Properties of a Complex Logistic Equation and Its Application to Economic Dynamics

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1. Introduction

It is well-known that the solution of the real logistic equation with the initial condition is said to be the logistic curve (See Section 3. Remark). This logistic curve represents many economic phenomena, for instance, the development of a saturation level of consumer durables in product markets (See Figure 3.2 and Figure 4.1).

To shed some light on economic dynamic phenomena, we will define a "complex" logistic equation and obtain the explicit solution of this equation. We then consider the absolute value squared of this solution to investigate a change of population size and environment. Futhermore, we discover the properties of this solution in Section 2. In Section 3, we make use of many examples to show the graph of the absolute value squared of the solution. By this way, we will show the dynamic relationship between economic activities and the phenomena induced by the solution of the complex logistic equation in Section 4. More precisely, we remember the Diffusion Theory developed by E.M.Rogers (1962, 5th ed. 2003). This theory means that the movement of the saturation level of consumer durables shares the basic dynamic behavior of a typical logistic curve. Second, we remember the Product Life Cycle Theory developed by R.Vernon and L.T.Wells (1966). In this paper, we construct logistic curves of movements of the production level of steel pipes and pig iron in Japan by using the solution of the complex logistic equation. Moreover, we illustrate logistic curves of the debt accumulation in Japan and Germany by using our method.

It is clear that we can illustrate many examples of economic phenomena by our method.

2. Complex logistic equation

Let α, T and z be complex numbers. We have real representations of each complex numbers:

$$\alpha = a + b\sqrt{-1},\ T = t + s\sqrt{-1},\ \text{and } z = x + y\sqrt{-1}$$

We assume α is non-zero constant. We define a complex differential equation :

$$\frac{dz}{dT} = \alpha z(1-z)\ . \tag{2.1}$$

We call (2.1) the complex logistic equation. It is not so apparent that this equation (2.1) obtained by extending the real logistic equation of $\frac{dx}{dt} = ax(1-x)$, includes representations of two phenomena, for instance, a relation of a change of population size and environment. However, we obtain the explicit solution of (2.1) with the initial condition $z_0 \neq 0, 1$:

$$Z(T) = \frac{1}{\frac{1-z_0}{z_0} \cdot e^{-\alpha T} + 1}\ . \tag{2.2}$$

Here we remember the exponential expression of $\frac{1-z_0}{z_0} = e^{u_0 + v_0\sqrt{-1}}$, where $u_0 = \log_e \left|\frac{1-z_0}{z_0}\right|$ and $v_0 = \arg\left(\frac{1-z_0}{z_0}\right)$. We note that $\left|\frac{1-z_0}{z_0}\right|$ is the absolute value of $\frac{1-z_0}{z_0}$ and $\arg\left(\frac{1-z_0}{z_0}\right)$ is the argument of $\frac{1-z_0}{z_0}$.

After the coordinate transformation $\alpha T = U = u + v\sqrt{-1}$, i.e. $u = at - bs$ and $v = bt + as$, (2.2) is rewritten by U as follows:

$$\tilde{Z}(U) = \frac{1}{e^{(u_0-u)} \cdot e^{(v_0-v)\sqrt{-1}} + 1}\ . \tag{2.3}$$

In the following analysis, it is convenient to define the absolute value squared of $\tilde{Z}(U)$:

$$\left|\tilde{Z}(U)\right|^2 = \frac{1}{e^{2(u_0-u)} + 2e^{(u_0-u)} \cdot \cos(v_0 - v) + 1}\ . \tag{2.4}$$

Here, it becomes definitely clear that the complex logistic equation represents phenomena of the change of population size and environment.

We envisage that, regerding the two real variables u and v in equation (2.4), the former represents a change of population size and the latter presents that environmental variable such as the changes of the seasons and business-cycles, etc.

To find a relationship between u and v, we take three points B_1, B_2 and B_3 on U-plane. When a point U moves along a line segment $\overline{B_1B_2}$ or a polygonal line $\overline{B_1B_2} \cup \overline{B_2B_3}$, we define $\varphi(\theta)$ as the absolute value squared of $\tilde{Z}(U)$: $\left|\tilde{Z}(U)\right|^2$. $\overline{B_1B_2}$ and $\overline{B_1B_2} \cup \overline{B_2B_3}$ are parametrized by θ as follows:

Figure 2.1 The polygonal line $\overline{B_1B_2} \cup \overline{B_2B_3}$

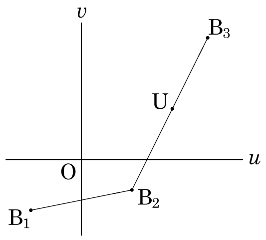
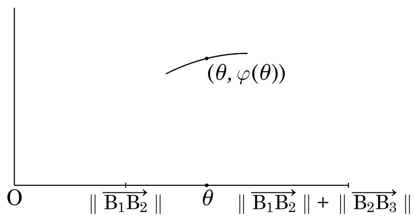


Figure 2.2 The logistic curve on $\overline{B_1B_2} \cup \overline{B_2B_3}$



(2.5) $$\overline{B_1B_2} = \left\{ \overrightarrow{OU} = \theta \cdot \frac{1}{L_1} \cdot \overrightarrow{B_1B_2} + \overrightarrow{OB_1} \,\middle|\, 0 \leqq \theta \leqq L_1 \right\}$$

(2.6) $$\begin{aligned} \overline{B_1B_2} \cup \overline{B_2B_3} &= \left\{ \overrightarrow{OU} = \theta \cdot \frac{1}{L_1} \cdot \overrightarrow{B_1B_2} + \overrightarrow{OB_1} \,\middle|\, 0 \leqq \theta \leqq L_1 \right\} \\ &\cup \left\{ \overrightarrow{OU} = (\theta - \| \overrightarrow{B_1B_2} \|) \cdot \frac{1}{L_2} \cdot \overrightarrow{B_2B_3} + \overrightarrow{OB_2} \,\middle|\, L_1 \leqq \theta \leqq L_1 + L_2 \right\} \end{aligned}$$

where O is the origin on U-plane, and $L_1 = \| \overrightarrow{B_1B_2} \|$, $L_2 = \| \overrightarrow{B_2B_3} \|$

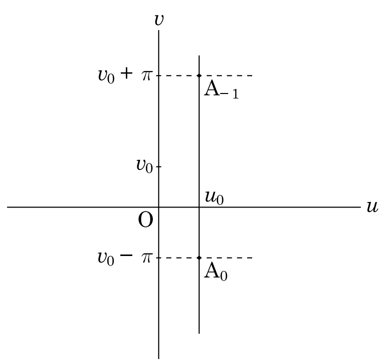
Then we can draw the graph of $\varphi = \{(\theta, \varphi(\theta))\}$ and in this paper we call this graph the logistic curve.

Under the above notation, the line segment $\overline{B_1B_2}$ or the polygonal line $\overline{B_1B_2} \cup \overline{B_2B_3}$ is defined with graph $\{(\theta, \varphi(\theta))\}$ depicted by numerical analysis which simulate the movements of the data used in Section 4.

Finally, we discover properties of the solution $\tilde{Z}(U)$ such that (1) periodic property: $\tilde{Z}(U + 2\pi\sqrt{-1}) = \tilde{Z}(U)$ and (2) existence of pole points : We denote by A the set of pole points of $\tilde{Z}(U)$ which is defined by

(2.7) $$A = \{ u_0 - u = 0 \text{ and } v_0 - v = \pi + 2\pi k \mid k \in \mathbf{Z} \} = \{ A_k = (u_0, v_0 - \pi - 2\pi k) \mid k \in \mathbf{Z} \}.$$

Figure 2.3 U-plane ((u, v)-plane)



Moreover, we will illustrate some graphs $\{(\theta, \varphi(\theta))\}$ such as Section 3.

3. Some examples of the graph of $\varphi(\theta)$

In this section, we study the graph of $\varphi(\theta)$ by given concrete examples. In all examples, we fix complex numbers:

$$\alpha = 1.0 + 0.5\sqrt{-1} \quad \text{and} \quad z_0 = 0.2 + 0.3\sqrt{-1}$$

Then we get the real numbers $u_0 = 0.86$ and $v_0 = -1.34$ by the solution of the equation:

$$\frac{1-z_0}{z_0} = e^{u_0 + v_0\sqrt{-1}}$$

By (2.7), the pole points A_0 and A_{-1} are $A_0 = (0.86, -4.48)$ and $A_{-1} = (0.86, 1.80)$ respectively.

Example 3.1 Let the points B_1 and B_2 be $(-2.0, -1.34)$ and $(8.0,\ -1.34)$ respectively. The two segments $\overline{B_1B_2}$ and $\overline{A_0A_{-1}}$ perpendicularly intersect at the point $\dfrac{A_0 + A_{-1}}{2}$. Then we write the graph of $\varphi(\theta)$ by moving the point U along the line segment $\overline{B_1B_2}$ as follows:

Figure 3.1 The line segment $\overline{B_1B_2}$

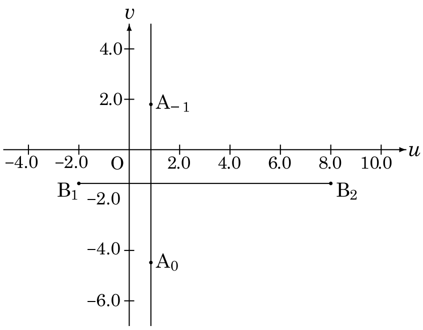
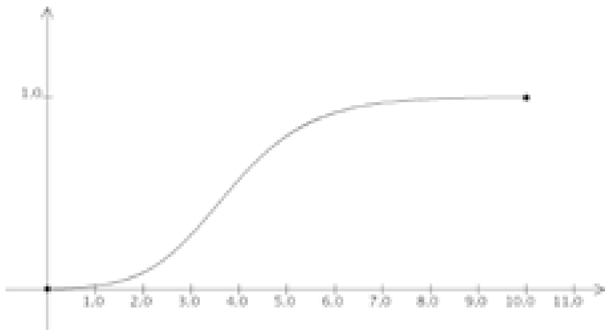


Figure 3.2 The logistic curve on $\overline{B_1B_2}$



Remark Remember the real logistic equation and its solution.

$$\frac{dx}{dt} = x(1-x)$$

The solution with the initial condition $x_0 = 0.2$ is $x(t) = \dfrac{1}{4 \cdot e^{-t} + 1}$.
This graph is ordinarily said to be the logistic curve.

Example 3.2 Let the points C_1 and C_2 be $(-2.0, -3.5)$ and $(8.0, -3.5)$ respectively. Consider the line segment $\overline{C_1C_2}$ such that $\overline{C_1C_2}$ is parallel to $\overline{B_1B_2}$ and is nearer to the point A_0 than $\overline{B_1B_2}$. Then we draw the graph of $\varphi(\theta)$ on the line segment $\overline{C_1C_2}$ as follows:

Figure 3.3 The line segment $\overline{C_1C_2}$

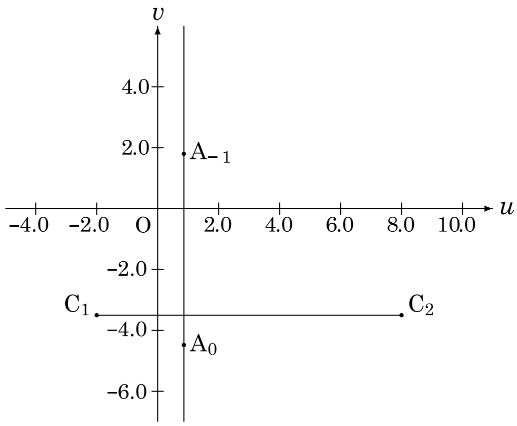
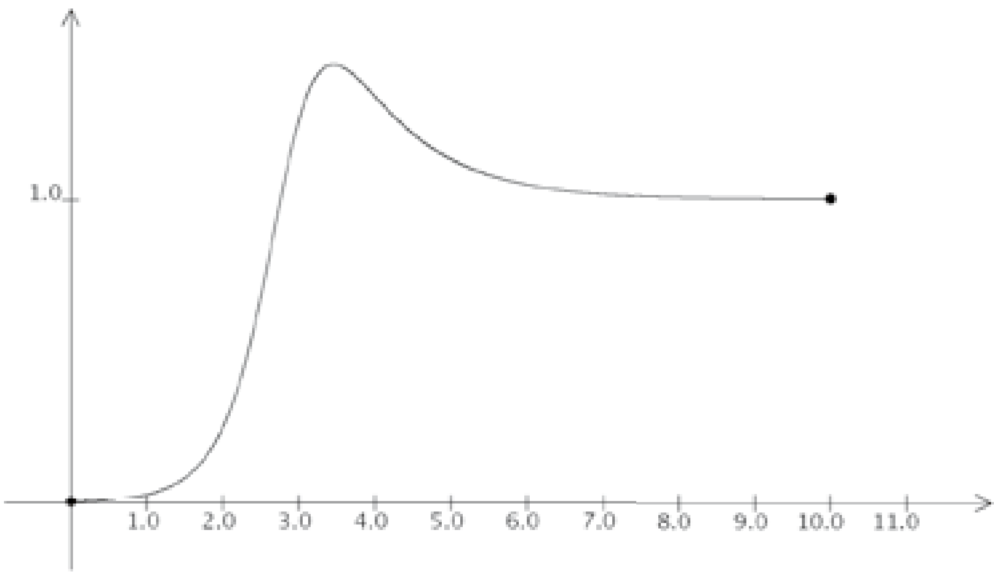


Figure 3.4 The logistic curve on $\overline{C_1C_2}$



Example 3.3 Let the points D_1 and D_2 be $(-2.25, -6.26)$ and $(5.64, 1.90)$ respectively. Then we draw the graph of $\varphi(\theta)$ on the line segment $\overline{D_1D_2}$ as follows:

Figure 3.5 The line segment $\overline{D_1D_2}$

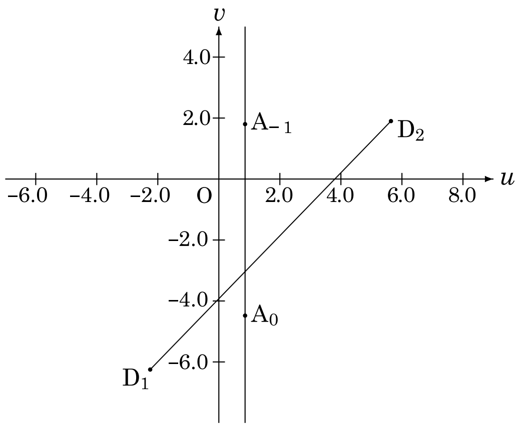
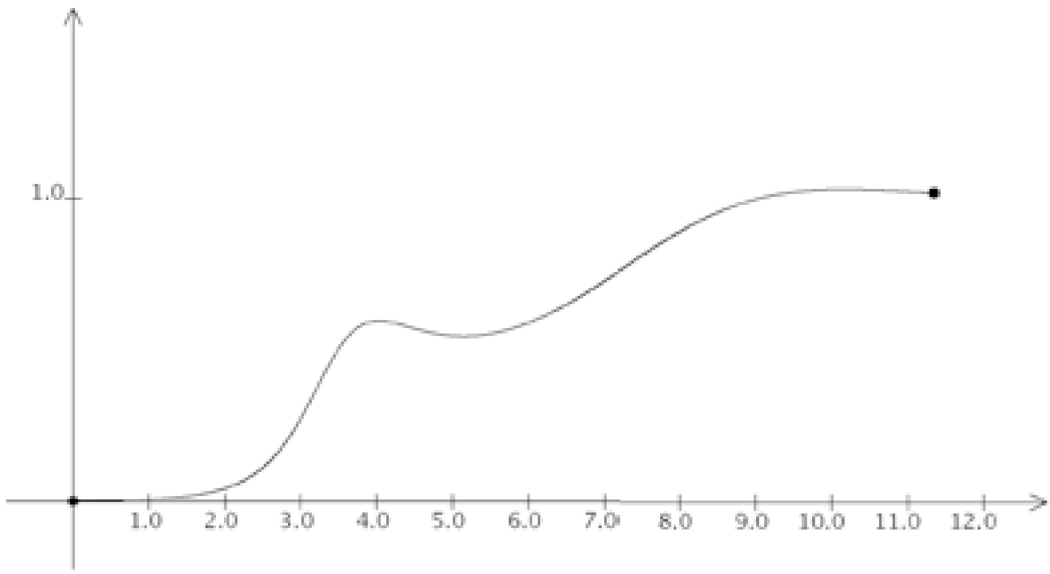


Figure 3.6 The logistic curve on $\overline{D_1D_2}$



Example 3.4 Let the points E_1, E_2 and E_3 be $(-2.24, -6.24)$, $(2.21, -1.41)$ and $(2.54, 4.78)$ respectively. We consider the polygonal line $\overline{E_1E_2} \cup \overline{E_2E_3}$. We study $\varphi(\theta)$ when the point U moves along $\overline{E_1E_2} \cup \overline{E_2E_3}$.

Figure 3.7 The polygonal line $\overline{E_1E_2} \cup \overline{E_2E_3}$

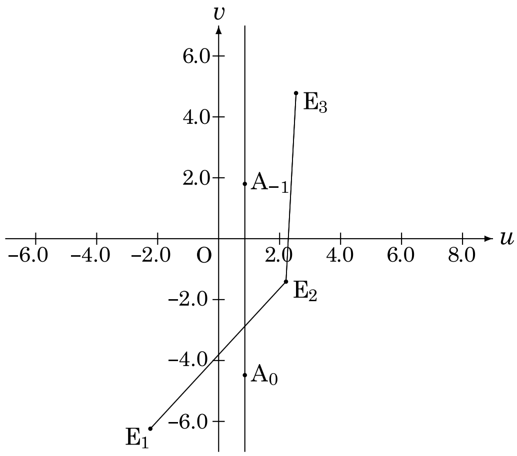
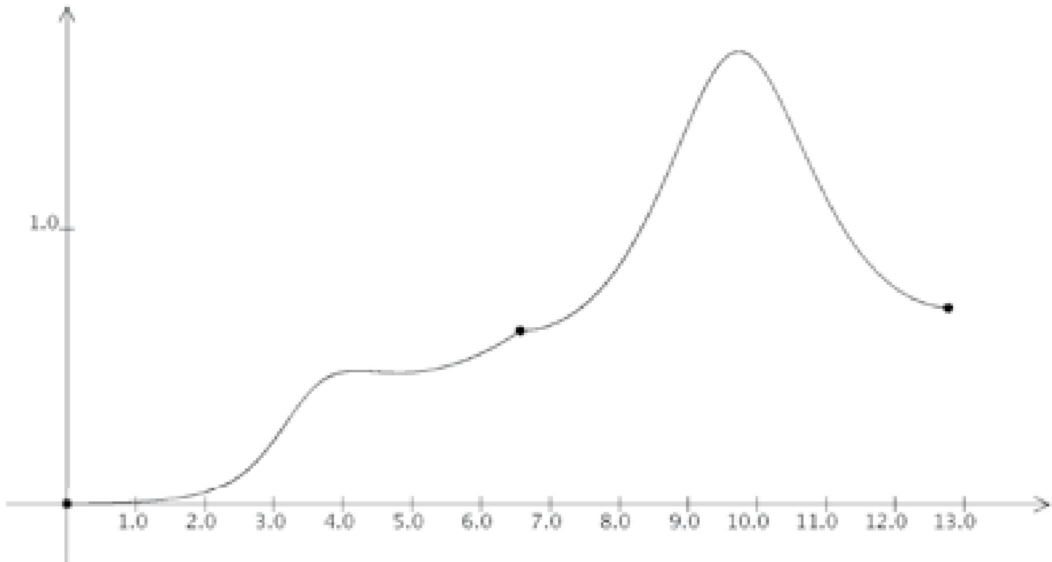


Figure 3.8 The logistic curve on $\overline{E_1E_2} \cup \overline{E_2E_3}$



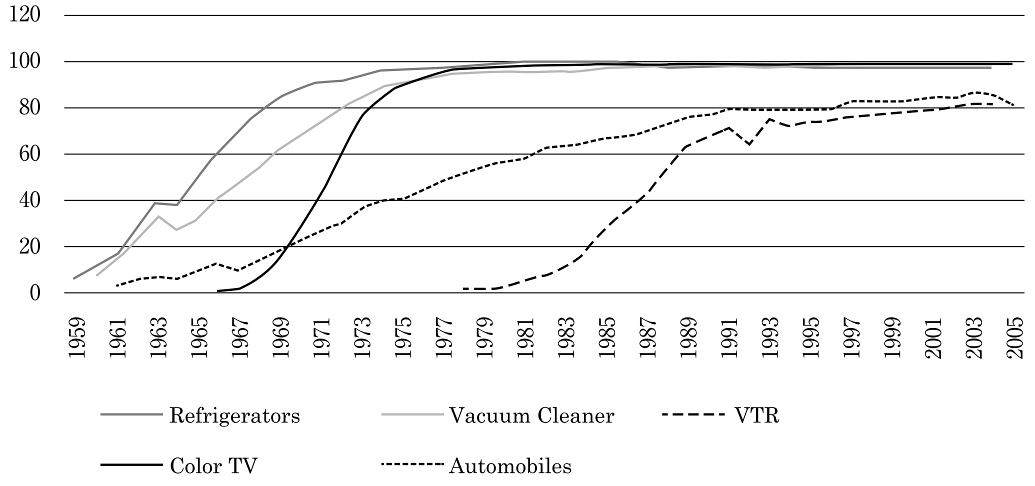
4. Dynamic behavior and movements of economic variables and its expressions with the complex logistic equation

In this section, we will consider the dynamic behavior and movements of economic variables which are closely related to the dynamics induced in the last section. Especially we will draw attention to Figure 3.2, 3.4, 3.6 and 3.8 as typical examples and study the dynamic behavior of the economy that lies behind the movements of the variables. In this way, we will show the similarities of the dynamics that underlie economic activities and the phenomena induced by our complex logistic equation.

4.1 Diffusion Theory

Figure 4.1 shows the developments of the saturation level of consumer durables in Japan. The movements of the saturation level of these five consumer durables share the basic dynamic behavior of the ordinal logistic curve shown in the last section. The basic dynamics lay behind the development of the mature levels are explained by the Diffusion of Innovation Theory by E.M.Rogers (1962, 5th ed. 2003).

Figure 4.1 Saturation Level of Consumer Durables in Japan



(Cabinet Office, Consumer Confidence Survey, 1960-2006.)

Diffusion Theory, developed by Rogers in 1962, explains how, over time, an idea or product gains momentum and diffuses (or spreads) through a specific population or social system. Adoption means that a person perceives something different than what they had previously i.e., purchase or use a new product, acquire and perform a new behavior, etc.

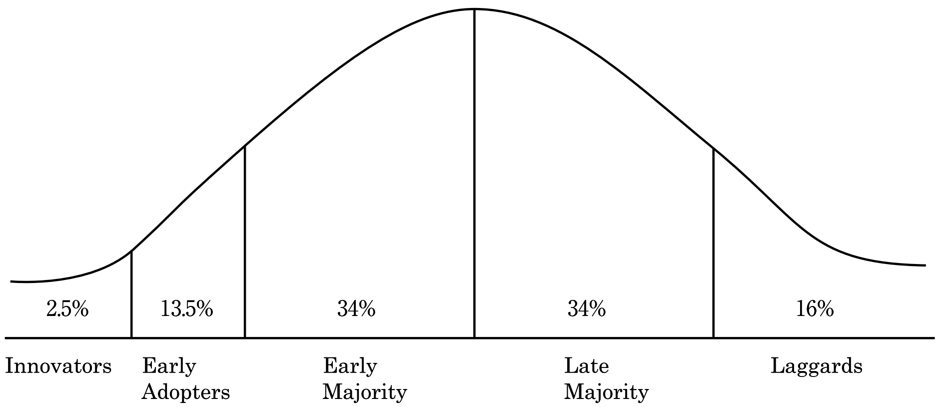
According to Rogers, adoption of a new idea, behavior, or product does not happen simultaneously; rather it has a process whereby there are some people who are more apt to adopt the innovation than others.

There are five established adopter categories.

1. Innovators: People who want to be the first to try the innovation or innovative products. They are venturesome and interested in new ideas. These people are very few and very willing to take risks, and are often the first to develop new ideas.
2. Early Adopters: Opinion leaders who enjoy leadership roles, and embrace change opportunities. They are very sensitive to the need to change and so are very comfortable adopting new ideas.
3. Early Majority: These people are rarely leaders, but they do adopt new ideas before the average person. It is said, they need to see evidence that the innovation or innovative product works before they are willing to adopt it.
4. Late Majority: These people are skeptical of change, and will only adopt an innovation or innovative product after it has been tried by the majority of the society.
5. Laggards: These people are very conservative and bound by their custom or tradition. They are very skeptical of change and are the hardest group to persuade to try innovation or innovative products.

According to Rogers (1962, 5th ed. 2003), the shares of these categories differ from each other and they are shown in Figure 4.2.

Figure 4.2 Adopter Categories

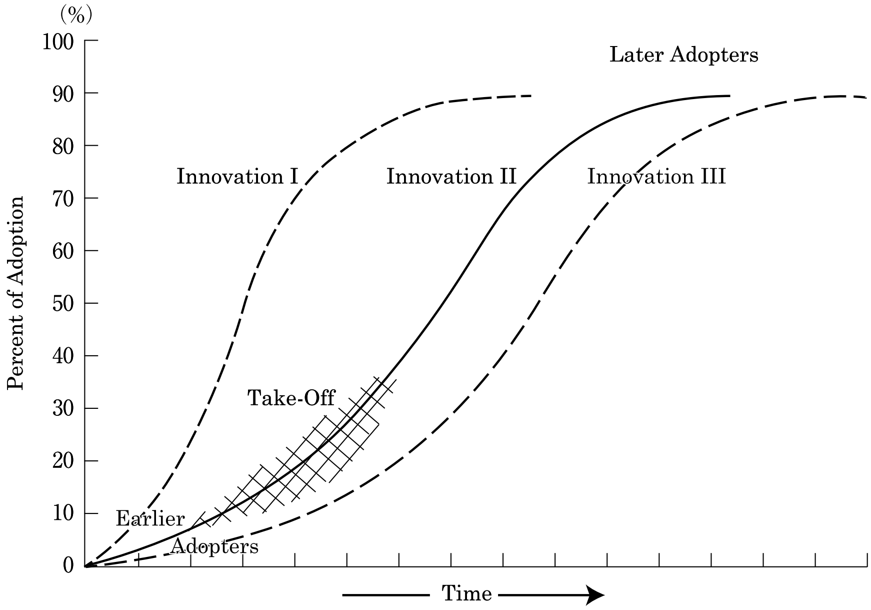


(Rogers (1962, 5th ed. 2003))

Rogers defined diffusion as the process by which an innovation is communicated through certain channels over time among the members of a social system. The four main elements are the innovation, communication channels, time, and the social system.

He also pointed out that the duration of the process for each five categories are not always same and sometimes it takes longer time to change over from one category to the next one. Therefore, times needed to attain certain level of maturity differ from one innovation to others (Figure 4.3).

Figure 4.3 The Diffusion Process

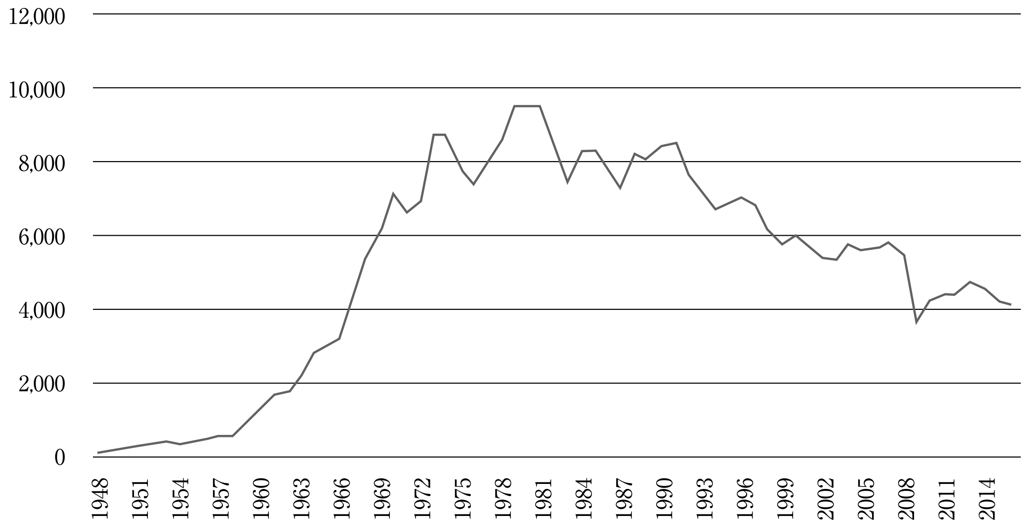


(Rogers (1962, 5th ed. 2003))

4.2 Product life cycle theory

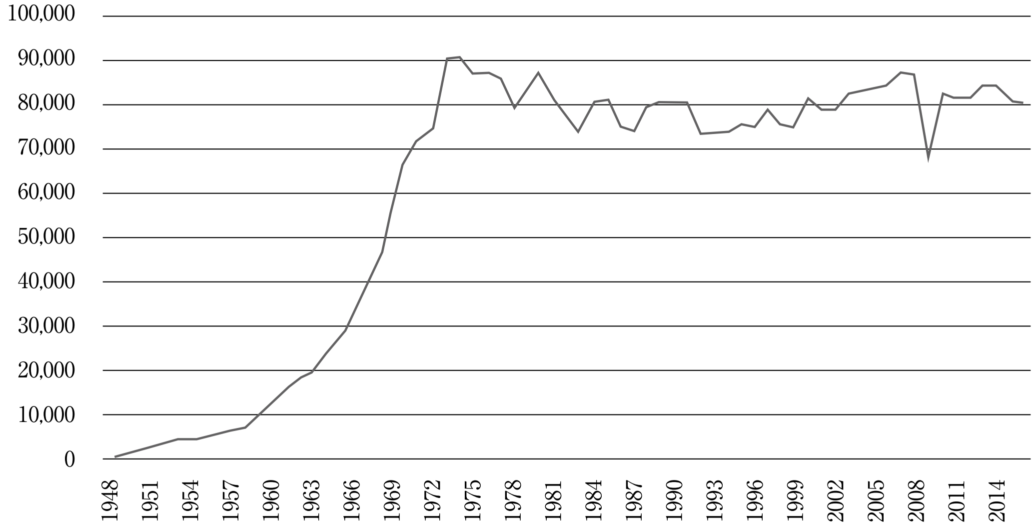
Figure 4.4 and 4.5 show the movements of the level of production of steel pipes and pig iron in Japan. The movements of the level of production have a special feature which is common to these two products. The dynamics that underlie that is explained by the Product Life Cycle Theory.

Figure 4.4 Steel Pipe Production in Japan



(Ministry of Economy, Trade and Industry, Current Survey of production, 1960-2018.)

Figure 4.5 Pig Iron Production in Japan



(Ministry of Economy, Trade and Industry, Current Survey of production, 1960-2018.)

The Product Life Cycle Theory is still a widely used model to explain life cycle fluctuations of the production level of typical products in economics and marketing. It states that Products have a life cycle and older, long-established products become less popular and gradually disappear. According to R. Vernon (1979), each product has a certain life cycle that begins with its development and introduction to the market, followed by rapid growth of production and demand, and ends with its decline.

4.2.1 Product life cycle stages

According to Vernon there are four very clearly defined stages in a product's life cycle: **introduction**, **growth**, **maturity** and **decline**. The length of a Product Life Cycle stage varies for different products, one stage may last some weeks while others even last decades. The life span of a product and how fast it goes through the entire cycle depends on for instance market demand, its diffusion process, durations and how marketing instruments are used.

The introduction stage: When a company has developed a product successfully, it will be introduced into the national and international market. In order to create demand, investments are made to enhance consumer awareness and promotion of the new product in order to increase sales. At this stage, as cost of R&D, consumer testing and marketing is high, profits are low and there are only few competitors. When more items of the product are sold, it will enter the next stage automatically.

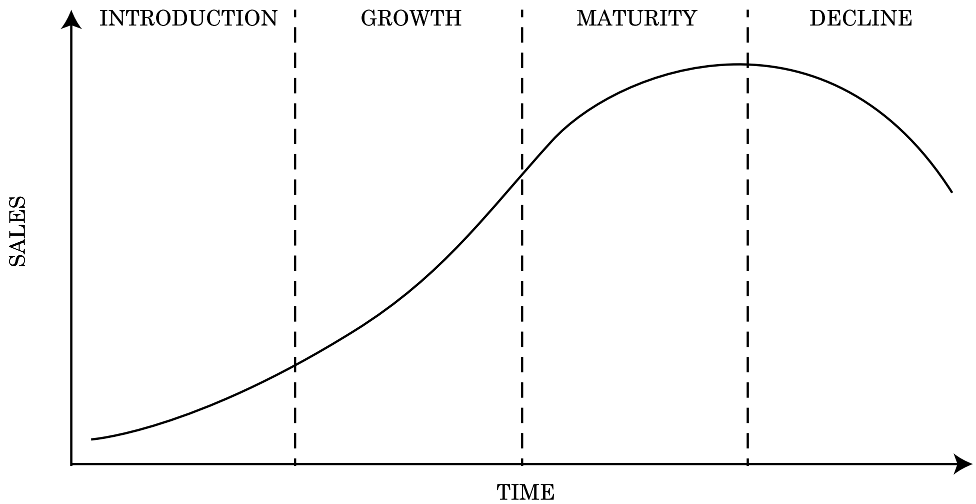
The growth stage: In this stage, the strong growth of demand for the product increases its sales. As a result, economy of scale decreases the cost of production and high profits are generated from the growing sales. The product becomes widely known, and competitors will enter the market with their own version of the product. Usually, they offer the product at a much lower sales price because of follower's benefits. To attract as many consumers as possible, the company that developed the original product will still increase its promotional spending. When many potential new customers have bought the product and the rate of its diffusion reaches certain level, it will enter the next stage.

The maturity stage: In this stage, the product is widely known and is bought by many consumers. Competition is intense and a company will make much effort to maintain a stable market share. This is why the product is sold at record low prices. Also, the company will start looking for other commercial opportunities such as improvements or innovations to the product and the production of by-products. Furthermore, consumers will also be encouraged to replace their current product with a new one. There is fear of decline of the product and therefore every step will be taken in order to boost sales. The marketing and promotion costs are therefore very high in this stage.

The declining stage: At some point, however, the market becomes saturated and the product is no longer sells well and becomes unpopular. This stage can occur as a natural result but can also be stimulated by the introduction of new and innovative products. Despite its decline in sales, it may still be possible for companies to maintain some profits by using less-expensive technology of production. Companies usually continue to offer the product as a service to their loyal customers.

By passing through all these four stages, the changes of the sales form a logistic growth curve shown in Figure 4.6.

Figure 4.6 Four stages in a product's life cycle



(Vernon (1979))

Figure 4.7 U-plane for Steel Pipe Production

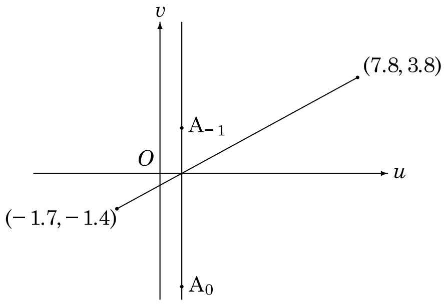


Figure 4.8 The logistic curve for "Steel Pipe Production in Japan"

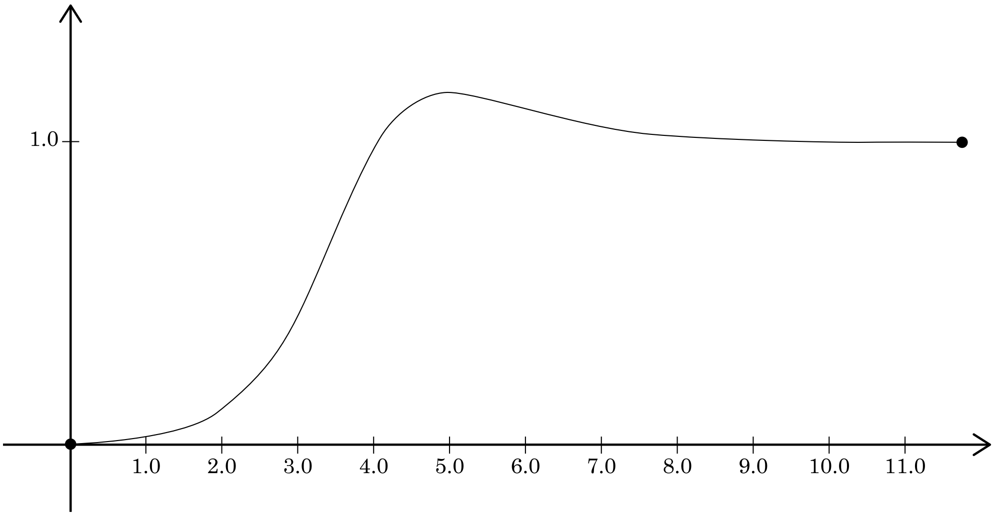


Figure 4.9 U-plane for Pig Iron Production

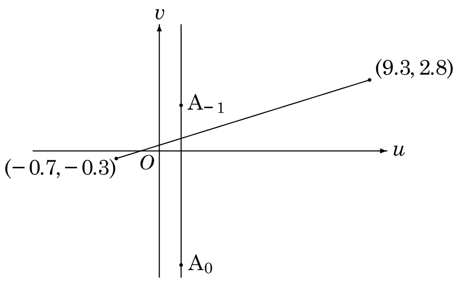
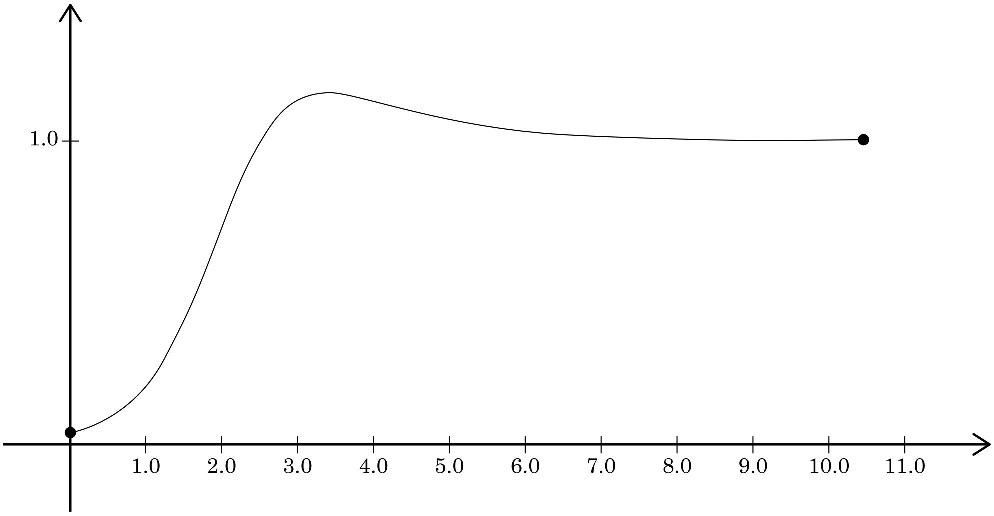


Figure 4.10 The logistic curve for "Pig Iron Production in Japan"



First, it is just a myth that every product has to go through each of the four stages of the product life cycle. There are products that never get beyond the introduction stage, whereas other products remain in the maturity stage for a considerable length of time.

The duration of each stage depends on demand, production costs and revenues. Low production costs and high demand will ensure a longer product life. When production costs are high and there is low demand for the product, it will not be offered on the market for a long time and, eventually, it will be withdrawn from the market via the decline stage.

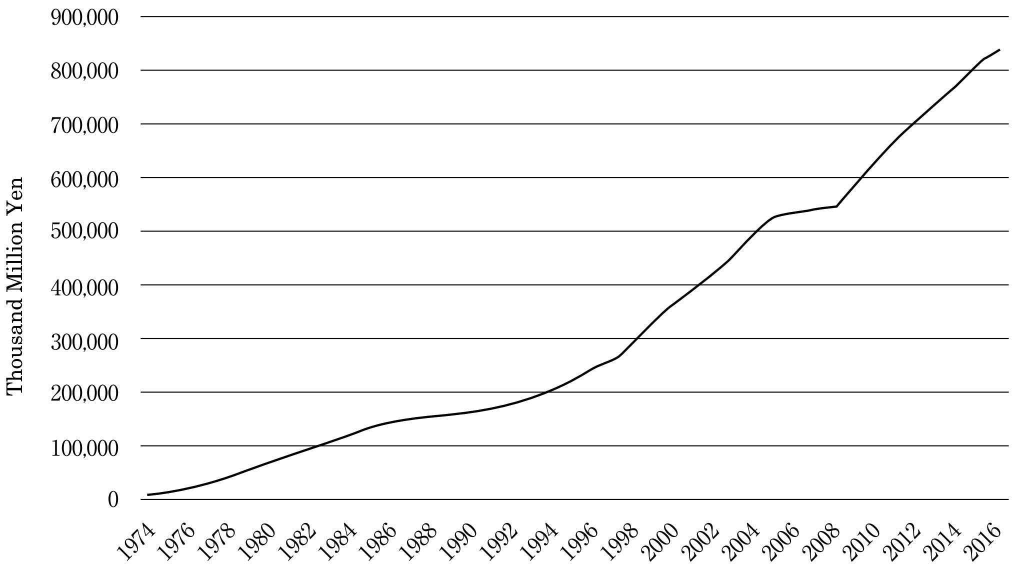
4.3 Debt Accumulation in Japan

Figure 4.11 shows the accumulation of Japanese debt from 1974 to 2017. As we can see, it exhibits "three stages of rapid accumulation" starting from 1974 to 1986, from 1994 to 2005, and from 2009 to 2017.

To see the reason for these swift increases of outstanding public debt, we will examine the economic fluctuations and the management of Japanese macro-economic policy. Japanese government deficit after world war first become a substantial problem after the first and second oil crises. In the decreases of the tax revenue because of the serious recession, Japanese government took the action of demand stimulating fiscal policy by issuing substantial debt (from 1974 to 1985).

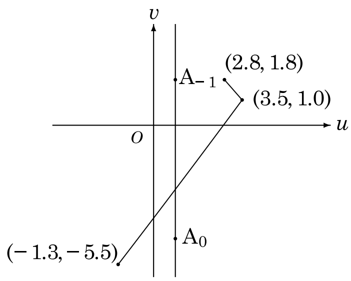
The second steep increase of outstanding debt started in the recession caused by the collapse of the Japanese bubble (after 1994). Especially, a rapid increase of debt issue arose after the financial crisis of Asian countries (after 1997 to 2004) which caused the swift increase of the debt accumulation after 1997. The third rapid increase of the debt accumulation was caused by the large scale demand stimulating policy taken in the world economic

Figure 4.11 Debt Accumulation in Japan



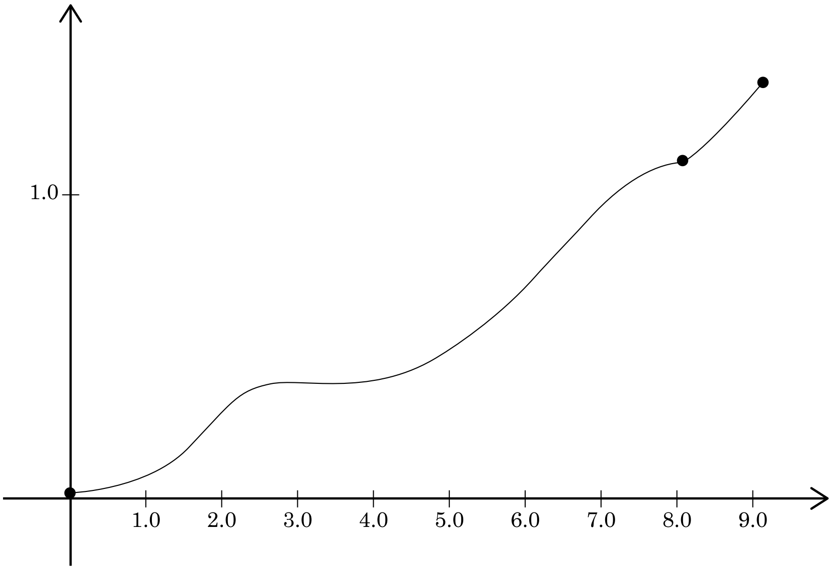
(OECD, National Account Statistics, 1997-2019.)

Figure 4.12 *U*-plane for Debt Accumulation in Japan



depression caused by the collapse of the housing bubbles in the U.S.A. In 2009, Japanese government launched a large scale fiscal policy finance by issuing substantial debt which amounted to more than 5 trillion a year.

Figure 4.13 The logistic curve for "Debt Accumulation in Japan"



4.4 Debt Accumulation in Germany

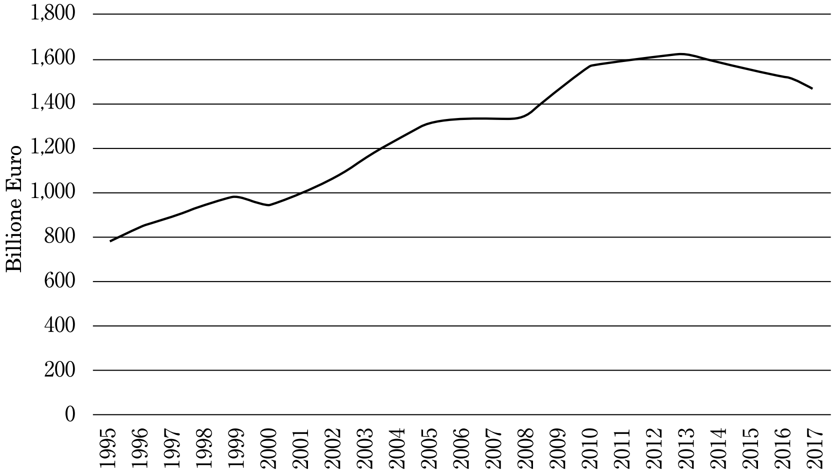
Figure 4.14 shows the outstanding debt of Germany from 1995 to 2017. As we can see, it exhibits "three stages of swift accumulation" starting from 1995 to 1999, from 2001 to 2005, and from 2009 to 2010.

The major reasons for these swift increases of outstanding public debt are also explained by economic fluctuations and the management of German macro-economic policy. German government deficit after the union of West and East Germany came to be a substantial problem in the mid 1990ies. The increases of the expenditure of financial aid are observed for recovering and reconstructing former East Germany Area.

The second steep increase of outstanding debt started in the recession caused by the Asian financial crisis (after 1998), which caused the swift increase of the debt accumulation after 2000. The third rapid increase of the debt accumulation was caused by the large scale demand stimulating policy taken in the world economic depression caused by the collapse of the housing bubbles in the U.S.A. In 2009, German government launched a largest scale fiscal policy finance by issuing substantial debt which amounted to more than 500 Billion Euro a year.

In 2009, the German government decided to reduce the yearly fiscal deficit to below 0.35% of GDP until 2016, and has decreased its outstanding debt by more than 600 billion so far.

Figure 4.14 Debt Accumulation in Germany



(Ministry of Finance, Budget Statistics, 1975-2019.)

Figure 4.15 U-plane for Debt accumulation in Germany

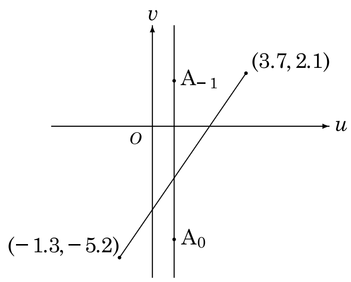
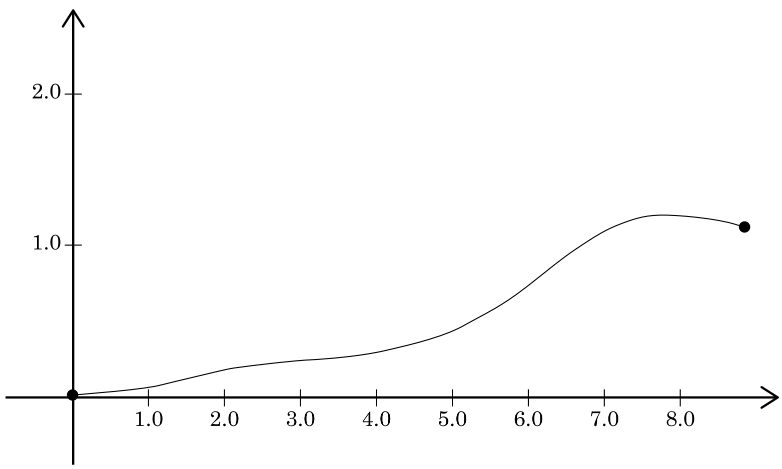


Figure 4.16 the logistic curve for "Debt accumulation in Germany"



5. Conclusion

In this paper, we defined a complex logistic equation and investigated the properties of its solution. The most interesting characteristic of the solution was that it has countable infinite pole points. We considered the square of the absolute value of the solution. Then in the neighborhood of the pole points, the square value of the solution perceives sudden and sharp changes. By using these properties we discovered typical phenomena in economy such as production life cycle curves (logistic growth curve) and accumulation curves of the public debt in Japan and Germany.

We will illustrate other examples of economic phenomena by our method.

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