中 央 大 学 理 工 学 研 究 所 論 文 集　第 7 号　2 0 0 1 年
Journal of the Institute of Science and Engineering, Chuo University

SOME REMARKS ON THE ARTIN-TATE FORMULA FOR DIAGONAL HYPERSURFACES

NORIYUKI SUWA*)

Abstract

In this note, we give some remarks on the Artin-Tate-Milne formula for diagonal hypersurfaces, which was studied by [3]. Detailed account will appear elsewhere.

1. Prelinminaries

1.1. Let k be a field, X be a smooth projective variety over k. Let G be a finite subgroup of $\mathrm{Aut}_k(X)$ and χ a $\mathbb{Q}$-character of G. Put $N = |G|$. Define an idempotent element p_χ of $\mathbb{Z}[\frac{1}{N}][G]$ by

$$p_\chi = \frac{1}{N}\sum_{g\in G}\chi(g^{-1})\, g.$$

1.2. Let R be a ring, in which N is invertible, and let $\boldsymbol{F}$ be a contravariant functor from a category of varieties over k to the category of R-modules. For a $\mathbb{Q}$-character of χ of G, define

$$\boldsymbol{F}(X)^{(\chi)} = \mathrm{Im}[p_\chi^*: \boldsymbol{F}(X) \to \boldsymbol{F}(X)].$$

Hence we obtain a decomposition

$$\boldsymbol{F}(X) = \bigoplus_\chi \boldsymbol{F}(X)^{(\chi)},$$

where the summation is taken over all the $\mathbb{Q}$-irreducible characters of G.

We apply terminologies defined for varieities to the pair (X, p_χ).

Example 1.3. Let l be a prime number different from the characteristic of k. The l-adic étale cohomology group $H^*(X, \mathbb{Q}_l(i))$ defines a contravariant functor from a category of varieties over k to the category of $\mathbb{Q}_l$-linear spaces; moreover, if l is prime to N, $H^*(X, \mathbb{Z}/l^r\mathbb{Z}(i))$, $H^*(X, \mathbb{Z}_l(i))$ and $H^*(X, \mathbb{Q}_l/\mathbb{Z}_l(i))$ define contravariant functors from a category of varieties over k to the category of $\mathbb{Z}_l$-modules. Recall that $\dim_{\mathbb{Q}_l} H^j(X_{\bar{k}}, \mathbb{Q}_l)$ is called the j-th Betti number of X and denoted by $B_j(X)$.

Example 1.4. If N is invertible in k, the de Rham cohomology group $H^*_{DR}(X/k)$ defines a contravariant functor from a category of varieties over k to the category of k-linear spaces. Moreover the Hodge spectral sequence

$$E_1^{ij} = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{DR}(X/k)$$

is functorial. Recall that $\dim_k H^j(X, \Omega^i_X)$ is called the (i,j)-th Hodge number of X and denoted by $H^{ij}(X)$.

For the examples 1.5 and 1.6, we assume that k is perfect of characteristic $p>0$ and that p does

*) Partially supported by Grant-in-Aid for Scientific Research No.12640041

1991 *Mathematics Subject Classification* Primary 14L05; Secondary 13K05, 20G10.

not divide N.

Example 1.5. (cf.[5],Ch.II, [6],Ch.IV.) The crystalline cohomology group $H^*(X/W)_K$ defines a contravariant functor from a category of varieties over k to the category of K-linear spaces, and $H^*(X/W_n)$ and $H^*(X/W)$ define contravariant functors from a category of varieties over k to the category of W-modules. Here W denotes the ring of Witt vectors with components in k, and K denotes the field of fractions of W. Moreover the slope spectral sequences

$$E_1^{ij}=H^j(X,W_n\Omega_X^i)\Rightarrow H^*(X/W_n)$$

and

$$E_1^{ij}=H^j(X,W\Omega_X^i)\Rightarrow H^*(X/W)$$

are functorial.

Recall the definitions

$$T^{ij}(X)=\dim \mathrm{Domino} H^j(X,W\Omega_X^*)^i$$

and

$$h_W^{ij}(X)=\dim_k \mathrm{H}^j(X,W\Omega_X^i)/(\mathrm{tors}+V)+\dim_k H^{j+1}(X,W\Omega_X^{i-1})/(\mathrm{tors}+F)$$
$$+\mathrm{T}^{i,j}(X)-2T^{i-1,j+1}(X)+\mathrm{T}^{i-2,j+2}(X)$$

([6], Ch. I .2.16 and [3]).

Recall also the following terminologies:

(1) X is of Hodge-Witt type in degree n if $H^j(X, W\Omega_X^i)$ is of finite type for all (i,j) with $i+j=n$;

(2) X is ordinary if X is of Hodge-Witt type in degree n and $H^j(X, BW\Omega_X^i)=0$ for all (i,j) with $i+j=n+1$ ([6],Ch.IV.4.6 and 4.12).

We shall say that:

(3) X is supersingular in degree n if the F-isocrystal $H^n(X/W)_K$ is purely of slope $n/2$.

If X is an ordinary (resp. supersingular) abelian variety, X is an ordinary (resp. supersingular) in each degree n in the above sense.

Example 1.6. (cf.[6],Ch.IV.3,[4],Ch. I) The logarithmic Hodge-Witt cohomology groups

$$H^*(X,\mathbb{Z}/p^r\mathbb{Z}(i))=H^{*-i}(X,W_r\Omega^i_{X,\log}),$$
$$H^*(X,\mathbb{Z}_p(i))=\varprojlim H^*(X,\mathbb{Z}/p^r\mathbb{Z}(i)),$$
$$H^*(X,\mathbb{Q}_p/\mathbb{Z}_p(i))=\varinjlim H^*(X,\mathbb{Z}/p^r\mathbb{Z}(i)).$$

define contravariant functors from a category of varieties over k to the category of $\mathbb{Z}_p$-modules. Moreover the perfect groups

$$\underline{D}^*(X,\mathbb{Z}/p^n\mathbb{Z}(i)),\ \underline{U}^*(X,\mathbb{Z}/p^n\mathbb{Z}(i)),$$
$$\underline{D}^*(X,\mathbb{Z}_p(i)),\ \underline{U}*(X,\mathbb{Z}_p(i)),$$
$$\underline{D}^*(X,\mathbb{Q}_p/\mathbb{Z}_p(i)),\ \underline{U}^*(X,\mathbb{Q}_p/\mathbb{Z}_p(i)).$$

define contravariant functors from a category of varieties over k to the category of pro-algebraic groups. Recall the equality

$$\dim \mathrm{Domino}^i H^j(X, W\Omega_X^*) = \dim \underline{U}^{i+j}(X, \mathbb{Z}_p(i-1)).$$

Example 1.7. Let $CH^r(X)$ and $CH^r(X_{\bar{k}})$ denote the Chow group of rational equivalence classes of algebraic cycles of codimension r on X and $X_{\bar{k}}$, respectively. Recall that there is defined a cycle map $CH^r(X_{\bar{k}}) \to H^n(X_{\bar{k}}, \mathbb{Z}_l(r))$ for each prime l. The Tate conjecture ([18]) asserts that, if k is finitely generated over $\mathbb{Q}$ or $\mathbb{F}_p$, $H^n(X_{\bar{k}}, \mathbb{Q}_l(r))^{\Gamma}$ is spanned by the image of the composite $CH^r(X) \to CH^r(X_{\bar{k}}) \to H^n(X_{\bar{k}}, \mathbb{Q}_l(r))$.

Let $N^r(X_{\bar{k}})$ denote the group of numerical equivalence classes of algebraic cycles on $X_{\bar{k}}$ of codimension r. Then $N^r(X_{\bar{k}})$ is a free $\mathbb{Z}$-module of finite rank and equipped with a nondegenerate symmetric bilinear from induced by the intersection pairing. Let $N^r(X)$ denote the image of the composite $CH^r(X) \to CH^r(X_{\bar{k}}) \to N^r(X_{\bar{k}})$.

Lemma 1.8. *Let l be a prime different from the characteristic of k. Let χ be a $\mathbb{Q}$-irreducible character of G. Assume that* $\dim_{\mathbb{Q}_l} H^{2r}(X_{\bar{k}}, \mathbb{Q}_l)^{(\chi)} \leq \deg \chi$. *Then the cycle map* $[CH^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l]^{(\chi)} \to H^{2r}(X_{\bar{k}}, \mathbb{Q}_l)^{(\chi)}$ *is surjective or* $[N^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}]^{(\chi)} = 0$.

Corollary 1.9. *Under the assumption of 1.8,*
(1) *If k is of characteristic 0 and* $h^{rr}(X,\chi) \neq B_{2r}(X,\chi)$, $[N^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}]^{(\chi)} = 0$;
(2) *If k is of characteristic $p > 0$ and (X, χ) is not supersingular in the degree $2r$,* $[N^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}]^{(\chi)} = 0$.

2. Diagonal hypersurfaces

2.1. Let n and m be integers ≥ 1. Let k be a field and X be a diagonal hypersurface of $\mathbb{P}_k^{n+1}$ defined by

$$c_0 T_0^m + c_1 T_1^m + \cdots + c_{n+1} T_{n+1}^m = 0,$$

$(c_0, c_1, \cdots, c_{n+1}) \in k^{\times}$. If $c_0 = c_1 = \ldots = c_{n+1} = 1$, X is nothing but the Fermat variety of dimension n and of degree m.

We assume that k contains all the m-th roots of unity and that $(m, p) = 1$ if k is of characteristic $p > 0$. Let μ_m denote the group of m-th roots of unity in k. The group $G = (\mu_m)^{n+2}$/(diagonal) acts on X by

$$(\zeta_0, \zeta_1, \ldots, \zeta_{n+1})(t_0, t_1, \ldots, t_{n+1}) = (\zeta_0 t_0, \zeta_1 t_1, \ldots, \zeta_{n+1} t_{n+1}).$$

The character group $\hat{G}$ of G is identified with the set

$$\{\mathbf{a} = (a_0, a_1, \ldots a_{n+1})\,;\, a_i \in \mathbb{Z}/m\mathbb{Z}\,,\ \sum_{i=0}^{n+1} a_i = 0\};$$

Let $(\mathbb{Z}/m\mathbb{Z})^{\times}$ act on $\hat{G}$ by $t\mathbf{a} = (ta_0, \ldots, ta_{n+1}) \in \hat{G}$ for any $\mathbf{a} \in \hat{G}$ and $t \in (\mathbb{Z}/m\mathbb{Z})^{\times}$. Let ζ_m be a fixed primitive m-th root of unity in $\overline{\mathbb{Q}}$. For the $(\mathbb{Z}/m\mathbb{Z})^{\times}$-orbit A of $\mathbf{a} = (a_0, \ldots, a_{n+1}) \in \hat{G}$, define a $\mathbb{Q}$-character χ_A of G by $\chi_A(g) = \dfrac{1}{m^{n+1}} \sum_{g \in G} \mathrm{Tr}_{\mathbb{Q}(\zeta_m^d)/\mathbb{Q}}(\mathbf{a}(g)^{-1})$. Here $d = \gcd(m, a_0, \ldots, a_{n+1})$. Note that χ_A is $\mathbb{Q}$-irreducible.

Let $X(n,m)$ denote the Fermat variery of dimension n and of degree m. Then

$$(t_0, t_1, \ldots t_{n+1}) \to (\sqrt[m]{c_0}\, t_0, \sqrt[m]{c_1}\, t_1, \ldots, \sqrt[m]{c_{n+1}}\, t_{n+1})$$

defines an $\bar{k}$-isomorphism of X to $X(n,m)$, which is compatible the actions of G on X and on $X(n,m)$. Hence we see that (X, χ_A) is isomorphic to $(X(n,m), \chi_A)$ over $\bar{k}$. It follows that the Betti numbers, Hodge numbers and the Newton polygons, if k is characteristic $p > 0$, of (X, χ_A) do not depend on $(c_0, c_1, \cdots, c_{n+1})$, but depend only on A.

2.2. Now assume that $n=2r$. Then we have a diagram

$$\begin{array}{ccc} \left[N^r(X)\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)} & \longrightarrow & \left[N^r(X_{\bar{k}})\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)} \\ & & \downarrow\wr \\ \left[N^r(X(n,m))\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)} & \longrightarrow & \left[N^r(X(n,m)_{\bar{k}})\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)}. \end{array}$$

Proposition 2.2.1. *Let k be a field and X a diagonal hypersuface of degree m of $\mathbb{P}_k^{2r+1}$. Under the identification $\left[N^r(X_{\bar{k}})\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)} = \left[N^r(X(n,m)_{\bar{k}})\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)}$, we have*

$$\left[N^r(X)\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)} = \left[N^r(X(n,m))\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)}$$

or

$$\left[N^r(X)\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{m}]\right]^{(\chi_A)} = 0.$$

Proof. It is enough to note that X is isomorphic to $X(n,m)$ over an extension of k of degree dividing a power of m.

Corollary 2.3. *Let k be a field and X a diagonal hypersuface of degree m of $\mathbb{P}_k^{2r+1}$. If m is prime, then $B_n(X) - \mathrm{rk} N^r(X)$ is divisivle by $m-1$.*

Proposition 2.4. *Let k be a perfect field of characteristic $p \geq 0$ and X a diagonal hypersuface of degree m of $\mathbb{P}_k^{2r+1}$. Assume that:*

(1) $n=2$;

(2) $n \geq 4$, *and m is not divisible by any prime less than $n+2$;*

or (3) $n \geq 4$, *and m is a prime or 4.*

If $p = 0$ or $p \equiv 1 \bmod m$, $\det N^r(X)$ divides a power of m.

Proof. The case of Fermat varieties is verified as in [15]. The proposition follows from, together with Proposition 2.2.1, from the assertion for Fermat varieties.

Remark 2.4.1. Under the assumptions of 2.4, the numerical equivalence coincides with the homological equivalence.

Remark 2.4.2. [3](p.8) proved the assertion for $n = 2r \geq 4$ when m is prime and k is finite under the assumption on the existence of Lichtenbaum complexes $\mathbb{Z}(r)$.

3. The Artin-Tate formula

3.1. Let $k = \mathbb{F}_q$ and X a smooth projective variety over k of dimension n. Put $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let Φ denote the geometric Frobenius of X over k and $P_i(X;T) = \det(1-\Phi^* T; H^i(X_{\bar{k}}, \mathbb{Q}_l))$, where l is a prime different from the characteristic of k. By Deligne, $P_j(X;T) \in \mathbb{Q}[T]$ and independent of l.

3.2. Now we recall the Artin-Tate-Milne formula ([9],Cor.6.4, Prop.6.6, Th.0,1,[10],Th.6.6) in a modified form. We refer to [7] and [10] on the formalism of Lichtenbaum complexes $\mathbb{Z}(r)$.

Define

$$H^i(X_{\bar{k}}, \hat{\mathbb{Z}}(r)) = \prod_l H^i(X_{\bar{k}}, \mathbb{Z}_l(r))$$

and

$$H^i(X,\hat{\mathbb{Z}}(r))=\prod_l H^i(X,\mathbb{Z}_l(r))$$

We define also

$$e^i(r)=T^{r-1,i-r}(X)+T^{r-1,i-r+1}(X)-\sum_{j=0}^{r-1}(r-j)h_W^{j,i-j}(X)\ ,$$

$$\alpha^i(X)=-2T^{r-1,r}(X)+\sum_{j=0}^{r-1}(r-j)h_W^{j,2r-j}(X)$$

and

ρ^r=the multiplicity of q^r as a reciprocal root of $P_{2r}(X;T)=0$.

Let ε^{2r}: $H^{2r}(X,\hat{\mathbb{Z}})\to H^{2r+1}(X,\hat{\mathbb{Z}})$ denote a map defined by the cup-product with $1\ \in\hat{\mathbb{Z}}=H^1(k,\hat{\mathbb{Z}})$.

3.2.1.(Artin-Tate-Milne formula Ⅰ) *Assume that* $i\neq 2r$. *Then* $H^i(X_{\bar{k}},\hat{\mathbb{Z}}(r))^{\Gamma}$ *and* $H^i(X_{\bar{k}},\hat{\mathbb{Z}}(r))_{\Gamma}$ *are finite. Moreover,*

$$P_i(X;q^{-r})=\pm q^{e^i(r)}\frac{|\ H^i(X_{\bar{k}},\hat{\mathbb{Z}}(r))_{\Gamma}|}{|\ H^i(X_{\bar{k}},\hat{\mathbb{Z}}(r))^{\Gamma}|}.$$

In the following assertions, *we assume that* $n\ =\ 2r$.

3.2.2. (Artin-Tate-Milne formula IIa) $H^{2r}(X_{\bar{k}},\hat{\mathbb{Z}}(r))^{\Gamma}_{\mathrm{tors}}$ *and* $H^{2r}(X_{\bar{k}},\hat{\mathbb{Z}}(r))_{\Gamma,\mathrm{tors}}$, *are finite. Moreover, assume that for all l, the action of* Φ^* *on* $H^{2r}(X_{\bar{k}},\mathbb{Q}_l(r))$ *is semi-simple. Then* det (ε^{2r}) *is defined and*

$$\left.\frac{P_{2r}(X;T)}{(1-q^rT)^{\rho^r}}\right|_{T=q-r}=\pm q^{e^{2r}(r)}\det(\varepsilon^{2r})\frac{|\ H^{2r+1}(X,\hat{\mathbb{Z}}(r))_{\mathrm{tors}}|}{|\ H^{2r}(X_{\bar{k}},\hat{\mathbb{Z}}(r))^{\Gamma}_{\mathrm{tors}}|\,|\ H^{2r+1}(X_{\bar{k}},\hat{\mathbb{Z}}(r))^{\Gamma}_{\mathrm{tors}}|}.$$

3.2.3. (Artin-Tate-Milne formula IIb) *Assume that for all l, the action of* Φ^* *on* $H^i(X_{\bar{k}},\mathbb{Q}_l(r))$ *is semi-simple. Then*

$$\left.\frac{P_{2r}(X;T)}{(1-q^rT)^{\rho^r}}\right|_{T=q-r}=\pm\frac{\det(\varepsilon^{2r})}{q^{\alpha^r(X)}}\frac{|\ H^{2r+1}(X,\hat{\mathbb{Z}}(r))_{\mathrm{tors}}|}{|\ H^{2r}(X_{\bar{k}},\hat{\mathbb{Z}}(r))^{\Gamma}_{\mathrm{tors}}|^2}.$$

3.2.4. (Artin-Tate-Milne formula IIc) *Assume that:*

(1) *There exists a Lichtenbaum complex* $\mathbb{Z}(r)$;

(2) *the Tate conjecture holds for r and all l;*

(3) *the cycle map* $CH^r(X)\to H^{2r}(X,\mathbb{Z}(r))$ *is surjective.*

Then.

$$\left.\frac{P_{2r}(X;T)}{(1-q^rT)^{\rho^r}}\right|_{T=q-r}=\pm\frac{|\ \mathrm{Br}^r(X)\ |\ \det[C^r(X)/\mathrm{tors}]}{q^{\alpha^r(X)}|\ C^r(X)_{\mathrm{tors}}|^2}.$$

Here $\mathrm{Br}^r(X) = H^{2r+1}(X, \mathbb{Z}(r))$ *and* $C^r(X)$ *denotes the image of the cycle map* $CH^r(X) \to H^{2r}(X_{\bar{k}}, \hat{\mathbb{Z}}(r))$.

Remark 3.2.5. Milne [9] defined $e^j(r)$ and $\alpha^r(X)$ in a different form:

$$e^i(r) = T^{r-1,i-r} - \sum_{\upsilon(\alpha)<r} (r - \upsilon(\alpha))$$

and

$$\alpha^r(X) = T^{r-1,r+1} - 2T^{r-1,r} + \sum_{\upsilon(\alpha)<r} (r - \upsilon(\alpha)),$$

where the summation is taken over all the reciprocal roots α of $P_r(X;T)=0$ and υ is the p-adic order nomalized by $\upsilon(q)=1$. Now we verify

$$\sum_{\upsilon(\alpha)<r} (r - \upsilon(\alpha)) = \sum_{j=0}^{r-1} (r-j) h_W^{j,i-j}(X) - T^{r-1,i-r+1}(X).$$

Indeed, by the definition,

$$h_W^{j,i-j}(X) = \dim_k H^{i-j}(X, W\Omega_X^j)/(\mathrm{tors}+V) + \dim_k H^{i-j+1}(X, W\Omega_X^{j+1})/(\mathrm{tors}+F)$$
$$+ T^{j,i-j}(X) - 2T^{j-1,i-j+1}(X) + T^{j-2,i-j+2}(X).$$

Therefore,

$$\sum_{j=0}^{r-1} (r-j) h_W^{j,i-j}(X) - T^{r-1,i-r+1}(X)$$
$$= \sum_{j=0}^{r-1} \left\{ (r-j) \dim_k H^{i-j}(X, W\Omega_X^j)/(\mathrm{tors}+V) + (r-j-1) \dim_k H^{i-j}(X, W\Omega_X^j)/(\mathrm{tors}+F) \right\}$$
$$= \sum_{j=0}^{r-1} \left\{ (r-j) \sum_{j \le \upsilon(\alpha) < j+1} (j+1-\upsilon(\alpha)) + (r-j-1) \sum_{j \le \upsilon(\alpha) < j+1} (\upsilon(\alpha) - j) \right\}$$
$$= \sum_{\upsilon(\alpha)<r} (r - \upsilon(\alpha)).$$

Example 3.3. Let X be a smooth complete intersection in a projective space over k. Let $n = \dim X$. Then it is known that

$$h_W^{ij} = h^{ij} \text{ for all } (i,j)$$

and

$$T^{ij} = 0 \text{ if } i+j \neq n.$$

Hence

$$e^n(r) = T^{r-1,n-r+1} - \sum_{j=0}^{r-1} (r-j) h^{j,n-j}$$

and, if $n=2r$,

$$\alpha^r(X) = \sum_{j=0}^{r-1} (r-j) h^{j,2n-j}.$$

Moreover, if $n \neq 2r$,

$$H^{n-1}(X_{\bar{k}}, \hat{\mathbb{Z}}(r))_{\Gamma} \xrightarrow{\sim} H^{n}(X, \mathbb{Z}(r))$$

and

$$H^{n}(X_{\bar{k}}, \hat{\mathbb{Z}}(r))^{\Gamma} = 0;$$

if $n = 2r$,

$$H^{n}(X, \hat{\mathbb{Z}}_l(r)) = H^{n}(X_{\bar{k}}, \hat{\mathbb{Z}}(r))^{\Gamma}_{\mathrm{tors}} = 0, \ C^{r}(X)_{\mathrm{tors}} = 0$$

and

$$H^{n}(X_{\bar{k}}, \hat{\mathbb{Z}}(r))_{\Gamma,\mathrm{tors}} \xrightarrow{\sim} H^{n+1}(X, \hat{\mathbb{Z}}(r))_{\mathrm{tors}}.$$

Remark 3.3.1. Let X be a smooth complete intersection in a projective space of dimension n over a perfect field of characteristic $p>0$. Assume that X is supersingular. Then we have

$$T^{i,n-i} = T^{n-i-2,i+2} = \sum_{k=0}^{i} (i+1-k)h^{k,n-j}$$

for $0 \leq i \leq r-1$ if $n = 2r$ or $2r+1$.

Remark 3.3.2. [3](Prop.8.4) proved the assertions of 3.2 when X is a diagonal hypersurface of Hodge-Witt type.

Example 3.4. Let X be an abelian variety over k. Let $n = \dim X$. Then it is known that

$$h^{ij}_{W} = h^{ij} \text{ for all } (i,j).$$

Hence we obtain

$$e^{i}(r) = T^{r-1,i-r} + T^{r-1,i-r+1} - \sum_{j=0}^{r-1} (r-j)h^{j,i-j}$$

and

$$\alpha^{r}(X) = -2T^{r-1,r} + \sum_{j=0}^{r-1} (r-j)h^{j,2r-j}.$$

Remark 3.5. Here are some easy consequences of the hypotheses, from which Milne [10] deduced the Artin-Tate-Milne formula. Assume that:

(1) there exists a Lichtenbaum complex $\mathbb{Z}(r)$;

(2) $\mathrm{Br}^{r}(X) = H^{2r+1}(X, \mathbb{Z}(r))$ is finite,

Then we have gotten bijections

$$\begin{array}{l} H^{2r}(X, \mathbb{Q}_l/\mathbb{Z}_l(r))_{\mathrm{cotors}} \xrightarrow{\sim} \mathrm{Br}^{r}(X)_{l-\mathrm{tors}} \\ \qquad \downarrow \wr \\ H^{2r+1}(X, \mathbb{Z}_l(r))_{\mathrm{tors}} \end{array}$$

for all l. It follows that, if $n = 2r$, $|\mathrm{Br}^{r}(X)|$ is a square up to 2. Moreover, assume that $CH^{r}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^{2r}(X_{\bar{k}}, \mathbb{Q}_l(r))$ is surjective for some $l \neq p$. Then we have an exact sequence

$$H^{2r}(X_{\bar{k}}, \mathbb{Z}_l(r))_{\mathrm{tors},\Gamma} \to \mathrm{Br}^r(X)_{l-\mathrm{tors}} \to H^{2r+1}(X_{\bar{k}}, \mathbb{Z}_l(r))^{\Gamma}_{\mathrm{tors}}$$

for all l. In particular, if $H^{2r}(X_{\bar{k}}, \mathbb{Z}_l(r))_{\mathrm{tors}} = H^{2r+1}(X_{\bar{k}}, \mathbb{Z}_l(r))_{\mathrm{tors}} = 0$ for all $l \neq p$, then $\mathrm{Br}^r(X)$ is a p-group.

Remark 3.6. Let k be a field and X a smooth projective variety over k of dimension n. Let $2r \leq n$. Assume that the cycle map $CH^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^{2r}(X_{\bar{k}}, \mathbb{Q}_l(r))$ is surjective for some l. Then the numerical equivalence coincides with the homological equivalence up to torsion for algebraic cycles of $X_{\bar{k}}$ of codimension r and $n-r$.

Remark 3.7. Let k be a field of characteristic $p > 0$ and X a smooth projective variety over k of dimension $n = 2r$. Assume that $CH^r(X) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to H^{2r}(X_{\bar{k}}, \mathbb{Z}_l(r))$ is surjective for all $l \neq p$. Then $\det N^r(X)$ is a power of p.

Remark 3.8. Let G be a finite subgroup of $\mathrm{Aut}_k(X)$ and $N = |G|$. Let χ be a $\mathbb{Q}$-character of G. Put $P_i(X, \chi; T) = \det(1-\Phi^* T; H^i(X_{\bar{k}}, \mathbb{Q}_l)^{(\chi)})$. Then $P_i(X, \chi; T) \in \mathbb{Q}[T]$. Moreover, we have gotten a factorization $P_i(X;T) = \prod_{\chi} P_j(X, \chi; T)$, where the summation is taken over all the $\mathbb{Q}$-irreducible characters of G. There is no difficulty to pass the Artin-Tate-Milne formula for X to the pair (X, p_χ) up to the prime factors of N. For example, the case of Fermat varieties is studied in [15],[17], and the case of diagonal hypersurfaces in [3].

References

[1] N.Aoki-On some arithmetic problems related to the Hodge cycles on the Fermat varieties. Math. Ann. 266 (1983) 23-54

[2] T. Ekedahl - Diagonal camplexes and F-gauge structures. Hermann (1986)

[3] F. Gouvêa, N. Yui - Arithmetic of diagonal hypersurfaces over finite fields. Max Planck Institut Preprint MPI 94-36 (1994)

[4] M. Gros, N. Suwa-Application d'Abel-Jacobi p-adique et cycles algébriques. Duke Math. J. 57 (1988) 579-613

[5] L. Illusie-Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. Ecole Norm. Sup. 4^e serie 12 (1979) 501-661

[6] L. Illusie, M. Raynaud-Les suites spectrales associées au complexe de de Rham-Witt. Publ.Math. IHES 57 (1983) 73-212

[7] S. Lichtenbaum-The construction of weight-two arithmetic cohomology. Invent. Math. 88 (1987) 183-215

[8] J-S. Milne-On a conjecture of Artin-Tate. Ann. of Math. 102 (1975) 517-533

[9] J-S. Milne-Values of zeta functions of varieties over finite fields. Amer. J. Math. 108 (1986) 297-360

[10] J-S. Milne-Motivic cohomology and values of zeta functions. Compositio Math. 68 (1988) 59-102

[11] T. Shioda-The Hodge conjecture and the Tate conjecture for Fermat varieties. Proc. of Japan Acad. 55 Ser. A (1979) 111-114

[12] T. Shioda-The Hodge conjecture for Fermat varieties. Math. Ann. 245 (1979) 175-184

[13] T. Shioda-Some observations on Jacobi sums. Advanced Studies in Pure Math. 12, North Holland-Kinokuniya (1987) 119-135

[14] T. Shioda, T. Katsura-On Fermat varieties. Tohoku Math. J. 31 (1979) 97-115

[15] N. Suwe-Fermat motives and the Artin-Tate formula Ⅰ,Ⅱ. Proc. Japan Acad. 67 (1991) 104-107, 135-138

[16] N. Suwa-Hodge-Witt cohomology of complete intersections. J. Math. Soc. Japan 45 (1993) 295-300

[17] N. Suwa N. Yui-Arithmetic of Fermat varieties I: Fermat motives and p-adic cohomologies. MSRI Preprint Series, Berkeley (1988)

[18] J. Tate-Algebraic cycles and poles of zeta functions. Arithmetic algebraic geometry, Harper and Row (1965) 93-110

[19] J. Tate-Endomorphisms of abelian varieties over finite fields. Invent. Math. 2 (1966) 133-144

[20] J. Tate-On the conjecture of Birch and Swinnerton-Dyer and a geometric analog. Sém. Bourbaki exposé 306, in Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam (1986) 189-214

DEPARTMENT OF MATHEMATICS, CHUO UNIVERSITY, 1-13-27 KASUGA,
BUNKYO-KU, TOKYO 112-8551, JAPAN