regular solutions is to obtain boundedness of local solutions in $H^{3/2}$-norm at the life span. On one hand, the main difficulty lies in controlling an intensive oscillation of the coefficient $\varphi(\|\nabla u(t)\|^2_{L^2})$. On the other hand, when data are very small, one can avoid such an oscillation problem to get global solutions (again see [12] and the references therein). For data without any smallness condition in Sobolev spaces, no one has any ideas to control $H^{3/2}$-norm of solutions.

The aim in this paper is to give an affirmative answer to these open problems. If the standard energy method is employed to get a priori estimates, one faces an estimate involving time-derivative of $\varphi(\|\nabla u(t)\|^2_{L^2})$. However, this kind of estimate is no use to control $H^{3/2}$-norm of solutions. Our crucial tool for control of time-derivative of $\varphi(\|\nabla u(t)\|^2_{L^2})$ is a uniform energy estimate for global in time analytic solutions to (1.1), which is proved by a contradiction argument. This estimate enables us to derive an absolute integrability of the time-derivative of $\varphi(\|\nabla u(t)\|^2_{L^2})$ on the maximal interval of existence of solutions. Hence, this allows us to obtain the boundedness of $H^{3/2}$-norm of solutions at the life span, and as a consequence, the solution globally exists.

We conclude this section by stating our plan. In Section 2 we state main results. In Section 3 local existence theorems together with H^σ-well-posedness in the sense of Hadamard are discussed. In Section 4 a uniform energy estimate for global in time analytic solutions in a suitable Sobolev space is proved. After that, absolute integrability of time-derivative of the coefficient of equation is proved. Section 5 is devoted to proving main theorems: Theorems 2.1 and 2.2. In Section 6 energy estimates are derived. In Section 7 the global existence of low regular solutions is proved.

2. Statement of results

In this section we state main results. These consist of the Cauchy problem and the initial-boundary value problem.

2.1. **The Cauchy problem.** To begin with, let us consider the problem (1.1) in the case when Ω is the whole space $\mathbb{R}^n$. We recall the definition of fractional Sobolev spaces

$$H^\sigma(\mathbb{R}^n) = (1-\Delta)^{-\frac{\sigma}{2}} L^2(\mathbb{R}^n), \quad \sigma \in \mathbb{R},$$

and their homogeneous version is

$$\dot{H}^\sigma(\mathbb{R}^n) = (-\Delta)^{-\frac{\sigma}{2}} L^2(\mathbb{R}^n), \quad \sigma \in \mathbb{R}.$$

We shall prove the following.

Theorem 2.1. *Assume that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $\sigma \geq 3/2$. Then for any $(u_0, u_1) \in H^\sigma(\mathbb{R}^n) \times H^{\sigma-1}(\mathbb{R}^n)$, the Cauchy problem (1.1) admits a unique global solution $u(t,x)$ such that*

$$u \in C([0,\infty); H^\sigma(\mathbb{R}^n)) \cap C^1([0,\infty); H^{\sigma-1}(\mathbb{R}^n)).$$

We have also the theorem on the global existence of H^σ-solutions for $1 \leq \sigma < 3/2$. This topic is postponed until Section 7.

Next, we shall state a result on global solvability in Gevrey spaces. We recall the definition of Gevrey class of L^2 type. For $s \geq 1$, we denote by $\gamma^s_{L^2}(\mathbb{R}^n)$ the Gevrey space of order s:

$$\gamma^s_{L^2}(\mathbb{R}^n) = \bigcup_{\eta>0} \gamma^s_{\eta,L^2}(\mathbb{R}^n).$$

Here, f belongs to $\gamma^s_{\eta,L^2}(\mathbb{R}^n)$ if $\|f\|_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} e^{\eta|\xi|^{\frac{1}{s}}} |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}},$$

and $\widehat{f}(\xi)$ stands for the Fourier transform of f. The class $\gamma^s_{L^2}(\mathbb{R}^n)$ is endowed with the inductive limit topology. In particular, we have

$$\gamma^1_{L^2}(\mathbb{R}^n) = \mathcal{A}_{L^2}(\mathbb{R}^n),$$

where $\mathcal{A}_{L^2}(\mathbb{R}^n)$ is the space of real analytic functions f such that

$$\|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)} \leq CA^{|\alpha|}\alpha!$$

for all $\alpha \in \mathbb{N}^n \cup \{0\}$ and for some constants $A, C \geq 0$.

We shall prove the following.

Theorem 2.2. *Assume that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $s > 1$. Then for any $(u_0, u_1) \in \gamma^s_{L^2}(\mathbb{R}^n) \times \gamma^s_{L^2}(\mathbb{R}^n)$, the Cauchy problem (1.1) admits a unique global solution $u(t,x)$ such that*

$$u \in C^1([0,\infty); \gamma^s_{L^2}(\mathbb{R}^n)).$$

2.2. **Extension to the initial-boundary value problem.** We extend Theorems 2.1 and 2.2 to the initial-boundary value problem. Replacing the Fourier transform with the Fourier series or the generalized Fourier transform, and applying exactly the same arguments of proofs of Theorems 2.1 and 2.2, we can prove similar results for the initial-boundary value problem (1.1) on $[0,\infty)\times\Omega$ with the boundary condition

$$u|_{[0,\infty)\times\partial\Omega}=0$$

(see Theorems 2.3 and 2.4).

Let us recall the definition of Sobolev spaces of fractional order over a bounded domain Ω with smooth boundary $\partial\Omega$. Let $\{w_k\}_{k=1}^{\infty}$ be a complete orthonormal system of eigenfunctions of the Laplace operator $-\Delta$ whose domain is $H^2(\Omega)\cap H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^1(\Omega)$-norm. Let λ_k be eigenvalues corresponding to w_k, i.e., $\{w_k,\lambda_k\}$ satisfy the elliptic equations:

$$\begin{cases} -\Delta w_k=\lambda_k w_k & \text{in } \Omega,\\ w_k=0 & \text{on } \partial\Omega. \end{cases}$$

Then we have

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots \quad \text{and} \quad \lambda_k\to\infty.$$

Let $\sigma\ge 0$. Then we say that $f\in H^\sigma(\Omega)$ if

$$\|f\|_{H^\sigma(\Omega)}:=\left(\sum_{k=1}^{\infty}\lambda_k^{2\sigma}\left|(f,w_k)_{L^2(\Omega)}\right|^2\right)^{\frac12}<\infty.$$

Here $(f,w_k)_{L^2(\Omega)}$ stands for the inner product of f and w_k in $L^2(\Omega)$.

Next, let Ω be an exterior domain with a smooth compact boundary. Denoting by Δ_D the Dirichlet Laplacian on Ω, we define inhomogeneous and homogeneous Sobolev spaces over Ω as

$$H^\sigma(\Omega)=(1-\Delta_D)^{-\frac{\sigma}{2}}L^2(\Omega),$$

$$\dot H^\sigma(\Omega)=(-\Delta_D)^{-\frac{\sigma}{2}}L^2(\Omega)$$

for $\sigma\ge 0$, respectively. Here, the operators $(1-\Delta_D)^{-\sigma/2}$ and $(-\Delta_D)^{-\sigma/2}$ are defined via the generalized Fourier transform $\mathscr{F}$, which maps unitarily $L^2(\Omega)$ to $L^2(\mathbb{R}^n)$. For its definition we refer to, e.g., our previous paper [13]. Hereafter, for the sake of simplicity, we denote Δ_D by Δ.

Finally, we recall the definition of Gevrey class of L^2 type. For $s\ge 1$, we denote by $\gamma^s_{L^2}(\Omega)$ the Gevrey space of order s on Ω:

$$\gamma^s_{L^2}(\Omega)=\bigcup_{\eta>0}\gamma^s_{\eta,L^2}(\Omega).$$

Here, f belongs to $\gamma^s_{\eta,L^2}(\Omega)$ if $\|f\|_{\gamma^s_{\eta,L^2}(\Omega)} < \infty$, where

$$\|f\|_{\gamma^s_{\eta,L^2}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{k=1}^{\infty} e^{\eta\lambda_k^{\frac{1}{s}}} \left|(f, w_k)_{L^2(\Omega)}\right|^2 \right)^{\frac{1}{2}}, & \text{when } \Omega \text{ is a bounded domain,} \\ \left(\displaystyle\int_{\mathbb{R}^n} e^{\eta|\xi|^{\frac{1}{s}}} |(\mathscr{F} f)(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}, & \text{when } \Omega \text{ is an exterior domain.} \end{cases}$$

The space $\gamma^s_{L^2}(\Omega)$ is endowed with the inductive limit topology. In particular, we have

$$\gamma^1_{L^2}(\Omega) = \mathcal{A}_{L^2}(\Omega),$$

where $\mathcal{A}_{L^2}(\Omega)$ is the space of real analytic functions f such that

$$\|\partial_x^\alpha f\|_{L^2(\Omega)} \le C A^{|\alpha|} \alpha!$$

for all $\alpha \in \mathbb{N}^n \cup \{0\}$ and for some constants $A, C \ge 0$.

We need the compatibility condition on data.

Compatibility condition. Let $\sigma \ge 1$. Then $f \in H^\sigma(\Omega)$ is said to satisfy the compatibility condition if

$$\Delta^k f \in H^1_0(\Omega) \quad \text{for } 0 \le k \le \frac{\sigma - 1}{2}.$$

We have the following.

Theorem 2.3. *Assume that Ω is an analytic domain with a compact boundary. Suppose that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $\sigma \ge 3/2$. Then for any $(u_0, u_1) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$ satisfying the compatibility condition, the initial-boundary value problem (1.1) admits a unique global solution $u(t,x)$ such that*

$$u \in C([0,\infty); H^\sigma(\Omega)) \cap C^1([0,\infty); H^{\sigma-1}(\Omega)).$$

Let us remark that the analyticity of the domain is necessary for the theorem. For, we approximate local solutions in Sobolev spaces by a series of analytic solutions, which is possible if Ω is analytic. For further details, see the proof of Proposition 4.2 in Section 4.

We have also the global solvability in Gevrey spaces. In this case we have to impose the analytic compatibility condition on initial data.

Analytic compatibility condition. $f \in \gamma^s_{L^2}(\Omega)$ $(s \ge 1)$ is said to satisfy the analytic compatibility condition if f is analytic in some neighbourhood of $\overline{\Omega}$ such that

$$\Delta^k f = 0 \quad \text{on } \partial\Omega \quad \text{for } k = 0, 1, \cdots.$$

We have the following.

Theorem 2.4. *Assume that Ω is an analytic domain. Suppose that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $s > 1$. Then for any $(u_0, u_1) \in \gamma^s_{L^2}(\Omega) \times \gamma^s_{L^2}(\Omega)$ satisfying the analytic compatibility condition, the initial-boundary value problem (1.1) admits a unique global solution $u(t,x)$ such that*

$$u \in C^1([0,\infty); \gamma^s_{L^2}(\Omega)).$$

3. Well-posedness in Sobolev spaces

In this section, assuming that Ω is the whole space $\mathbb{R}^n$, or an open set of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, we present local existence theorems for the problem (1.1) (see Propositions 3.2 and 3.3). After those, we state H^σ-well-posedness for (1.1) in the sense of Hadamard, i.e., the continuity of solutions in Sobolev spaces H^σ with respect to data (see Propositions 3.6 and 3.7). When Ω is the whole space, the compatibility condition is not required in all of results of this section, and statements are given without any comment on this condition.

The Kirchhoff equation has a first integral. Namely, we have:

Lemma 3.1. *Suppose that $u \in \bigcap_{j=0}^{1} C^j([0,T]; H^{(3/2)-j}(\Omega))$ is the solution to (1.1). Then we have*

$$\mathscr{H}(u;t) = \mathscr{H}(u;0) \tag{3.1}$$

for all $t \in [0,T]$, where we put

$$\mathscr{H}(u;t) := \|\partial_t u(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^{\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)}} \varphi(\rho)\, d\rho.$$

Proof. Multiplying Equation (1.1) by $\partial_t u$ and integrating, we get

$$\frac{d}{dt}\mathscr{H}(u;t) = 0,$$

which implies (3.1). Integrating it with respect to t, we get (3.1). The proof of Lemma 3.1 is complete. □

We introduce a local existence theorem in Sobolev spaces. Let us define a functional

$$c(t) = c_u(t) := \varphi\left(\int_\Omega |\nabla u(t,x)|^2\, dx\right) \tag{3.2}$$

and a σ-energy

$$\mathcal{E}_\sigma(u;t) = \left\|(-\Delta)^{\frac{\sigma-1}{2}} \partial_t u(t,\cdot)\right\|^2_{L^2(\Omega)} + c(t)\left\|(-\Delta)^{\frac{\sigma}{2}} u(t,\cdot)\right\|^2_{L^2(\Omega)} \tag{3.3}$$

for $\sigma \ge 1$.

The following result is our starting point.

Proposition 3.2 (Arosio and Galavaldi ([1]))**.** *Assume that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $\sigma \ge 3/2$. Then for any $(u_0, u_1) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$ satisfying the* *compatibility condition, there exists a life span* $T_m = T_m(u_0, u_1)$ *depending only on* $\mathscr{H}(u;0)$ *and* $\mathcal{E}_{3/2}(u;0)$ *such that the problem* (1.1) *admits a unique maximal solution*

$$u \in C([0, T_m); H^\sigma(\Omega)) \cap C^1([0, T_m); H^{\sigma-1}(\Omega)),$$

and at least one of the following statements is valid:

(i) $T_m = \infty$;
(ii) $T_m < \infty$ *and* $\limsup_{t\to T_m-0} \mathcal{E}_{3/2}(u;t) = \infty$.

We remark that the life span T_m is to be understood as follows:

$$T_m = \sup\left\{t: \ H^{\frac{3}{2}}\text{-norm of the solution } u(\tau,\cdot) \text{ to (1.1) exists for } 0 \le \tau < t\right\}.$$

It should be noted that, however big the regularity of data is, T_m depends only on the norm of data in $H^{3/2} \times H^{1/2}$. This means that if one would show the global existence of smooth, or even Gevrey space solutions to (1.1), it suffices to obtain that the norm of solutions in $H^{3/2} \times H^{1/2}$ is bounded on $[0, T_m)$. Based on this observation, we shall introduce a local existence theorem for Gevrey spaces.

Proposition 3.3. *Assume that* Ω *is an analytic domain with a compact boundary. Suppose that* $\varphi(\rho)$ *is* C^1 *on* $\mathbb{R}$ *and satisfies* (1.2). *Let* $s > 1$. *Then for any* $(u_0, u_1) \in \gamma^s_{L^2}(\Omega) \times \gamma^s_{L^2}(\Omega)$ *satisfying the analytic compatibility condition, there exists a life span* $T_m = T_m(u_0, u_1)$ *depending only on* $\mathscr{H}(u;0)$ *and* $\mathcal{E}_{3/2}(u;0)$ *such that the problem* (1.1) *admits a unique solution*

$$u \in C^1([0, T_m); \gamma^s_{L^2}(\Omega)),$$

and one of the following statements is valid:

(i) $T_m = \infty$;
(ii) $T_m < \infty$ *and* $\limsup_{t\to T_m-0} \mathcal{E}_{3/2}(u;t) = \infty$.

In the rest of this section, we shall discuss several results on H^σ-well-posedness in the sense of Hadamard. Given a constant $\Lambda > 0$, let (u_0, u_1) be satisfied with

$$\nu_0^{-1}\mathscr{H}(u;0) \le \Lambda.$$

Combining (1.2) and (3.1), we see that the local solution $u(t,x)$ to the problem (1.1) satisfies

$$\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} \le \nu_0^{-1}\mathscr{H}(u;t) = \nu_0^{-1}\mathscr{H}(u;0) \le \Lambda$$

for any $t \in [0, T_m)$. This means that $[0, \Lambda]$ is the actual domain of $\varphi(\rho)$ which the H^σ-solution $u(t,x)$ to (1.1) exists on $[0, T_m)$. Let us define a quantity

$$M = \sup\{\varphi(\rho) : 0 \le \rho \le \Lambda\}. \tag{3.4}$$

Then we have

$$\nu_0 \le \varphi(\rho) \le M \tag{3.5}$$

for any $\rho \in [0, \Lambda]$. Also, we put

$$M_1 = \sup\left\{|\varphi'(\rho)| : 0 \le \rho \le \Lambda\right\}. \tag{3.6}$$

When $M_1 = 0$, Equation (1.1) is reduced to the classical wave equation. Hence, we may assume that $M_1 > 0$.

For the moment, we discuss a property related to the life span. For $\sigma \geq 3/2$, let $u(t,x)$ be the maximal solution to (1.1) with data $(u_0,u_1) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$ satisfying the compatibility condition in the sense of Proposition 3.2, and let T_m be the life span of $u(t,x)$. Recalling the notation $\mathcal{E}_\sigma(u;t)$ (see (3.3)), we put

$$T_1 = T_1(u_0,u_1) = \begin{cases} \dfrac{\nu_0^{3/2}}{M_1 \mathcal{E}_{3/2}(u;0)}, & \text{if } (u_0,u_1) \neq (0,0); \\ \infty, & \text{if } (u_0,u_1) = (0,0), \end{cases} \tag{3.7}$$

where ν_0 is the lower bound of φ (see (1.2)) and M_1 is the constant defined by (3.6). Then Arosio and Panizzi proved that the problem (1.1) admits a unique solution $u(t,x)$ such that

$$u \in C([0,T_1); H^\sigma(\Omega)) \cap C^1([0,T_1); H^{\sigma-1}(\Omega)), \tag{3.8}$$

and that (1.1) is H^σ-well-posed on $[0,T_1)$ (see Theorem 2.1 from [2]). The precise statement is given in Lemma 3.5 below. It is easy to see that T_m is bounded from below like

$$T_m \geq T_1. \tag{3.9}$$

For, suppose that

$$T_m < T_1. \tag{3.10}$$

Then we see from (3.8) that $\mathcal{E}_{3/2}(u;t)$ is bounded at $t = T_m$. Hence, we conclude from Proposition 3.2 that $T_m = \infty$, which actually contradicts (3.10). Thus we get (3.9).

We shall prove here the following.

Lemma 3.4. *Assume that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $\sigma \geq 3/2$. Suppose that $(u_0,u_1) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$ satisfy the compatibility condition. Let $u(t,x)$ be a maximal solution to (1.1) with data (u_0,u_1) in the sense of Proposition 3.2. Let T_1 be as in (3.7). If $T_1 < T_m$, then there exists a non-decreasing sequence $\{T_k\}_{k=1}^\infty$ such that*

$$T_0 = 0, \quad T_k - T_{k-1} = \frac{\nu_0^{3/2}}{M_1 \mathcal{E}_{3/2}(u;T_{k-1})}, \quad k = 1,2,\cdots. \tag{3.11}$$

Furthermore, if $T_m < \infty$, we have

$$T_k \underset{k\to\infty}{\longrightarrow} T_m. \tag{3.12}$$

Before proving Lemma 3.4, we remark that Kajitani and Satoh obtained a similar result to Lemma 3.4 for $\mathcal{E}_2(u;t)$ (see [9]). However, it is not clear whether the corresponding sequence $\{T_k\}$ is convergent or not.

Proof of Lemma 3.4. We see from (3.7) that (3.11) holds true for $k = 1$. As to $k = 2$, we regard the solution $u(t,x)$ to the problem (1.1) as that with data

$$(u(T_1,\cdot), \partial_t u(T_1,\cdot)) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega) \tag{3.13}$$

on $(T_1,T_m) \times \Omega$. Introducing a new time variable s as

$$s = t - T_1,$$

and defining a function

$$\widetilde{u}(s,x) := u(t,x) = u(s+T_1,x),$$

we can write the problem (1.1) with data (3.13) as

$$\begin{cases} \partial_s^2 \widetilde{u} - \varphi\left(\displaystyle\int_\Omega |\nabla \widetilde{u}(s)|^2\,dx\right)\Delta\widetilde{u} = 0, & s>0, \quad x\in\Omega, \\ \widetilde{u}(0,x) = u(T_1,x), \quad \partial_s\widetilde{u}(0,x) = \partial_t u(T_1,x), & x\in\Omega \end{cases} \tag{3.14}$$

and $\widetilde{u}$ satisfies the Dirichlet boundary condition

$$\widetilde{u}|_{[0,\infty)\times\partial\Omega} = 0.$$

Applying Theorem 2.1 from Arosio and Panizzi [2] to the problem (3.14), we find a time $S_1 \in (0, T_m - T_1]$ fulfilling

$$S_1 = \frac{\nu_0^{3/2}}{M_1\mathcal{E}_{3/2}(\widetilde{u};0)} = \frac{\nu_0^{3/2}}{M_1\mathcal{E}_{3/2}(u;T_1)} \tag{3.15}$$

and $\widetilde{u}$ satisfies

$$\widetilde{u} \in C([0,S_1);H^\sigma(\Omega)) \cap C^1([0,S_1);H^{\sigma-1}(\Omega)).$$

Hence, putting

$$T_2 := S_1 + T_1,$$

we deduce from (3.15) that

$$T_2 - T_1 = \frac{\nu_0^{3/2}}{M_1\mathcal{E}_{3/2}(u;T_1)}.$$

Since $S_1 \in (0, T_m - T_1]$, it follows that $T_2 \in (T_1, T_m]$.

If $T_2 = T_m$, we put $T_k = T_m$ for $k \geq 2$. Then (3.12) holds true. If $T_2 < T_m$, then the sequence $\{T_k\}_{k=1}^\infty$ satisfying (3.11) is constructed step by step. If there exists an integer $\ell > 2$ such that $T_\ell = T_m$, we put $T_k = T_m$ for $k \geq \ell$. Then (3.12) holds true. Otherwise, $\{T_k\}_{k=1}^\infty$ is an infinite series. We show the convergence (3.12) in this case. Suppose that

$$T_* := \lim_{k\to\infty} T_k < T_m.$$

Since the sequence $\{T_k\}$ is convergent, it follows that

$$T_k - T_{k-1} \underset{k\to\infty}{\longrightarrow} 0.$$

We notice that $\mathcal{E}_{3/2}(u;t)$ is continuous in $t \in [0, T_m)$. Then, letting $k \to \infty$ in (3.11), we get

$$\mathcal{E}_{3/2}(u;T_*) = \lim_{k\to\infty} \mathcal{E}_{3/2}(u;T_{k-1}) = \infty,$$

which leads to a contradiction, since $\mathcal{E}_{3/2}(u;t)$ is finite at $t = T_*$. Thus we must have $T_* = T_m$. The proof of Lemma 3.4 is complete. □

We now turn to the proof of H^σ-well-posedness result on the interval $[0, T_m)$. For this purpose, we prepare the preliminary result on $[0, T_1)$, which is proved by Arosio and Panizzi (see [2]). Since $\varphi(\rho)$ is C^1 and bounded (see (3.5)), the assumptions in

Theorems 5.1 and 5.2 from [2] are fulfilled. Let us define a variant norm of $\mathcal{E}_\sigma(u;t)$ as

$$\widetilde{\mathcal{E}}_\sigma(u;t) = \big\|\partial_t u(t,\cdot)\big\|^2_{H^{\sigma-1}(\Omega)} + c(t)\big\|u(t,\cdot)\big\|^2_{H^\sigma(\Omega)}.$$

We note that when Ω is a bounded domain, $\widetilde{\mathcal{E}}_\sigma(u;t)$ coincides with $\mathcal{E}_\sigma(u;t)$. Then we have the following.

Lemma 3.5 ([2]). *Let $\sigma \geq 3/2$. Assume that φ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $u(t,x)$ be a maximal solution to (1.1) with data $(u_0,u_1) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$ satisfying the compatibility condition in the sense of Proposition 3.2. Let T_1 be as in (3.7). Then the following assertions hold for every $T \in (0,T_1)$:*

(i) (*Theorem* 2.1 *from* [2]) *The mapping*

$$\begin{array}{ccc} (u_0,u_1) & \longmapsto & u \\ \Cap & & \Cap \\ H^\sigma(\Omega)\times H^{\sigma-1}(\Omega) & \longrightarrow & C([0,T];H^\sigma(\Omega))\cap C^1([0,T];H^{\sigma-1}(\Omega)) \end{array}$$

is continuous at the point (u_0,u_1).

More precisely, we have:

(ii) (*Theorems* 5.1 *and* 5.2 *from* [2]) *Let M and M_1 be the constants as in (3.4) and (3.6), respectively. For every $\varepsilon > 0$, there exists a real $\delta > 0$ depending on ε, ν_0, M, M_1, u_0, u_1, T and T_1 such that*

$$\widetilde{\mathcal{E}}_{\sigma_0}(u-v;0) < \delta \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u-v;t) < \varepsilon \quad (t \in [0,T])$$

for every $\sigma_0 \in [1,\sigma]$, where

$$v \in C([0,T];H^\sigma(\Omega))\cap C^1([0,T];H^{\sigma-1}(\Omega))$$

solves (1.1) with data $(v(0,\cdot),\partial_t v(0,\cdot)) \in H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)$ satisfying the compatibility condition.

Remark 3.1. In Theorem 5.1 from [2], given a bounded set W in $H^{3/2}(\Omega)\times H^{1/2}(\Omega)$, Arosio and Panizzi introduced the limit of existence time of solutions as follows:

$$T^* = \frac{\nu_0^{3/2}}{M_1 \sup\{\mathcal{E}_{3/2}(u;0);(u_0,u_1)\in W\}}.$$

This means that T^* is less than T_1 in Lemma 3.5. However, it is possible to take W as a singleton $\{(u_0,u_1)\}$ in the proof of [2], and as a result, it is sufficient for our purpose to adopt the statement of Lemma 3.5.

For the convenience of terminology, let us give a notion of H^σ-well-posedness in the sense of Hadamard.

Definition 3.1. Let $u(t,x)$ be an H^σ-solution to the problem (1.1) on $[0,T]$ with data $(u_0,u_1) \in H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)$ for $\sigma \geq 3/2$. Then we say that the problem (1.1) is H^σ-well-posed at (u_0,u_1) on $[0,T]$ if $u(t,x)$ satisfies the assertion (ii) from Lemma 3.5 on $[0,T]$.

It is possible to extend the interval $[0, T_1)$ in Lemma 3.5 to $[0, T_m)$. More precisely, we have the following.

Proposition 3.6. *Let $\sigma \geq 3/2$. Suppose that φ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $u(t,x)$ be a maximal solution to the problem (1.1) with data $(u_0, u_1) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$ satisfying the compatibility condition in the sense of Proposition 3.2. Assume that $T_m < \infty$. Then the problem (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, T]$ for every $T \in (0, T_m)$.*

Proof. If $T_1 = T_m$, the proposition is entirely Lemma 3.5. We consider the case when $T_1 < T_m$. It follows from Lemma 3.5 that the problem (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, T_1)$. We claim that the problem (1.1) is H^σ-well-posed even at $t = T_1$. In fact, thanks to Lemma 3.5, for any $\varepsilon > 0$ there exists a real $\delta > 0$ depending on $\varepsilon, M, M_1, \nu_0$, u_0, u_1 and T_1 such that if

$$\widetilde{\mathcal{E}}_{\sigma_0}(u - v; 0) < \delta \tag{3.16}$$

for every $\sigma_0 \in [1, \sigma]$, then

$$\widetilde{\mathcal{E}}_{\sigma_0}(u - v; t) < \varepsilon \quad (t \in [0, T_1)). \tag{3.17}$$

Here, the function

$$v \in C([0, T_1); H^\sigma(\Omega)) \cap C^1([0, T_1); H^{\sigma-1}(\Omega))$$

solves (1.1) with data $(v(0, \cdot), \partial_t v(0, \cdot)) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$ satisfying the compatibility condition. It is possible that the H^σ-solution $v(t, \cdot)$ is extended beyond $t = T_1$. Indeed, we suppose that

$$\lim_{t \to T_1 - 0} \widetilde{\mathcal{E}}_\sigma(v; t) = \infty.$$

Then $\widetilde{\mathcal{E}}_\sigma(u - v; t)$ is unbounded near $t = T_1$. However, this contradicts (3.17). Hence, $v(t, \cdot)$ exists beyond $t = T_1$. Therefore, we deduce that

$$v \in C([0, T_1]; H^\sigma(\Omega)) \cap C^1([0, T_1]; H^{\sigma-1}(\Omega)).$$

Now, we regard the problem (1.1) as that with data

$$(u(T_1 - \eta, \cdot), \partial_t u(T_1 - \eta, \cdot)) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega) \tag{3.18}$$

for sufficiently small $\eta > 0$. We use the idea of the proof of Lemma 3.4. Introducing a new time variable s as

$$s = t - (T_1 - \eta),$$

and defining a function

$$\widetilde{u}(s, x) := u(t, x) = u(s + T_1 - \eta, x),$$

we can write the problem (1.1) with data (3.18) as

$$\begin{cases} \partial_s^2 \widetilde{u} - \varphi\left(\displaystyle\int_\Omega |\nabla \widetilde{u}(s)|^2 \, dx\right) \Delta \widetilde{u} = 0, & s > 0, \quad x \in \Omega, \\ \widetilde{u}(0, x) = u(T_1 - \eta, x), \quad \partial_s \widetilde{u}(0, x) = \partial_t u(T_1 - \eta, x) & x \in \Omega \end{cases} \tag{3.19}$$

and $\widetilde{u}$ satisfies the Dirichlet boundary condition

$$\widetilde{u}|_{[0,\infty) \times \partial\Omega} = 0.$$

Applying Lemma 3.5 to the problem (3.19), we find a time $S_1 \in (0, T_m - T_1]$ fulfilling

$$S_1 = \frac{\nu_0^{3/2}}{M_1 \mathcal{E}_{3/2}(\widetilde{u}; 0)} = \frac{\nu_0^{3/2}}{M_1 \mathcal{E}_{3/2}(u; T_1 - \eta)}$$

and $\widetilde{u}$ satisfies

$$\widetilde{u} \in C([0, S_1); H^\sigma(\Omega)) \cap C^1([0, S_1); H^{\sigma-1}(\Omega)).$$

Furthermore, for any $\varepsilon > 0$ there exists a real $\widetilde{\delta} > 0$ depending on $\varepsilon, \nu_0, M, M_1, \widetilde{u}(0)$, $\partial_s \widetilde{u}(0)$ and S_1 such that

$$\widetilde{\mathcal{E}}_{\sigma_0}(\widetilde{u} - \widetilde{v}; 0) < \widetilde{\delta} \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(\widetilde{u} - \widetilde{v}; s) < \varepsilon \quad (s \in [0, S_1)) \tag{3.20}$$

for every $\sigma_0 \in [1, \sigma]$, where we put

$$\widetilde{v}(s, x) = v(s + T_1 - \eta, x).$$

We notice that $\widetilde{\delta} \leq \varepsilon$. If we define

$$\widetilde{T} := S_1 + (T_1 - \eta),$$

the assertion (3.20) is written as

$$\widetilde{\mathcal{E}}_{\sigma_0}(u - v; T_1 - \eta) < \widetilde{\delta} \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u - v; t) < \varepsilon \quad (t \in [T_1 - \eta, \widetilde{T})) \tag{3.21}$$

for every $\sigma_0 \in [1, \sigma]$. By the arbitrariness of ε, it is possible to choose ε in (3.17) as $\widetilde{\delta}$, and some δ_1 as δ in (3.16), respectively. Namely, we find a real $\delta_1 \in (0, \delta)$ such that

$$\widetilde{\mathcal{E}}_{\sigma_0}(u - v; 0) < \delta_1 \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u - v; t) < \widetilde{\delta} \quad (t \in [0, T_1 - \eta]) \tag{3.22}$$

for every $\sigma_0 \in [1, \sigma]$. Thus, recalling $\widetilde{\delta} \leq \varepsilon$, and combining (3.21) and (3.22), we deduce that

$$\widetilde{\mathcal{E}}_{\sigma_0}(u - v; 0) < \delta_1 \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u - v; t) < \varepsilon \quad (t \in [0, \widetilde{T})).$$

Since $T_1 < \widetilde{T}_1$, we conclude that (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, T_1]$.

Summarizing the above argument, we arrive at the following: For any $\varepsilon > 0$ there exists a real $\delta_1 > 0$ such that

$$\widetilde{\mathcal{E}}_{\sigma_0}(u - v; 0) < \delta_1 \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u - v; t) < \varepsilon \quad (t \in [0, T_1]) \tag{3.23}$$

for every $\sigma_0 \in [1, \sigma]$, where $v(t, x)$ is the H^σ-solution to the problem (1.1) with data

$$(v(0, \cdot), \partial_t v(0, \cdot)) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$$

satisfying the compatibility condition.

Next, we extend the time interval in which (1.1) is H^σ-well-posed at (u_0, u_1). Let $\{T_k\}_{k=1}^\infty$ be the sequence satisfying (3.11) and (3.12) in Lemma 3.4. We consider the problem (1.1) on $(T_1, T_2) \times \Omega$ with data

$$(u(T_1, \cdot), \partial_t u(T_1, \cdot)) \in H^\sigma(\Omega) \times H^{\sigma-1}(\Omega).$$

We may assume that $T_2 < T_m$. By Lemma 3.5 together with the time translation method to get (3.20), we have the corresponding assertion to (3.21): There exists a real $\widetilde{\delta}_1 > 0$ such that

$$\widetilde{\mathcal{E}}_{\sigma_0}(u - v; T_1) < \widetilde{\delta}_1 \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u - v; t) < \varepsilon \quad (t \in [T_1, T_2)) \tag{3.24}$$

for every $\sigma_0 \in [1,\sigma]$. Actually, by the continuation argument as above, the assertion (3.24) holds even at the end point T_2. We notice that $\widetilde{\delta}_1 \le \varepsilon$. Next, let us choose ε in (3.23) as $\widetilde{\delta}_1$. Then there exists a real $\delta_2 \in (0,\delta_1)$ such that

$$\widetilde{\mathcal{E}}_{\sigma_0}(u-v;0) < \delta_2 \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u-v;t) < \widetilde{\delta}_1 \quad (t \in [0,T_1]) \tag{3.25}$$

for every $\sigma_0 \in [1,\sigma]$. Combining (3.24) and (3.25), we have the following: For any $\varepsilon > 0$, there exist a real $\delta_2 > 0$ such that

$$\widetilde{\mathcal{E}}_{\sigma_0}(u-v;0) < \delta_2 \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u-v;t) < \varepsilon \quad (t \in [0,T_2])$$

for every $\sigma_0 \in [1,\sigma]$. Therefore, by the successive argument, for every $k \ge 3$ there exists a real $\delta_k \in (0,\delta_{k-1})$ such that

$$\widetilde{\mathcal{E}}_{\sigma_0}(u-v;0) < \delta_k \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u-v;t) < \varepsilon \quad (t \in [0,T_k]) \tag{3.26}$$

for every $\sigma_0 \in [1,\sigma]$.

Let $T \in (0,T_m)$, and take an integer $k = k(T)$ such that $T \in [T_{k-1},T_k]$. Then, thanks to (3.26), there exists a real $\delta_T > 0$ such that

$$\widetilde{\mathcal{E}}_{\sigma_0}(u-v;0) < \delta_T \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma_0}(u-v;t) < \varepsilon \quad (t \in [0,T])$$

for every $\sigma_0 \in [1,\sigma]$. Thus, we conclude that (1.1) is H^σ-well-posed at (u_0,u_1) on every interval $[0,T]$. This completes the proof of Proposition 3.6. $\square$

H^σ-well-posedness holds also for analytic solutions under an assumption that Ω is an analytic domain. We shall prove here the following.

Proposition 3.7. *Assume that Ω is an analytic domain with a compact boundary. Suppose that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let M and M_1 be the constants as in (3.4) and (3.6), respectively. Let $u(t,x)$ be a global in time analytic solution to the problem (1.1) with data $(u_0,u_1) \in \mathcal{A}_{L^2}(\Omega)\times\mathcal{A}_{L^2}(\Omega)$ satisfying the analytic compatibility condition. Then for every $T>0$, the mapping*

$$\begin{array}{ccc} (u_0,u_1) & \overset{U(t)}{\longmapsto} & u \\ \Cap & & \Cap \\ \mathcal{A}_{L^2}(\Omega)\times\mathcal{A}_{L^2}(\Omega) & \longrightarrow & C([0,T];H^\sigma(\Omega))\cap C^1([0,T];H^{\sigma-1}(\Omega)) \end{array}$$

is continuous at the point (u_0,u_1) for every $\sigma \ge 1$, where $\mathcal{A}_{L^2}(\Omega)\times\mathcal{A}_{L^2}(\Omega)$ is endowed with the induced topology of $H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)$. More precisely, for every $\varepsilon>0$ there exists a $\delta>0$ depending on ε, ν_0, M, M_1, u_0, u_1 and T such that

$$\widetilde{\mathcal{E}}_{\sigma}(u-v;0) < \delta \quad \Longrightarrow \quad \widetilde{\mathcal{E}}_{\sigma}(u-v;t) < \varepsilon \quad (t \in [0,T])$$

for every $\sigma \ge 1$, where

$$v \in C^1([0,T];\mathcal{A}_{L^2}(\Omega))$$

solves (1.1) with data $(v(0,\cdot),\partial_t v(0,\cdot)) \in \mathcal{A}_{L^2}(\Omega)\times\mathcal{A}_{L^2}(\Omega)$ satisfying the analytic compatibility condition.

Proof. The proof is similar to that of Proposition 3.6. To make the argument self-contained, we perform it carefully by the repetition of the previous lemmas and propositions. We fix data $(u_0, u_1) \in \mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega)$. Let σ_1 be an arbitrary real such that

$$\sigma_1 \geq \max\left(\sigma, \frac{3}{2}\right).$$

We construct a sequence $\{\overline{T}_k\}_{k=1}^{\infty}$ fulfilling a similar property to that in Lemma 3.4. Since

$$\mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega) \subsetneqq H^{\sigma_1}(\Omega) \cap H^{\sigma_1 - 1}(\Omega),$$

thanks to Theorem 2.1 from Arosio and Panizzi [2] (cf. (3.8)), we deduce that the global in time analytic solution $u(t,x)$ to the problem (1.1) satisfies

$$u \in C([0, \overline{T}_1); H^{\sigma_1}(\Omega)) \cap C([0, \overline{T}_1); H^{\sigma_1 - 1}(\Omega)),$$

where $\overline{T}_1$ is defined as in (3.7):

$$\overline{T}_1 = \frac{\nu_0^{3/2}}{M_1 \mathcal{E}_{3/2}(u; 0)}.$$

Actually, $u(t, \cdot)$ exists at $t = \overline{T}_1$. Next, as in the proof of Lemma 3.4, there exists a $\overline{T}_2 \in (\overline{T}_1, \infty)$ such that the global in time analytic solution $u(t,x)$ satisfies

$$u \in C([\overline{T}_1, \overline{T}_2); H^{\sigma_1}(\Omega)) \cap C([\overline{T}_1, \overline{T}_2); H^{\sigma_1 - 1}(\Omega))$$

and

$$\overline{T}_2 - \overline{T}_1 = \frac{\nu_0^{3/2}}{M_1 \mathcal{E}_{3/2}(u; \overline{T}_1)}.$$

Actually, $u(t, \cdot)$ exists at $t = \overline{T}_2$. Hence, by the successive argument, it is possible to construct an increasing sequence $\{\overline{T}_k\}_{k=1}^{\infty}$ satisfying

$$\overline{T}_0 = 0, \quad \overline{T}_k - \overline{T}_{k-1} = \frac{\nu_0^{3/2}}{M_1 \mathcal{E}_{3/2}(u; \overline{T}_{k-1})}, \quad k = 1, 2, \cdots. \tag{3.27}$$

Now, we claim that

$$\overline{T}_k \underset{k \to \infty}{\longrightarrow} \infty. \tag{3.28}$$

The proof is similar to that of Lemma 3.4. Suppose that $\{\overline{T}_k\}$ is convergent. Then we see that

$$\overline{T}_k - \overline{T}_{k-1} \underset{k \to \infty}{\longrightarrow} 0, \tag{3.29}$$

and there exists the finite limit:

$$T^* := \lim_{k \to \infty} \overline{T}_k < \infty.$$

By the continuity of $\mathcal{E}_{3/2}(u; t)$ in t, we deduce from (3.27) and (3.29) that

$$\mathcal{E}_{3/2}(u; T^*) = \lim_{k \to \infty} \mathcal{E}_{3/2}(u; \overline{T}_{k-1}) = \infty.$$

This contradicts that $\mathcal{E}_{3/2}(u; t)$ is finite at $t = T^*$, since $u(t,x)$ is the global in time analytic solution. Thus (3.28) is true.

We turn to the proof of well-posedness. If $\overline{T}_1 = \infty$, then $(u_0, u_1) = (0, 0)$, and hence, the proposition is entirely Lemma 3.5. We consider the case when $\overline{T}_1 < \infty$.

We claim that for every $\varepsilon > 0$ there exists a real $\delta_1 > 0$ depending on $\varepsilon, \nu_0, M, M_1$, u_0, u_1 and $\overline{T}_1$ such that if $v \in C^1([0,\infty); \mathcal{A}_{L^2}(\Omega))$ solves (1.1) with data

$$(v(0,\cdot), \partial_t v(0,\cdot)) \in \mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega)$$

satisfying the analytic compatibility condition and

$$\widetilde{\mathcal{E}}_\sigma(u-v;0) < \delta_1$$

for every $\sigma \in [1, \sigma_1]$, then

$$\widetilde{\mathcal{E}}_\sigma(u-v;t) < \varepsilon \quad (t \in [0, \overline{T}_1]).$$

In fact, by using Lemma 3.5, we deduce that (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, \overline{T}_1)$ in the above sense. Furthermore, employing the continuation argument as in the proof of Proposition 3.6, we conclude that H^σ-well-posedness holds even at $t = \overline{T}_1$. Next, we consider the problem (1.1) on $(\overline{T}_1, \overline{T}_2) \times \Omega$ with data

$$(u(\overline{T}_1,\cdot), \partial_t u(\overline{T}_1,\cdot)) \in \mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega) \subsetneqq H^{\sigma_1}(\Omega) \times H^{\sigma_1 - 1}(\Omega).$$

Then, from Lemma 3.5 together with the time translation method, we deduce that the problem (1.1) is H^σ-well-posed at $(u(\overline{T}_1,\cdot), \partial_t u(\overline{T}_1,\cdot))$ on $[\overline{T}_1, \overline{T}_2]$ for every $\sigma \in [1, \sigma_1]$. Hence, by adjusting the smallness of $\widetilde{\mathcal{E}}_\sigma(u-v;0)$, we conclude that the problem (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, \overline{T}_2]$ for every $\sigma \in [1, \sigma_1]$. Therefore, by the successive argument, the problem (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, \overline{T}_k]$ for all positive integer k and every $\sigma \in [1, \sigma_1]$. In conclusion, the above argument implies that the problem (1.1) is H^σ-well-posed at (u_0, u_1) on every interval $[0, T]$. The proof of Proposition 3.7 is now finished. □

4. Absolute integrability of time-derivative of the coefficient

In this section we prove that time-derivative of the coefficient $\varphi(\|\nabla u\|_{L^2}^2)$ of principal term is absolutely integrable on the maximal interval of existence of solutions. For this purpose, we need a uniform energy estimate for analytic solutions.

We shall prove here the following.

Lemma 4.1. *Assume that Ω is the whole space $\mathbb{R}^n$, or an analytic domain with a compact boundary. Suppose that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let M and M_1 be the constants as in (3.4) and (3.6), respectively. Let $\sigma \geq 1$, $T > 0$, and let $\mathcal{K}$ be a compact subset of $H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$. Then there exists a positive constant $C_{\mathcal{K}}(M, M_1, \nu_0, T)$ depending on M, M_1, ν_0, T and $\mathcal{K}$ such that*

$$\|(u(t,\cdot), \partial_t u(t,\cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} \leq C_{\mathcal{K}}(M, M_1, \nu_0, T)\|(u_0, u_1)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} \tag{4.1}$$

for any $t \in [0, T]$, and for any $(u_0, u_1) \in (\mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega)) \cap \mathcal{K}$ satisfying the analytic compatibility condition, where $u(t, x)$ is a global in time analytic solution to (1.1) with data (u_0, u_1).

Proof. Obviously, (4.1) is true for $(u_0, u_1) = (0, 0)$. Hence, we have only to prove (4.1) for $(u_0, u_1) \neq (0, 0)$. We divide the proof into two cases:

(i) $\mathcal{K}$ is a finite set.
(ii) $\mathcal{K}$ is an infinite set.

Case (i). We consider the case when $\mathcal{K}$ is a finite set. We may assume that $\mathcal{K}$ is a singleton, i.e.,

$$\mathcal{K} = \{(u_0, u_1)\}$$

without loss of generality. Suppose that (4.1) is not true. Then for every $k = 1, 2, \dots$, there exists a sequence $\{t_k\} \subsetneqq [0, T]$ such that

$$\|(u(t_k, \cdot), \partial_t u(t_k, \cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} > k\|(u_0, u_1)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}. \tag{4.2}$$

By the compactness of $[0, T]$ we can extract a subsequence $\{t_{k'}\}$ such that

$$t_{k'} \underset{k'\to\infty}{\longrightarrow} t^*$$

for some $t^* \in [0, T]$. Since the sequence

$$\{k'\|(u_0, u_1)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\}$$

is unbounded, it follows from (4.2) that

$$\{\|(u(t_{k'}, \cdot), \partial_t u(t_{k'}, \cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\}$$

is unbounded, which implies that the function

$$\|(u(t, \cdot), \partial_t u(t, \cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}$$

is unbounded near a neighbourhood of $t = t^*$. This contradicts the fact that

$$u \in C([0, T]; H^\sigma(\Omega) \cap C^1([0, T]; H^{\sigma-1}(\Omega)).$$

Hence, (4.1) is true in this case.

Case (ii). We consider the case when $\mathcal{K}$ is an infinite set. Suppose that (4.1) is not true. Then for every $k = 1, 2, \dots$, there exists a pair of non-trivial functions (u_0^k, u_1^k) in $(\mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega)) \cap \mathcal{K}$ satisfying the analytic compatibility condition and a sequence $\{t_k\} \subsetneqq [0, T]$ such that

$$\|(u^k(t_k, \cdot), \partial_t u^k(t_k, \cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} > k\|(u_0^k, u_1^k)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}, \tag{4.3}$$

where $u^k(t, x)$ are global in time analytic solutions to (1.1) with data (u_0^k, u_1^k). By the compactness of $[0, T]$ we can extract a subsequence $\{t_{k'}\}$ such that

$$t_{k'} \underset{k'\to\infty}{\longrightarrow} t_*$$

for some $t_* \in [0, T]$. Let $U(t)$ be the solution operator associated to (1.1), i.e., $U(t)$ is the continuous mapping introduced in Proposition 3.7. Here, $\mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega)$ is endowed with the induced topology of $H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$ for $\sigma \geq 1$. Since

$$\{(u_0^{k'}, u_1^{k'})\} \subset (\mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega)) \cap \mathcal{K},$$

it follows that the image $\{u^{k'}(t, \cdot)\} = \{U(t)(u_0^{k'}, u_1^{k'})\}$ is bounded in the solution space

$$C([0, T]; H^\sigma(\Omega)) \cap C^1([0, T]; H^{\sigma-1}(\Omega))$$

for every $T > 0$. For the sake of convenience, k' is denoted by k.

We claim that the sequence

$$\{k\|(u_0^k, u_1^k)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\}$$

is bounded. In fact, if this is not true, it follows from (4.3) that the sequence

$$\{\|(u^k(t_k, \cdot), \partial_t u^k(t_k, \cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\}$$

is unbounded. Therefore, the sequence of functions

$$\{\|(u^k(t,\cdot),\partial_t u^k(t,\cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\}$$

is unbounded at $t=t_*$. This contradicts the boundedness of $\{u^k(t,\cdot)\}$ in the space

$$C([0,T];H^\sigma(\Omega))\cap C^1([0,T];H^{\sigma-1}(\Omega)).$$

Therefore, there exists a positive constant A such that

$$k\|(u_0^k,u_1^k)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\le A \tag{4.4}$$

for all k.

Put

$$(\widetilde{u}_0^k,\widetilde{u}_1^k)=\frac{1}{k\|(u_0^k,u_1^k)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}}(u_0^k,u_1^k)\quad \text{for } k=1,2,\cdots.$$

Then we have

$$\|(\widetilde{u}_0^k,\widetilde{u}_1^k)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\underset{k\to\infty}{\longrightarrow}0. \tag{4.5}$$

Let us define

$$\widetilde{u}^k(t,x)=\frac{1}{k\|(u_0^k,u_1^k)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}}u^k(t,x).$$

Then each $\widetilde{u}^k(t,x)$ satisfies the Kirchhoff equation with a new coefficient $\widetilde{\varphi}_k$:

$$\begin{cases}\partial_t^2\widetilde{u}^k-\widetilde{\varphi}_k(\|\nabla\widetilde{u}^k(t,\cdot)\|^2_{L^2(\Omega)})\Delta\widetilde{u}^k=0, & t\in[0,T],\quad x\in\Omega,\\ \widetilde{u}^k(0,x)=\widetilde{u}_0^k(x),\quad \partial_t\widetilde{u}^k(0,x)=\widetilde{u}_1^k(x), & x\in\Omega\end{cases} \tag{4.6}$$

with the boundary condition

$$\widetilde{u}^k|_{[0,T]\times\partial\Omega}=0,$$

where

$$\widetilde{\varphi}_k(\|\nabla\widetilde{u}^k(t,\cdot)\|^2_{L^2(\Omega)}):=\varphi\left(k^2\|(u_0^k,u_1^k)\|^2_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\|\nabla\widetilde{u}^k(t,\cdot)\|^2_{L^2(\Omega)}\right).$$

We observe that functions $\widetilde{\varphi}_k(\rho)$ have a common domain $[0,\nu_0^{-1}C_0]$ for all k, where C_0 is a positive constant satisfying

$$\mathscr{H}(\widetilde{u}_k;0)\le C_0\quad \text{for } k=1,2,\cdots, \tag{4.7}$$

which is possible on account of (4.5). In fact, we see from Lemma 3.1 that

$$\mathscr{H}(\widetilde{u}^k;t)=\|\partial_t\widetilde{u}^k(t,\cdot)\|^2_{L^2(\Omega)}+\int_0^{\|\nabla\widetilde{u}^k(t,\cdot)\|^2_{L^2(\Omega)}}\widetilde{\varphi}_k(\rho)\,d\rho=\mathscr{H}(\widetilde{u}^k;0) \tag{4.8}$$

for all k. It follows from assumption (1.2) that

$$\widetilde{\varphi}_k(\rho)=\varphi\left(k^2\|(u_0^k,u_1^k)\|^2_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\rho\right)\ge\nu_0$$

for any $\rho\ge 0$ and all k. Hence, this inequality together with (4.7) and (4.8) imply that

$$\|\nabla\widetilde{u}^k(t,\cdot)\|^2_{L^2(\Omega)}\le\nu_0^{-1}\mathscr{H}(\widetilde{u}^k;t)=\nu_0^{-1}\mathscr{H}(\widetilde{u}^k;0)\le\nu_0^{-1}C_0$$

for all k and $t\in[0,T]$. This proves the assertion.

We need to check the boundedness and Lipschitz continuity of $\widetilde{\varphi}_k(\rho)$ uniformly in k to use Proposition 3.7. Indeed, we can show that $\{\widetilde{\varphi}_k'(\rho)\}$ is uniformly bounded on $[0,\nu_0^{-1}C_0]$. Namely, we claim that

(i) $\{\widetilde{\varphi}_k(\rho)\}$ is uniformly bounded on $[0,\nu_0^{-1}C_0]$, and satisfies

$$\nu_0 \le \widetilde{\varphi}_k(\rho) \le \widetilde{M} \tag{4.9}$$

for any $\rho \in [0,\nu_0^{-1}C_0]$ and all k, where we put

$$\widetilde{M} = \sup\{\varphi(\rho) : 0 \le \rho \le A^2\nu_0^{-1}C_0\}.$$

Here A is the constant appearing in (4.4).

(ii) $\{\widetilde{\varphi}_k'(\rho)\}$ is uniformly bounded on $[0,\nu_0^{-1}C_0]$, and satisfies

$$|\widetilde{\varphi}_k'(\rho)| \le \widetilde{M}_1 A^2 \tag{4.10}$$

for any $\rho \in [0,\nu_0^{-1}C_0]$ and all k, where we put

$$\widetilde{M}_1 = \sup\left\{|\varphi'(\rho)| : 0 \le \rho \le A^2\nu_0^{-1}C_0\right\}. \tag{4.11}$$

In fact, (4.9) is an immediate consequence of assumption (1.2) and the following:

$$\sup\{\widetilde{\varphi}_k(\rho) : 0 \le \rho \le \nu_0^{-1}C_0\} \le \sup\{\varphi(\rho) : 0 \le \rho \le A^2\nu_0^{-1}C_0\}.$$

This proves the assertion (i). As to the assertion (ii), it follows from (4.4) and (4.11) that

$$|\widetilde{\varphi}_k'(\rho)| = \left|\varphi'\left(k^2\|(u_0^k,u_1^k)\|^2_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\rho\right)\right| \cdot k^2\|(u_0^k,u_1^k)\|^2_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} \le \widetilde{M}_1 A^2$$

for any $\rho \in [0,\nu_0^{-1}C_0]$ and all k. This proves (4.10).

We are now in a position to lead to a contradiction. Obviously, the zero function $w(t,x)=0$ solves the problems (4.6) with zero initial condition:

$$\begin{cases} \partial_t^2 w - \widetilde{\varphi}_k(\|\nabla w(t,\cdot)\|^2_{L^2(\Omega)})\Delta w = 0, & t\in[0,T], \quad x\in\Omega, \\ w(0,x)=0, \quad \partial_t w(0,x) = 0, & x\in\Omega, \end{cases} \tag{4.12}$$

and satisfies the Dirichlet boundary condition

$$w|_{[0,T]\times\partial\Omega} = 0$$

for all k, where

$$\widetilde{\varphi}_k(\|\nabla w(t,\cdot)\|^2_{L^2(\Omega)}) = \varphi\left(k^2\|(u_0^k,u_1^k)\|^2_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)}\|\nabla w(t,\cdot)\|^2_{L^2(\Omega)}\right).$$

Now, thanks to the convergence (4.5), applying Proposition 3.7 to the problems (4.12), we find an integer k_0 such that

$$\|(\widetilde{u}^k(t,\cdot),\partial_t\widetilde{u}^k(t,\cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} \le 1 \tag{4.13}$$

on $[0,T]$ for all $k\ge k_0$. However, it follows from (4.3) that the sequence $\{\widetilde{u}^k(t_k,\cdot)\}$ satisfies

$$\|(\widetilde{u}^k(t_k,\cdot),\partial_t\widetilde{u}^k(t_k,\cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} > 1, \quad k=1,2,\cdots.$$

This inequality contradicts (4.13). Thus, the estimate (4.1) is true. The proof of Lemma 4.1 is now finished. □

We conclude this section by proving the absolute integrability of time-derivative of the functional $c(t)$ defined by (3.2).

Proposition 4.2. *Let Ω, M, M_1 and ν_0 be as in Lemma 4.1, and let $u(t,x)$ be the maximal solution to the problem (1.1) in the sense of Proposition 3.2 with data $(u_0,u_1) \in H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)$ for $\sigma \geq 3/2$. Suppose that $T_m < \infty$. Then for the functional $c(t)$ defined by (3.2), the time-derivative $c'(t)$ is absolutely integrable on $[0,T_m]$, and there exists a positive constant $L(M,M_1,\nu_0,T_m)$ depending on M, M_1, ν_0 and T_m such that*

$$\int_0^t |c'(\tau)|\,d\tau \leq L(M,M_1,\nu_0,T_m)\|(u_0,u_1)\|^2_{H^{\frac{3}{2}}(\Omega)\times H^{\frac{1}{2}}(\Omega)} t \tag{4.14}$$

for any $t\in[0,T_m]$.

Proof. We note that $\mathcal{A}_{L^2}(\Omega)$ is dense in $H^\sigma(\Omega)$ for any $\sigma \geq 0$ (see Lemma A.1 in appendix A). Let $u^j(t,x)$ be global in time analytic solutions to (1.1) with data $(u_0^j,u_1^j) \in \mathcal{A}_{L^2}(\Omega)\times\mathcal{A}_{L^2}(\Omega)$, $j=1,2,\dots$, satisfying the analytic compatibility condition and

$$(u_0^j,u_1^j) \underset{j\to\infty}{\longrightarrow} (u_0,u_1) \quad \text{in } H^\sigma(\Omega)\times H^{\sigma-1}(\Omega) \tag{4.15}$$

for $\sigma\geq 3/2$. We note that $\{(u_0^j,u_1^j)\}\cup\{(u_0,u_1)\}$ is an infinite compact subset of $H^{\sigma_0}(\Omega)\times H^{\sigma_0-1}(\Omega)$ for every $\sigma_0\in[1,\sigma]$. Define a sequence of functionals

$$c_j(t) = \varphi\left(\int_\Omega |\nabla u^j(t,x)|^2\,dx\right), \quad j=1,2,\cdots.$$

For every $T\in(0,T_m)$, thanks to Proposition 3.6, the convergence (4.15) implies that

$$\begin{aligned} c_j'(t) &= \varphi'\left(\int_\Omega |\nabla u^j(t,x)|^2\,dx\right)\cdot 2\mathrm{Re}\left((-\Delta)^{\frac14}\partial_t u^j(t,\cdot),(-\Delta)^{\frac34}u^j(t,\cdot)\right)_{L^2(\Omega)} \\ &\underset{j\to\infty}{\longrightarrow} \varphi'\left(\int_\Omega |\nabla u(t,x)|^2\,dx\right)\cdot 2\mathrm{Re}\left((-\Delta)^{\frac14}\partial_t u(t,\cdot),(-\Delta)^{\frac34}u(t,\cdot)\right)_{L^2(\Omega)} \\ &= c'(t) \end{aligned} \tag{4.16}$$

for all $t\in[0,T]$. Here, there exists a positive constant C_1 such that

$$\|(u_0^j,u_1^j)\|_{H^{\frac32}(\Omega)\times H^{\frac12}(\Omega)} \leq C_1\|(u_0,u_1)\|_{H^{\frac32}(\Omega)\times H^{\frac12}(\Omega)}$$

for all j. By using the upper bound M_1 of φ' (see (3.6)), and by applying the estimate (4.1) from Lemma 4.1 for $\mathcal{K}=\{(u_0^j,u_1^j)\}\cup\{(u_0,u_1)\}$ to $u^j(t,x)$ on $[0,T_m]$, we find a positive constant $C(M,M_1,\nu_0,T_m)$, independent of j, such that

$$\begin{aligned} |c_j'(t)| &\leq 2\left|\varphi'(\|\nabla u^j(t,\cdot)\|^2_{L^2(\Omega)})\right|\|\partial_t u^j(t,\cdot)\|_{H^{\frac12}(\Omega)}\|u^j(t,\cdot)\|_{H^{\frac32}(\Omega)} \\ &\leq 2M_1 C(M,M_1,\nu_0,T_m)^2\|(u_0^j,u_1^j)\|^2_{H^{\frac32}(\Omega)\times H^{\frac12}(\Omega)} \\ &\leq 2C_1M_1C(M,M_1,\nu_0,T_m)^2\|(u_0,u_1)\|^2_{H^{\frac32}(\Omega)\times H^{\frac12}(\Omega)} \end{aligned}$$

for all $t\in[0,T_m]$ and j. Thus, thanks to (4.16), Lebesgue's dominated convergence theorem implies that $c'(t)$ is absolutely integrable on $[0,T]$ and satisfies

$$\int_0^t |c'(\tau)|\,d\tau \leq 2C_1M_1C(M,M_1,\nu_0,T_m)^2\|(u_0,u_1)\|^2_{H^{\frac32}(\Omega)\times H^{\frac12}(\Omega)} t$$

for all $t \in [0, T]$. Since $T \in (0, T_m)$ is arbitrary, we conclude (4.14). The proof of Proposition 4.2 is finished. $\square$

5. Proofs of Theorems 2.1 and 2.2

In this section we prove Theorems 2.1 and 2.2. Our goal is to show that the local H^σ-solution $u(t,x)$ is bounded in $H^{3/2}(\mathbb{R}^n)$ at $t = T_m$; this allows us that part (ii) in Proposition 3.2 never occurs, and hence, $u(t,x)$ exists globally on $[0,\infty)$.

Proof of Theorem 2.1. Let $u(t,x)$ be the maximal H^σ-solution in Proposition 3.2 with data $(u_0,u_1) \in H^\sigma(\mathbb{R}^n) \times H^{\sigma-1}(\mathbb{R}^n)$. Suppose that $T_m < \infty$. We adopt an energy density as

$$E(t,\xi) = \left\{|\widehat{u}'(t)|^2 + c(t)|\xi|^2|\widehat{u}(t)|^2\right\}|\xi|,$$

where $\widehat{u}(t) = \widehat{u}(t,\xi)$ stands for the Fourier transform of $u(t,x)$ and we put

$$c(t) = \varphi\left(\int_{\mathbb{R}^n} |\nabla u(t,x)|^2\,dx\right).$$

Then we can write

$$\mathcal{E}_{3/2}(u;t) = \int_{\mathbb{R}^n} E(t,\xi)\,d\xi.$$

We notice that $\widehat{u}(t)$ satisfies the equation

$$\widehat{u}''(t) + c(t)|\xi|^2\widehat{u}(t) = 0. \tag{5.1}$$

By using the equation (5.1), we compute the time-derivative of $E(t,\xi)$:

$$\begin{aligned}
E'(t,\xi) &= \left[2\mathrm{Re}\left\{\widehat{u}''(t)\overline{\widehat{u}'(t)}\right\} + c'(t)|\xi|^2|\widehat{u}(t)|^2 + 2c(t)|\xi|^2\mathrm{Re}\left\{\widehat{u}'(t)\overline{\widehat{u}(t)}\right\}\right]|\xi| \\
&= c'(t)|\xi|^3|\widehat{u}(t)|^2 \\
&\le \frac{|c'(t)|}{c(t)}E(t,\xi).
\end{aligned}$$

Hence, we find from Gronwall's lemma that

$$\mathcal{E}_{3/2}(u;t) \le \mathcal{E}_{3/2}(u;0)e^{\int_0^t \frac{|c'(\tau)|}{c(\tau)}\,d\tau} \tag{5.2}$$

for any $t \in [0, T_m)$. Therefore, it follows from Proposition 4.2 that

$$\limsup_{t\to T_m-0} \mathcal{E}_{3/2}(u;t) \le \mathcal{E}_{3/2}(u;0)e^{\nu_0^{-1}L(M,M_1,\nu_0,T_m)\|(u_0,u_1)\|^2_{H^{3/2}(\mathbb{R}^n)\times H^{1/2}(\mathbb{R}^n)}T_m} < \infty.$$

Thus, the assertion (ii) in Proposition 3.2 never occurs, and hence, we conclude that $T_m = \infty$. The proof of Theorem 2.1 is complete. $\square$

We prove Theorem 2.2.

Proof of Theorem 2.2. Let $u(t,x)$ be a maximal solution to (1.1) with data $(u_0,u_1) \in \gamma^s_{L^2}(\mathbb{R}^n) \times \gamma^s_{L^2}(\mathbb{R}^n)$ in the sense of Proposition 3.3. It is sufficient to prove that

$$\limsup_{t\to T_m-0} \mathcal{E}_{3/2}(u;t) < \infty.$$

The finiteness of this superior limit is proved in the completely same way as in proof of Theorem 2.1. Thus we conclude from Proposition 3.3 that the Gevrey class solution $u(t,x)$ exists globally on $[0,\infty)$. The proof of Theorem 2.2 is complete. $\square$

6. Energy estimates

In this section we prove energy estimates for global solutions to the Cauchy problem (1.1) obtained in Theorems 2.1 and 2.2. These kinds of estimates for global solutions to the initial-boundary value problem are proved in a completely similar manner to the Cauchy problem.

We prepare global H^σ-well-posedness for (1.1).

Proposition 6.1. *Let $\sigma \geq 3/2$. Assume that φ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $u(t,x)$ be a global in time H^σ-solution to the Cauchy problem (1.1) with data $(u_0, u_1) \in H^\sigma(\mathbb{R}^n) \times H^{\sigma-1}(\mathbb{R}^n)$ in the sense of Theorem 2.1. Then the problem (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, T]$ for every $T > 0$.*

Proof. In Lemma 3.4 we constructed a sequence $\{T_k\}$ satisfying

$$T_0 = 0, \quad T_k - T_{k-1} = \frac{\nu_0^{2/3}}{M_1 \mathcal{E}_{3/2}(u; T_{k-1})}, \quad k = 1, 2, \cdots .$$

Then, according to the proof of Proposition 3.6, (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, T_k]$ for all k. Hence, if we prove that $\{T_k\}$ is divergent, then (1.1) is H^σ-well-posed at (u_0, u_1) on $[0, T]$ for every $T > 0$. Thus, all we have to do is to show that $\{T_k\}$ is divergent. However, the proof is completely the same as that of Proposition 3.7, and we omit it. The proof of Proposition 6.1 is complete. $\square$

We shall prove here the following.

Theorem 6.2. *Suppose that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let M and M_1 be as in (3.4) and (3.6), respectively. Let $u(t,x)$ be the global H^σ-solution in Theorem 2.1 for $\sigma \geq 3/2$. Then for every $T > 0$, there exists a positive constant $C(M, M_1, \nu_0, T)$ depending on M, M_1, ν_0 and T such that*

$$\begin{aligned} &\|u(t,\cdot)\|_{H^\sigma(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|_{H^{\sigma-1}(\mathbb{R}^n)} \\ &\leq C(M, M_1, \nu_0, T) \left(\|u_0\|_{H^\sigma(\mathbb{R}^n)} + \|u_1\|_{H^{\sigma-1}(\mathbb{R}^n)} \right) \end{aligned} \tag{6.1}$$

for any $t \in [0, T]$.

Proof. For any $(u_0, u_1) \in H^\sigma(\mathbb{R}^n) \times H^{\sigma-1}(\mathbb{R}^n)$ let us take $(u_0^k, u_1^k) \in \mathcal{A}_{L^2}(\mathbb{R}^n) \times \mathcal{A}_{L^2}(\mathbb{R}^n)$ such that

$$(u_0^k, u_1^k) \underset{k\to\infty}{\longrightarrow} (u_0, u_1) \quad \text{in } H^\sigma(\mathbb{R}^n) \times H^{\sigma-1}(\mathbb{R}^n). \tag{6.2}$$

Since $\{(u_0^k, u_1^k)\} \cup \{(u_0, u_1)\}$ is an infinite compact subset of $H^\sigma(\mathbb{R}^n) \times H^{\sigma-1}(\mathbb{R}^n)$, thanks to Lemma 4.1, there exists a positive constant $C(M, M_1, \nu_0, T)$, independent of k, such that

$$\begin{aligned} &\|(u^k(t,\cdot), \partial_t u^k(t,\cdot))\|_{H^\sigma(\mathbb{R}^n)\times H^{\sigma-1}(\mathbb{R}^n)} \\ &\leq C(M, M_1, \nu_0, T) \|(u_0^k, u_1^k)\|_{H^\sigma(\mathbb{R}^n)\times H^{\sigma-1}(\mathbb{R}^n)} \end{aligned} \tag{6.3}$$

for any $t \in [0, T]$ and all k, where $u^k(t,x)$ are global in time analytic solutions to (1.1) with data (u_0^k, u_1^k). Thanks to (6.2), we deduce from Proposition 6.1 that

$$u^k \underset{k\to\infty}{\longrightarrow} u \quad \text{strongly in } C([0,T]; H^\sigma(\mathbb{R}^n)) \cap C^1([0,T]; H^{\sigma-1}(\mathbb{R}^n))$$

for every $T > 0$. Thus, taking the limit in (6.3) as $k \to \infty$, we get the required estimate (6.1). The proof of Theorem 6.2 is complete. $\square$

We have also an energy estimate in Gevrey spaces.

Theorem 6.3. *Let $\varphi(\rho)$, M and M_1 be as in Theorem 6.2. Let $s > 1$ and $T > 0$. Suppose that there exists an $\eta > 0$ such that $(u_0, u_1) \in (-\Delta)^{1/2}\gamma^s_{\eta,L^2}(\mathbb{R}^n) \times \gamma^s_{\eta,L^2}(\mathbb{R}^n)$. Let $u(t,x)$ be the global solution with data (u_0, u_1) in the sense of Theorem 2.2. Then there exists a constant $\eta_0(M, M_1, \nu_0, T)$ depending on M, M_1, ν_0 and T such that*

$$\begin{aligned}
&\varphi(\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)})\|\nabla u(t,\cdot)\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} \\
&\le e^{\eta_0(M,M_1,\nu_0,T)\|(u_0,u_1)\|^2_{H^{3/2}(\mathbb{R}^n)\times H^{1/2}(\mathbb{R}^n)}t} \times \\
&\qquad \left(\varphi(\|\nabla u_0\|^2_{L^2(\mathbb{R}^n)})\|\nabla u_0\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} + \|u_1\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)}\right)
\end{aligned} \tag{6.4}$$

for any $t \in [0, T]$.

Proof. Defining the energy density as

$$E(t,\xi) = \left\{|\widehat{u}'(t)|^2 + c(t)|\xi|^2|\widehat{u}(t)|^2\right\} e^{\eta|\xi|^{\frac{1}{s}}},$$

we can write

$$\|\partial_t u(t,\cdot)\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} + c(t)\|\nabla u(t,\cdot)\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} E(t,\xi)\,d\xi.$$

Then, by the same argument of the derivation of (5.2), we deduce that

$$\begin{aligned}
&\|\partial_t u(t,\cdot)\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} + c(t)\|\nabla u(t,\cdot)\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} \\
&\le \left(\|u_1\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)} + c(0)\|\nabla u_0\|^2_{\gamma^s_{\eta,L^2}(\mathbb{R}^n)}\right) e^{\int_0^t \frac{|c'(\tau)|}{c(\tau)}\,d\tau}
\end{aligned}$$

for any $t \in [0, T]$. This estimate together with Proposition 4.2 for T_m replaced by T imply (6.4). The proof of Theorem 6.3 is complete. $\square$

7. Global existence of low regular solutions

In this section we prove the global existence theorem for low regular solutions to the problem (1.1). To begin with, we define a notion of low regular solutions.

Definition 7.1. Let $\sigma \in [1, 3/2)$. The function $u(t,x)$ is said to be an H^σ-solution to (1.1) iff it satisfies the following:

$$u \in C([0,\infty); H^\sigma(\Omega)) \cap C^1([0,\infty); H^{\sigma-1}(\Omega))$$

and

$$_X\langle \partial_t u(t), \psi\rangle_{X'} - {}_X\langle u_1, \psi\rangle_{X'} + \int_0^t \varphi(\|\nabla u(\tau)\|^2_{L^2(\Omega)})\,{}_X\langle \nabla u(\tau), \nabla\psi\rangle_{X'}\,d\tau = 0$$

for any $\psi \in C_0^\infty(\Omega)$ and $t \ge 0$. Here, $_X\langle f, g\rangle_{X'}$ denotes the duality pair of $f \in X$ and $g \in X'$, and we put

$$X = H^{\sigma-1}(\Omega) \quad \text{and} \quad X' = H^{-(\sigma-1)}(\Omega).$$

We shall prove here the following.

Theorem 7.1. *Assume that Ω is the whole space $\mathbb{R}^n$, or an analytic domain with a compact boundary. Suppose that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let $\sigma \in [1, 3/2)$. Then for any $(u_0, u_1) \in (H^\sigma(\Omega) \cap H_0^1(\Omega)) \times H^{\sigma-1}(\Omega)$, the problem (1.1) admits a unique global solution $u(t,x)$ such that*

$$u \in C([0,\infty); H^\sigma(\Omega)) \cap C^1([0,\infty); H^{\sigma-1}(\Omega)).$$

We have also an energy estimate.

Theorem 7.2. *Suppose that $\varphi(\rho)$ is C^1 on $\mathbb{R}$ and satisfies (1.2). Let M and M_1 be as in (3.4) and (3.6), respectively. Let $u(t,x)$ be the global H^σ-solution in Theorem 7.1 for $\sigma \in [1, 3/2)$. Then for every $T > 0$, there exists a positive constant $C(M, M_1, \nu_0, T)$ depending on M, M_1, ν_0 and T such that*

$$\|(u(t,\cdot), \partial_t u(t,\cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} \le C(M, M_1, \nu_0, T)\|(u_0,u_1)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} \tag{7.1}$$

for all $t \in [0,T]$. In particular, we have

$$\mathscr{H}(u;t) = \mathscr{H}(u;0) \tag{7.2}$$

for all $t \ge 0$.

We prove these theorems.

Proof of Theorem 7.1. Let $u^j(t,x)$ be global in time analytic solutions to (1.1) with data $(u_0^j, u_1^j) \in \mathcal{A}_{L^2}(\Omega) \times \mathcal{A}_{L^2}(\Omega)$, $j = 1, 2, \dots$, satisfying the analytic compatibility condition and

$$(u_0^j, u_1^j) \underset{j\to\infty}{\longrightarrow} (u_0, u_1) \quad \text{in } H^\sigma(\Omega) \times H^{\sigma-1}(\Omega) \tag{7.3}$$

for $\sigma \in [1, 3/2)$. Since $\{(u_0^j, u_1^j)\}$ is the Cauchy sequence in $H^\sigma(\Omega) \times H^{\sigma-1}(\Omega)$, it follows from Proposition 3.7 that $\{u^j(t,x)\}$ is also a Cauchy sequence:

$$\widetilde{\mathcal{E}}_\sigma(u^j - u^k; t) \underset{j,k\to\infty}{\longrightarrow} 0$$

for all $t \in [0,T]$. Hence, there exists the limit

$$u := \mathrm{s} - \lim_{j\to\infty} u^j \quad \text{in } C([0,T]; H^\sigma(\Omega)) \cap C^1([0,T]; H^{\sigma-1}(\Omega)). \tag{7.4}$$

Obviously, $u^j(t,x)$ satisfy the following identity:

$${}_X\langle \partial_t u^j(t), \psi\rangle_{X'} - {}_X\langle u_1^j, \psi\rangle_{X'} + \int_0^t \varphi(\|\nabla u^j(\tau)\|^2_{L^2(\Omega)})\, {}_X\langle \nabla u^j(\tau), \nabla\psi\rangle_{X'}\, d\tau = 0 \tag{7.5}$$

for any $\psi \in C_0^\infty(\Omega)$ and $t \ge 0$. Letting $j \to \infty$ in (7.5), thanks to (7.4), we conclude that

$${}_X\langle \partial_t u(t), \psi\rangle_{X'} - {}_X\langle u_1, \psi\rangle_{X'} + \int_0^t \varphi(\|\nabla u(\tau)\|^2_{L^2(\Omega)})\, {}_X\langle \nabla u(\tau), \nabla\psi\rangle_{X'}\, d\tau = 0$$

for any $\psi \in C_0^\infty(\Omega)$ and $t \ge 0$. Thus, $u(t,x)$ is a unique H^σ-solution to the problem (1.1). The proof of Theorem 7.1 is finished. □

Proof of Theorem 7.2. Let $u^j(t,x)$ be as in the proof of Theorem 7.1. Since $\mathcal{K} := \{(u_0^j,u_1^j)\} \cup \{(u_0,u_1)\}$ is an infinite compact subset of $H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)$, applying Lemma 4.1 for $u^j(t,x)$ and $\mathcal{K}$, we find a positive constant $C(M,M_1,\nu_0,T)$, independent of j, such that

$$\begin{aligned} &\|(u^j(t,\cdot),\partial_t u^j(t,\cdot))\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} \\ \le\, &C(M,M_1,\nu_0,T)\|(u_0^j,u_1^j)\|_{H^\sigma(\Omega)\times H^{\sigma-1}(\Omega)} \end{aligned} \tag{7.6}$$

for any $t\in[0,T]$ and j. Therefore, thanks to (7.3) and (7.4), letting $j\to\infty$ in (7.6), we get the required estimate (7.1).

For analytic solutions $u^j(t,x)$, we have, by using (3.1),

$$\mathscr{H}(u^j;t) = \mathscr{H}(u^j;0) \tag{7.7}$$

for all $t\ge 0$ and j. Hence, letting $j\to\infty$ in (7.7), we conclude from (7.3) and (7.4) that (7.2) holds. The proof of Theorem 7.2 is complete. $\square$

Appendix A.

Lemma A.1. *Let Ω be the whole space $\mathbb{R}^n$, or an exterior domain with analytic boundary. Then $\mathcal{A}_{L^2}(\Omega)$ is dense in $\dot{H}^\sigma(\Omega)$ for any $\sigma\ge 0$. In particular, $\mathcal{A}_{L^2}(\Omega)$ is dense in $H^\sigma(\Omega)$.*

Proof. It is sufficient to prove the case when $\Omega=\mathbb{R}^n$. If Ω is an exterior domain, we can perform the argument by using generalized Fourier transform in place of Fourier transform on $\mathbb{R}^n$. If Ω is a bounded domain, Fourier series expansion method is employed.

Define

$$\mathcal{A}_c(\mathbb{R}^n) = \{f\in L^2(\mathbb{R}^n) : \operatorname{supp}\widehat{f}(\xi) \text{ is compact in } \mathbb{R}^n\}.$$

We claim that $\mathcal{A}_c(\mathbb{R}^n)$ is dense in $\dot{H}^\sigma(\mathbb{R}^n)$. In fact, let $f\in\dot{H}^\sigma(\mathbb{R}^n)$, and put

$$B_j = \{\xi\in\mathbb{R}^n : |\xi|<j\}, \quad j=1,2,\cdots,$$

and

$$f_j = \mathbb{1}_{B_j}(D)f, \quad j=1,2,\cdots,$$

where $\mathbb{1}_E(D)$ is an operator with symbol $\mathbb{1}_E(\xi)$ which is a characteristic function of a measurable set E in $\mathbb{R}^n$. Then $\{f_j\}$ is a sequence in $\mathcal{A}_c(\mathbb{R}^n)$ such that

$$\|f-f_j\|_{\dot{H}^\sigma(\mathbb{R}^n)} = \left\||\xi|^\sigma\big(\widehat{f}-\mathbb{1}_{B_j}\widehat{f}\big)\right\|_{L^2(\mathbb{R}^n)} \underset{j\to\infty}{\longrightarrow} 0.$$

In a similar way, let $g\in\mathcal{A}_{\eta,L^2}(\mathbb{R}^n)$ for some $\eta>0$, and

$$g_j = \mathbb{1}_{B_j}(D)g, \quad j=1,2,\cdots.$$

Then $\{g_j\}$ is a sequence in $\mathcal{A}_c(\mathbb{R}^n)$ such that

$$\|g-g_j\|_{\mathcal{A}_{\eta,L^2}(\mathbb{R}^n)} = \left\|e^{\frac{\eta|\xi|}{2}}\big(\widehat{g}-\mathbb{1}_{B_j}\widehat{g}\big)\right\|_{L^2(\mathbb{R}^n)} \underset{j\to\infty}{\longrightarrow} 0.$$

Hence, $\mathcal{A}_c(\mathbb{R}^n)$ is dense in $\mathcal{A}_{L^2}(\mathbb{R}^n)$. Since

$$\mathcal{A}_c(\mathbb{R}^n) \subsetneqq \mathcal{A}_{L^2}(\mathbb{R}^n) \subsetneqq H^\sigma(\mathbb{R}^n) \subsetneqq \dot{H}^\sigma(\mathbb{R}^n),$$

we conclude the proof of Lemma A.1. $\square$

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