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# An Improvement of Yannakakis' Algorithm for MAX SAT*

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**abstract**

MAX SAT (maximum satisfiability problem) is stated as follows: given a set of clauses with weights, find a truth assignment that maximizes the sum of the weights of the satisfied clauses. In this paper, we consider approximation algorithms for MAX SAT proposed by Yannakakis and Goemans-Williamson and present an approximation algorithm which is an improvement of Yannakakis' algorithm. Although Yannakakis' original algorithm has no better performance guarantee than Goemans-Williamson, our improved algorithm has a better performance guarantee and leads to a 0.770-approximation algorithm.

## 1 Introduction

MAX SAT (maximum satisfiability problem) is stated as follows: given a set of clauses with weights, find a truth assignment that maximizes the sum of the weights of the satisfied clauses. MAX SAT is well known to be NP-hard and many researchers have proposed approximation algorithms for MAX SAT. Johnson [9] proposed a 0.5-approximation algorithm for MAX SAT based on the probabilistic method. Since then a lot of works had been done for MAX SAT and Yannakakis [12] and Goemans-Williamson [7] finally proposed 0.75-approximation algorithms. On the other hand, Goemans-Williamson proposed, based on semidefinite programming [6], a 0.878-approximation algorithm for MAX 2SAT, the restricted version of MAX SAT where each clause has at most 2 literals, and showed that their algorithm, if combined with Johnson's algorithm and Goemans-Williamson's 0.75-approximation algorithm, leads to a 0.7584-approximation algorithm for MAX SAT [8]. Asano-Ono-Hirata also proposed a semidefinte programming approach to MAX SAT [3] and obtained a 0.765-approximation algorithm by combining it with Yannakakis' 0.75-approximation algorithm as well as the algorithms of Johnson and Goemans-Williamson. More recently, Asano-Hori-Ono-Hirata [2] presented a refinement of Yannakakis' algorithm based on network flows, and suggested that it might lead to a 0.767-approximation algorithm.

In this paper, we present a further refinement of the 0.75-approximation algorithm of Yannakakis for MAX SAT and show that it has a better bound and leads to a 0.770-approximation algorithm.[1] To explain our result more precisely, we need some notations.

An instance of MAX SAT is defined by $(\mathcal{C}, w)$, where $\mathcal{C}$ is a set of boolean clauses such that each clause $C \in \mathcal{C}$ is a disjunction of literals with a positive weight $w(C)$. We sometimes write an instance $\mathcal{C}$ instead of $(\mathcal{C}, w)$ if the weight function $w$ is clear from the context. Let $X = \{x_1, \ldots, x_n\}$ be the

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1 Several progresses have been made since this paper was presented, and the current best one is a 0.7877-approximation algorithm (see [1,4]), however, we believe the method proposed in this paper will be used as a building block in making improvement of approximation algorithms for MAX SAT.

set of variables in the weighted clauses of $(\mathcal{C}, w)$. For each $x_i \in X$, let $x_i = 1$ ($x_i = 0$, resp.) if $x_i$ is true (false, resp.). Then, $\bar{x}_i = 1 - x_i$ and a clause $C_j \in \mathcal{C}$ can be considered to be a function of $\boldsymbol{x} = (x_1, \ldots, x_n)$ as follows:

$$C_j = C_j(\boldsymbol{x}) = 1 - \prod_{x_i \in X_j^+} (1 - x_i) \prod_{x_i \in X_j^-} x_i, \tag{1}$$

where $X_j^+$ ($X_j^-$, resp.) denotes the set of variables appearing unnegated (negated, resp.) in $C_j$. Thus, $C_j = C_j(\boldsymbol{x}) = 0$ or 1 for any *truth assignment* $\boldsymbol{x} \in \{0,1\}^n$, and $C_j$ is *satisfied* if $C_j(\boldsymbol{x}) = 1$. The *value* of a truth assignment $\boldsymbol{x}$ is defined to be

$$F_{\mathcal{C}}(\boldsymbol{x}) = \sum_{C_j \in \mathcal{C}} w(C_j) C_j(\boldsymbol{x}). \tag{2}$$

That is, the value of $\boldsymbol{x}$ is the sum of the weights of the clauses in $\mathcal{C}$ satisfied by $\boldsymbol{x}$. Thus, MAX SAT is to find an optimal truth assignment, i.e., a truth assignment of maximum value.

Let $A$ be an algorithm for MAX SAT and let $F_{\mathcal{C}}(\boldsymbol{x}^A(\mathcal{C}))$ be the value of a truth assignment $\boldsymbol{x}^A(\mathcal{C})$ produced by $A$ for an instance $\mathcal{C}$. If $F_{\mathcal{C}}(\boldsymbol{x}^A(\mathcal{C}))$ is at least $\alpha$ times the value $F_{\mathcal{C}}(\boldsymbol{x}^*(\mathcal{C}))$ of an optimal truth assignment $\boldsymbol{x}^*(\mathcal{C})$ for any instance $\mathcal{C}$, then $A$ is called an approximation algorithm with *performance guarantee* $\alpha$. A polynomial time approximation algorithm $A$ with performance guarantee $\alpha$ is called an $\alpha$*-approximation algorithm*.

The 0.75-approximation algorithm of Yannakakis is based on the probabilistic method. Let $\boldsymbol{x}^p$ be a *random truth assignment* with $0 \le x_i^p = p_i \le 1$, that is, $\boldsymbol{x}^p$ is obtained by setting independently each variable $x_i \in X$ to be true with probability $p_i$. Then the probability of a clause $C_j \in \mathcal{C}$ satisfied by the assignment $\boldsymbol{x}^p$ is

$$C_j(\boldsymbol{x}^p) = 1 - \prod_{x_i \in X_j^+} (1 - p_i) \prod_{x_i \in X_j^-} p_i. \tag{3}$$

Thus, the expected value of the random truth assignment $\boldsymbol{x}^p$ is

$$F_{\mathcal{C}}(\boldsymbol{x}^p) = \sum_{C_j \in \mathcal{C}} w(C_j) C_j(\boldsymbol{x}^p). \tag{4}$$

The probabilistic method assures that there is a truth assignment $\boldsymbol{x}^q \in \{0,1\}^n$ of value at least $F_{\mathcal{C}}(\boldsymbol{x}^p)$. Such a truth assignment $\boldsymbol{x}^q$ can be obtained by the method of conditional probability [7], [12]. The 0.75-approximation algorithm of Yannakakis [12] finds, for a given instance $(\mathcal{C}, w)$, a random truth assignment $\boldsymbol{x}^p$ of value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ at least

$$0.75W_1^* + 0.75W_2^* + 0.75W_3^* + 0.765W_4^* + 0.762W_5^* + 0.822W_6^* + \sum_{k \ge 7} (1 - (0.75)^k) W_k^* \tag{5}$$

where

$$W_k^* = \sum_{C \in \mathcal{C}_k} w(C) C(\boldsymbol{x}^*)$$

for an optimal truth assignment $\boldsymbol{x}^*$ of $\mathcal{C}_k$, the set of clauses in $\mathcal{C}$ with $k$ literals, and thus,

$$F_{\mathcal{C}}(\boldsymbol{x}^*) = \sum_{k \ge 1} W_k^*.$$

On the other hand, the 0.75-approximation algorithm of Goemans-Williamson [7] finds a random truth assignment of value at least

$$0.75W_1^* + 0.75W_2^* + 0.789W_3^* + 0.810W_4^* + 0.820W_5^* + 0.824W_6^* + \sum_{k \ge 7} \beta_k W_k^* \tag{6}$$

where

$$2\beta_k = 2 - \frac{1}{2^k} - \left(1 - \frac{1}{k}\right)^k.$$

Note that $\beta_k < 1 - (0.75)^k$ for $k \geq 7$. Thus, for two algorithms of Yannakakis and Goemans-Williamson, we cannot say that one is better than the other. In fact, for MAX 3SAT, Goemans-Williamson's algorithm is better than Yannakakis' one and used to obtain a better performance guarantee [11], while both are used for MAX SAT in [3] to obtain a performace guarantee 0.765.

In this paper, we will give an algorithm, an improvement of Yannakakis' algorithm, for finding a random truth assignment $\boldsymbol{x}^p = (p_1, p_2, ..., p_n)$ with value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ at least

$$0.75W_1^* + 0.75W_2^* + 0.791W_3^* + 0.811W_4^* + \ 0.823W_5^* + 0.850W_6^* + \sum_{k\geq 7}(1 - (0.75)^k)W_k^*. \quad (7)$$

Note that this bound is better than the bounds of Goemans-Williamson and Yannakakis. Our algorithm also leads to a 0.770-approximation algorithm if it is combined with the algorithms in [3], [11].

## 2 Outline of an Improvement

The 0.75-approximation algorithm of Yannakakis divides the variables $X = \{x_1, \ldots, x_n\}$ of a given instance $(\mathcal{C}, w)$ into three groups $P$, $P'$ and $P''$ based on maximum network flows (some variables will be negated appropriately). Then it sets independently each variable $x_i \in X$ to be true with probability $p_i$ such that $p_i = 3/4$ if $x_i \in P$, $p_i = 5/9$ if $x_i \in P'$ and $p_i = 1/2$ if $x_i \in P''$. The expected value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ of this random truth assignment $\boldsymbol{x}^p = (p_1, p_2, ..., p_n)$ is at least the bound in (5).

To divide the variables $X$ of a given instance $(\mathcal{C}, w)$ into three groups $P$, $P'$ and $P''$, Yannakakis transformed $(\mathcal{C}, w)$ into an equivalent instance $(\mathcal{C}', w')$ of the weighted clauses with some nice property by using network flows. Note that two sets $(\mathcal{C}, w)$, $(\mathcal{C}', w')$ of weighted clauses over the same set of variables are called *equivalent* if, for every truth assignment, $(\mathcal{C}, w)$ and $(\mathcal{C}', w')$ have the same value. Based on [2], we call $(\mathcal{C}, w)$,$(\mathcal{C}', w')$ are *strongly equivalent*, if, for every *random* truth assignment, $(\mathcal{C}, w)$ and $(\mathcal{C}', w')$ have the same expected value. Clearly, if $(\mathcal{C}, w)$,$(\mathcal{C}', w')$ are strongly equivalent then they are also equivalent since a truth assignment is always a random truth assignment (the converse is not true). Our notion of equivalence will be required when we try to obtain an improved bound 0.770. The following lemma [2] plays a central role throughout this paper.

**Lemma 1** *Let all clauses below have the same weight. Then* $\mathcal{A} = \{\bar{x}_i \vee x_{i+1} | i = 1, ..., k\}$ *and* $\mathcal{A}' = \{x_i \vee \bar{x}_{i+1} | i = 1, ..., k\}$ *are strongly equivalent (we consider* $k + 1 = 1$*) . Furthermore,* $\mathcal{B} = \{x_1\} \cup \{\bar{x}_i \vee x_{i+1} | i = 1, ..., \ell\}$ *and* $\mathcal{B}' = \{x_i \vee \bar{x}_{i+1} | i = 1, ..., \ell\} \cup \{x_{\ell+1}\}$ *are strongly equivalent.*

**Proof.** We can assume weights are all equal to 1. For a random truth assignment $\boldsymbol{x}^p$ with $x_i^p = p_i$, let $F_{\mathcal{D}}(\boldsymbol{x}^p) \equiv \sum_{C\in\mathcal{D}} C(\boldsymbol{x}^p)$ be the expected value of $\boldsymbol{x}^p$ for $\mathcal{D}$ ($\mathcal{D} = \mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$). Then, we have

$$F_{\mathcal{A}}(\boldsymbol{x}^p) = \sum_{i=1}^{k}(1 - p_i(1 - p_{i+1})) = k - \sum_{i=1}^{k} p_i + \sum_{i=1}^{k} p_i p_{i+1},$$

$$F_{\mathcal{A}'}(\boldsymbol{x}^p) = \sum_{i=1}^{k}(1 - p_{i+1}(1 - p_i)) = k - \sum_{i=1}^{k} p_i + \sum_{i=1}^{k} p_i p_{i+1} \text{ by } k + 1 = 1,$$

$$F_{\mathcal{B}}(\boldsymbol{x}^p) = p_1 + \sum_{i=1}^{\ell}(1 - p_i(1 - p_{i+1})) = \ell - \sum_{i=2}^{\ell} p_i + \sum_{i=1}^{\ell} p_i p_{i+1},$$

$$F_{\mathcal{B}'}(\boldsymbol{x}^p) = p_{\ell+1} + \sum_{i=1}^{\ell}(1 - p_{i+1}(1 - p_i)) = \ell - \sum_{i=2}^{\ell} p_i + \sum_{i=1}^{\ell} p_i p_{i+1}.$$

Thus, $F_{\mathcal{A}}(\boldsymbol{x}^p) = F_{\mathcal{A}'}(\boldsymbol{x}^p)$ and $F_{\mathcal{B}}(\boldsymbol{x}^p) = F_{\mathcal{B}'}(\boldsymbol{x}^p)$ for any random truth assignment $\boldsymbol{x}^p$ and we have the lemma. Q.E.D.

In this section, we present a brief outline of an improvement of the 0.75-approximation algorithm of Yannakakis for MAX SAT. Our algorithm consists of 8 steps (Steps $0-7$ below) based on network flows and divides the variables $X$ into four groups. (Yannakakis' algorithm consists of only 4 steps and all steps below except Step 0 are different from those in Yannakakis' one. We believe Yannakakis' algorithm is simple from the network theoretical point of view, although most people think it is very complicated. For those people, our algorithm below might be much more complicated.)

In each step except for Step 7, we output a set of weighted clauses which is strongly equivalent to a set of weighted clauses given as an input of that step. The output of Step $i$ ($i = 1, 2, ..., 6$) consists of groups of weighted clauses and all but one group are set aside (we call such a group being split off). The remaining group becomes an input of Step $i+1$. After Step 6, we obtain a partition of $X$ into $R_6, Q_6, P_6, Z_6$ and in Step 7, we obtain a random truth assignment $\boldsymbol{x}^p = (p_1, p_2, ..., p_n)$ by setting each variable $x_i$ to be true with probability $p_i$ such that $p_i = 0.75$ if $x_i \in R_6$, $p_i = 0.629$ if $x_i \in Q_6$, $p_i = 0.557$ if $x_i \in P_6$ and $p_i = 0.5$ if $x_i \in Z_6$. Then, all groups of weighted clauses split off in Steps $1-6$ and the remaining group $(\mathcal{D}^{(6)}, w_6)$ of weighted clauses after Step 6 have the expected values at least the bound in (7). Since the set of all split groups together with $(\mathcal{D}^{(6)}, w_6)$ is strongly equivalent to a given instance $(\mathcal{C}, w)$ in Step 0, we have thus obtain the bound in (7). More specifically, $(\mathcal{D}^{(6)}, w_6)$ has the following property.

**Property $\pi$.**

(a) $x \in R_6$ for each $C = x \in \mathcal{D}^{(6)}$.

(b) For each $C = \bar{x} \vee y \in \mathcal{D}^{(6)}$, $y \in R_6$ if $x \in R_6$, $y \in Q_6 \cup R_6$ if $x \in Q_6$ and $y \in P_6 \cup Q_6 \cup R_6$ if $x \in P_6$.

(c) For $k = 3, 4, 5, 6$, there is no clause in $\mathcal{D}^{(6)}$ with $k$ literals such that $k_1$ ($k_1 \geq 2k-6$) literals are contained in $\bar{R}_6$ and all the remaining literals are in $\bar{Q}_6$.

(d) For a clause in $\mathcal{D}^{(6)}$ with $k$ literals ($k = 3, 4$) of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$ such that $x_1, ..., x_{k-2} \in R_6$, $a \in R_6 \cup Q_6$ if $x_{k-1} \in R_6$ and $a \in R_6 \cup Q_6 \cup P_6$ if $x_{k-1} \in Q_6$.

(e) For a clause in $\mathcal{D}^{(6)}$ of form $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3$, $x_3 \notin R_6 \cup Q_6 \cup P_6$ if $x_1, x_2 \in Q_6$ and $x_3 \notin R_6$ if $x_1, x_2 \in P_6$.

It is easily observed that if $(\mathcal{D}^{(6)}, w_6)$ satisfies property $\pi$ then the random truth assignment $\boldsymbol{x}^p = (p_1, p_2, ..., p_n)$ in Step 7 has the expected value at least the bound in (7). All the split groups also have some nice properties assuring the bound in (7).

## 3 Improving Yannakakis' Algorithm

Now we will go into details. Let $\mathcal{C}_{1,2} \equiv \mathcal{C}_1 \cup \mathcal{C}_2$ (the set of clauses in $\mathcal{C}$ with one or two literals). As Yannakakis did, we first construct a network $N(\mathcal{C})$ which represents the weighted clauses in $(\mathcal{C}_{1,2}, w)$ as follows. The set of nodes of $N(\mathcal{C})$ consists of the set of literals in $\mathcal{C}$ and two new nodes $s$ and $t$ which represent true ($T$) and false ($F$) respectively. The (directed) arcs of $N(\mathcal{C})$ are corresponding to the clauses in $\mathcal{C}_{1,2}$. Each clause $C = x \vee y \in \mathcal{C}_2$ corresponds to two arcs $(\bar{x}, y)$ and $(\bar{y}, x)$ with capacity $cap(\bar{x}, y) = cap(\bar{y}, x) = w(C)/2$ ($\bar{\bar{x}} = x$). Similarly, each clause $C = x \in \mathcal{C}_1$ corresponds to two arcs $(s,x)$ and $(\bar{x},t)$ with capacity $cap(s,x) = cap(\bar{x},t) = w(C)/2$. Thus, we can regard a clause $C = x \in \mathcal{C}_1$ as $x \vee F$ when considering a network as above. Note that $N(\mathcal{C})$ is a naturally defined network since $x \vee y = \bar{x} \rightarrow y = \bar{y} \rightarrow x$.

Two arcs $(\bar{x},y)$ and $(\bar{y},x)$ are called *symmetric arcs*. If each symmetric two arcs in a network are of the same capacity, then the network is called *symmetric*. By the above correspondence of a clause and two symmetric arcs, a symmetric network $N$ exactly corresponds to a set $\mathcal{C}(N)$ of weighted clauses with one or two literals. In the case of $N = N(\mathcal{C})$ defined above, we have $\mathcal{C}(N(\mathcal{C})) = (\mathcal{C}_{1,2}, w)$. Thus, we can always construct the set $\mathcal{C}(N)$ of weighted clauses with one or two literals from a symmetric network $N$ and we sometimes use the term "the set of weighted clauses of a symmetric network" below. Then we consider a symmetric flow $f_0$ of maximum value $v(f_0)$ in $N_0 \equiv N(\mathcal{C})$ from source node $s$ to sink node $t$ (flow $f$ is called *symmetric* if $f(\bar{x},y) = f(\bar{y},x)$ for each symmetric arcs $(\bar{x},y),(\bar{y},x)$). Let $L_0$ be the network obtained from the residual network $N_0(f_0)$ of $N_0$ with respect to $f_0$ by deleting all arcs into $s$ and all arcs from $t$. Then $L_0$ is clearly symmetric since $N_0$ is a symmetric network and $f_0$ is a symmetric flow.

Let $(\mathcal{C}'_{1,2}, w')$ be the set of weighted clauses of the symmetric network $L_0$ ($(\mathcal{C}'_{1,2}, w') = \mathcal{C}(L_0)$) and let $(\mathcal{C}', w')$ be the set of weighted clauses obtained from $(\mathcal{C}, w)$ by replacing $(\mathcal{C}_{1,2}, w)$ with $(\mathcal{C}'_{1,2}, w')$. Then, for each truth assignment $\boldsymbol{x}$,

$$F_{\mathcal{C}}(\boldsymbol{x}) = F_{\mathcal{C}'}(\boldsymbol{x}) + v(f_0). \tag{8}$$

Note that (8) holds even if $\boldsymbol{x}$ is a random truth assignment. This can be obtained by Lemma 1 using an observation similar to the one in [12]. Note also that, for $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$ in Lemma 1, $\mathcal{A}$ corresponds to a cycle and $\mathcal{A}'$ corresponds to the reverse cycle. Similarly, $\mathcal{B}$ corresponds to a path from $x_1$ to $x_{\ell+1}$ (plus $(s,x_1)$) and $\mathcal{B}'$ corresponds to the reverse path from $x_{\ell+1}$ to $x_1$ (plus $(s,x_{\ell+1})$).

Since $f_0$ is a maximum flow, there is no path from $s$ to $t$ in $L_0$. Let $R$ be the set of nodes that are reachable from $s$ in $L_0$ and let $\bar{Y} = \{\bar{y} | y \in Y\}$ for $Y \subseteq X$. Then, there is no arc from a node in $R$ to a node not in $R$ and the set of nodes that can reach $t$ is $\bar{R}$ (in a symmetric network, $x_1, x_2, ..., x_{k-1}, x_k$ is a path if and only if $\bar{x}_k, \bar{x}_{k-1}, ..., \bar{x}_2, \bar{x}_1$ is a path) and $R$ does not contain any complementary literals, since $L_0$ is a symmetric network and $f_0$ is a maximum flow ($x, \bar{x} \in R$ implies that there is a path from $s$ to $t$ since $L_0$ is symmetric and there are paths from $s$ to $x$ (by $x \in R$) and $x$ to $t$ (by $\bar{x} \in R$), which contradicts the maximality of $f_0$). This implies that every clause of form $\bar{x} \vee y$ with $x \in R$ satisfies $y \in R$. Thus, we can set all literals of $R$ to be true consistently and, for each truth assignment $\boldsymbol{x}$ in which all literals of $R$ are true, every clause in $\mathcal{C}'_{1,2}$ that contains a literal in $R \cup \bar{R}$ is satisfied. From now on we assume that all literals in $R$ are unnegated ($R \subseteq X$ and thus all literals in $\bar{R}$ are negated).

By the argument above we can summarize Step 0 of our algorithm as follows.

**Step 0.** Find $R$ and $(\mathcal{C}', w')$ from $(\mathcal{C}, w)$ using the network $N_0$, a symmetric flow $f_0$ of $N_0$ of maximum value and the network $L_0$ defined above.

Note that, by (8), if we have an $\alpha$-approximation algorithm for $(\mathcal{C}', w')$, then it will also be an $\alpha$-approximation algorithm for $(\mathcal{C}, w)$. Thus, for simplicity, we can assume from now on $(\mathcal{C}', w') = (\mathcal{C}, w)$ (and thus, $f_0 = 0$ and $L_0 = N_0$) and have the following assumption.

**Assumption.** $\mathcal{C}$ and $N_0 = N(\mathcal{C})$ satisfy the following:
(a) $R \subseteq X$ and $x \in R$ for each $C = x \in \mathcal{C}$ (there are arcs $(s,x),(\bar{x},t)$).
(b) $y \in R$ for each $C = \bar{x} \vee y \in \mathcal{C}$ with $x \in R$ (there is no arc going outside from a node in $R$).

Let $\gamma_k$ be the coeffient of $W_k^*$ in (7), i.e.,

$$\gamma_k = \begin{cases} 0.75 & (k=1,2) \\ 0.791 & (k=3) \\ 0.811 & (k=4) \\ 0.823 & (k=5) \\ 0.850 & (k=6) \\ 1-0.75^k & (k \geq 7). \end{cases} \tag{9}$$

To obtain a 0.75-approximation algorithm, Yannakakis tried to set each variable in $R$ to be true with probability 0.75 and each variable in $Z_0 \equiv X - R$ to be true with probability 0.5. Then the probability of a clause in $\mathcal{C}_{1,2}$ being satisfied is at least $\gamma_1 = \gamma_2 = 0.75$. Similarly, the probability of a clause in $\mathcal{C}$ with five or more literals being satisfied is at least 0.75. Clauses satisfied with probability less than 0.75 have 3 or 4 literals and are of form $\bar{x} \vee \bar{y} \vee \bar{z}$ with $x, y, z \in R$ or of form $\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{u}$ with $x, y, z, u \in R$ or of form $\bar{x} \vee \bar{y} \vee a$ with $x, y \in R$ and $a \in Z_0 \cup \bar{Z}_0$. Similarly, clauses of form $C = \bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k$ with $x_1, x_2, \ldots, x_k \in R$ $(k = 5, 6)$ are satisfied with probability less than $\gamma_k$. To delete such clauses, let $\mathcal{A}_k^{(1)}$ be the set of clauses $C$ of form $C = \bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k$ with $x_1, x_2, \ldots, x_k \in R$ $(k = 3, 4, 5, 6)$, i.e.,

$$\mathcal{A}_k^{(1)} = \{C = \bar{x}_1 \vee \cdots \vee \bar{x}_k \in \mathcal{C} | x_1, \ldots, x_k \in R\} \tag{10}$$

To split off clauses in $\mathcal{A}_3^{(1)} \cup \mathcal{A}_4^{(1)} \cup \mathcal{A}_5^{(1)} \cup \mathcal{A}_6^{(1)}$, we consider the network $N_1$ obtained from $M_0 \equiv N_0$ as follows. Let $M_0^-$ be the network obtained from $M_0$ by deleting all arcs from $\bar{R}$ to $R$, all arcs from $\bar{R}$ to $Z_0 \cup \bar{Z}_0$ and all arcs from $Z_0 \cup \bar{Z}_0$ to $R$. Let $(\mathcal{C}_{1,2}^-, w) = \mathcal{C}(M_0^-)$ (the set of weighted clauses of the symmetric network $M_0^-$). $N_1$ is the network obtained from $M_0^-$ as follows. For each clause $C = \bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k \in \mathcal{A}_k^{(1)}$ with $x_1, x_2, \ldots, x_k \in R$ $(k = 3, 4, 5, 6)$, we consider two new nodes $C, \bar{C}$ and let $E_A(C)$ be the set of arcs from $x_i$ $(i = 1, 2, ..., k)$ to $C$ and from $C$ to $t$ and their symmetric arcs. Thus, $E_A(C)$ contains $2k + 2$ arcs and

$$E_A(C) = \{(s, \bar{C}), (C, t)\} \cup \cup_{i=1}^k \{(x_i, C), (\bar{C}, \bar{x}_i)\} \tag{11}$$

We add $C, \bar{C}$ and $E_A(C)$ for all $C = \bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k \in \mathcal{A}_k^{(1)}$ with $x_1, x_2, \ldots, x_k \in R$ $(k = 3, 4, 5, 6)$. We set the arcs $(s, \bar{C}), (C, t)$ to have capacity $w(C)$ and all remaining arcs of forms $(x_i, C)$ and $(\bar{C}, \bar{x}_i)$ to have capacity $w(C)/a_k^{(1)}$ with

$$a_k^{(1)} = \begin{cases} 6 & (k=3) \\ 10 & (k=4) \\ 14 & (k=5) \\ 22 & (k=6). \end{cases} \tag{12}$$

$N_1$ is the network obtained from $M_0^-$ in this way. Then, we find a symmetric flow $f_1$ of maximum value from $s$ to $t$ in $N_1$ such that

$$f_1(x_1, C) = f_1(x_2, C) = \cdots = f_1(x_k, C)$$

for each clause $C = \bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k \in \mathcal{A}_k^{(1)}$ $(k = 3, 4, 5, 6)$. Such a flow $f_1$ can be obtained in a polynomial time by [10]. Let $L_1$ be the network obtained from the residual network $N_1(f_1)$ of $N_1$ with respect to $f_1$ by deleting all arcs into $s$, all arcs from $t$ and all nodes $C, \bar{C}$ (and incident arcs) with $C \in \mathcal{A}_3^{(1)} \cup \mathcal{A}_4^{(1)} \cup \mathcal{A}_5^{(1)} \cup \mathcal{A}_6^{(1)}$.

Now we can split off clauses in $\mathcal{A}_3^{(1)} \cup \mathcal{A}_4^{(1)} \cup \mathcal{A}_5^{(1)} \cup \mathcal{A}_6^{(1)}$. For each $C = \bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k \in \mathcal{A}_k^{(1)}$ $(k = 3, 4, 5, 6)$, let $f_1(C) = f_1(x_1, C)$ and let

$$\mathcal{J}_{1,k}^{(1)}(C) = \{x_1, x_2, \ldots, x_k, C\} \tag{13}$$

with weights $w_1(x_1) = w_1(x_2) = \cdots = w_1(x_k) = 2f_1(C)$ and $w_1(C) = a_k^{(1)} f_1(C) \geq 2kf_1(C)$. Let

$$\mathcal{J}_{1,k}^{(1)} = \cup_{C \in \mathcal{A}_k^{(1)}} \mathcal{J}_{1,k}^{(1)}(C), \quad \mathcal{J}^{(1)} = \cup_{k=3}^{6} \mathcal{J}_{1,k}^{(1)}. \tag{14}$$

Let $(\mathcal{D}'^{(1)}_{1,2}, w_1) = \mathcal{C}(L_1)$ (i.e., $(\mathcal{D}'^{(1)}_{1,2}, w_1)$ is the set of weighted clauses with 1 or 2 literals of the symmetric network $L_1$) and let $(\mathcal{D}^{(1)}, w_1)$ be the set of clauses with weight function $w_1$ obtained from $(\mathcal{C}, w)$ by replacing $(\mathcal{C}^-_{1,2}, w)$ with $(\mathcal{D}'^{(1)}_{1,2}, w_1)$ and by replacing the weight $w(C)$ of each clause $C \in \mathcal{A}_k^{(1)}$ $(k = 3, 4, 5, 6)$ with

$$w_1(C) = w(C) - a_k^{(1)} f_1(C)$$

(note that $w_1(C) \geq 0$ since $f_1(C) \leq w(C)/a_k^{(1)}$ and we assume clauses with weight 0 are not included in $\mathcal{D}^{(1)}$). Then $(\mathcal{C}, w)$ and $(\mathcal{C}^1 \equiv \mathcal{D}^{(1)} \cup \mathcal{J}^{(1)}, w_1)$ are shown to be strongly equivalent based on Lemma 1 (note that a clause $C \in \mathcal{C}_k$ with $k = 3, 4, 5, 6$ may be split off and appear in two groups of $\mathcal{C}^1$, for example, in $\mathcal{D}^{(1)}$ and $\mathcal{J}_{1,3}^{(1)}$, but the total weight of $C$ is not changed). Let $R_1$ be the set of nodes reachable from $s$ in $L_1$ (thus, $y \in R_1$ for each $y \in \mathcal{D}^{(1)}$ and for each $\bar{x} \vee y \in \mathcal{D}^{(1)}$ with $x \in R_1$). Clearly, $R_1 \subseteq R$ $(\bar{R}_1 \subseteq \bar{R})$. Furthermore, there are no clauses in $\mathcal{D}^{(1)}$ with $k$ $(k = 3, 4, 5, 6)$ literals all contained in $\bar{R}_1$ by the maximality of $f_1$.

By the argument above, we can summarize Step 1 of our algorithm and have a lemma as follows.

**Step 1.** Find $R_1$ and $(\mathcal{D}^{(1)} \cup \mathcal{J}^{(1)}, w_1)$ using the network $N_1$, a symmetric flow $f_1$ of $N_1$ of maximum value and the network $L_1$ defined above.

**Lemma 2** *$(\mathcal{C}, w)$ and $(\mathcal{D}^{(1)} \cup \mathcal{J}^{(1)}, w_1)$ are strongly equivalent and the following statements hold.*
(a) *$x \in R_1$ for each $C = x \in \mathcal{D}^{(1)}$.*
(b) *$y \in R_1$ for each $C = \bar{x} \vee y \in \mathcal{D}^{(1)}$ with $x \in R_1$.*
(c) *There is no clause in $\mathcal{D}^{(1)}$ with 3,4,5 or 6 literals all contained in $\bar{R}_1$.*
(d) *$R_1 \subseteq R$.*

Next we will split off clauses $C_k \in \mathcal{D}^{(1)}$ of $k$ $(k = 3, 4)$ literals such that $C_k = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$ with $x_1, ..., x_{k-1} \in R_1$ and $a \in Z_1 \cup \bar{Z}_1$ $(Z_1 \equiv X - R_1)$. Let $\mathcal{B}_k^{(2)}$ be the set of such clauses $C_k$ in $\mathcal{D}^{(1)}$, i.e.,

$$\mathcal{B}_k^{(2)} = \{C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(1)} \mid x_1, \ldots, x_{k-1} \in R_1, \ a \in Z_1 \cup \bar{Z}_1\} \tag{15}$$

Let $M_1^-$ be the network obtained from the network $M_1 \equiv N(\mathcal{D}^{(1)})$ representing the set of weighted clauses in $\mathcal{D}^{(1)}$ with one or two literals by deleting all arcs from $\bar{X} \cup Z_1$ to $R_1$ and all arcs from $\bar{R}_1$ to $Z_1 \cup \bar{Z}_1$. Let $(\mathcal{D}^{(1)-}_{1,2}, w_1) = \mathcal{C}(M_1^-)$. Let $N_2$ be the network obtained from $M_1^-$ as follows. For each clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(2)}$ with $x_1, ..., x_{k-1} \in R_1$ and $a \in Z_1 \cup \bar{Z}_1$, we add two nodes $C, \bar{C}$ and $2k + 2$ arcs

$$E_B(C) \equiv \{(C, t), (s, \bar{C}), (\bar{a}, \bar{C}), (C, a)\} \ \cup \ \cup_{i=1}^{k-1} \{(x_i, C), (\bar{C}, \bar{x}_i)\} \tag{16}$$

Two arcs $(s, \bar{C})$,$(C, t)$ have capacity $w_1(C)$ and all the remaining arcs have capacity $w_1(C)/b_k^{(2)}$ with

$$b_k^{(2)} = \begin{cases} 6 & (k = 3) \\ 10 & (k = 4). \end{cases} \tag{17}$$

$N_2$ is the network obtained from $M_1^-$ in this way. Then, we find a symmetric flow $f_2$ of maximum value from $s$ to $t$ in $N_2$ such that $f_2(x_1, C) = \cdots = f_2(x_{k-1}, C) = f_2(C, a)$ for each clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(2)}$. Let $L_2$ be the network obtained from the residual network $N_2(f_2)$ with respect to $f_2$ by deleting all arcs into $s$, all arcs from $t$ and all nodes $C, \bar{C}$ (and incident arcs) with $C \in \mathcal{B}_3^{(2)} \cup \mathcal{B}_4^{(2)}$.

Now we can split off clauses $C \in \mathcal{B}_3^{(2)} \cup \mathcal{B}_4^{(2)}$. For each clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(2)}$ with $x_1, ..., x_{k-1} \in R_1$ and $a \in Z_1 \cup \bar{Z}_1$, using $f_2(C) \equiv f_2(x_1, C)$, let

$$\mathcal{K}_{1,k}^{(2)}(C) = \{x_1, ..., x_{k-1}, \bar{a}, C, x_0, \bar{x}_0\} \tag{18}$$

with weights $w_2(x_1) = \cdots = w_2(x_{k-1}) = w_2(\bar{a}) = 2f_2(C)$, $w_2(x_0) = w_2(\bar{x}_0) = -f_2(C)$ and $w_2(C) = b_k^{(2)} f_2(C)$ ($x_0$ is any variable in $X$ and the negative weights are accepted in this case). Let

$$\mathcal{K}_{1,k}^{(2)} = \cup_{C \in \mathcal{B}_k^{(2)}} \mathcal{K}_{1,k}^{(2)}(C), \quad \mathcal{K}^{(2)} = \mathcal{K}_{1,3}^{(2)} \cup \mathcal{K}_{1,4}^{(2)}. \tag{19}$$

Let $(\mathcal{D}_{1,2}'^{(2)}, w_2) = \mathcal{C}(L_2)$ (the set of weighted clauses of the symmetric network $L_2$) and let $(\mathcal{D}^{(2)}, w_2)$ be the set of weighted clauses obtained from $(\mathcal{D}^{(1)}, w_1)$ by replacing $(\mathcal{D}_{1,2}^{(1)-}, w_1)$ with $(\mathcal{D}_{1,2}'^{(2)}, w_2)$ and by replacing the weight $w_1(C)$ of each clause $C \in \mathcal{B}_k^{(2)}$ $(k = 3, 4)$ with

$$w_2(C) = w_1(C) - b_k^{(2)} f_2(C) \geq 0$$

(we assume clauses with weight 0 are not included in $\mathcal{D}^{(2)}$). Then, by the same argument as before, $(\mathcal{D}^{(1)}, w_1)$ and $(\mathcal{D}^{(2)} \cup \mathcal{K}^{(2)}, w_2)$ are shown to be strongly equivalent based on Lemma 1. Let $R_2$ be the set of nodes reachable from $s$ in $L_2$. Clearly, $R_2 \subseteq R_1$ $(\bar{R}_2 \subseteq \bar{R}_1)$.

A node $a \in Z_1 \cup \bar{Z}_1 \cup (R_1 - R_2)$ is called an *entrance* if there is a clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(2)}$ with $x_1, ..., x_{k-1} \in R_2$ ($w_2(C) > 0$ and $k = 3, 4$). Let $Q_2$ be the set of nodes in $Z_1 \cup \bar{Z}_1 \cup (R_1 - R_2) \cup (\bar{R}_1 - \bar{R}_2)$ that are reachable from an entrance by a path in $M_2 \equiv N(\mathcal{D}^{(2)})$. Note that $M_2$ is also obtained from $L_2$ by adding all the arcs in $M_1 - M_1^-$ and that there is no arc from a node in $R_1 - R_2$ to a node in $(X - R_1) \cup \bar{X}$. Thus, $Q_2 \subset Z_1 \cup \bar{Z}_1 \cup (R_1 - R_2)$ and $Q_2$ contains no complementary literals by the symmetry and maximality of $f_2$, and we can assume all literals in $Q_2$ are unnegated. Note that some variable in $R - R_1$ will be in $\bar{Q}_2$ and we have to correct the previous assumption that $R \subseteq X$. However, it suffices to assume that $R_1 \subseteq X$ (not $R \subseteq X$) in the argument below.

By the argument above we can summarize Step 2 of our algorithm and have a lemma as follows.

**Step 2.** Find $R_2$, $Q_2$ and $(\mathcal{D}^{(2)} \cup \mathcal{K}^{(2)}, w_2)$ from $(\mathcal{D}^{(1)}, w_1)$ using the network $M_1^-$, $N_2$, a symmetric flow $f_2$ of $N_2$ of maximum value and the network $L_2$ defined above.

**Lemma 3** $(\mathcal{D}^{(1)}, w_1)$ *and* $(\mathcal{D}^{(2)} \cup \mathcal{K}^{(2)}, w_2)$ *are strongly equivalent. Furthermore, the following statements hold.*

(a) $x \in R_2$ *for each* $C = x \in \mathcal{D}^{(2)}$.

(b) *For each* $C = \bar{x} \vee y \in \mathcal{D}^{(2)}$, $y \in R_2$ *if* $x \in R_2$ *and* $y \in R_2 \cup Q_2$ *if* $x \in Q_2$.

(c) *There is no clause in* $\mathcal{D}^{(2)}$ *with 3,4,5 or 6 literals all contained in* $\bar{R}_2$.

(d) $a \in Q_2 \cup R_2$ *for each* $C \in \mathcal{D}^{(2)}$ *with* $C = \bar{x} \vee \bar{y} \vee a$ *and* $x, y \in R_2$ *or with* $C = \bar{x} \vee \bar{y} \vee \bar{z} \vee a$ *and* $x, y, z \in R_2$.

(e) $R_2 \subseteq R_1$ *and* $Q_2 \subseteq X - R_2$.

Now we would like to set each variable in $R_2$ to be true with probability 0.75, each variable in $Q_2$ to be true with probability 0.629 and each variable in $Z_2 \equiv X - (Q_2 \cup R_2)$ to be true with probability 0.5. Then, each clause $C_j$ in $\mathcal{D}^{(2)}$ of $j$ literals except for a clause $C$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_k$ with $k = 3, 4, 5$, $x_i \in R_2 \cup Q_2$ $(i = 1, 2, ..., k-1)$ and $x_k \in Q_2$ is satisfied with probability at least $\gamma_j$ defined in (9), the coefficient of $W_j^*$ in (7).

Thus, we will try to split off such clauses. Let $\mathcal{A}_k^{(3)}$ $(k=3,4)$ be the set of clauses $C \in \mathcal{D}^{(2)}$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_k$ with $x_1, x_{k-2} \in R_2$ and $x_{k-1}, x_k \in Q_2$. Similarly, let $\mathcal{A}_5^{(3)}$ be the set of clauses $C \in \mathcal{D}^{(2)}$ of form $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4 \vee \bar{x}_5$ with $x_1, x_2, x_3, x_4 \in R_2$ and $x_5 \in Q_2$. Thus, for $k = 3, 4, 5$,

$$\mathcal{A}_k^{(3)} = \{C = \bar{x}_1 \vee \cdots \vee \bar{x}_k \in \mathcal{D}^{(2)} \mid x_1, \ldots, x_{2^{k-3}} \in R_2,\ x_{2^{k-3}+1}, \ldots, x_k \in Q_2\}. \tag{20}$$

Let $\mathcal{B}_k^{(3)}$ $(k=3,4)$ be the set of clauses $C \in \mathcal{D}^{(2)}$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$ with $x_1, \ldots, x_{k-1} \in R_2$ and $a \in Q_2$, i.e.,

$$\mathcal{B}_k^{(3)} = \{C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(2)} \mid\ x_1, \ldots, x_{k-1} \in R_2, a \in Q_2\}. \tag{21}$$

Let $M_2^-$ be the network obtained from $M_2 \equiv N(\mathcal{D}^{(2)})$ by deleting all arcs from $\bar{X} \cup Q_2 \cup Z_2$ to $R_2$, all arcs from $\bar{X} \cup Z_2$ to $Q_2$ and their symmetric arcs. Let $(\mathcal{D}_{1,2}^{(2)-}, w_2) = \mathcal{C}(M_2^-)$ and let $N_3$ be the network obtained from $M_2^-$ as follows. For each clause $C \in \mathcal{B}_k^{(3)}$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$ with $x_1, \ldots, x_{k-1} \in R_2$ and $a \in Q_2$, we add two nodes $C, \bar{C}$ and $(2k+2)$ arcs $E_B(C)$ defined in (16) (i.e., $E_B(C) = \{(C,t), (s,\bar{C}), (\bar{a},\bar{C}), (C,a)\} \cup \cup_{i=1}^{k-1}\{(x_i, C), (\bar{C}, \bar{x}_i)\}$). Two arcs $(s,\bar{C}), (C,t)$ have capacity $w_2(C)$ and all the remaining arcs have capacity $w_2(C)/b_k^{(3)}$ with

$$b_k^{(3)} = \begin{cases} 7 & (k=3) \\ 12 & (k=4). \end{cases} \tag{22}$$

For each clause $C = \bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k \in \mathcal{A}_k^{(3)}$ $(k=3,4,5)$, we add two nodes $C, \bar{C}$ and $2k+2$ arcs $E_A(C)$ defined in (11) (i.e., $E_A(C) = \{(C,t), (s,\bar{C})\} \cup \cup_{i=1}^{k}\{(x_i, C), (\bar{C}, \bar{x}_i)\}$). Two arcs $(s,\bar{C})$,$(C,t)$ have capacity $w_2(C)$ and all the remaining arcs have capacity $w_2(C)/a_k^{(3)}$ with

$$a_k^{(3)} = \begin{cases} 6 & (k=3) \\ 10 & (k=4) \\ 12 & (k=5). \end{cases} \tag{23}$$

Then, we find a symmetric flow $f_3$ of maximum value from $s$ to $t$ in $N_3$ such that $f_3(x_1, C) = \cdots = f_3(x_{k-1}, C) = f_3(C, a)$ for each clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(3)}$ $(k=3,4)$ and $f_3(x_1, C) = \cdots = f_3(x_k, C)$ for each clause $C = \bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k \in \mathcal{A}_k^{(3)}$ $(k=3,4,5)$. Let $L_3$ be the network obtained from the residual network $N_3(f_3)$ with respect to $f_3$ by deleting all arcs into $s$, all arcs from $t$ and all nodes $C, \bar{C}$ (and incident arcs) with $C \in \mathcal{B}_3^{(3)} \cup \mathcal{B}_4^{(3)} \cup \mathcal{A}_3^{(3)} \cup \mathcal{A}_4^{(3)} \cup \mathcal{A}_5^{(3)}$.

Now we can split off clauses $C \in \mathcal{B}_3^{(3)} \cup \mathcal{B}_4^{(3)} \cup \mathcal{A}_3^{(3)} \cup \mathcal{A}_4^{(3)} \cup \mathcal{A}_5^{(3)}$. For each $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(3)}$ with $x_1, \ldots, x_{k-1} \in R_2$ and $a \in Q_2$, let

$$\mathcal{K}_{1,k}^{(3)}(C) = \{x_1, \ldots, x_{k-1}, \bar{a}, C, x_0, \bar{x}_0\} \tag{24}$$

with weights $w_3(x_1) = \cdots = w_3(x_{k-1}) = w_3(\bar{a}) = 2f_3(C)$, $w_3(x_0) = w_3(\bar{x}_0) = -2f_3(C)$ and $w_3(C) = b_k^{(3)} f_3(C)$ using $f_3(C) \equiv f_3(x_1, C)$ ($x_0$ is any variable in $X$). Let

$$\mathcal{K}_{1,k}^{(3)} = \cup_{C \in \mathcal{B}_k^{(3)}} \mathcal{K}_{1,k}^{(3)}(C), \qquad \mathcal{K}^{(3)} = \mathcal{K}_{1,3}^{(3)} \cup \mathcal{K}_{1,4}^{(3)}. \tag{25}$$

For each clause $C \in \mathcal{A}_k^{(3)}$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_k$, let

$$\mathcal{J}_{1,k}^{(3)}(C) = \{x_1, \ldots, x_k, C\} \tag{26}$$

with weights $w_3(x_1) = \cdots = w_3(x_k) = 2f_3(C)$ and $w_3(C) = a_k^{(3)} f_3(C)$ using $f_3(C) \equiv f_3(x_1, C)$. Let

$$\mathcal{J}_{1,k}^{(3)} = \cup_{C \in \mathcal{A}_k^{(3)}} \mathcal{J}_{1,k}^{(3)}(C), \qquad \mathcal{J}^{(3)} = \cup_{k=3}^{5} \mathcal{J}_{1,k}^{(3)}. \tag{27}$$

Let $(\mathcal{D}'^{(3)}_{1,2}, w_3) = \mathcal{C}(L_3)$ (the set of weighted clauses of the symmetric network $L_3$) and let $(\mathcal{D}^{(3)}, w_3)$ be the set of weighted clauses obtained from $(\mathcal{D}^{(2)}, w_2)$ by replacing $(\mathcal{D}^{(2)-}_{1,2}, w_2)$ with $(\mathcal{D}'^{(3)}_{1,2}, w_3)$ and by replacing the weight $w_2(C)$ of each clause $C \in \mathcal{B}^{(3)}_3 \cup \mathcal{B}^{(3)}_4 \cup \mathcal{A}^{(3)}_3 \cup \mathcal{A}^{(3)}_4 \cup \mathcal{A}^{(3)}_5$ with

$$w_3(C) = \begin{cases} w_2(C) - a^{(3)}_k f_3(C) & (C \in \mathcal{A}^{(3)}_k) \\ w_2(C) - b^{(3)}_k f_3(C) & (C \in \mathcal{B}^{(3)}_k) \end{cases}$$

($w_3(C) \geq 0$ and we assume clauses with weight 0 are not included in $\mathcal{D}^{(3)}$). Then, by the same argument as before, $(\mathcal{D}^{(2)}, w_2)$ and $(\mathcal{D}^{(3)} \cup \mathcal{J}^{(3)} \cup \mathcal{K}^{(3)}, w_3)$ are shown to be strongly equivalent. Let $R_3$ be the set of nodes reachable from $s$ in $L_3$. Clearly, $R_3 \subseteq R_2$ ($\bar{R}_3 \subseteq \bar{R}_2$). A node $a \in Q_2 \cup (R_2 - R_3)$ is called an *entrance* if there is a clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(3)}$ $(k = 3, 4)$ such that $x_1, ..., x_{k-1} \in R_3$ $(w_3(C) > 0)$. Let $Q_3$ be the set of nodes in $Q_2 \cup (R_2 - R_3)$ that are reachable from an entrance by a path in $M_3 \equiv N(\mathcal{D}^{(3)})$ ($M_3$ is also obtained from $L_3$ by adding all arcs in $M_2 - M_2^-$). Then, by the symmetry and maximality of $f_3$, $Q_3$ contains no complementary literals and all literals in $Q_3$ are unnegated.

By the argument above we can summarize Step 3 of our algorithm and have a lemma as follows.

**Step 3.** Find $R_3$, $Q_3$ and $(\mathcal{D}^{(3)} \cup \mathcal{J}^{(3)} \cup \mathcal{K}^{(3)}, w_3)$ from $(\mathcal{D}^{(2)}, w_2)$ using the network $M_2^-$, $N_3$, a symmetric flow $f_3$ of $N_3$ of maximum value and the network $L_3$ defined above.

**Lemma 4** *$(\mathcal{D}^{(2)}, w_2)$ and $(\mathcal{D}^{(3)} \cup \mathcal{J}^{(3)} \cup \mathcal{K}^{(3)}, w_3)$ are strongly equivalent and the following statements hold.*

(a) *$x \in R_3$ for each $C = x \in \mathcal{D}^{(3)}$.*

(b) *For each $C = \bar{x} \vee y \in \mathcal{D}^{(3)}$, $y \in R_3$ if $x \in R_3$ and $y \in Q_3 \cup R_3$ if $x \in Q_3$.*

(c) *There is no clause in $\mathcal{D}^{(3)}$ with 3, 4, 5 or 6 literals all contained in $\bar{R}_3$.*

(d) *$a \in Q_3 \cup R_3$ for each clause of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(3)}$ with $x_1, ..., x_{k-1} \in R_3$ $(k = 3, 4)$.*

(e) *There is no clause $C \in \mathcal{D}^{(3)}$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_k$ with $x_1, ..., x_{2^{k-3}} \in R_3$, $x_{2^{k-3}+1}, ..., x_k \in Q_3$ for $k = 3, 4, 5$.*

(f) *$R_3 \subseteq R_2$ and $Q_3 \subseteq Q_2 \cup R_2 - R_3$.*

Step 4 below is almost similar to Step 3 above. Let

$$\mathcal{A}^{(4)}_3 = \{\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \in \mathcal{D}^{(3)} \mid x_1, x_2, x_3 \in Q_3\}, \tag{28}$$

$$\mathcal{B}^{(4)}_k = \{\bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(3)} \mid x_1, \ldots, x_{k-1} \in R_3,\ a \in Q_3\} \tag{29}$$

for $k = 3, 4$. Let $M_3^-$ be the network obtained from $M_3 \equiv N(\mathcal{D}^{(3)})$ by deleting all arcs from $\bar{X} \cup Q_3 \cup Z_3$ to $R_3$, all arcs from $\bar{X} \cup Z_3$ to $Q_3$ and their symmetric arcs. Let $(\mathcal{D}^{(3)-}_{1,2}, w_3) = \mathcal{C}(M_3^-)$ and let $N_4$ be the network obtained from $M_3^-$ as follows. For each clause $C \in \mathcal{B}^{(4)}_k$ $(k = 3, 4)$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$, we add two nodes $C, \bar{C}$ and $(2k + 2)$ arcs $E_B(C)$ defined in (16). Two arcs $(s, \bar{C}), (C, t)$ have capacity $w_3(C)$ and all the remaining arcs have capacity $w_3(C)/b^{(4)}_k$ with

$$b^{(4)}_k = \begin{cases} 7 & (k = 3) \\ 12 & (k = 4). \end{cases} \tag{30}$$

For each clause $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \in \mathcal{A}^{(4)}_3$, we add two nodes $C, \bar{C}$ and 8 arcs $E_A(C)$ defined in (11). Two arcs $(s, \bar{C})$,$(C, t)$ have capacity $w_3(C)$ and all the remaining arcs have capacity $w_3(C)/a^{(4)}_3$ with

$$a^{(4)}_3 = 6. \tag{31}$$

Then, we find a symmetric flow $f_4$ of maximum value such that $f_4(x_1, C) = \cdots = f_4(x_{k-1}, C) = f_4(C, a)$ for each clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(4)}$ $(k = 3, 4)$ and $f_4(x_1, C) = f_4(x_2, C) = f_4(x_3, C)$ for each clause $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \in \mathcal{A}_3^{(4)}$. Let $L_4$ be the network obtained from the residual network $N_4(f_4)$ with respect to $f_4$ by deleting all arcs into $s$, all arcs from $t$ and all nodes $C, \bar{C}$ (and incident arcs) with $C \in \mathcal{B}_3^{(4)} \cup \mathcal{B}_4^{(4)} \cup \mathcal{A}_3^{(4)}$.

Now we can split off clauses $C \in \mathcal{B}_3^{(4)} \cup \mathcal{B}_4^{(4)} \cup \mathcal{A}_3^{(4)}$. For each $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(4)}$, let

$$\mathcal{K}_{1,k}^{(4)}(C) = \{x_1, ..., x_{k-1}, \bar{a}, C, x_0, \bar{x}_0\} \tag{32}$$

with weights $w_4(x_1) = \cdots = w_4(x_{k-1}) = w_4(\bar{a}) = 2f_4(C)$, $w_4(x_0) = w_4(\bar{x}_0) = -2f_4(C)$ and $w_4(C) = b_k^{(4)} f_4(C)$ using $f_4(C) \equiv f_4(x_1, C)$ ($x_0$ is any variable in $X$). Let

$$\mathcal{K}_{1,k}^{(4)} = \cup_{C \in \mathcal{B}_k^{(4)}} \mathcal{K}_{1,k}^{(4)}(C), \qquad \mathcal{K}^{(4)} = \mathcal{K}_{1,3}^{(4)} \cup \mathcal{K}_{1,4}^{(4)}. \tag{33}$$

For each clause $C \in \mathcal{A}_3^{(4)}$ of form $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3$, let

$$\mathcal{J}_{1,3}^{(4)}(C) = \{x_1, x_2, x_3, C\} \tag{34}$$

with weights $w_4(x_1) = w_4(x_2) = w_4(x_3) = 2f_4(C)$ and $w_4(C) = a_3^{(4)} f_4(C)$ using $f_4(C) \equiv f_4(x_1, C)$. Let

$$\mathcal{J}^{(4)} = \mathcal{J}_{1,3}^{(4)} = \cup_{C \in \mathcal{A}_3^{(4)}} \mathcal{J}_{1,3}^{(4)}(C). \tag{35}$$

Let $(\mathcal{D}'^{(4)}_{1,2}, w_4) = \mathcal{C}(L_4)$ (the set of weighted clauses of the symmetric network $L_4$) and let $(\mathcal{D}^{(4)}, w_4)$ be the set of weighted clauses obtained from $(\mathcal{D}^{(3)}, w_3)$ by replacing $(\mathcal{D}_{1,2}^{(3)-}, w_3)$ with $(\mathcal{D}'^{(4)}_{1,2}, w_4)$ and by replacing the weight $w_3(C)$ of each clause $C \in \mathcal{B}_3^{(4)} \cup \mathcal{B}_4^{(4)} \cup \mathcal{A}_3^{(4)}$ with

$$w_4(C) = \begin{cases} w_3(C) - a_3^{(4)} f_4(C) & (C \in \mathcal{A}_3^{(4)}) \\ w_3(C) - b_k^{(4)} f_4(C) & (C \in \mathcal{B}_k^{(4)},\ k = 3, 4) \end{cases}$$

($w_4(C) \geq 0$ and clauses with weight 0 are not included in $\mathcal{D}^{(4)}$). Then, by the same argument as before, $(\mathcal{D}^{(3)}, w_3)$ and $(\mathcal{D}^{(4)} \cup \mathcal{J}^{(4)} \cup \mathcal{K}^{(4)}, w_4)$ are shown to be strongly equivalent. Let $R_4$ be the set of nodes reachable from $s$ in $L_4$. Clearly, $R_4 \subseteq R_3$ ($\bar{R}_4 \subseteq \bar{R}_3$). A node $a \in Q_3 \cup (R_3 - R_4)$ is called an *entrance* again if there is a clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(4)}$ $(k = 3, 4)$ such that $x_1, ..., x_{k-1} \in R_4$ ($w_4(C) > 0$). Let $Q_4$ be the set of nodes in $Q_3 \cup (R_3 - R_4)$ that are reachable from an entrance by a path in $M_4 \equiv N(\mathcal{D}^{(4)})$ ($M_4$ is also obtained from $L_4$ by adding all arcs in $M_3 - M_3^-$). Then, by the symmetry and maximality of $f_4$, $Q_4$ contains no complementary literals and all literals in $Q_4$ are unnegated.

By the argument above we can summarize Step 4 of our algorithm and have a lemma as follows.

**Step 4.** Find $R_4$, $Q_4$ and $(\mathcal{D}^{(4)} \cup \mathcal{J}^{(4)} \cup \mathcal{K}^{(4)}, w_4)$ from $(\mathcal{D}^{(3)}, w_3)$ using the network $M_3^-$, $N_4$, a symmetric flow $f_4$ of $N_4$ of maximum value and the network $L_4$ defined above.

**Lemma 5** $(\mathcal{D}^{(3)}, w_3)$ *and* $(\mathcal{D}^{(4)} \cup \mathcal{J}^{(4)} \cup \mathcal{K}^{(4)}, w_4)$ *are strongly equivalent and the following statements hold.*

(a) $x \in R_4$ *for each* $C = x \in \mathcal{D}^{(4)}$.

(b) *For each* $C = \bar{x} \vee y \in \mathcal{D}^{(4)}$, $y \in R_4$ *if* $x \in R_4$ *and* $y \in Q_4 \cup R_4$ *if* $x \in Q_4$.

(c) *There is no clause in* $\mathcal{D}^{(4)}$ *with 3, 4, 5 or 6 literals all contained in* $\bar{R}_4$.

(d) $a \in Q_4 \cup R_4$ *for each clause of form* $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(4)}$ *with* $x_1, ..., x_{k-1} \in R_4$ $(k = 3, 4)$.

(e) *There is no clause* $C \in \mathcal{D}^{(4)}$ *of form* $C = \bar{x}_1 \vee \cdots \vee \bar{x}_k$ *with* $x_1, x_2, x_3 \in Q_4$ *for* $k = 3$ *or with* $x_1, ..., x_{2^{k-3}} \in R_4$, $x_{2^{k-3}+1}, ..., x_k \in Q_4$ *for* $k = 3, 4, 5$.

(f) $R_4 \subseteq R_3$ *and* $Q_4 \subseteq Q_3 \cup R_3 - R_4$.

Now we would like to set each variable in $R_4$ to be true with probability 0.75, each variable in $Q_4$ to be true with probability 0.629 and each variable in $Z_4 \equiv X - (Q_4 \cup R_4)$ to be true with probability 0.5. Then, each clause in $\mathcal{D}^{(4)}$ except for a clause $C$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$ ($k = 3, 4$) with $x_1, x_{k-2} \in R_4$, $x_{k-1} \in Q_4$ and $a \in Z_4 \cup \bar{Z}_4$ ($Z_4 \equiv X - (R_4 \cup Q_4)$) is satisfied with probability at least $\gamma_k$ in (9).

We will split off such clauses. For $k = 3, 4$, let

$$\mathcal{B}_k^{(5)} = \{\bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(4)} \mid x_1, \ldots, x_{k-1} \in R_4,\ a \in Q_4\} \quad (36)$$

$$\mathcal{B}_k'^{(5)} = \{\bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(4)} \mid x_1, \ldots, x_{k-2} \in R_4,\ x_{k-1} \in Q_4, a \in Z_4 \cup \bar{Z}_4\} \quad (37)$$

Let $M_4^-$ be the network obtained from $M_4 \equiv N(\mathcal{D}^{(4)})$ by deleting all arcs from $\bar{X} \cup Q_4 \cup Z_4$ to $R_4$, all arcs from $\bar{X} \cup Z_4$ to $Q_4$ and their symmetric arcs. Let $(\mathcal{D}_{1,2}^{(4)-}, w_3) = \mathcal{C}(M_4^-)$ and let $N_5$ be the network obtained from $M_4^-$ as follows.

For each clause $C \in \mathcal{B}_k^{(5)} \cup \mathcal{B}_k'^{(5)}$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$, we add two nodes $C, \bar{C}$ and $(2k+2)$ arcs $E_B(C)$ defined by (16). Two arcs $(s, \bar{C}), (C, t)$ have capacity $w_4(C)$ and all the remaining arcs have capacity $w_4(C)/b_k''^{(5)}$ with

$$b_k''^{(5)} = \begin{cases} 6.8 & (C \in \mathcal{B}_k^{(5)},\ k = 3) \\ 12 & (C \in \mathcal{B}_k^{(5)},\ k = 4) \\ 6.5 & (C \in \mathcal{B}_k'^{(5)},\ k = 3) \\ 10 & (C \in \mathcal{B}_k'^{(5)},\ k = 4) \end{cases} \quad (38)$$

Then, we find a symmetric flow $f_5$ of maximum value from $s$ to $t$ in $N_5$ such that $f_5(x_1, C) = \cdots = f_5(x_{k-1}, C) = f_5(C, a)$ for each clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(5)} \cup \mathcal{B}_k'^{(5)}$. Let $L_5$ be the network obtained from the residual network $N_5(f_5)$ with respect to $f_5$ by deleting all arcs into $s$, all arcs from $t$ and all nodes $C, \bar{C}$ (and incident arcs) with $C \in \mathcal{B}_3^{(5)} \cup \mathcal{B}_4^{(5)} \cup \mathcal{B}_3'^{(5)} \cup \mathcal{B}_4'^{(5)}$.

Now we can split off clauses $C \in \mathcal{B}_3^{(5)} \cup \mathcal{B}_4^{(5)} \cup \mathcal{B}_3'^{(5)} \cup \mathcal{B}_4'^{(5)}$. For each $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(5)} \cup \mathcal{B}_k'^{(5)}$ ($k = 3, 4$), let

$$\mathcal{K}_{1,k}''^{(5)}(C) = \{x_1, ..., x_{k-1}, \bar{a}, C, x_0, \bar{x}_0\} \quad (39)$$

with weights

$$w_5(x_1) = \cdots = w_5(x_k) = w_5(\bar{a}) = 2f_5(C),$$

$$w_5(x_0) = w_5(\bar{x}_0) = \begin{cases} -2f_5(C) & (C \in \mathcal{B}_k^{(5)}) \\ -f_5(C) & (C \in \mathcal{B}_k'^{(5)}), \end{cases}$$

$$w_5(C) = b_k''^{(5)} f_4(C) \qquad (C \in \mathcal{B}_k^{(5)} \cup \mathcal{B}_k'^{(5)})$$

using $f_5(C) \equiv f_5(x_1, C)$ ($x_0$ is any variable in $X$). Let

$$\mathcal{K}_{1,k}^{(5)} = \cup_{C \in \mathcal{B}_k^{(5)}} \mathcal{K}_{1,k}''^{(5)}(C), \qquad \mathcal{K}^{(5)} = \mathcal{K}_{1,3}^{(5)} \cup \mathcal{K}_{1,4}^{(5)}, \quad (40)$$

$$\mathcal{K}_{1,k}'^{(5)} = \cup_{C \in \mathcal{B}_k'^{(5)}} \mathcal{K}_{1,k}''^{(5)}(C), \quad \mathcal{K}'^{(5)} = \mathcal{K}_{1,3}'^{(5)} \cup \mathcal{K}_{1,4}'^{(5)}. \quad (41)$$

Let $(\mathcal{D}_{1,2}'^{(5)}, w_5) = \mathcal{C}(L_5)$ and let $(\mathcal{D}^{(5)}, w_5)$ be the set of weighted clauses obtained from $(\mathcal{D}^{(4)}, w_4)$ by replacing $(\mathcal{D}_{1,2}^{(4)-}, w_4)$ with $(\mathcal{D}_{1,2}'^{(5)}, w_5)$ and by replacing the weight $w_4(C)$ of each clause $C \in \mathcal{B}_k^{(5)} \cup \mathcal{B}_k'^{(5)}$ ($k = 3, 4$) with

$$w_5(C) = w_4(C) - b_k''^{(5)} f_5(C) \qquad (C \in \mathcal{B}_k^{(5)} \cup \mathcal{B}_k'^{(5)})$$

($w_5(C) \geq 0$ and we assume clauses with weight 0 are not included in $\mathcal{D}^{(5)}$). Then, by the same argument as before, $(\mathcal{D}^{(4)}, w)$ and $(\mathcal{D}^{(5)} \cup \mathcal{K}^{(5)} \cup \mathcal{K}'^{(5)}, w_5)$ are strongly equivalent. Let $R_5$ be the set of nodes reachable from $s$ in $L_5$. Clearly, $R_5 \subseteq R_4$ ($\bar{R}_5 \subseteq \bar{R}_4$). A node $a \in Q_4 \cup (R_4 - R_5)$ is called an *entrance1* if there is a clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(5)}$ ($k = 3, 4$) with $x_1, ..., x_{k-1} \in R_5$ ($w_5(C) > 0$). Let $Q_5$ be the set of nodes in $Q_4 \cup (R_4 - R_5)$ that are reachable from an entrance1 by a path in $M_5 \equiv N(\mathcal{D}^{(5)})$. Similarly, a node $a \in ((R_4 \cup Q_4) - (R_5 \cup Q_5)) \cup Z_4 \cup \bar{Z}_4$ is called an *entrance2* if there is a clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(5)}$ ($k = 3, 4$) with $x_1, x_{k-2} \in R_5$, $x_{k-1} \in Q_5$ ($w_5(C) > 0$). Let $P_5$ be the set of nodes in $((R_4 \cup Q_4) - (R_5 \cup Q_5)) \cup Z_4 \cup \bar{Z}_4$ that are reachable from an entrance2 by a path in $M_5$. Then, $Q_5$ and $P_5$ contain no complementary literals by the symmetry and maximality of $f_5$ and we can assume all literals in $Q_5 \cup P_5$ are unnegated.

By the argument above we can summarize Step 5 of our algorithm and have a lemma as follows.

**Step 5.** Find $R_5$, $Q_5$, $P_5$ and $(\mathcal{D}^{(5)} \cup \mathcal{K}^{(5)} \cup \mathcal{L}^{(5)}, w_5)$ from $(\mathcal{D}^{(4)}, w_4)$ using the network $M_4^-$, $N_5$, a symmetric flow $f_5$ of $N_5$ of maximum value and the network $L_5$ defined above.

**Lemma 6** *$(\mathcal{D}^{(4)}, w_4)$ and $(\mathcal{D}^{(5)} \cup \mathcal{K}^{(5)} \cup \mathcal{K}'^{(5)}, w_5)$ are strongly equivalent and the following statements hold.*

(a) *$x \in R_5$ for each $C = x \in \mathcal{D}^{(5)}$.*

(b) *For each $C = \bar{x} \vee y \in \mathcal{D}^{(5)}$, $y \in R_5$ if $x \in R_5$, $y \in Q_5 \cup R_5$ if $x \in Q_5$ and $y \in P_5 \cup Q_5 \cup R_5$ if $x \in P_5$.*

(c) *For $k = 3, 4, 5, 6$, there is no clause in $\mathcal{D}^{(5)}$ with $k$ literals such that $k_1$ ($k_1 \geq 2k - 6$) literals are contained in $\bar{R}_5$ and the remaining literals are in $\bar{Q}_5$.*

(d) *A clause in $\mathcal{D}^{(5)}$ with 3 or 4 literals all except one contained in $\bar{R}_5$ has a literal in $R_5 \cup Q_5$.*

(e) *A clause in $\mathcal{D}^{(5)}$ with $k$ literals ($k = 3, 4$) of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$ such that $x_1, ..., x_{k-2} \in R_5$ and $x_{k-1} \in Q_5$ satisfies $a \in R_5 \cup Q_5 \cup P_5$.*

(f) *$R_5 \subseteq R_4$, $Q_5 \subseteq Q_4 \cup R_4 - R_5$ and $P_5 \subseteq X - (R_5 \cup Q_5)$.*

Now we would like to set each variable in $R_5$ to be true with probability 0.75, each variable in $Q_5$ to be true with probability 0.629 and each variable in $P_5$ to be true with probability 0.557 and each variable in $Z_5 \equiv X - (P_5 \cup Q_5 \cup R_5)$ to be true with probability 0.5. Then, each clause $C_k$ in $\mathcal{D}^{(5)}$ of $k$ literals except for a clause $C$ of form $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3$ with $x_1 \in R_5$ and $x_2, x_3 \in P_5$ or with $x_1, x_2 \in Q_5$ and $x_3 \in P_5$ is satisfied with probability at least $\gamma_k$ in (9). We will split off such clauses. Let

$$\mathcal{A}_3^{(6)} = \{\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \in \mathcal{D}^{(5)} \mid (x_1 \in R_5,\ x_2, x_3 \in P_5) \ \text{ or } \ (x_1, x_2 \in Q_5,\ x_3 \in P_5)\}, \tag{42}$$

$$\mathcal{B}_k^{(6)} = \{\bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(5)} \mid x_1, \ldots, x_{k-1} \in R_5,\ a \in Q_5\} \tag{43}$$

$$\mathcal{B}_k'^{(6)} = \{\bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(5)} \mid x_1, \ldots, x_{k-2} \in R_5,\ x_{k-1} \in Q_5,\ a \in P_5\} \tag{44}$$

for $k = 3, 4$. Let $M_5^-$ be the network obtained from $M_5 \equiv N(\mathcal{D}^{(5)})$ by deleting all arcs from $\bar{X} \cup Q_5 \cup P_5$ to $R_5$, all arcs from $\bar{X} \cup P_5$ to $Q_5$, all arcs from $\bar{X}$ to $P_5$ and their symmetric arcs. Let $(\mathcal{D}_{1,2}^{(5)-}, w_5) = \mathcal{C}(M_5^-)$ and let $N_6$ be the network obtained from $M_5^-$ as follows.

For each clause $C \in \mathcal{B}_k^{(6)} \cup \mathcal{B}_k'^{(6)}$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$, we add two nodes $C, \bar{C}$ and $(2k+2)$ arcs $E_B(C)$ defined by (16). Two arcs $(s, \bar{C}), (C, t)$ have capacity $w_5(C)$ and all the remaining arcs have capacity $w_5(C)/b_k''^{(6)}$ with

$$b_k''^{(6)} = \begin{cases} 6.8 & (C \in \mathcal{B}_k^{(6)},\ k = 3) \\ 12 & (C \in \mathcal{B}_k^{(6)},\ k = 4) \\ 6.5 & (C \in \mathcal{B}_k'^{(6)},\ k = 3) \\ 10 & (C \in \mathcal{B}_k'^{(6)},\ k = 4) \end{cases} \tag{45}$$

For each clause in $\mathcal{A}_3^{(6)}$ of form $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3$, we add two nodes $C, \bar{C}$ and 8 arcs $E_A(C)$ defined by (11). Two arcs $(s, \bar{C}), (C, t)$ have capacity $w_5(C)$ and all the remaining arcs have capacity $w_5(C)/a_3^{(6)}$ with

$$a_3^{(6)} = 6. \tag{46}$$

Then, we find a symmetric flow $f_6$ of maximum value from $s$ to $t$ in $N_6$ such that $f_6(x_1, C) = \cdots = f_6(x_{k-1}, C) = f_6(C, k)$ for each clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{B}_k^{(6)} \cup \mathcal{B}_k'^{(6)}$ and $f_6(x_1, C) = f_6(x_2, C) = f_6(x_3, C)$ for each clause $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \in \mathcal{A}_3^{(6)}$. Let $L_6$ be the network obtained from the residual network $N_6(f_6)$ with respect to $f_6$ by deleting all arcs into $s$, all arcs from $t$ and all nodes $C, \bar{C}$ (and incident arcs) with $C \in \mathcal{B}_3^{(6)} \cup \mathcal{B}_4^{(6)} \cup \mathcal{B}_3'^{(6)} \cup \mathcal{B}_4'^{(6)} \cup \mathcal{A}_3^{(6)}$.

Now we can split off clauses $C \in \mathcal{B}_3^{(6)} \cup \mathcal{B}_4^{(6)} \cup \mathcal{B}_3'^{(6)} \cup \mathcal{B}_4'^{(6)} \cup \mathcal{A}_3^{(6)}$. For each clause $C \in \mathcal{B}_k^{(6)} \cup \mathcal{B}_k'^{(6)}$ $(k = 3, 4)$ of form $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$ with $x_1, ..., x_{k-1} \in R_5$ and $a \in Q_5$ $(C \in \mathcal{B}_k^{(6)})$ or with $x_1, x_{k-2} \in R_5$, $x_{k-1} \in Q_5$ and $a \in P_5$ $(C \in \mathcal{B}_k'^{(6)})$, let

$$\mathcal{K}_{1,k}''^{(6)}(C) = \{x_1, ..., x_{k-1}, \bar{a}, C, x_0, \bar{x}_0\} \tag{47}$$

with weights $w_6(x_1) = \cdots = w_6(x_k) = w_6(\bar{a}) = 2f_6(C)$, $w_6(x_0) = w_6(\bar{x}_0) = -2f_6(C)$ and $w_5(C) = b_k''^{(6)} f_5(C)$ using $f_6(C) \equiv f_6(x_1, C)$ ($x_0$ is any variable in $X$). Let

$$\mathcal{K}_{1,k}^{(6)} = \cup_{C \in \mathcal{B}_k^{(6)}} \mathcal{K}_{1,k}''^{(6)}(C), \qquad \mathcal{K}^{(6)} = \mathcal{K}_{1,3}^{(6)} \cup \mathcal{K}_{1,4}^{(6)}, \tag{48}$$

$$\mathcal{K}_{1,k}'^{(6)} = \cup_{C \in \mathcal{B}_k'^{(6)}} \mathcal{K}_{1,k}''^{(6)}(C), \quad \mathcal{K}'^{(6)} = \mathcal{K}_{1,3}'^{(6)} \cup \mathcal{K}_{1,4}'^{(6)}. \tag{49}$$

For each clause $C = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \in \mathcal{A}_3^{(6)}$, let

$$\mathcal{J}_{1,3}^{(6)}(C) = \{x_1, x_2, x_3, C\} \tag{50}$$

with weights $w_6(x_1) = w_6(x_2) = w_6(x_3) = 2f_6(C)$, and $w_6(C) = a_3^{(6)} f_6(C)$ using $f_6(C) \equiv f_6(x_1, C)$. Let

$$\mathcal{J}^{(6)} = \mathcal{J}_{1,3}^{(6)} = \cup_{C \in \mathcal{A}_3^{(6)}} \mathcal{J}_{1,3}^{(6)}(C). \tag{51}$$

Let $(\mathcal{D}_{1,2}'^{(6)}, w_6) = \mathcal{C}(L_6)$ and let $(\mathcal{D}^{(6)}, w_6)$ be the set of weighted clauses obtained from $(\mathcal{D}^{(5)}, w_5)$ by replacing $(\mathcal{D}_{1,2}^{(5)-}, w_5)$ with $(\mathcal{D}_{1,2}'^{(6)}, w_6)$ and by replacing the weight $w_5(C)$ of each clause $C \in \mathcal{B}_3^{(6)} \cup \mathcal{B}_4^{(6)} \cup \mathcal{B}_3'^{(6)} \cup \mathcal{B}_4'^{(6)} \cup \mathcal{A}_3^{(6)}$ with

$$w_6(C) = \begin{cases} w_5(C) - a_3^{(6)} f_6(C) & (C \in \mathcal{A}_3^{(6)}) \\ w_5(C) - b_k''^{(6)} f_6(C) & (C \in \mathcal{B}_k^{(6)} \cup \mathcal{B}_k'^{(6)}) \end{cases}$$

($w_6(C) \geq 0$ and we assume clauses with weight 0 are not included in $\mathcal{D}^{(6)}$). Then, by the same argument as before, $(\mathcal{D}^{(5)}, w_5)$ and $(\mathcal{D}^{(6)} \cup \mathcal{J}^{(6)} \cup \mathcal{K}^{(6)} \cup \mathcal{K}'^{(6)}, w_6)$ are strongly equivalent. Let $R_6$ be the set of nodes reachable from $s$ in $L_6$. Clearly, $R_6 \subseteq R_5$ $(\bar{R}_6 \subseteq \bar{R}_5)$. A node $a \in Q_5 \cup (R_5 - R_6)$ is called an *entrance1* again if there is a clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(6)}$ $(k = 3, 4)$ with $x_1, ..., x_{k-1} \in R_6$ $(w_6(C) > 0)$. Let $Q_6$ be the set of nodes in $Q_5 \cup (R_5 - R_6)$ that are reachable from an entrance1 by a path in $M_6 \equiv N(\mathcal{D}^{(6)})$. A node $a \in ((R_5 \cup Q_5) - (R_6 \cup Q_6)) \cup P_5$ is called an *entrance2* if there is a clause $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a \in \mathcal{D}^{(6)}$ $(k = 3, 4)$ with $x_1, x_{k-2} \in R_6$, $x_{k-1} \in Q_6$ $(w_6(C) > 0)$. Let $P_6$ be the set of nodes in$((R_5 \cup Q_5) - (R_6 \cup Q_6)) \cup P_5$ that are reachable from an entrance2 by a path in $M_6$. Then, by the symmetry and maximality of $f_6$, $Q_6 \cup P_6$ contains no complementary literals and all literals in $Q_6 \cup P_6$ are unnegated.

By the argument above we can summarize Step 6 of our algorithm and have a lemma as follows.

**Step 6.** Find $R_6$, $Q_6$, $P_6$ and $(\mathcal{D}^{(6)} \cup \mathcal{J}^{(6)} \cup \mathcal{K}^{(6)} \cup \mathcal{K}'^{(6)}, w_6)$ from $(\mathcal{D}^{(5)}, w_5)$ using the network $M_5^-$, $N_6$, a symmetric flow $f_6$ of $N_6$ of maximum value and the network $L_6$ defined above.

**Lemma 7** *$(\mathcal{D}^{(5)}, w_5)$ and $(\mathcal{D}^{(6)} \cup \mathcal{J}^{(6)} \cup \mathcal{K}^{(6)} \cup \mathcal{K}'^{(6)}, w_6)$ are strongly equivalent and $R_6 \subseteq R_5$, $Q_6 \subseteq Q_5 \cup R_5 - R_6$ and $P_6 \subseteq (P_5 \cup Q_5 \cup R_5) - (Q_6 \cup R_6)$. Furthermore, $(\mathcal{D}^{(6)}, w_6)$ satisfies property $\pi$ described in Section 2.*

Now we are ready to set the probability for each variable to be true.

**Step 7.** Obtain a random truth assignment $\boldsymbol{x}^p$ by setting independently each variable $x_i$ to be true with probability $p_i$ as follows ($Z_6 \equiv X - (R_6 \cup Q_6 \cup P_6)$):

$$p_i = \begin{cases} 0.75 & (x_i \in R_6) \\ 0.629 & (x_i \in Q_6) \\ 0.557 & (x_i \in P_6) \\ 0.5 & (x_i \in Z_6). \end{cases}$$

Then find a truth assignment $\boldsymbol{x}^A \in \{0,1\}^n$ with value $F_{\mathcal{C}}(\boldsymbol{x}^A) \geq F_{\mathcal{C}}(\boldsymbol{x}^p)$ by the probabilistic method.

We will give an analysis of the expected value of the random truth assingment $\boldsymbol{x}^p$ in the next section, where the following lemma plays an important role.

**Lemma 8** *The probability $p_i$ of variable $x_i$ in Step 7 satisfies the following.*

$$p_i \in \begin{cases} [0.371, 0.75] & (x_i \in R) \\ [0.443, 0.75] & (x_i \in R_j,\ j = 1,2,3) \\ [0.5, 0.75] & (x_i \in R_j\ j = 4,5) \\ [0.443, 0.629] & (x_i \in Q_j,\ j = 2,3) \\ [0.5, 0.629] & (x_i \in Q_j,\ j = 4,5) \\ [0.5, 0.557] & (x_i \in P_5) \\ [0.371, 0.629] & (x_i \in Z_j,\ j = 0,1) \\ [0.443, 0.557] & (x_i \in Z_j,\ j = 2,3,4) \\ [0.5, 0.5] & (x_i \in Z_5). \end{cases}$$

The above lemma can be obtained by Lemmas 2-7. For example, $p_i \in [0.443, 0.75]$ $(x_i \in R_1)$ is obtained by $R_1 \cap \bar{R}_6 = \emptyset$ and $R_1 \cap \bar{Q}_6 = \emptyset$ since $R_6 \subseteq R_5 \subseteq R_4 \subseteq R_3 \subseteq R_2 \subseteq R_1$ $(\bar{R}_6 \subseteq \bar{R}_1)$ and $Q_6 \subseteq Q_5 \cup R_5 \subseteq Q_4 \cup R_4 \subseteq Q_3 \cup R_3 \subseteq Q_2 \cup R_2 \subseteq Z_1 \cup \bar{Z}_1 \cup R_1 = X \cup \bar{Z}_1$ $(\bar{Q}_6 \subseteq Z_1 \cup \bar{X})$. The other cases are similarly obtained.

## 4 Analysis

In this section we consider the expected value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ of the random truth assignment $\boldsymbol{x}^p$ obtained by Step 7. Let $\boldsymbol{x}^*$ be an optimal truth assignment for $(\mathcal{C}, w)$. Then, the random truth assignment $\boldsymbol{x}^p$ satisfies (7), which will be shown below.

Let $(\mathcal{C}^6, w_6) = (\mathcal{D}^{(6)} \cup \mathcal{J}^{(1)} \cup \mathcal{K}^{(2)} \cup \mathcal{J}^{(3)} \cup \mathcal{K}^{(3)} \cup \mathcal{J}^{(4)} \cup \mathcal{K}^{(4)} \cup \mathcal{K}^{(5)} \cup \mathcal{K}'^{(5)} \cup \mathcal{J}^{(6)} \cup \mathcal{K}^{(6)} \cup \mathcal{K}'^{(6)}, w_6)$ (we assume $w_i = w_6$ for $i = 1, ..., 5$). Let $\boldsymbol{x}^r$ be any random truth assignment and let $W_k^r(\mathcal{L})$ be the expected value of $\boldsymbol{x}^r$ for the weighted clauses in $(\mathcal{L}, w_6)$ with $k$ literals. Thus, $W_k^r(\mathcal{C}^6) = \sum W_k^r(\mathcal{L})$, where the summation is taken over for all $\mathcal{L} = \mathcal{D}^{(6)}$, $\mathcal{J}^{(1)}$, $\mathcal{K}^{(2)}$, $\mathcal{J}^{(3)}$, $\mathcal{K}^{(3)}$, $\mathcal{J}^{(4)}$, $\mathcal{K}^{(4)}$, $\mathcal{K}^{(5)}$, $\mathcal{K}'^{(5)}$, $\mathcal{J}^{(6)}$, $\mathcal{K}^{(6)}$, $\mathcal{K}'^{(6)}$. Similarly, let $W_k^r = W_k^r(\mathcal{C})$ be the expected value of $\boldsymbol{x}^r$ for the weighted clauses in $(\mathcal{C}, w)$ with $k$ literals. $W_k^*(\mathcal{L})$ is the value of the optimal truth assignment $\boldsymbol{x}^*$ for weighted clauses in $(\mathcal{L}, w_6)$ with $k$ literals and $W_k^* = W_k^*(\mathcal{C})$ is the value of $\boldsymbol{x}^*$ for weighted clauses in $(\mathcal{C}, w)$ with $k$ literals.

Then we have the following lemmas since $(\mathcal{C}, w)$ and $(\mathcal{C}^6, w_6)$ are strongly equivalent by Lemmas 2-7.

**Lemma 9** *For any random truth assignment $\boldsymbol{x}^r$, the following statements hold.*

(a) $W_k^r = W_k^r(\mathcal{C}^6)$ *for all* $k \geq 3$.

(b) $W_2^r(\mathcal{C}^6) = W_2^r(\mathcal{D}^{(6)})$ *and* $W_1^r(\mathcal{C}^6) = \sum W_1^r(\mathcal{L})$ *where the summation is taken over for all* $\mathcal{L} = \mathcal{D}^{(6)}, \mathcal{J}^{(1)}, \mathcal{K}^{(2)}, \mathcal{J}^{(3)}, \mathcal{K}^{(3)}, \mathcal{J}^{(4)}, \mathcal{K}^{(4)}, \mathcal{K}^{(5)}, \mathcal{K}'^{(5)}, \mathcal{J}^{(6)}, \mathcal{K}^{(6)}, \mathcal{K}'^{(6)}$. *Furthermore,* $W_{1,2}^r = W_{1,2}^r(\mathcal{C}^6)$ *where* $W_{1,2}^r \equiv W_1^r + W_2^r$ *and* $W_{1,2}^r(\mathcal{C}^6) \equiv W_1^r(\mathcal{C}^6) + W_2^r(\mathcal{C}^6)$.

**Lemma 10** *For $\boldsymbol{x}^p$ obtained in Step 7 in Section 3 and an optimal truth assignment $\boldsymbol{x}^*$, if*

$$F_{\mathcal{L}}(\boldsymbol{x}^p) \geq \sum_{k\geq 1} \gamma_k W_k^*(\mathcal{L}) \tag{52}$$

*for $\mathcal{L} = \mathcal{C}^6$, then $F_{\mathcal{C}}(\boldsymbol{x}^p)$ satisfies (7) (i.e., $F_{\mathcal{C}}(\boldsymbol{x}^p) \geq \sum_{k\geq 1} \gamma_k W_k^*(\mathcal{C})$).*

This lemma is obtained as follows. By Lemma 9 (for $\boldsymbol{x}^r = \boldsymbol{x}^*$), we have $W_1^* + W_2^* = W_1^*(\mathcal{C}^6) + W_2^*(\mathcal{C}^6)$ and $W_k^* = W_k^*(\mathcal{C}^6)$ for all $k \geq 3$. Thus, $F_{\mathcal{C}}(\boldsymbol{x}^p)$ satisfies (7) since (52) for $\mathcal{L} = \mathcal{C}^6$ implies $F_{\mathcal{C}}(\boldsymbol{x}^p) = F_{\mathcal{C}^6}(\boldsymbol{x}^p) \geq \gamma_1(W_1^* + W_2^*) + \sum_{k\geq 3} \gamma_k W_k^*$ by Lemma 9 and $\gamma_1 = \gamma_2$.

By Lemma 10, we have only to show that (52) is true for $\mathcal{L} = \mathcal{C}^6$. Furthermore, it suffices to show that each group $\mathcal{L}$ satisfies (52) for $\mathcal{L} = \mathcal{D}^{(6)}, \mathcal{J}_{1,k}^{(i)}, \mathcal{K}_{1,k}^{(i)}, \mathcal{K}_{1,k}'^{(i)}$ defined in Section 3. Similarly, if each $\mathcal{L}(C)$ satisfies (52) then $\mathcal{L}$ satisfies (52). For simplicity, we first assume $\mathcal{L}(C) = \mathcal{L}$. Thus, for example,
$\mathcal{J}_{1,k}^{(1)} = \{x_1, ..., x_k, C\}$ with $x_1, ..., x_k \in R$ of weight $2f_1(C)$ and $C = \bar{x}_1 \vee \cdots \vee \bar{x}_k$ of weight $a_k^{(1)} f_1(C)$,
$\mathcal{K}_{1,k}^{(2)} = \{x_1, ..., x_{k-1}, \bar{a}, x_0, \bar{x}_0, C\}$ with $x_1, ..., x_{k-1} \in R_1$ and $a \in Z_1 \cup \bar{Z}_1$ of weight $2f_2(C)$, $x_0, \bar{x}_0$ with weight $-f_2(C)$ and $C = \bar{x}_1 \vee \cdots \vee \bar{x}_{k-1} \vee a$ of weight $b_k^{(2)} f_2(C)$.

Now we will find a lower bound on the expected value of $F_{\mathcal{L}}(\boldsymbol{x}^p)$ for each $(\mathcal{L}, w_6)$ based on the assumption above (for simplicity, we first assume $f_1(C) = \cdots = f_6(C) = 1$ and $a = x_k$).

**A.** $F_{\mathcal{J}_{1,k}^{(1)}}(\boldsymbol{x}^p) = 2(p_1 + \cdots + p_k) + a_k^{(1)}(1 - p_1 \cdots p_k)$ $(k = 3, 4, 5, 6)$.

Let $p = \sqrt[k]{p_1 p_2 \cdots p_k}$ and $g(\mathcal{J}_{1,k}^{(1)}) = 2kp + a_k^{(1)}(1 - p^k)$. Then $F_{\mathcal{J}_{1,k}^{(1)}}(\boldsymbol{x}^p) \geq g(\mathcal{J}_{1,k}^{(1)})$ by the arithmetic/geometric mean inequality. Since $x_i \in R$, we have $p_i \in [0.371, 0.75]$ by Lemma 8 and $p \in [0.371, 0.75]$. In this interval, it can be easily shown that $g(\mathcal{J}_{1,k}^{(1)})$ takes a minimum value at $p = 0.371$ for $k = 3, 4, 5, 6$. Thus,

$$F_{\mathcal{J}_{1,k}^{(1)}}(\boldsymbol{x}^p) \geq g(\mathcal{J}_{1,k}^{(1)}) \geq 2(0.371k) + a_k^{(1)}(1 - 0.371^k) = \begin{cases} 7.9196 & (k=3) \\ 12.7785 & (k=4) \\ 17.6115 & (k=5) \\ 26.3946 & (k=6). \end{cases}$$

On the other hand, $W_1^*(\mathcal{J}_{1,k}^{(1)}) = 2\sum_{i=1}^k x_i^*$ and $W_k^*(\mathcal{J}_{1,k}^{(1)}) = a_k^{(1)}(1 - \prod_{i=1}^k x_i^*)$. Using the inequality

$$1 - \prod_{i=1}^{k} x_i^* \leq \min\{1, k - \sum_{i=1}^{k} x_i^*\} \tag{53}$$

for $x_i^* = 0, 1$ (this inequality holds even for $0 \leq x_i^* \leq 1$ and will also be used below) and $\gamma_1 < \gamma_k$, we have

$$\gamma_1 W_1^*(\mathcal{J}_{1,k}^{(1)}) + \gamma_k W_k^*(\mathcal{J}_{1,k}^{(1)}) \le 2\gamma_1 \sum_{i=1}^{k} x_i^* + a_k^{(1)}\gamma_k \min\{1, k - \sum_{i=1}^{k} x_i^*\}$$

$$\le 2(k-1)\gamma_1 + a_k^{(1)}\gamma_k = \begin{cases} 4\gamma_1 + 6\gamma_3 = 7.746 & (k=3) \\ 6\gamma_1 + 10\gamma_4 = 12.61 & (k=4) \\ 8\gamma_1 + 14\gamma_5 = 17.522 & (k=5) \\ 10\gamma_1 + 22\gamma_6 = 26.2 & (k=6) \end{cases}$$

and $F_{\mathcal{J}_{1,3}^{(1)}}(\boldsymbol{x}^p) \ge \gamma_1 W_1^*(\mathcal{J}_{1,k}^{(1)}) + \gamma_k W_k^*(\mathcal{J}_{1,k}^{(1)})$.

**B.** $F_{\mathcal{K}_{1,k}^{(2)}}(\boldsymbol{x}^p) = 2(p_1 + \cdots + p_{k-1} + 1 - p_k) - 1 + \ b_k^{(2)}(1 - p_1 \cdots p_{k-1}(1-p_k))\ (k=3,4)$.

Let $p = \sqrt[k-1]{p_1 p_2 \cdots p_{k-1}}$ and $g(\mathcal{K}_{1,k}^{(2)}) = 2(k-1)p + 2(1-p_k) - 1 + b_k^{(2)}(1 - p^{k-1}(1-p_k))$. Then $F_{\mathcal{K}_{1,k}^{(2)}}(\boldsymbol{x}^p) \ge g(\mathcal{K}_{1,k}^{(2)})$. Since $x_i \in R_1$ $(i = 1, ..., k-1)$ and $x_k \in Z_1 \cup \bar{Z}_1$, we have $p_i, p \in [0.443, 0.75]$ and $p_k \in [0.371, 0.629]$ by Lemma 8. In these intervals, $g(\mathcal{K}_{1,k}^{(2)})$ takes a minimum value at $p = 0.443$ and $p_k = 0.629$ for $k = 3, 4$. Thus,

$$F_{\mathcal{K}_{1,k}^{(2)}}(\boldsymbol{x}^p) \ge g(\mathcal{K}_{1,k}^{(2)})$$

$$\ge 2(0.443(k-1) + (1-0.629)) - 1 + \ b_k^{(2)}(1 - 0.443^{k-1}(1-0.629)) = \begin{cases} 7.077 & (k=3) \\ 12.077 & (k=4). \end{cases}$$

Since $W_1^*(\mathcal{K}_{1,k}^{(2)}) = 2(x_1^* + \cdots + x_{k-1}^* + 1 - x_k^*) - 1$ and $W_k^*(\mathcal{K}_{1,k}^{(2)}) = b_{1,k}^{(2)}(1 - x_1^* \cdots x_{k-1}^*(1-x_k^*))$, we also have

$$\gamma_1 W_1^*(\mathcal{K}_{1,k}^{(2)}) + \gamma_k W_k^*(\mathcal{K}_{1,k}^{(2)}) \le \gamma_1(2(\sum_{i=1}^{k-1} x_i^* + 1 - x_k^*) - 1) + \ b_k^{(2)}\gamma_k \min\{1, k - (\sum_{i=1}^{k-1} x_i^* + 1 - x_k^*)\}$$

$$\le (2(k-1)-1)\gamma_1 + b_k^{(2)}\gamma_k = \begin{cases} 3\gamma_1 + 6\gamma_3 = 6.996 & (k=3) \\ 5\gamma_1 + 10\gamma_4 = 11.86 & (k=4) \end{cases}$$

and $F_{\mathcal{K}_{1,k}^{(2)}}(\boldsymbol{x}^p) \ge \gamma_1 W_1^*(\mathcal{K}_{1,k}^{(2)}) + \gamma_k W_k^*(\mathcal{K}_{1,k}^{(2)})$.

**C.** $F_{\mathcal{J}_{1,k}^{(3)}}(\boldsymbol{x}^p) = 2\sum_{i=1}^{k} p_i + a_k^{(3)}(1 - \prod_{i=1}^{k} p_i)\ (k = 3, 4, 5)$.

Let $k_1 = 2^{k-3}$, $p = \sqrt[k_1]{p_1 \cdots p_{k_1}}$, $p' = \sqrt[k-k_1]{p_{k_1+1} \cdots p_k}$ and $g(\mathcal{J}_{1,k}^{(3)}) = 2k_1 p + 2(k-k_1)p' + a_k^{(3)}(1 - p^{k_1}p'^{k-k_1})$. Then $F_{\mathcal{J}_{1,k}^{(3)}}(\boldsymbol{x}^p) \ge g(\mathcal{J}_{1,k}^{(3)})$. Since $x_i \in R_2$ $(i = 1, ..., k_1)$ and $x_j \in Q_2$ $(j = k_1+1, ..., k)$, we have $p_i \in [0.443, 0.75]$ and $p_j \in [0.443, 0.629]$ by Lemma 8. This implies $p \in [0.443, 0.75]$ and $p' \in [0.443, 0.629]$. In these intervals, $g(\mathcal{J}_{1,k}^{(3)})$ takes a minimum value at $p = p' = 0.443$. Thus,

$$F_{\mathcal{J}_{1,k}^{(3)}}(\boldsymbol{x}^p) \ge g(\mathcal{J}_{1,k}^{(3)}) \ge 2(0.443k) + a_k^{(3)}(1 - 0.443^k) = \begin{cases} 8.1363 & (k=3) \\ 13.1588 & (k=4) \\ 16.2252 & (k=5). \end{cases}$$

Since $W_1^*(\mathcal{J}_{1,k}^{(3)}) = 2\sum_{i=1}^{k} x_i^*$ and $W_k^*(\mathcal{J}_{1,k}^{(3)}) = a_k^{(3)}(1 - \prod_{i=1}^{k} x_i^*)$, we also have

$$\gamma_1 W_1^*(\mathcal{J}_{1,k}^{(3)}) + \gamma_k W_k^*(\mathcal{J}_{1,k}^{(3)}) \le 2\gamma_1 \sum_{i=1}^{k} x_i^* + a_k^{(3)}\gamma_k \min\{1, k - \sum_{i=1}^{k} x_i^*\}$$

$$\le 2(k-1)\gamma_1 + a_k^{(3)}\gamma_k = \begin{cases} 4\gamma_1 + 6\gamma_3 = 7.746 & (k=3) \\ 6\gamma_1 + 10\gamma_4 = 12.61 & (k=4) \\ 8\gamma_1 + 12\gamma_5 = 15.876 & (k=5) \end{cases}$$

and $F_{\mathcal{J}^{(3)}_{1,k}}(\boldsymbol{x}^p) \geq \gamma_1 W_1^*(\mathcal{J}^{(3)}_{1,k}) + \gamma_k W_k^*(\mathcal{J}^{(3)}_{1,k}) + 0.349$.

**D.** $F_{\mathcal{K}^{(3)}_{1,k}}(\boldsymbol{x}^p) = 2(p_1 + \cdots + p_{k-1} + 1 - p_k) - 2 + b_k^{(3)}(1 - p_1 \cdots p_{k-1}(1 - p_k))$ $(k = 3, 4)$.

Let $p = \sqrt[k-1]{p_1 p_2 \cdots p_{k-1}}$ and $g(\mathcal{K}^{(3)}_{1,k}) = 2(k-1)p + 2(1 - p_k) - 2 + b_k^{(3)}(1 - p^{k-1}(1 - p_k))$. Then $F_{\mathcal{K}^{(3)}_{1,k}}(\boldsymbol{x}^p) \geq g(\mathcal{K}^{(3)}_{1,k})$. Since $x_i \in R_2$ $(i = 1, ..., k-1)$ and $x_k \in Q_2$, we have $p_i, p \in [0.443, 0.75]$ and $p_k \in [0.443, 0.629]$ by Lemma 8. In these intervals, $g(\mathcal{K}^{(3)}_{1,k})$ takes a minimum value at $p = 0.75$ and $p_k = 0.443$ for $k = 3, 4$. Thus,

$$\begin{aligned} F_{\mathcal{K}^{(3)}_{1,k}}(\boldsymbol{x}^p) &\geq g(\mathcal{K}^{(3)}_{1,k}) \\ &\geq 2(0.75(k-1) + (1 - 0.443)) - 1 + \ b_k^{(3)}(1 - 0.75^{k-1}(1 - 0.443)) \\ &= \begin{cases} 6.9208 & (k = 3) \\ 12.7941 & (k = 4). \end{cases} \end{aligned}$$

Since $W_1^*(\mathcal{K}^{(3)}_{1,k}) = 2(x_1^* + \cdots + x_{k-1}^* + 1 - x_k^*) - 2$ and $W_k^*(\mathcal{K}^{(3)}_{1,k}) = b_{1,k}^{(3)}(1 - x_1^* \cdots x_{k-1}^*(1 - x_k^*))$, we also have

$$\gamma_1 W_1^*(\mathcal{K}^{(3)}_{1,k}) + \gamma_k W_k^*(\mathcal{K}^{(3)}_{1,k}) \leq (2(k-1) - 2)\gamma_1 + b_k^{(3)}\gamma_k = \begin{cases} 2\gamma_1 + 7\gamma_3 = 7.037 & (k = 3) \\ 4\gamma_1 + 12\gamma_4 = 12.732 & (k = 4) \end{cases}$$

and $F_{\mathcal{K}^{(3)}_{1,4}}(\boldsymbol{x}^p) \geq \gamma_1 W_1^*(\mathcal{K}^{(3)}_{1,4}) + \gamma_4 W_4^*(\mathcal{K}^{(3)}_{1,4})$ and $F_{\mathcal{K}^{(3)}_{1,3}}(\boldsymbol{x}^p) \geq \gamma_1 W_1^*(\mathcal{K}^{(3)}_{1,3}) + \gamma_3 W_3^*(\mathcal{K}^{(3)}_{1,3}) - 0.1162$.

By similar arguments we have the following.

**E.** $F_{\mathcal{J}^{(4)}_{1,3}}(\boldsymbol{x}^p) = 2(p_1 + p_2 + p_3) + a_3^{(4)}(1 - p_1 p_2 p_3)$.

$g(\mathcal{K}^{(4)}_{1,3}) \equiv 6p + a_3^{(4)}(1 - p^3)$ with $p \equiv \sqrt[3]{p_1 p_2 p_3}$ takes a minimum value at $p = p' = 0.443$ since $x_i \in Q_3$ $(i = 1, 2, 3)$ and $p_i, p \in [0.443, 0.629]$ by Lemma 8. Thus, $F_{\mathcal{J}^{(4)}_{1,3}}(\boldsymbol{x}^p) \geq g(\mathcal{J}^{(4)}_{1,3}) \geq 6(0.443) + a_3^{(4)}(1 - 0.443^3) = 8.13637$. On the other hand, since $\gamma_1 W_1^*(\mathcal{J}^{(4)}_{1,3}) + \gamma_k W_k^*(\mathcal{J}^{(4)}_{1,3}) \leq 4\gamma_1 + a_3^{(4)}\gamma_3 = 7.746$, we have $F_{\mathcal{J}^{(4)}_{1,3}}(\boldsymbol{x}^p) \geq \gamma_1 W_1^*(\mathcal{J}^{(4)}_{1,3}) + \gamma_k W_k^*(\mathcal{J}^{(4)}_{1,3}) + 0.390$.

**F.** $F_{\mathcal{K}^{(4)}_{1,k}}(\boldsymbol{x}^p) = 2(p_1 + \cdots + p_{k-1} + 1 - p_k) - 2 + b_k^{(4)}(1 - p_1 \cdots p_{k-1}(1 - p_k))$ $(k = 3, 4)$.

By the same argument as for $F_{\mathcal{K}^{(3)}_{1,k}}(\boldsymbol{x}^p)$, we have

$$F_{\mathcal{K}^{(4)}_{1,k}}(\boldsymbol{x}^p) \geq \begin{cases} 6.9208 & (k = 3) \\ 12.7941 & (k = 4), \end{cases}$$

$$\gamma_1 W_1^*(\mathcal{K}^{(4)}_{1,k}) + \gamma_k W_k^*(\mathcal{K}^{(4)}_{1,k}) \leq \begin{cases} 2\gamma_1 + 7\gamma_3 = 7.037 & (k = 3) \\ 4\gamma_1 + 12\gamma_4 = 12.732 & (k = 4) \end{cases}$$

and $F_{\mathcal{K}^{(4)}1,4}(\boldsymbol{x}^p) \geq \gamma_1 W_1^*(\mathcal{K}^{(4)}_{1,4}) + \gamma_4 W_4^*(\mathcal{K}^{(4)}_{1,4})$ and $F_{\mathcal{K}^{(4)}1,3}(\boldsymbol{x}^p) \geq \gamma_1 W_1^*(\mathcal{K}^{(4)}_{1,3}) + \gamma_3 W_3^*(\mathcal{K}^{(4)}_{1,3}) - 0.1162$.

**G.** $F_{\mathcal{K}^{(5)}_{1,k}}(\boldsymbol{x}^p) = 2(p_1 + \cdots + p_{k-1} + 1 - p_k) - 2 + b_k^{\prime\prime(5)}(1 - p_1 \cdots p_{k-1}(1 - p_k))$ $(k = 3, 4)$.

By an argument similar to one above, $g(\mathcal{K}^{(5)}_{1,k}) \equiv 2(k-1)p + 2(1 - p_k) - 2 + b_k^{\prime\prime(5)}(1 - p^{k-1}(1 - p_k))$ with $p \equiv \sqrt[k-1]{p_1 p_2 \cdots p_{k-1}}$ takes a minimum value at $p = 0.75$ and $p_k = 0.5$ for $k = 3, 4$ since $x_i \in R_4$ $(i = 1, ..., k-1)$, $x_k \in Q_4$ and thus $p_i, p \in [0.5, 0.75]$ $(i = 1, ..., k-1)$ and $p_k \in [0.5, 0.631]$ by Lemma

8. Thus, we have

$$\begin{aligned} F_{\mathcal{K}^{(5)}_{1,k}}(\boldsymbol{x}^p) &\geq g(\mathcal{K}^{(5)}_{1,k}) \\ &\geq 2(0.75(k-1)+(1-0.5))-2+\ b''^{(5)}_k(1-0.75^{k-1}(1-0.5)) \\ &= \begin{cases} 6.8875 & (k=3) \\ 12.96875 & (k=4), \end{cases} \end{aligned}$$

$$\gamma_1 W_1^*(\mathcal{K}^{(5)}_{1,k})+\gamma_k W_k^*(\mathcal{K}^{(5)}_{1,k}) \leq (2(k-1)-2)\gamma_1 + b''^{(5)}_k\gamma_k = \begin{cases} 2\gamma_1+6.8\gamma_3 = 6.8788 & (k=3) \\ 4\gamma_1+12\gamma_4 = 12.732 & (k=4) \end{cases}$$

and $F_{\mathcal{K}^{(5)}_{1,k}}(\boldsymbol{x}^p) \geq \gamma_1 W_1^*(\mathcal{K}^{(5)}_{1,k})+\gamma_k W_k^*(\mathcal{K}^{(5)}_{1,k})$.

**H.** $F_{\mathcal{K}'^{(5)}_{1,k}}(\boldsymbol{x}^p) = 2(p_1+\cdots+p_{k-1}+1-p_k)-1+b''^{(5)}_k(1-p_1\cdots p_{k-1}(1-p_k))$ $(k=3,4)$.

Let $p=p_1$ if $k=3$ and $p=\sqrt{p_1p_2}$ if $k=4$. Then $g(\mathcal{K}'^{(5)}_{1,k}) \equiv 2(k-2)p+2p_{k-1}+2(1-p_k)-1+b''^{(5)}_k(1-p^{k-2}p_{k-1}(1-p_k))$ takes a minimum value at $p=p_{k-2}=0.5$, and $p_k=0.557$ for $k=3,4$, since $x_i \in R_4$ $(i=1,...,k-2)$, $x_{k-1}\in Q_4$ and $x_k \in Z_4 \cup \bar{Z}_4$ and $p_i, p \in [0.5, 0.75]$, $p_{k-1}\in[0.5,0.629]$ and $p_k \in [0.443, 0.557]$ by Lemma 8. Thus, we have

$$\begin{aligned} F_{\mathcal{K}'^{(5)}_{1,k}}(\boldsymbol{x}^p) &\geq g(\mathcal{K}'^{(5)}_{1,k}) \\ &\geq 2(0.5(k-1)+(1-0.557))-1+\ b''^{(5)}_k(1-0.5^{k-1}(1-0.557)) \\ &= \begin{cases} 7.66612 & (k=3) \\ 12.3322 & (k=4), \end{cases} \end{aligned}$$

$$\gamma_1 W_1^*(\mathcal{K}'^{(5)}_{1,k})+\gamma_k W_k^*(\mathcal{K}'^{(5)}_{1,k}) \leq (2(k-1)-1)\gamma_1 + b''^{(5)}_k\gamma_k = \begin{cases} 3\gamma_1+6.5\gamma_3 = 7.3915 & (k=3) \\ 5\gamma_1+10\gamma_4 = 11.86 & (k=4) \end{cases}$$

and $F_{\mathcal{K}'^{(5)}_{1,k}}(\boldsymbol{x}^p) \geq \gamma_1 W_1^*(\mathcal{K}'^{(5)}_{1,k})+\gamma_k W_k^*(\mathcal{K}'^{(5)}_{1,k})$.

**I.** $F_{\mathcal{J}^{(6)}_{1,3}}(\boldsymbol{x}^p) = 2(p_1+p_2+p_3)+a^{(6)}_3(1-p_1p_2p_3)$.

Let $g(\mathcal{J}^{(6)}_{1,3}) = 4p+2p'+a^{(6)}_3(1-p^2p')$, where $p=\sqrt{p_2p_3}$ and $p'=p_1$ if $x_1\in R_5$ and $x_2,x_3 \in P_5$ and $p=\sqrt{p_1p_2}$ and $p'=p_3$ if $x_1,x_2\in Q_5$ and $x_3\in P_5$ . Then $g(\mathcal{J}^{(6)}_{1,3})$ takes a minimum value at $p=p'=0.5$, since $p'\in[0.5,0.75]$, $p\in[0.5,0.557]$ or $p\in[0.5,0.629]$, $p'\in[0.5,0.557]$ by Lemma 8, and we have

$$\begin{aligned} F_{\mathcal{J}^{(6)}_{1,3}}(\boldsymbol{x}^p) &\geq g(\mathcal{J}^{(6)}_{1,3}) \\ &\geq 6(0.5)+a^{(6)}_k(1-0.5^3) = 8.25 \\ &\geq 7.746 \geq \gamma_1 W_1^*(\mathcal{J}^{(6)}_{1,3})+\gamma_k W_k^*(\mathcal{J}^{(6)}_{1,3}). \end{aligned}$$

**J.** $F_{\mathcal{K}^{(6)}_{1,k}}(\boldsymbol{x}^p) = 2(p_1+\cdots+p_{k-1}+1-p_k)-2+b''^{(6)}_k(1-p_1\cdots p_{k-1}(1-p_k))$ $(k=3,4)$.

By the same argument as for $F_{\mathcal{K}^{(5)}_{1,k}}(\boldsymbol{x}^p)$, we have

$$F_{\mathcal{K}^{(6)}_{1,k}}(\boldsymbol{x}^p) \geq \begin{cases} 6.8875 & (k=3) \\ 12.96875 & (k=4), \end{cases}$$

$$\gamma_1 W_1^*(\mathcal{K}_{1,k}^{(6)}) + \gamma_k W_k^*(\mathcal{K}_{1,k}^{(6)}) \le \begin{cases} 2\gamma_1 + 6.8\gamma_3 = 6.8788 & (k=3) \\ 4\gamma_1 + 12\gamma_4 = 12.732 & (k=4) \end{cases}$$

and $F_{\mathcal{K}_{1,k}^{(6)}}(\boldsymbol{x}^p) \ge \gamma_1 W_1^*(\mathcal{K}_{1,k}^{(6)}) + \gamma_k W_k^*(\mathcal{K}_{1,k}^{(6)})$.

**K.** $F_{\mathcal{K}_{1,k}^{\prime(6)}}(\boldsymbol{x}^p) = 2(p_1 + \cdots + p_{k-1} + 1 - p_k) - 2 + b_k^{\prime\prime(6)}(1 - p_1 \cdots p_{k-1}(1-p_k))$ $(k = 3, 4)$.

By an argument similar to one for $F_{\mathcal{K}_{1,k}^{\prime(5)}}(\boldsymbol{x}^p)$, we have

$$\begin{aligned} F_{\mathcal{K}_{1,k}^{\prime(6)}}(\boldsymbol{x}^p) &\ge 2(0.5(k-1) + (1-0.557)) - 2 + b_k^{\prime\prime(6)}(1 - 0.5^{k-1}(1-0.557)) \\ &= \begin{cases} 6.66612 & (k=3) \\ 11.3322 & (k=4), \end{cases} \end{aligned}$$

$$\gamma_1 W_1^*(\mathcal{K}_{1,k}^{\prime(6)}) + \gamma_k W_k^*(\mathcal{K}_{1,k}^{\prime(6)}) \le \begin{cases} 2\gamma_1 + 6.5\gamma_3 = 6.6415 & (k=3) \\ 4\gamma_1 + 10\gamma_4 = 11.11 & (k=4) \end{cases}$$

and $F_{\mathcal{K}_{1,k}^{\prime(6)}}(\boldsymbol{x}^p) \ge \gamma_1 W_1^*(\mathcal{K}_{1,k}^{\prime(6)}) + \gamma_k W_k^*(\mathcal{K}_{1,k}^{\prime(6)})$.

**L.** $F_{\mathcal{D}_k^{(6)}}(\boldsymbol{x}^p)$.

Let $C = y_1 \vee y_2 \vee \cdots \vee y_k \in \mathcal{D}_k^{(6)}$ and let $p(y_i)$ be the probability of literal $y_i$ being true obtained in Step 7. Then $C(\boldsymbol{x}^p) = 1 - \prod_{i=1}^k (1 - p(y_i)) \ge 1 - 0.75^k = \gamma_k$ for $k \ge 7$. Similarly, if $k \le 6$, then it is easily shown that $C(\boldsymbol{x}^p) = 1 - \prod_{i=1}^k (1 - p(y_i)) \ge \gamma_k$ by Lemma 7. Thus, by $W_k(\mathcal{D}^{(6)}) = \sum_{C \in \mathcal{D}_k^{(6)}} w_6(C)$ $\ge W_k^*(\mathcal{D}^{(6)}) = \sum_{C \in \mathcal{D}_k^{(6)}} w_6(C) C(x^*)$, $F_{\mathcal{D}_k^{(6)}}(\boldsymbol{x}^p)$ satisfies (52).

We have shown that each group $\mathcal{L}$ satisfies (52) for $\mathcal{L} \ne \mathcal{K}_{1,3}^{(i)}$ $(i = 3, 4)$. Note that, such $\mathcal{K}_{1,3}^{(i)}$ exists only if $\mathcal{J}_{1,k}^{(i)}$ exists. Furthermore, a unit flow on $(\bar{x}_k, C_k)$ with $C = \bar{x}_1 \vee \cdots \vee \bar{x}_k \in \mathcal{A}_{1,k}^{(3)}$ $(k = 3, 4, 5)$ such that $x_1, \ldots, x_{2^{k-3}} \in R$ and $x_{2^{k-3}+1}, \ldots, x_k \in Q_2$ comes from a unit flow on $(C_j, a)$ with $C_j = \bar{y}_1 \vee \cdots \vee \bar{y}_{j-1} \vee a \in \mathcal{B}_j^{(3)}$ $(j = 3, 4)$ such that $y_1, \ldots, y_{j-1} \in R_2$ and $a \in Q_2$ by the construction of $N_3$. Thus, at worst, two units of $F_{\mathcal{K}_{1,j}^{(3)}}(\boldsymbol{x}^p)$ corresponds to one unit of $F_{\mathcal{J}_{1,3}^{(3)}}(\boldsymbol{x}^p)$, two units of $F_{\mathcal{K}_{1,j}^{(3)}}(\boldsymbol{x}^p)$ corresponds to one unit of $F_{\mathcal{J}_{1,4}^{(3)}}(\boldsymbol{x}^p)$ and one unit of $F_{\mathcal{K}_{1,j}^{(3)}}(\boldsymbol{x}^p)$ corresponds to one unit of $F_{\mathcal{J}_{1,5}^{(3)}}(\boldsymbol{x}^p)$. Thus, for $j = 3$,

$$\begin{aligned} 2F_{\mathcal{K}_{1,3}^{(3)}}(\boldsymbol{x}^p) + F_{\mathcal{J}_{1,3}^{(3)}}(\boldsymbol{x}^p) &\ge 2(6.9208) + 8.1363 \\ &\ge 2(7.037) + 7.746 \\ &\ge 2\gamma_1 W_1^*(\mathcal{K}_{1,3}^{(3)}) + 2\gamma_3 W_3^*(\mathcal{K}_{1,3}^{(3)}) + \gamma_1 W_1^*(\mathcal{J}_{1,3}^{(3)}) + \gamma_3 W_3^*(\mathcal{J}_{1,3}^{(3)}). \end{aligned}$$

Similarly, $2F_{\mathcal{K}_{1,3}^{(3)}}(\boldsymbol{x}^p) + F_{\mathcal{J}_{1,4}^{(3)}}(\boldsymbol{x}^p) \ge 2\gamma_1 W_1^*(\mathcal{K}_{1,3}^{(3)}) + 2\gamma_3 W_3^*(\mathcal{K}_{1,3}^{(3)}) + \gamma_1 W_1^*(\mathcal{J}_{1,4}^{(3)}) + \gamma_4 W_4^*(\mathcal{J}_{1,4}^{(3)})$ and $F_{\mathcal{K}_{1,3}^{(3)}}(\boldsymbol{x}^p) + F_{\mathcal{J}_{1,5}^{(3)}}(\boldsymbol{x}^p) \ge \gamma_1 W_1^*(\mathcal{K}_{1,3}^{(3)}) + \gamma_3 W_3^*(\mathcal{K}_{1,3}^{(3)}) + \gamma_1 W_1^*(\mathcal{J}_{1,5}^{(3)}) + \gamma_5 W_5^*(\mathcal{J}_{1,5}^{(3)})$. Thus, we have (52) for $\mathcal{K}_{1,3}^{(3)}$ and $\mathcal{J}_{1,k}^{(3)}$. Similarly we have (52) for $\mathcal{J}_{1,3}^{(4)}$ and $\mathcal{K}_{1,3}^{(4)}$. By the argument above $F_{\mathcal{C}^6}(\boldsymbol{x}^p)$ of $\boldsymbol{x}^p$ satisfies (52) and, by Lemma 10, we have (7).

## 5 Concluding Remarks

We have presented a refinement of Yannakakis' algorithm with a better bound than Goemans-Williamson. It leads to a 0.770-approximation algorithm if it is combined with the algorithms in [3],

[11]. In fact, for an instance $(\mathcal{C}, w)$, if we choose the better solution bewteen two solutions obtained by our algorithm in this paper and the algorithm in [3], it has the value at least $0.770F_{\mathcal{C}}(x^*)$ (the expected value of a solution obtained by using our algorithm with probability 0.8427 and the algorithm in [3] with probability 0.1573 can be shown to be at least $0.770F_{\mathcal{C}}(x^*)$). Since a refinement of Yannakakis' algorithm in this paper is not optimized yet, we believe further refinements can be done and the performance guarantee for MAX SAT can be improved. Furthemore, if the refinement of Yannakakis' algorithm in this paper is combined with the techniques proposed in 0.931-approximation algorithm for MAX 2SAT by Feige-Goemans [5], it will lead to a better approximation algorithm.

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