CHUO UNIVERSITY

DOCTORAL THESIS

Estimation of Value at Risk and Conditional Value at Risk

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Declaration of Authorship

I, Andres Mauricio Molina Barreto , declare that this thesis titled, "Estimation of Value at Risk and Conditional Value at Risk" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment."

Carl Friedrich Gauss

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Abstract

Faculty of Commerce

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Doctor of Philosophy in Commerce

Estimation of Value at Risk and Conditional Value at Risk

by Andres Mauricio Molina Barreto

Several investors and operation researchers have been interested in modeling the risk of financial operations. VaR estimates the risk that can be taken from a financial position over a specific horizon of time. It is used specially to manage portfolio risk. However, there is strong empirical evidence of problems when VaR is calculated. This evidence shows that financial returns have a heavy tail and excess of kurtosis, demonstrating that portfolio return is not always normal.

On other hand, VaR gives little importance to extreme losses. Assuming elliptical distribution for the losses leads VaR to be a coherent risk measure. Generally, VaR is not a coherent risk measure because it does not satisfies the sub-addivity properties. Conditional Value at Risk (CVaR) was introduced as a coherent risk measure. CVaR is a function of VaR defined as the conditional expected value of losses that exceed VaR. This measure has desirable properties of convexity and sub-addivity. A parametric approach is used to obtain analytical expressions for computing these risk measures.

Based in previous works, estimating the VaR and CVaR using different mixtures of probability distributions is desired. This thesis is concerned with the estimation of VaR for the portfolio problem. The considered portfolio is formulated as a linear combination of several random variables. It is noted that VaR is commonly defined for a single random variable. The innovative point of this research is that these random variables are not necessarily assumed to be independent, but may have a possible nonlinear relationship. Although the assumption of independence is very often used in financial analysis, it is believed that many phenomena show some dependence features. To deal with nonlinear dependence, Copula functions methods are used.

The main focus is to analyze previous approaches of estimation of VaR and CVaR for the portfolio problem with numerical experiments and compare these with the alternative of Copulas method with normal mixture distributed margins. These empirical studies show the effectiveness of copula-based methods compared with some benchmarks methodologies.

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Contents

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List of Abbreviations

VaR	Value at Risk
CVaR	Conditional Value at Risk
ES	Expected Shortfall
pdf	Probability distribution function
edf	Empirical distribution function
ARMA	Auto-regressive Moving Average
GARCH	Generalized Auto-regressive Conditional Heteroscedastic
EM	Expectation-maximization
MVaR	Multivariate Conditional Value at Risk
CCVaR	Copula Conditional Value at Risk

List of Symbols

S	Stock price	Monetary units
X	Return process	No units

Dedicated to my parents

Chapter 1

Introduction

1.1 Introduction

Several crises around the world have produced disastrous results for many financial institutions impacting consumers as well as entire nations. To name a few, the subprime mortgage crisis in the USA around 2008, the European debt crisis since late 2009, or the recent global situation caused by COVID-19 pandemic. As a result of that, the study of risk caused by unexpected movements in the financial markets increasingly important.

From investors to operation researchers, people have been interested in modeling the risk of financial operations. The problem of measuring this risk has been widely studied. In this frame, it was proposed to measure financial risk in order to prevent huge losses, or at least to protect the investors from losing all their investment. As a result of this, the Basel Committee on Banking Supervision (BCBS) was established in 1988, under the auspices of the Bank for International Settlements (BIS), which stated recommendations to banks and financial institutions for minimum capital requirements. In 1993, JPMorgan proposed a market risk measure in its RiskMetrics methodology. It was based on an optimization portfolio model by Markowitz (1952) which tries to maximize the profit given a preset risk level. This risk measure is known as Value at Risk (VaR). This measure has become a standard in all industry thanks to its easy understanding.

VaR estimates the risk that can be taken from a financial position over a specific horizon of time. It is used specially to manage portfolio risk. Because of its simplicity and readiness of use, VaR is now well recognized as one of the principal risk measures in financial risk management. It is defined as the worst loss over a target horizon such that there is a low, pre-specified probability that the actual loss will be larger (Jorion et al., 2010). Theoretical study as well as various practical estimation methods have deepened its computation and properties. In 1993, Basel Committee stated that financial entities may adopt its own proprietary models for VaR. The usual calculation of VaR includes historical, analytical method and Monte Carlo simulation. However, there is strong empirical evidence of problems when VaR is calculated with these methods. For instance, historical and Monte Carlo methods assume that the return pattern will repeat over a time horizon which may not be true for certain risk assets. In Duffie and Pan (1997), a portfolio consisting of derivatives using first and second order approximations is considered. Fat tails and skewness can be observed in several markets such as equity, exchange rates, interest rates and commodity. But maybe the biggest flaw of VaR is its lack of sub-additivity, meaning that the total risk of a sum of two portfolios may not be less than or equal to sum of the risk for each individual portfolio.

Also, one limitation of VaR measure is that it gives little importance to the most extreme losses since the skewness and kurtosis of the distribution are not adequately reflected. On the other hand, the assumption of normality overestimates the VaR for very high percentile values while it is underestimated for low percentile values, which correspond to extreme events. VaR is not a coherent risk measure since it has undesirable mathematical characteristics such as the lack of sub-additivity and convexity, which are only satisfied under the assumption of normality and whose absence can lead to contradictory results in portfolio optimization processes.

Therefore, it is necessary to implement a coherent risk measure in terms of its mathematical characterization, which allows greater precision to estimate the assumed level of market risk. Assuming elliptical distribution for the losses (e.g. normal distribution) leads VaR to be a coherent risk measure. Artzner et al. (1999) states the properties of a good indicator of market risk. Conditional Value at Risk (CVaR) was introduced as a coherent risk measure. CVaR is a function of VaR defined as the conditional expected value of losses that exceed VaR. This measure has desirable properties as convexity and sub-additivity regardless of the functional form of the lost profit distribution is.

Diverse methods are used to estimate VaR and CVaR of a portfolio. The Variance Co-variance method is an easy form to estimate VaR given by Baumol (1963). Unfortunately, this method is not convenient due to the assumption of normality in the distribution of portfolio losses. Cornish Fisher estimation approximates the probability distribution percentiles using asymmetry and kurtosis coefficients for a confidence degree and it is used to calculate VaR proposed by Zangari (1996). In the historical simulation method, an enormous amount of relevant data is required for its estimation. In many occasions it is not enough for an adequate estimation in the future. Monte Carlo simulation method also fails in precision when not specifying a model for the distribution of losses in the period that requires to estimate the value at risk.

In order to overcome the non-normality in the distribution of portfolio, using a mixture of Gaussian distribution as a tool for modeling asymmetry and excess of kurtosis has been considered. Parameter fitting can be done quickly and efficiently thanks to Expected-Maximization methods. It can be remarked that this approach can also fit rare functional form as bi-modality.

In the previous methods, when considering the losses of portfolio return, the non-linear relationship of its component assets is not being taken into account. Although the assumption of independence is very often used in financial analysis, it is believed that many phenomena show some dependence features. When estimating risk, it is only being considered dependent through linear correlation. Thus, it is necessary to study the risk of the entire portfolio with another tool that also permits the study of the dependence structure.

Copulas are well recognized to provide a flexible tool for analyzing the dependence relation among random variables. Because of its readiness for applications, copulas are now usually employed in various areas. In fact, it is much easier to relate several individual margins with a copula than consider a specific multivariate distribution function. Copulas are very useful functions for modeling joint distributions because it is not necessary the hypothesis of joint normality and they allow to decompose any n-dimensional joint distribution into its n-marginal distributions. Copulas also indicate nonlinear relationships between their risk factors. That is why it is natural to think about using Copulas to estimate VaR. Also, the wide spectrum of families of copulas can give the estimation of the value at risk many more possibilities than the previous models. The estimation of VaR and CVaR can be implemented with Monte Carlo simulations.

Compared to the usual notion for the single random variable, a multivariate Value at Risk is concerned with several variables and thus the relation between each risk factors should be taken into account. A new definition of copula-based conditional Value at Risk (CCVaR) is presented, which is ready to be computed. VaR is defined for a single random variable, and there have been efforts for extending that the definition to involve multivariate random vectors. Indeed, the pioneering work of Prékopa (2012) considers a vector valued multivariate Value at risk (MVaR). It is natural to ask, however, whether MVaR really serves as a risk measure; in other words, whether MVaR characterizes effectively the risk structure of multiple random variables, especially, the non-linear dependence relation between each risk factors. The answer is partially positive and still under development.

With all the methods considered until now, it is important to measure the robustness of proposed methods for estimating VaR and CVaR. Through many calculation periods, we can check performance and determine which one is more accurate over a certain time window. The VaR is estimated and then compared to the actual losses at the end of the next day. Such methodology is known as **backtesting**. It can help to determine if it is necessary to re-calibrate a model or even to reject it. For example, Christoffersen and Pelletier (2004) develop a test for dependence between consecutive days of failures.

1.2 Objectives

The main goal of this thesis is to establish a method for estimating VaR and CVaR for the portfolio problem that overcomes deficiencies of classical approaches. The considered portfolio is formulated as the linear combination of several random variables. It is noted that VaR is commonly defined for a single random variable. The innovative point of this research is that these random variables are not necessarily assumed to be independent, but may have a possible nonlinear relationship. Also, consider the elements of portfolio to present asymmetry and excess of kurtosis. This can be modeled with mixture of Gaussian distributions.

The main contribution can be listed as follows:

- To try to establish a practical analytical formula to estimate VaR for certain families of copula and study its properties
- To make numerical computations with actual real data from market and estimate VaR and CVaR with copula with margins modeled by a mixture of Gaussian distribution
- To perform backtesting on the proposed copula model with classical approaches to determinate its validity

Then we conclude based on these empirical studies that effectiveness of copula-based methods compared with some benchmarks methodologies have strengths and weaknesses and try to conclude how the performance will be in the general setting.

1.3 Contents of thesis

In Chapter 2, preliminaries are described. It addresses an introduction to Value at Risk, Coherent risk measures, Conditional Value at Risk, properties and the basic approaches to estimate them for a portfolio.

In Chapter 3, the main tools for the realization of this work are presented. ARMA-GARCH models, mixture of Gaussian distributions, Copulas and their main properties.

In Chapter 4, analytical results for the copula-based VaR formula and the methodology for numerical estimation along with backtesting are discussed.

Chapter 5 presents experimental results with this thesis's approach and compares with classical methodologies through backtesting.

And finally, Chapter 6 presents our considerations and conclusions.

Chapter 2

Preliminaries

In this chapter, the definition for Value at Risk and its properties are presented. It also presents the definition for Coherent Risk Measure as Conditional Value at Risk and their properties in terms of VaR. Finally, classical methods for its estimation are introduced and brief explanations for Historical simulation, Variance-Covariance method, and Monte Carlo simulation are discussed.

2.1 Value at Risk

The stock price can be considered as a random variable. Let us describe its realization over a discrete period of time stating at $t = 0$ as initial day till $t = T$ as final day. From now, we will work with log-return of the stock price series as our starting point.

Definition 1 (Log-return of an asset)**.** *For an asset, consider the stock price $\{S_t\}_{t=0}^{T}$. Its daily geometric return X_t is represented by*

$$X_t = \log\left(\frac{S_t}{S_{t-1}}\right) \tag{2.1}$$

for $t = 1, 2, \ldots, T$.

From now, we write Y the loss process, defined as $Y = -X$.

Definition 2 (Value at Risk)**.** *Let $\{X_t\}$ be log-return of a stock price series. Let Y_t be the cost variable or the losses associated to X_t, generally described by $X = -Y$ and let F_Y be its distribution function, i.e $F_Y(u) = P(Y \leq u)$. Also consider $F_Y^{-1}(v)$ be its left continuous inverse, i.e. $F_Y^{-1}(v) = \inf\{u : F_Y(u) \geq v\}$. Value at Risk (VaR) is the maximum loss of X*

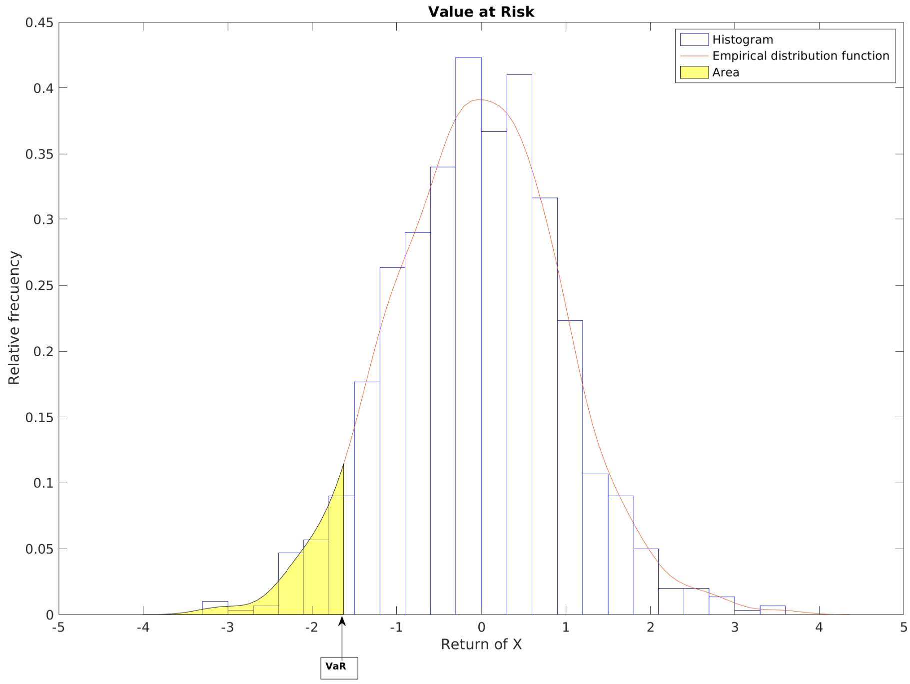

FIGURE 2.1: Graphic interpretation for Value at Risk

that will not be exceeded with a specified probability $\lambda \in (0,1)$ over a predetermined time horizon t. Then VaR_λ is defined as:

$$VaR_\lambda(Y) = F_Y^{-1}(\lambda)$$

If the loss distribution Y is continuous and strictly increasing, VaR will be uniquely determined by $F(\mathrm{VaR}) = \lambda$. Note that this same definition coincides with the λ-quantile function for the loss distribution. Usual values for λ, known as confidence level are $\lambda = 0.95$ or $\lambda = 0.99$. In this ambit, it represents the minimum loss of an asset given a significance level of λ. We note that some authors prefer to work with daily difference of prices $\Delta S_t = S_t - S_{t-1}$ in the definition of VaR. Also, some define VaR for a significance level (usually 5% or 1%), so VaR represents the worst expected loss given that significance level.

2.2 Classical methods for VaR

Some classical methods for estimating VaR will be briefly introduced. More detailed presentations are included in McNeil, Frey, and Embrechts (2015).

2.2.1 VaR under normal distribution

Suppose that the loss process is normally distributed with mean μ and standard deviation σ. Then, for a level of significance λ

$$\text{VaR}_{t,\lambda} = \mu + \sigma\Phi^{-1}(\lambda) \tag{2.2}$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function for a standard normal random variable.

2.2.2 Variance-Covariance Method

Supposing that for a set of n assets, their losses vector $\boldsymbol{Y_t} = (Y_{1,t}, Y_{2,t}, \ldots, Y_{n,t})'$ has a multivariate normal distribution named $\boldsymbol{Y_t} \sim N(\boldsymbol{\mu_t}, \boldsymbol{\Sigma_t})$, where $\boldsymbol{\mu_t}$ is the mean vector and $\boldsymbol{\Sigma_t}$ is the covariance matrix, both at time t. Also suppose that $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)'$ are the weight components for asset $\omega_i \geq 0$, $i(i = 1, \ldots, n)$ for the portfolio satisfying $\sum_{i=1}^{n} \omega_i = 1$. If the risk components are sufficiently small, the total loss of the portfolio at time t can be approximated by:

$$Y_{p,t} \approx \sum_{i=1}^{n} \omega_i Y_{i,t} \tag{2.3}$$

The variance of portfolio will be

$$\sigma_{p,t}^2 = \boldsymbol{\omega}' \boldsymbol{\Sigma_t} \boldsymbol{\omega} \tag{2.4}$$

Then, $Y_t \sim N(\boldsymbol{\omega}'\boldsymbol{\mu_t}, \sigma_{p,t}^2)$, and using the VaR formula for normal distribution, we have that

$$\text{VaR}_{t,\lambda} = \boldsymbol{\omega}'\boldsymbol{\mu_t} + \sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma_t}\boldsymbol{\omega}}\,\Phi^{-1}(\lambda) \tag{2.5}$$

Above methods have been very criticized due to the assumption of normally distributed returns, which in practice is not so feasible. Another problem with the variance-covariance method is the assumption of linearity of the assets conforming the portfolio.

2.2.3 Historical Simulation

Historical simulation requires data from the past values for assets. It tries to estimate a return distribution from empirical distribution function (*edf*) of data. Under this, *edf* can be used to sample a considerable number of independent realizations and sample the quantile for estimate VaR. In order to capture extreme scenarios, a good amount of past data has to be taken so estimation of VaR will be reliable. In this sense, the Historical method can be seen as a Monte Carlo simulation.

2.2.4 Monte Carlo simulation

The Monte Carlo method refers to multiple operations that try to model fit empirical distribution for the return of the underlying asset to another selected distribution. This simulation may or may not be of parametric type. For a certain period of time, m independent realizations are carried out. Once this is done, the empirical quantile for the estimation of VaR is extracted from the simulated data.

But for a portfolio with a large number of assets, the Monte Carlo method becomes pretty expensive in the computational sense. Even with a huge number of simulated realizations, it does not guarantee a reliable estimate.

2.3 Coherent Risk Measures

Under the assumption of normality, VaR is overestimated for high values and underestimated for low values in quantiles. Artzner et al. (1999) proposed a list of axioms that a coherent risk measure should satisfy.

Definition 3 (Coherent Risk Measures)**.** *A coherent risk measure is a risk indicator ρ, which satisfies the following axioms:*

- *Positive homogeneity:* $\rho(\lambda u) = \lambda \rho(u)$
- *Monotonicity:* $u \leq v$ *implies* $\rho(u) \leq \rho(v)$
- *Translation invariance:* $\rho(u + a) = \rho(u) + a$
- *Sub-additivity:* $\rho(u + v) \leq \rho(u) + \rho(v)$

One essential problem with VaR is that does not satisfy sub-additivity when the log-return series is not normally distributed, hence it is not a coherent risk measure. McNeil, Frey, and Embrechts (2015) explains each property on the following terms. The positive homogeneity has to hold if there is no diversification between the assets of the portfolio. The monotonicity is obvious from an economic viewpoint. Translation invariance states that by adding or taking a deterministic quantity a to a position leading to the loss, capital requirements are changed by exactly that amount. The sub-additivity says that the total risk of a sum of two portfolios may not be less than or equal to sum of the risk for each individual portfolio. The sum of two portfolios can not create additional risk because this is being diversified. The last property means that risk can not increase if the portfolio is compound. Also it is important to note that a risk measure that satisfies positive homogeneity and sub-additivity is convex. This important property is very desirable in the general context of optimization, and therefore in portfolio optimization theory.

2.4 Conditional Value at Risk

Knowing that VaR does not satisfy the sub-additivity property, another risk measure was suggested to be coherent. Acerbi and Tasche (2002) proposed Expected Shortfall as a substitute to overcome the deficiencies of VaR. The *Conditional Value at Risk*, also known as expected shortfall (ES); is defined as the conditional expected value of losses that excess VaR_λ.

Definition 4 (Conditional Value at Risk)**.** *Given a process $\{Y_t\}_{t=0}^{T}$ and a significance level $\lambda \in (0,1)$, the Conditional Value at Risk (Pflug, 2000) is defined as the solution of the following optimization problem:*

$$CVaR_\lambda(Y) := \inf\left\{a + \frac{1}{1-\lambda}\mathbb{E}[Y-a]^+ : a \in \mathbb{R}\right\}$$

where $[z]^+ = \max(z,0)$. Uryasev and Rockafellar (1999) show that for smooth F_Y, CVaR is equal to the conditional expectation of Y given that $Y > VaR_\lambda$, i.e.

$$CVaR_\lambda(Y) = \mathbb{E}\left[Y \mid Y > VaR_\lambda(Y)\right]$$

It is easily inferred that

$$\mathrm{CVaR}_\lambda(Y) \geq \mathrm{VaR}_\lambda(Y)$$

Note that CVaR measures the average loss of a portfolio, given that the loss is greater than a certain limit. CVaR is not a quantile measure like VaR, but a tail measure. It looks deeper on the tail, giving more information on extreme events (McNeil, Frey, and Embrechts, 2015).

If λ is in the range of F_Y, then an alternative representation of CVaR is given by:

$$\begin{aligned} \mathrm{CVaR}_\lambda(Y) &= \mathbb{E}\left[Y \mid Y > F^{-1}(\lambda)\right] \\ &= \frac{1}{1-\lambda}\int_\lambda^1 F^{-1}(v)dv \\ &= \frac{1}{1-\lambda}\int_{F^{-1}(\lambda)}^\infty u dF(u) \end{aligned} \tag{2.6}$$

CVaR was proved to be coherent in the sense of Artzner et al. (1999), and its applicability to portfolio optimization has been shown by Uryasev and Rockafellar (1999). More important properties are proven by Pflug (2000).

Proposition 1. *For $\lambda \in (0,1)$ and Y a random variable, then*

(i) If Y has a density,

$$\mathbb{E}(Y) = (1-\lambda)CVaR_\lambda(Y) - \lambda CVaR_{(1-\lambda)}(-Y)$$

(ii) CVaR is convex in the following sense: For arbitrary random variables Y_1 and Y_2 and $0 < \omega < 1$,

$$CVaR_\lambda\left(\omega Y_1 + (1-\omega)Y_2\right) \leq \omega CVaR_\lambda\left(Y_1\right) + (1-\omega)CVaR_\lambda\left(Y_2\right)$$

(iii) $VaR_\lambda(Y) = -VaR_{(1-\lambda)}(-Y)$

(iv) If Y is non-negative, then as $n \to \infty$,

$$\left[\frac{\mathbb{E}\left(Y^n\right) - (1-\lambda)CVaR_\lambda\left(Y^n\right)}{\lambda}\right]^{1/n} \to VaR_\lambda(Y)$$

2.4.1 Conditional Value at Risk for Normal loss distribution

Let's suppose that Y has a normal distribution with mean μ and standard deviation σ. For $\lambda \in (0,1)$, the CVaR is given by

$$\mathrm{CVaR}_\lambda(Y) = \mu + \sigma \frac{\phi\left(\Phi^{-1}(\lambda)\right)}{1-\lambda}$$

where ϕ represents the probability density function *(pdf)* of the standard normal distribution.

2.5 VaR and CVaR backtesting

It is important to measure the robustness of proposed methods for estimating VaR and CVaR. Through many calculation periods, we can check performance and determine which one is more accurate over a certain time window. The VaR is estimated and then compared to the actual losses at the end of the next day. Such methodology is known as **backtesting**. It can help to determine if it is necessary to re-calibrate a model or even to reject it. Basel Committee for Banking Supervision also suggests its own backtesting methodology for bank capital requirements for market risk.

2.5.1 Binomial Test

If T is the number of observations, $p = 1 - \lambda$ and N the number of failures, and if the failures are independent, then N is distributed as a binomial distribution with parameters T and p. The expected number of failures is Tp, and the standard deviation of the number of failures is

$$\sqrt{Tp(1-p)}$$

and the test statistic is the z-score, defined as

$$Z = \frac{(N - Tp)}{\sqrt{Tp(1-p)}}$$

The z-score approximately follows a standard normal distribution. We refer to Jorion et al. (2010) for details.

2.5.2 Unconditional Model Evaluation: Kupiec's test

This backtesting method was proposed by Kupiec (1995). Since VaR is based on a confidence level p, when we observe N losses in excess of VaR out of T observations, hence we observe N/T proportion of excessive losses: the Kupiec's determines whether N/T is statistically significantly different from p. Following binomial theory, the probability of observing N failures out of T observations is $(1-p)^{T-N}p^N$, so that the test of the null hypothesis that the expected exception frequency $N/T = p$ is given by a likelihood ratio test statistic:

$$LR_{UC} = -2\log[(1-p)^{T-N}p^N] + 2\log[(1-N/T)^{T-N}(N/T)^N]$$

which is distributed as $\chi^2(1)$ under H_0. This test can reject a model for both high and low failures.

2.5.3 Conditional Model Evaluation: Christoffersen's test

This test proposed by Christoffersen (1998) takes account of any conditionality in forecast: for example, if volatilities are low in some period and high in others, the VaR forecast should respond to this clustering event. The Christoffersen procedure enables us to separate clustering effects from distributional assumption effects.

$$LR_{CC} = -2\log[(1-p)^{T-N}p^N] + 2\log[(1-\pi_{01})^{n00}\pi_{01}^{n01}(1-\pi_{11})^{n10}\pi_{11}^{n11}]$$

where n_{ij} is the number of observations with value i followed by j for $i, j = 0, 1$ and $\pi_{ij} = \frac{n_{ij}}{\sum_i n_{ij}}$ are the corresponding probabilities. Under the H_0, this test is distributed as a $\chi^2(2)$

2.6 CVaR backtesting

Contrary to VaR backtesting where we find many types of tests, Conditional Value at Risk does not offer a large amount of these. This is due to the fact that we can see if the forecast amount is less than the realization. But for CVaR this is not so. Formally, it is said that VaR is *elicitable* but CVaR is not (Carver, 2013). This conclusion made many people think that the CVaR was not back-testable. But Acerbi and Szekely (2014) showed that elicitability has nothing to do with the backtesting and proposed some tests for CVaR which are non-parametric and can be applied to any estimate without worrying about the distribution of the losses.

2.6.1 Test 2 by Acerbi and Szekely

Test 2 by Acerbi and Szekely, also known as Unconditional test, scales the losses by the corresponding CVaR value. This test statistic is defined by:

$$Z_{Uncond} = \frac{1}{N(1-\lambda)} \sum_{t=1}^{N} \frac{Y_t \mathbf{1}_t}{\mathrm{CVaR}_\lambda} + 1 \tag{2.7}$$

where $\mathbf{1}_t$ represents the VaR failure indicator on period t with a value of 1 if $Y_t > \mathrm{VaR}_\lambda$ and 0 otherwise. Test 2 jointly evaluates frequency and magnitude of $1-\lambda$ tail. Critical values for this test statistic are based on Normal and t distribution with 3 degrees of freedom.

Chapter 3

Modeling tools

In this chapter the main tools for modeling random variables and their dependence structure are presented. For analyzing time series, the ARMA (autoregressive moving average) and GARCH (generalized autoregressive conditionally heteroscedastic) processes are introduced. Then for modeling the standardized residuals, the mixture of Gaussian distributions are presented. And finally we review copula functions and the Sklar's theorem.

3.1 ARMA-GARCH processes

Through several empirical observations drawn from financial time series, certain behaviors can be observed in the daily series of risk factors. This collection is known as *stylized facts*. For instance, the return series are not *independent identically distributed* (i.i.d), and show little serial correlation but squared returns show profound serial correlation (McNeil, Frey, and Embrechts, 2015). Also, conditional expected returns are close to zero, volatility varies over time and return series present leptokurtosis and asymmetry.

In Figure 3.1, volatility clustering can be seen, meaning that large absolute returns tend to follow large absolute returns and the same for small returns. The ARMA processes have been applied with success in the time series analysis. They are covariance-stationary processes and use white noise.

Definition 5 (ARMA). *An ARMA process of order $(p,q) \in \mathbb{N}$ for a series (X_t) is given by:*

$$X_t = a_0 + \sum_{i=1}^{p} a_i X_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j \varepsilon_{t-j}$$

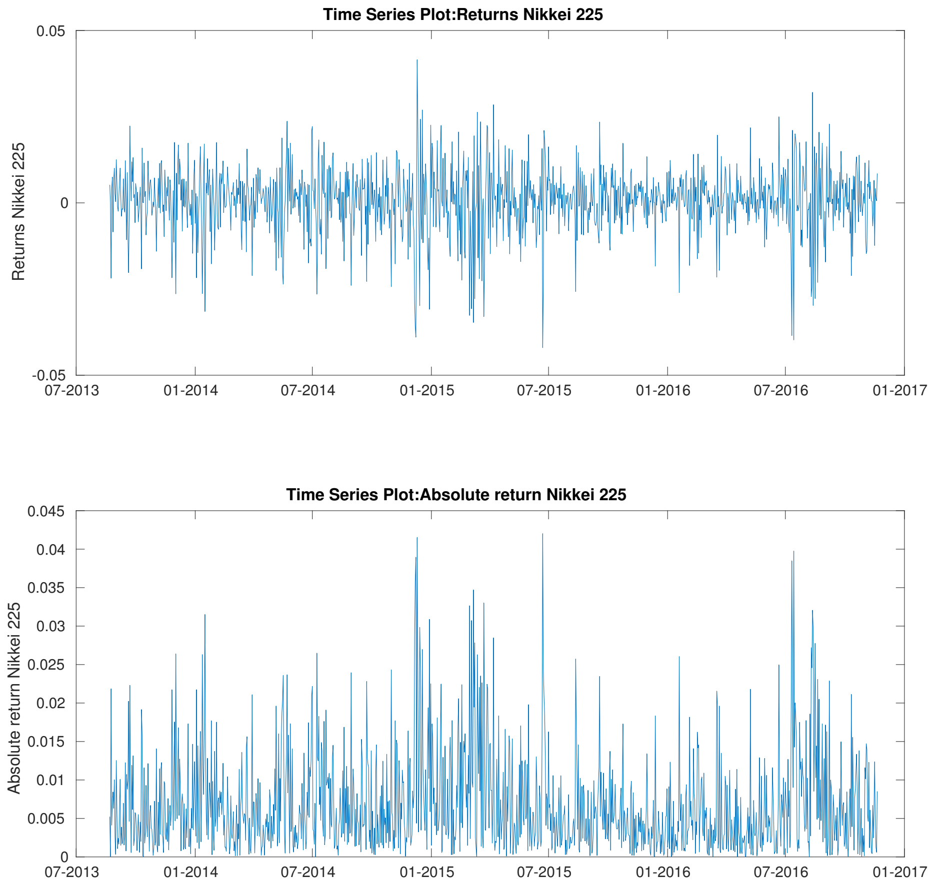

FIGURE 3.1: Daily returns and absolute return for Nikkei 225 stock index

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise process with zero mean and unit variance (Francq and Zakoian, 2019).

Note that volatility is not being considered yet on an ARMA model. We need to deal with the stylized fact of non-constant variance and add a model that generates volatility clusters. In practice, many authors have used GARCH to model high volatility.

Definition 6 (GARCH)**.** *A GARCH process of order $(r,s) \in \mathbb{N}$ for the variance σ_t^2 of $\{X_t\}$ is given by:*

$$\sigma_t^2 = c_0 + \sum_{i=1}^{r} c_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} d_j \sigma_{t-j}^2$$

where $\varepsilon_t = \sigma_t z_t$. Here $\{z_t\}_t$ is a sequence of independently and identically distributed (i.i.d.) random variables known as standardized innovation process and X_t is given by an ARMA(p,q) model.

The conditional distribution of the standardized innovations

$$z_t = \frac{\varepsilon_t}{\sigma_t} \mid \mathcal{F}_{t-1} \tag{3.1}$$

is usually modeled by standard normal or student t distributions. $\mathcal{F}_{t-1}$ represents a filtration generated by the information set at $t-1$.

GARCH processes allows the white noise ε_t to depend of its past values, so periods with huge fluctuations will be followed by the same fluctuations but different amplitude. Certain conditions must be imposed to the parameters $(a_0, a_1, \ldots, a_p, b_0, b_1, \ldots, b_q)$ and $(c_0, c_1, \ldots, c_r, d_0, d_1, \ldots, d_s)$ for the 'non-explosion' of the series. We refer to Francq and Zakoian (2019) for specific details.

3.2 Mixture of normal Gaussian distributions

ARMA-GARCH models have been used extensively to model high excessive kurtosis, dependence, and volatility clustering. But again, there is empirical evidence against the normality of the innovations. Lee and Lee (2011) proposed a new algorithm based on ARMA-GARCH models with Gaussian mixture innovations as a tool to estimate VaR in the presence of heavy tailed and skewed residuals. These

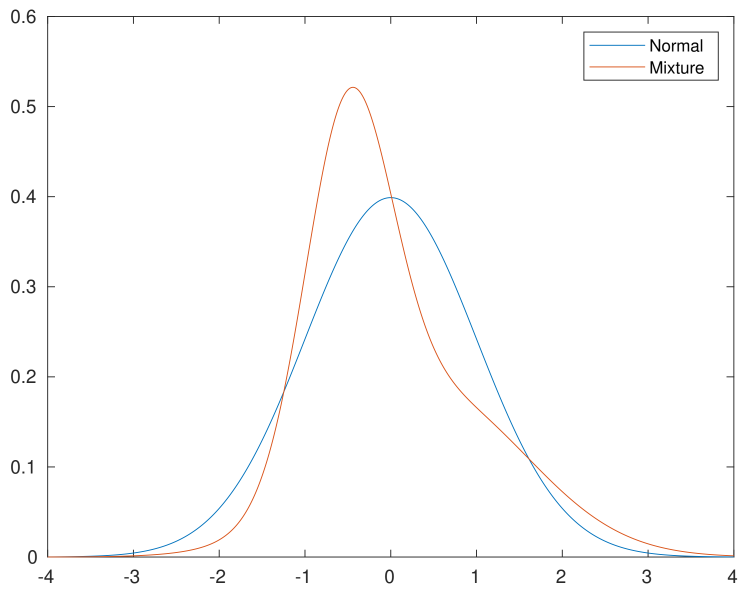

FIGURE 3.2: Probability density functions for a Normal and a mixture of two normal with parameters $\pi = 0.5$, $\mu_1 = -0.5$, $\mu_2 = 0.5$, $\sigma_1 = 0.5$, $\sigma_2 = 1.118$

mixture distributions allow also to model unusual forms for the return distribution as bimodality, for example.

Definition 7 (Gaussian Mixture). *A random variable Y is distributed with K component Gaussian mixture if its probability distribution function (pdf) is given by:*

$$f_\eta(y) = \sum_{i=1}^{K} \pi_i f(y; \mu_i, \sigma_i)$$

where

$$f(y; \mu_i, \sigma_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left\{ -\frac{1}{2} \left(\frac{y - \mu_i}{\sigma_i} \right)^2 \right\}$$

and $\sum_{i=1}^{K} \pi_i = 1$ *for* $0 \leq \pi_i \leq 1$ *and* $i = 1, 2, \ldots, K$

The mean and variance will be given by

$$\mathbb{E}[X] = \mu = \sum_{i=1}^{K} \pi_i \mu_i$$

$$\sigma^2 = \sum_{i=1}^{K} \pi_i \left(\sigma_i^2 + \mu_i^2 - \mu^2 \right)$$

Skewness and Kurtosis are given respectively by:

$$\alpha_3 = \frac{1}{\sigma^3} \sum_{i=1}^{K} \pi_i \left(\mu_i - \mu\right) \left[3\sigma_i^3 + \left(\mu_i - \mu\right)^2\right]$$

$$\alpha_4 = \frac{1}{\sigma^4} \sum_{i=1}^{K} \pi_j \left[3\sigma_i^4 + 6\left(\mu_i - \mu\right)^2 \sigma_i^2 + \left(\mu_i - \mu\right)^4\right]$$

3.3 Estimation of parameters for ARMA-GARCH with mixture of normal distributions

Lee and Lee (2011) describes the procedure for estimating model parameters for an ARMA-GARCH with mixture of Gaussian distributions. A Gaussian quasi-maximum likelihood estimator (QMLE) is necessary and then an (EM) algorithm is employed to find the estimators for the residual innovations distribution. Let the space of parameter be given by:

$$\Omega \subset \Big\{ \eta = (\pi_1, \ldots, \pi_K, \mu_1, \ldots, \mu_K, \sigma_1, \ldots, \sigma_K)' \in [0,1]^K \times \mathbb{R}^K \times (0,\infty)^K : \\ \sum_{i=1}^{K} \pi_i = 1, \sum_{i=1}^{K} \pi_i \mu_i = 0, \sum_{i=1}^{K} \pi_i (\mu_i^2 + \sigma_i^2) = 1 \Big\}$$

These conditions imply that the mixture has mean zero and unit variance, which are adequate for modeling standardized residuals $\{z_t\}$.

Let $\varphi = (\vartheta', \theta')' = \left(a_0, a_1, \ldots a_p, b_1, \ldots, b_q, \theta'\right)'$ be the real parameter vector. A Gaussian QMLE for the parameter φ, which is denoted by $\hat{\varphi}_T = \left(\hat{\theta}_1', \hat{\theta}_2'\right)$, where $\hat{\theta}_1 = \left(\hat{a}_0, \ldots, \hat{a}_p, \hat{b}_1, \ldots, \hat{b}_q\right)'$ and $\hat{\theta}_2 = \left(\hat{c}_0, \ldots, \hat{c}_r, \hat{d}_1, \ldots, \hat{d}_s\right)'$ is given by any solution of

$$\hat{\varphi}_n = \underset{\varphi \in \Phi}{\arg\min} \tilde{\mathbf{I}}_n(\varphi) \tag{3.2}$$

where $\tilde{\mathbf{I}}_n(\varphi) = n^{-1} \sum_{t=1}^{n} \tilde{\ell}_t$ and $\tilde{\ell}_t = \tilde{\ell}_t(\varphi) = \tilde{\epsilon}_t^2(\vartheta)/\tilde{\sigma}_t^2(\varphi) + \log \tilde{\sigma}_t^2(\varphi)$ Using the Gaussian QMLE $\hat{\varphi}_T$ we obtain the residuals for ARMA-GARCH

$$\tilde{z}_t = \frac{\tilde{\epsilon}_t\left(\hat{\theta}_1\right)}{\sqrt{\tilde{\sigma}_t^2\left(\hat{\varphi}_T\right)}}, \quad t = 1, \ldots, T \tag{3.3}$$

where $\tilde{\epsilon}_t\left(\hat{\theta}_1\right)$ and $\tilde{\sigma}_t^2\left(\hat{\varphi}_T\right), t = 1, \ldots, T$, are defined recursively by using

$$\tilde{\epsilon}_t\left(\hat{\theta}_1\right) = X_t - \hat{a}_0 - \sum_{i=1}^{p} \hat{a}_i X_{t-i} - \sum_{j=1}^{q} \hat{b}_j \tilde{\epsilon}_{t-j}\left(\hat{\theta}_1\right)$$

and

$$\tilde{\sigma}_t^2\left(\hat{\varphi}_T\right) = \hat{c}_0 + \sum_{i=1}^{r} \hat{c}_i \tilde{\epsilon}_{t-i}^2\left(\hat{\theta}_1\right) + \sum_{j=1}^{s} \hat{d}_j \tilde{\sigma}_{t-j}^2\left(\hat{\varphi}_T\right)$$

respectively.

Finally, we determine the parameter for density using the above residual and maximizing the log-likelihood function for Gaussian mixture parameter η as

$$\hat{\eta}_T = \left(\hat{\pi}_1, \ldots, \hat{\pi}_K, \hat{\mu}_1, \ldots, \hat{\mu}_K, \hat{\sigma}_1, \ldots, \hat{\sigma}_K\right)' := \arg\max_{\eta} \tilde{l}_T(\eta), \tag{3.4}$$

where

$$\tilde{l}_T(\eta) := \frac{1}{T} \sum_{t=1}^{\mathrm{T}} \log f_\eta\left(\tilde{z}_t\right)$$

This last step will take advantage of the efficiency of the EM algorithm for its simplicity and computational speed (Redner and Walker, 1984).

3.4 Copulas

Copulas are useful functions to describe the dependence structure between two or more random variables representing risk factor. Copulas are quite helpful for analyzing dependence of extreme events, and therefore its utility comes in the context of risk management being easy to interpret the quantile scale for use on the estimation of Value at Risk. We can think of Copulas being a multivariate cumulative distribution function which margins are uniformly distributed. So Copula will '*link*' these variables with a desirable dependence structure. While, the number of multivariate distributions are limited, Copulas function expand the possibilities for constructing joint distribution functions. Copulas have already been used in financial applications. Patton (2006) extended the application to conditional copula modeling for time varying conditional dependence.

In this section, the definition for a bivariate copula is given for simplicity purposes, but the higher dimensions case is straightforward (Nelsen, 2007).

Definition 8 (Copula). *A function C defined on* $\mathbb{I}^2 := [0,1] \times [0,1]$ *and valued in* $\mathbb{I}$ *is said to be a copula if the following conditions are satisfied:*

1. *For every* $(u, v) \in \mathbb{I}^2$,

$$C(u, 0) = C(0, v) = 0$$

$$C(u, 1) = u,\ C(1, v) = v$$

2. *For every* $(u_i, v_i) \in \mathbb{I}^2 (i = 1, 2)$ *with* $u_1 \leq u_2$ *and* $v_1 \leq v_2$,

$$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0$$

Hence any bivariate distribution function whose margins are standard uniform distributions is a copula. It is noted that a copula is a continuous function by its definition. The requirement (iv) is referred to as *the* 2*-increasing condition*. The Copula function C is a copula for the random vector $\boldsymbol{X} = (X_1, X_2)'$ if it is the joint distribution function of the random vector $\boldsymbol{U} = (U_1, U_2)'$ where $U_i = F_i(X_i)$ and F_i are the marginal distribution functions of X_i, $i = 1, 2$

This imply that:

$$H(x_1, x_2) = C(F_1(x_1), F_1(x_2))$$

where H in the joint distribution function for the vector $(X_1, X_2)'$

3.4.1 Sklar's Theorem

The most important result in Copula theory is the Sklar's theorem. It states that any group of univariate distribution can be linked with any copula and a valid multivariate distribution can be defined.

Theorem 1 (Sklar's Theorem). *Let H be a 2-dimensional joint distribution function with marginal distributions* F_1, F_2. *Then exist a copula C such that for all* $(x_1, x_2) \in \mathbb{R}^2$,

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

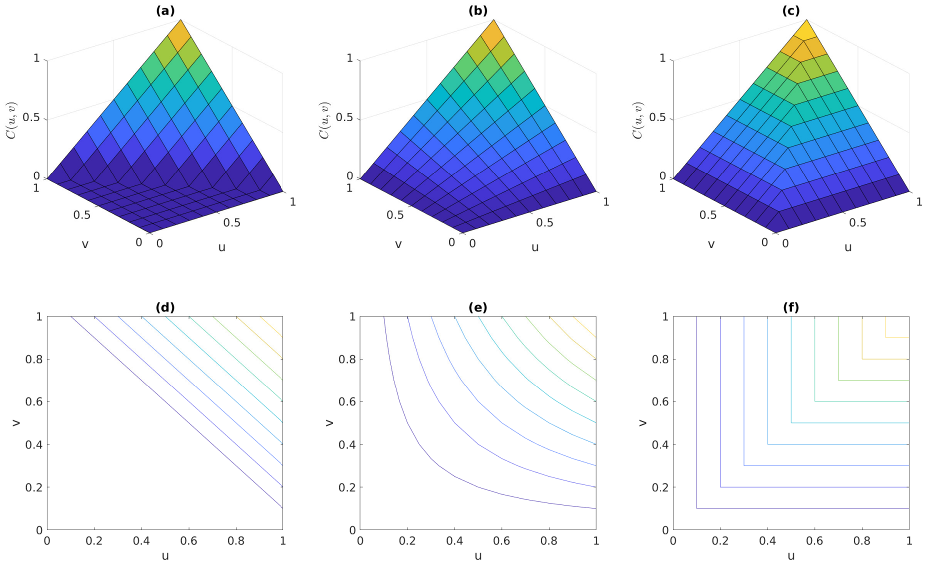

FIGURE 3.3: Plots and contour plots for the Frechet bounds copulas: (a),(d) countermonotonicity, (b),(e) independence, (c),(f) comonotonicity

If F_1, F_2 are continuous, then C is unique; otherwise, C is uniquely determined on $Ran(F_1) \times Ran(F_2)$. Otherwise, if C is a copula and F_1, F_2 are distribution functions, then the function H defined above, is a joint distribution function with margins F_1, F_2.

The Sklar theorem states that any group of univariate distribution can be linked with any copula and a valid multivariate distribution can be defined. The demonstration of this theorem can be found in Nelsen (2007).

Theorem 2 (Frechet bounds)**.** *For any copula C we have bounds*

$$\max\{u+v-1,0\} \leq C(u,v) \leq \min\{u,v\}$$

These are sometimes called Frechet-Hoeffding bounds.

Definition 9 (Empirical copula)**.** *The empirical copula $\hat{C}$ is defined as:*

$$\hat{C}\left(\frac{t_1}{T},\frac{t_2}{T}\right) = \frac{1}{T}\sum_{t=1}^{T}\mathbf{1}_{[x_{1,t}\leq x_{1(t_1)},x_{2,t}\leq x_{1(t_2)}]}$$

where $\mathbf{1}$ *is the indicator function defined as*

$$\mathbf{1}_A(x) := \begin{cases} 1 & \textit{if } x \in A \\ 0 & \textit{if } x \notin A \end{cases}$$

and $x_{i,(t_j)}$, $i = 1,2$, $j = 1,2$ *are the* t_j*-th order statistics of the i-th variable and* $t_1, t_2 \in \{1, \ldots, T\}$. *Given a sample of n variates* $X_1, \ldots, X_N$, *reorder them so that* $Y_1 < Y_2 < \ldots < Y_N$. *Then* Y_i *is called the i th order statistic sometimes also denoted* $X^{(i)}$

Definition 10 (Distance between two copulas)**.** *The quadratic distance between two copula* C_1 *and* C_2 *in a finite set of bivariate points* $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ *is defined as:*

$$\bar{d}(C_1, C_2) = \left[\sum_{i=1}^{m} (C_1(a_i) - C_2(a_i))^2\right]^{1/2}$$

For instance, when selecting a model in order to estimate VaR, the region of interest should be the lower tail.

The tail dependence is an important concept for modeling extreme losses in the context of Value at Risk. This measures the dependence between variables in the upper-right quadrant and in the lower-left quadrant of $\mathbb{I}^2$.

Definition 11 (Tail dependence)**.** *Let X and Y be continuous random variables with distribution functions F and G, respectively. The upper tail dependence parameter* τ_U *is the limit (if it exists) of the conditional probability that Y is greater than the t -th quantile of G given that X is greater than the t-th quantile of F as t approaches* 1, *i.e.*

$$\tau_U = \lim_{t \to 1^-} P\left[Y > G^{(-1)}(t) \mid X > F^{(-1)}(t)\right]$$

Similarly, the lower tail dependence parameter τ_L *is the limit (if it exists) of the conditional probability that Y is less than or equal to the t -th quantile of G given that X is less than or equal to the t -th quantile of F as t approaches* 0, *i.e.*

$$\tau_L = \lim_{t \to 0^+} P\left[Y \leq G^{(-1)}(t) \mid X \leq F^{(-1)}(t)\right]$$

Note that if C is copula for X and Y then tail dependence parameters can be expressed as:

$$\tau_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t,t)}{1-t}, \quad \tau_L = \lim_{t \to 0^+} \frac{C(t,t)}{t}$$

3.4.2 Some families of copulas

The independence copula is

$$\Pi(u,v) = uv \tag{3.5}$$

The Gaussian copula for $\mid \rho \mid < 1$ can be expressed as

$$\begin{aligned} &C_\rho^{\mathrm{Ga}}(u,v) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{ \frac{-(s^2 - 2\rho st + t^2)}{2(1-\rho^2)} \right\} \mathrm{d}s\mathrm{d}t \end{aligned} \tag{3.6}$$

The bivariate Student-t copula is the function

$$C^t_{R_{12,\nu}}(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{ 1 + \frac{s^2 - 2\rho^2 + t^2}{\nu(1-\rho^2)} \right\}^{(-(\nu+2)/2)} dsdt$$

where t_ν^{-1} is the inverse of univariate t distribution with ν degrees of freedom.

Plackett's copula is the function

$$\begin{aligned} C_\theta(u,v) = \frac{1}{2(\theta-1)} &\{1 + (\theta-1)(u+v) - \\ &([1+(\theta-1)(u+v)]^2 - 4uv\theta(\theta-1))^{1/2}\} \end{aligned}$$

for $\theta \neq 1$, and $C_\theta(u,v) = uv$ for $\theta = 1$, which is defined for $\theta > 0$. One disadvantage of the Plackett's copula is that it cannot be easily extended for dimensions larger than two.

The Joe-Clayton copula is given by

$$\begin{aligned} C_{JC}(u,v|\tau_U,\tau_L) = &1 - \left(\{[1-(1-u)^\kappa]^{-\gamma} \right. \\ &\left. [1-(1-v)^\kappa]^{-\gamma} - 1\}^{-1/\gamma} \right)^{1/\kappa} \end{aligned}$$

where

distribution

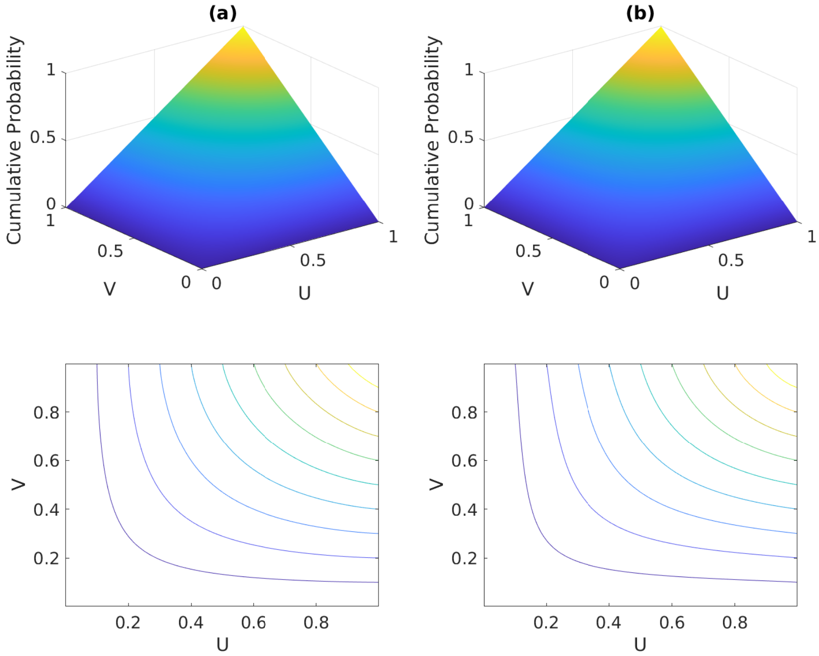

FIGURE 3.4: Plots and contour plots for Gaussian and t copula: (a),(c) Gaussian $\rho = 0.4$, (b),(d) t $\rho = 0.4$ and $\nu = 4$

$$\kappa = 1/\log_2(2 - \tau_U)$$
$$\gamma = -1/\log_2(\tau_L)$$
$$\tau_U, \tau_L \in (0, 1)$$

The parameter τ_U and τ_L are the coefficients of upper and low tail dependence, respectively. The Joe-Clayton copula still has a asymmetry when $\tau_U = \tau_L$. The symmetrized Joe-Clayton copula was proposed which is given by

$$C_{SJC}(u, v|\tau_U, \tau_L) = \frac{1}{2}C_{JC}(u, v|\tau_U, \tau_L) + \frac{1}{2}C_{JC}(1 - u, 1 - v|\tau_L, \tau_U) + u + v - 1$$

which is symmetric when $\tau_U = \tau_L$.

3.4.3 Archimedean copulas

An important class of copulas is given by the so-called Archimedean copulas. They have well algebraic properties and let the estimation of their parameters and simulation to be almost straightforward. We recall for completeness what are the Archimedean copulas.

Let $\varphi : \mathbb{I} \to [0, \infty]$ be a convex function such that φ is strictly decreasing and verifies $\varphi(1) = 0$. Let $\varphi^{(-1)}$ denote the pseudo-inverse of φ; that is, Dom $\varphi^{(-1)} = [0, \infty]$, Ran $\varphi^{(-1)} = \mathbb{I}$, and

$$\varphi^{(-1)}(t) = \begin{cases} \varphi^{-1}(t) & (0 \leq t \leq \varphi(0)) \\ 0 & (\varphi(0) \leq t \leq \infty). \end{cases}$$

It is then possible to prove that the function C defined on $\mathbb{I}^2$ by

$$C(u, v) = \varphi^{(-1)}(\varphi(u) + \varphi(v)) \tag{3.7}$$

provides a copula. Copulas of this form are called Archimedean copulas and the function φ is known as a generator of the copula.

The class of Archimedean copula finds a wide range of applications, because it is determined through single generator. For a general reference concerning Archimedean copulas, we refer for instance to a book by Nelsen (2007).

Clayton family copula is given by For $\theta \in (0, \infty)$, $\quad \varphi(t) = (1 + t)^{-1/\theta}$

$$C(u.v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta} \tag{3.8}$$

Frank family copula is given by For $\theta \in (0, \infty)$, $\quad \varphi(t) = -\log\left(1 - \left(1 - e^{-\theta}\right)\exp(-t)\right)/\theta$

$$C(u, v) = -\frac{1}{\theta}\log\left(1 + \frac{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right)}{\left(e^{-\theta} - 1\right)}\right)$$

Gumbel family copula for $\theta \in [1, \infty)$, with generator $\varphi(t) = \exp\left(-t^{1/\theta}\right)$ is expressed as

$$C(u, v) = \exp\left(-\left[(-\log u)^{\theta} + (-\log v)^{\theta}\right]^{1/\theta}\right) \tag{3.9}$$

The following proposition can be found in Nelsen (2007). It establishes the tail dependence parameters for Archimedean copulas involving generator and inverse functions.

Proposition 2. *Let C be an Archimedean copula with generator φ Then*

$$\tau_U = 2 - \lim_{t \to 1^-} \frac{1 - \varphi^{(-1)}(2\varphi(t))}{1 - t} = 2 - \lim_{x \to 0^+} \frac{1 - \varphi^{(-1)}(2x)}{1 - \varphi^{(-1)}(x)}$$

and

$$\tau_L = \lim_{t \to 0^+} \frac{\varphi^{(-1)}(2\varphi(t))}{t} = \lim_{x \to \infty} \frac{\varphi^{(-1)}(2x)}{\varphi^{(-1)}(x)}$$

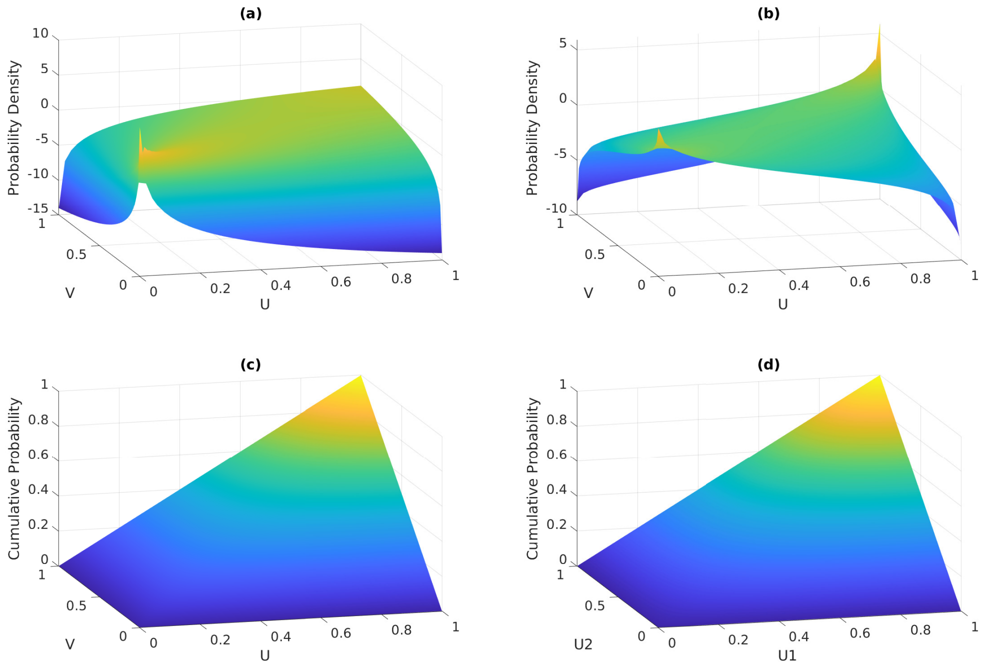

FIGURE 3.5: Plots of Clayton and Gumbel cumulative and density probabilities: (a),(c) Clayton $\theta = 2.2$, (b),(d) Gumbel $\theta = 2$

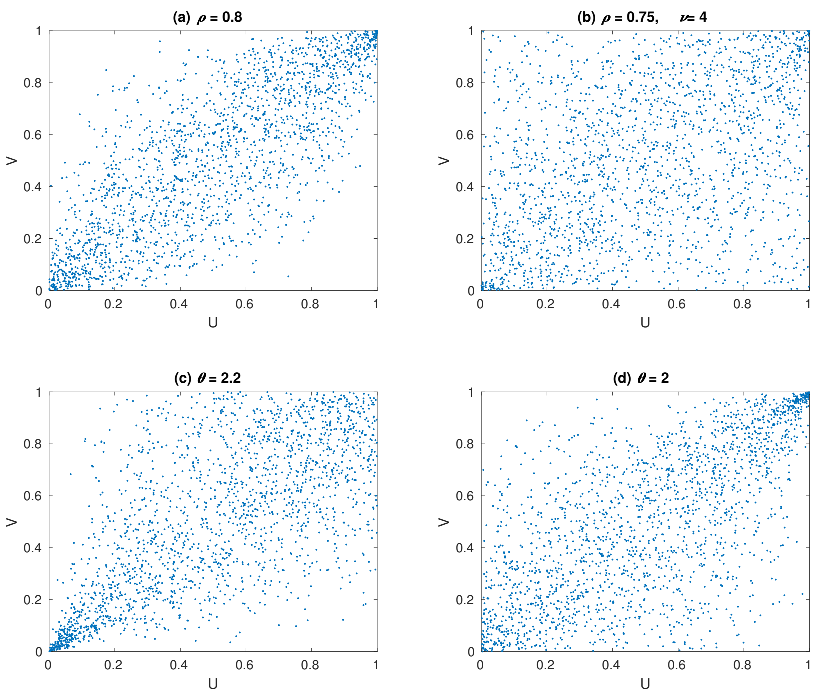

FIGURE 3.6: Simulated points for different copulas: (a) Gaussian ,(b) t Copula, (c) Clayton $\theta = 2.2$, (d) Gumbel $\theta = 2$

3.4.4 Estimating parameters for copula

Let (X_1, X_2), be a vector of two random variables with joint distribution function H and marginal distribution functions F_1 and F_2 respectively.Each function depends only on the parameter ϑ_i . Denote the ϑ unknown vector of parameters by $\vartheta = (\vartheta_1, \vartheta_2, \theta)$ where θ is the vector of parameters for the n-dimensional copula $\{C_\theta, \theta \in \Theta\}$ where the copula is known except for the parameter. Suppose that $\{(x_{1,t}, x_{2,t})\}_{t=1}^{T}$ is a sample of size T. Hence for Sklar's theorem it has

$$H(x_1, x_2) = C(F_1(x_1; \vartheta_1), F_2(x_2, \vartheta_2); \theta) \tag{3.10}$$

The density function differentiating the above expression can be obtained with respect to all variables

$$h(x_1, x_2) = c(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2) \tag{3.11}$$

where f_i is the density function associated to the marginal distribution F_i and c is the copula density, given by $c(u_1, u_2) = \dfrac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}$.

The log-likelihood function is given by

$$l(\vartheta) = \sum_{t=1}^{T} \log c(F_1(x_{1,t}; \vartheta_1), F_2(x_{2,t}, \vartheta_2); \theta) + \sum_{t=1}^{T}\sum_{i=1}^{2} \log f_i(x_{i,t}; \vartheta_i) \tag{3.12}$$

Thus, the maximum likelihood estimate $\hat{\vartheta}$ maximizes the above function, it is given by

$$\hat{\vartheta} = \arg\max_{\vartheta} l(\vartheta)$$

But this method results computationally quite expensive. It is better to use Inference Function for Margins (IFM) Method, where the parameters are estimated in two stages and it is computationally simpler than the maximum likelihood method.

Chapter 4

Copula-based Value at Risk

The main goal is to estimate Value at Risk for a portfolio of risk factors by exploiting the time series of daily stock returns for each asset. Fantazzini (2008) considered t-skewed distribution for marginal data and linked them together with a copula distribution function. However, his formula seems complicated to solve directly so his approach relies on Monte Carlo simulation. The specification in this thesis relies on mixture normal distribution for the margin and uses the same Monte Carlo approach as Fantazzini (2008). Also, a multivariate Conditional Value at Risk based on copula is proposed and its properties are discussed.

4.1 Estimation of copula-based VaR

Let X and Y denote risk factors at time t and be $\lambda \in (0,1)$ and F_t and G_t their cdf respectively. Consider $\omega \in (0,1)$ the weight of portfolio. If returns are sufficiently small, we can write the portfolio losses as $Z = \omega X + (1-\omega)Y$. The conditional joint distribution function at time $t-1$ is given by:

$$H_t\left(x, y \mid \mathcal{F}_{t-1}\right) = C_t\left(F_t\left(x \mid \mathcal{F}_{t-1}\right), G_t\left(y \mid \mathcal{F}_{t-1}\right) \mid \mathcal{F}_{t-1}\right) \tag{4.1}$$

and relative density function given by

$$h(x, y \mid \mathcal{F}_{t-1}) = c(F_1(x \mid \mathcal{F}_{t-1}), F_2(y \mid \mathcal{F}_{t-1})) f_1(x \mid \mathcal{F}_{t-1}) f_2(y \mid \mathcal{F}_{t-1}) \tag{4.2}$$

Hence, the cumulative distribution function for the portfolio loss is given by:

$$\begin{aligned} \zeta(z) &= \mathrm{P}\left(Z \leq z_t \mid \mathcal{F}_{t-1}\right) = \mathrm{P}\left(\omega X + (1-\omega)Y \leq z_t \mid \mathcal{F}_{t-1}\right) = \\ &= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{\frac{1}{\omega}z_t - \frac{1-\omega}{\omega}y} c_t\left(F_t\left(x \mid \mathcal{F}_{t-1}\right), G_t\left(y \mid \mathcal{F}_{t-1}\right) \mid \mathcal{F}_{t-1}\right) f_t\left(x \mid \mathcal{F}_{t-1}\right) dx \right\} g_t\left(y \mid \mathcal{F}_{t-1}\right) dy \end{aligned} \tag{4.3}$$

The one-step-ahead VaR computed in $t-1$ for the portfolio at level λ is the solution z^* of the equation $\zeta(z^*) = \lambda$, times the value of the financial position at $t-1$.

Note that resolving 4.3 seems quite complicated. In the case of Archimedean copula, however, this formula can be reduced to the following result which is presented in Molina Barreto and Ishimura (2020) and seems to be new in literature.

Theorem 3 (Determination formula for VaR)**.** *Suppose that X, Y be non-negative random variables, whose joint distribution function His represented by an Archimedean copula C_φ with generator φ; namely,*

$$H(x,y) = P(X \leq x, Y \leq y) = C_\varphi\left(F_X(x), F_Y(y)\right)$$

where $F_X(x) = P(X \leq x), F_Y(y) = P(Y \leq y)$ are marginal distribution functions of X, Y, respectively. Let $Z = \omega X + (1-\omega)Y (0 < \omega < 1)$ be a portfolio. Then, its Value at Risk at the confidence level $(0 < \lambda < 1)$, that is, $\mathrm{VaR}_\lambda(Z) = F_Z^{(-1)}(\lambda) = \inf\{t \mid F_Z(t) \geq \lambda\}$ can be attained as the solution z^* of the equation

$$\lambda = \frac{d}{dx}\int_0^x C_\varphi\left(F_X\left(\frac{1}{\omega}(x-y)\right), F_Y\left(\frac{y}{1-\omega}\right)\right) dy$$

In particular, we have

$$\lambda = \frac{d}{dx}\int_0^x C_\varphi\left(F_X\left(\frac{1}{\omega}(x-y)\right), F_Y\left(\frac{y}{1-\omega}\right)\right) dy\bigg|_{x=\mathrm{VaR}_\lambda(z)} \tag{4.4}$$

We have developed a substantially simple formula in the case of Archimedean copulas, which seems to be new in the literature. Proof of this result can be seen at Appendix A.

4.2 Steps for estimating VaR

Computation of VaR may result difficult from formula 3. However, it can be computed using a simple Monte Carlo simulation by taking sample random returns from conditional distribution 4.1 and reevaluating portfolio at time t. So Value at Risk can be determined by taking the empirical quantile at λ of the simulated loss portfolio. We present the general algorithm for simulate n random sample for a generic copula.

4.2.1 Random sampling for copula

Definition 12 (Conditional distributions of copulas)**.** *Let C be copula function for U and V random variables. The Conditional Copula of V given U is given by*

$$\begin{aligned} c_u(v) = C_{V|U}(v \mid u) = P(V \leqslant v \mid U = u) &= \lim_{\delta \to 0} \frac{C(u+\delta, v) - C(u,v)}{\delta} \\ &= \frac{\partial}{\partial u} C(u,v) \end{aligned}$$

We know that $c_u(v)$ is a non-decreasing function and exists for almost all $v \in [0,1]$. The following procedure is described in Cherubini, Luciano, and Vecchiato (2004). We can generate a pair (u, v) from copula C in the following way:

- Generate two independent uniform r.v.s $(u, w) \in [0,1].u$ is the first draw we are looking for.
- Compute the (quasi-)inverse function of $c_u(v)$. This will depend on the parameters of the copula and on u, which can be seen, in this context, as an additional parameter of $c_u(v)$. Set $v = c_u^{-1}(w)$ to obtain the second desired draw.

4.2.2 Algorithm for VaR and CVaR

Let us suppose we are interested in estimating VaR and CVaR at time t for a level λ for a portfolio $Z = \omega X + (1-\omega)Y$ for one day step-ahead. The following steps describes the procedure which is formulated in Fantazzini (2008) and we also employ with modification in this thesis.

1. For each risk factor (x_i, y_i), find the parameters for ARMA(p,q)-GARCH(r,s) with desired residual innovation distributions $\hat{F}_x$ and $\hat{F}_y$ with parameters $\hat{\theta}_{x,t-1}$ and $\hat{\theta}_{y,t-1}$. Transform each risk factor to uniform series using the density of estimated marginals in 3.1:
$$(U_{t-1}, V_{t-1})' = (\hat{F}_x(x_i), \hat{F}_y(y_i))' \tag{4.5}$$
2. Estimate parameter θ_{t-1} for copula C with maximum likelihood function with the uniform series in step 1.
3. Simulate a sample of N random draws from copula C with parameter θ_{t-1}. Denote this as $(u, v)'$.
4. Using the inverse functions of estimated marginals, revert the sample to its original scale
$$Q = (q_x, q_y)' = (\hat{F}_x^{-1}(u), \hat{F}_y^{-1}(v))' \tag{4.6}$$

5. Rescale the standardized residuals by using the forecasted means and variances given by ARMA-GARCH estimates in step 1.

$$\{x_{j,t}, y_{j,t}\} = \left(\hat{\mu}_{x,t} + q_{j,x} \cdot \hat{\sigma}_{x,t}, \quad \hat{\mu}_{y,t} + q_{j,y} \cdot \hat{\sigma}_{y,t}\right) \tag{4.7}$$

for $j = 1, \ldots, N$

6. Recalculate the loss of portfolio as

$$Z_j = \omega x_{j,t} + (1-\omega) y_{j,t}$$

7. The VaR is equal to the 100 λ%-th statistic order of the recalculated loss of portfolio, i.e the λ quantile for sample. To get CVaR, we compute the mean of sub-sample obtained to extract all values of loss greater than VaR.

In order to get an accurate estimates of the quantile, Fantazzini (2008) simulates $N = 100000$ random samples. One of the main differences between our method proposed here and Fantazzini's work rely on the specification for the margins. While Fantazzini uses AR(1)-Threshold GARCH(1,1) with Normal, Skew-Normal, the Student's T and Skew-T distribution, this work uses AR(1)-GARCH(1,1) with a mixture of two normal distributions. First make use of IFM algorithm for estimating parameters for copula and densities through the MLE. Their specification requires a modified logistic transformation in order to keep the conditional skewness and degrees of freedom in the desired interval. But our approach relies on the QMLE for estimating the parameters for the margins which can be easier and faster to implement than plain MLE. We do not make use of logistic transformation but thanks to the advantage of EM algorithm, we can readjust the parameters for the mixture in each iteration step to be inside the parameter space. Also, this specification is utilized with the log-return for the risk assets while Fantazzini uses the prices.

Evidence shows that this selection is adequate. In chapter 5, empirical studies are presented, comparing classical methods, Fantazinni's method (Normal, and t distributed innovations), Lee and Lee (2011) ARMA-GARCH normal mixture VaR, and this thesis approach method which consists of both methodologies (normal mixture innovations with copula).

4.3 Multivariate Conditional Value at Risk

VaR is defined for a single random variable, and there has been much effort such that the definition is extended to involve multivariate random vectors. Indeed, in the pioneering work of Prékopa (2012) considers a vector valued multivariate Value at risk (MVaR). We may wonder, however, whether MVaR really serves as a risk measure; in other words, whether MVaR characterizes effectively the risk structure of multiple random variables, especially, the nonlinear dependence relation between each risk factors. The answer is partially yes and partly still under development.

Copulas, we have to recall at this point, are well recognized functions, which provide a useful tool for understanding the dependence relation among random variables (see for example Genest and Favre (2007)). Because of their flexibility, copulas are now widely employed in the research of dependence structure of random variables. It is then natural and desirable to define MVaR through the formulation of copulas. Along this prospect, Krzemienowski and Szymczyk (2016) have introduced the concept of copula-based conditional Value at Risk. Their definition is, however, somewhat complicated and it seems that the computation is hard and requires much work.

Here we introduce a new definition of copula-based conditional Value at Risk (CCVaR), which is rather simple, easy to calculate, and also enjoys nice properties. It is noted that our CCVaR extends the multivariate conditional Value at Risk introduced by Lee and Prékopa (2013). Moreover, in the case of Archimedean copulas, a handy formula of CCVaR is obtained. Examples show that it works well to estimate the nonlinear relation between risk factors.

4.3.1 Definition of CCVaR

Now we turn our attention to the multivariate conditional Value at Risk, where a new definition is introduced. In the previous attempts to estimate VaR, it is only defined for a single random variable, the total loss of the portfolio. It is our intention that multivariate random variables should be incorporated into the definition of Value at Risk, which will be more useful to application. Several attempts have been already implemented. For example, Prékopa (2012) introduce a multivariate Value at Risk for random vector, which is vector valued. However, because of the fact that the measure is vector valued, the order relation becomes slightly indirect.

Recently, Krzemienowski and Szymczyk (2016) introduced a nice idea of copula-based conditional Value at Risk. Here their definition in the bivariate case is presented.

Let $\mathbf{X} = (X_1, X_2)$ be a bivariate random vector with the distribution functions $F_{X_j}(t) = P(X_j \leq t)$ $(j = 1, 2)$. Given a copula C, $H(x, y) = C(F_{X_1}(x), F_{X_2}(y))$ becomes a joint distribution function. Let

$$\mathcal{U}_\lambda^{KS} = \{(u, v) \in \mathbb{R}^2 \mid C(u, v) = \lambda\}.$$

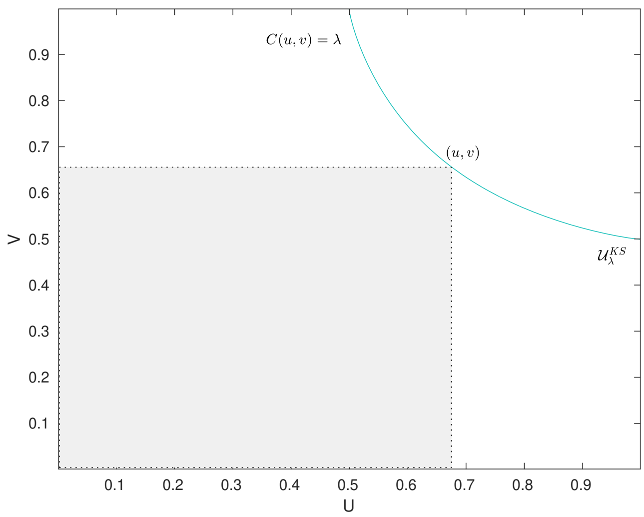

FIGURE 4.1: Region of integration of CCVaR in Krzemienowski and Szymczyk

A copula-based conditional Value at Risk ($\mathrm{CCVaR}_\lambda^{KS}(\mathbf{X})$) due to Krzemienowski and Szymczyk is then defined through

$$\mathrm{CCVaR}_\lambda^{KS}(\mathbf{X}) = \frac{1}{\lambda}\min_{(u,v)\in\mathcal{U}_\lambda^{KS}} \int_0^u \int_0^v (F_{X_1}^{(-1)}(p) + F_{X_2}^{(-1)}(q))dC(p,q). \tag{4.8}$$

However, since the risk measure involves the minimum procedure, the computation may become troublesome. For example, if $C(u,v) = \Pi(u,v) = uv$, namely, X_1 and X_2 are independent, then we see that

$$\begin{aligned}\mathrm{CCVaR}_\lambda^{KS}(\Pi) &= \frac{1}{\lambda}\min_{\lambda\le u\le 1} \int_0^u \int_0^{\frac{\lambda}{u}} (F_{X_1}^{(-1)}(p) + F_{X_2}^{(-1)}(q))dpdq \\ &= \frac{1}{\lambda}\min_{\lambda\le u\le 1}\Big(\frac{\lambda}{u}\int_0^u F_{X_1}^{(-1)}(p)dp + u\int_0^{\frac{\lambda}{u}} F_{X_2}^{(-1)}(q)dq\Big),\end{aligned}$$

taking into account of the fact

$$\mathcal{U}_\lambda^{KS}(\Pi) = \left\{\left(u, \frac{\lambda}{u}\right) \in \mathbb{R}^2 \mid \lambda \le u \le 1\right\}.$$

Here, another slightly different definition of copula-based multivariate conditional Value at Risk is proposed. We confine ourselves to the bivariate case as before for simplicity and let $\mathbf{X} = (X_1, X_2)$ be a random vector with the joint distribution function $H(x,y) = P(X_1 \le x, X_2 \le y)$ as well as marginal distribution functions $F_{X_j}(x) = P(X_j \le x)$ $(j = 1,2)$. Observing the definition of multivariate conditional Value at Risk introduced by Lee and Prékopa (2013), we now formulate our definition as follows:

Definition 13 (Copula-based Conditional Value at Risk). *For a random vector* $\mathbf{X} = (X_1, X_2)$, *a copula-based conditional Value at Risk* ($CCVaR_\lambda(\mathbf{X})$) *at the confidence level* λ $(0 \leq \lambda < 1)$ *is defined by*

$$\mathrm{CCVaR}_\lambda(\mathbf{X}) = \frac{\iint_{\mathcal{U}_\lambda} (\omega F_{X_1}^{(-1)}(u) + (1-\omega) F_{X_2}^{(-1)}(v)) dC(u,v)}{\iint_{\mathcal{U}_\lambda} dC(u,v)}, \tag{4.9}$$

where $0 < \lambda < 1$ *and we have put*

$$\mathcal{U}_\lambda := \{(u,v) \mid C(u,v) \geq \lambda\}.$$

The constant ω represents the portfolio weight of X_1 and X_2. If both X_1 and X_2 follows the same distribution, then the impact due to ω will be irrelevant. We also remark that for some C the denominator is zero and/or for some (X_1, X_2) the numerator is infinite.

It is to be noted that if we write temporally for abuse of notation

$$E^C[f] = \iint_{\mathbb{I}^2} f(u,v) dC(u,v),$$

then CCVaR can be written as

$$\mathrm{CCVaR}_\lambda(\mathbf{X}) = E^C[\boldsymbol{\omega}^t \mathbf{F}_{\mathbf{X}}^{(-1)} \mid \mathcal{U}_\lambda]$$

where $\boldsymbol{\omega}^t = (\omega, 1-\omega)$, which indicates that this Definition above extends Definition 3 of Lee and Prékopa (2013).

This CCVaR of (4.9) is simpler than CCVaR^{KS} of (4.8). Nevertheless, this definition seems to work well as a risk measure, which will be assured by the computation of examples in the next section.

4.3.2 Properties of CCVaR

First, basic properties of CCVaR are stated.

Proposition 3. *A copula-based conditional Value at Risk* CCVaR *defined by (4.9) verifies:*

(i) $\mathrm{CCVaR}_\lambda(\mathbf{0}) = 0$,

(ii) $\mathrm{CCVaR}_\lambda(\mathbf{X} + k\mathbf{e}) = \mathrm{CCVaR}_\lambda(\mathbf{X}) + k \qquad (k \in \mathbb{R}, \mathbf{e} = (1,1))$,

(iv) $\mathrm{CCVaR}_\lambda(s\mathbf{X}) = s\mathrm{CCVaR}_\lambda(\mathbf{X}) \quad (s > 0)$.

The proof is performed along the similar line of that for VaR.

Several remarks are in order. Concerning the monotonicity (iii), it has to be clarified the meaning of the order between $\mathbf{X}_1$ and $\mathbf{X}_2$; it is better avoid unfavorable assumption and not

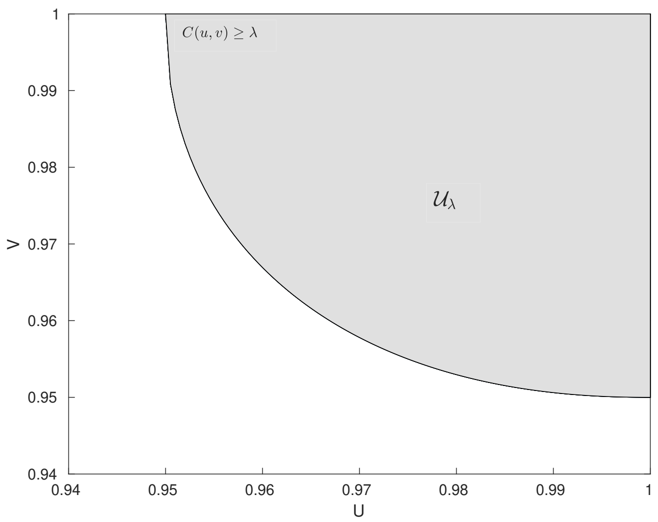

FIGURE 4.2: Region of integration of proposed CCVaR

treat it here. For the sub-additivity (v), it does not seem to be true in the general setting. Observe Theorem 8 and the example in Lee and Prékopa (2013).

In the case of Archimedean copula, the calculation of CCVaR seems to be ready to implement as it is stated in the following theorem.

Theorem 4. *Let* $\mathbf{X} = (X_1, X_2)$ *be a non-negative random vector, whose joint distribution function is provided by an Archimedean copula C of the form (3.7), where the generator* φ *is* C^1*-class. Then the proposed copula-based conditional Value at Risk* (CCVaR) *of (4.9) is expressed as*

$$\mathrm{CCVaR}_\lambda(\mathbf{X}) = \frac{\int_\lambda^1 (\omega F_{X_1}^{(-1)}(t) + (1-\omega) F_{X_2}^{(-1)}(t)) \left(1 - \frac{\varphi'(t)}{\varphi'(\lambda)}\right) dt}{1 - \lambda + \frac{\varphi(\lambda)}{\varphi'(\lambda)}}. \tag{4.10}$$

Proof of this theorem can be seen in Appendix A.

If the generator is $\varphi(t) = -\log t$, then the corresponding Archimedean copula is $\Pi(u,v) = uv$, that is, the product copula which represents the independence relation. In this case, the relevant CCVaR reduces to the multivariate conditional Value at Risk (MCVaR) due to Lee and Prékopa (2013), namely,

$$\mathrm{MCVaR}_\lambda(\mathbf{X}) = \frac{\int_\lambda^1 (\omega F_{X_1}^{(-1)}(t) + (1-\omega) F_{X_2}^{(-1)}(t)) \left(1 - \frac{\lambda}{t}\right) dt}{1 - \lambda + \lambda \log \lambda}.$$

We then obtain the next property immediately from Theorem 4.

Corollary 5. *If the generator φ verifies for $\lambda \le t \le 1$*

$$\begin{aligned} \frac{1-\frac{\varphi'(t)}{\varphi'(\lambda)}}{1-\lambda+\frac{\varphi(\lambda)}{\varphi'(\lambda)}} &\ge \frac{1-\frac{\lambda}{t}}{1-\lambda+\lambda\log\lambda} \\ \Big(\textit{resp.,}\ \frac{1-\frac{\varphi'(t)}{\varphi'(\lambda)}}{1-\lambda+\frac{\varphi(\lambda)}{\varphi'(\lambda)}} &\le \frac{1-\frac{\lambda}{t}}{1-\lambda+\lambda\log\lambda}\Big), \end{aligned} \tag{4.11}$$

then the corresponding CCVaR *is not less than (resp., not greater than) the one for the independent relation. Precisely stated, we have*

$$\begin{gathered} \mathrm{CCVaR}_\lambda(\mathbf{X}) \ge \mathrm{MCVaR}_\lambda(\mathbf{X}) \\ (\textit{resp.,} \mathrm{CCVaR}_\lambda(\mathbf{X}) \le \mathrm{MCVaR}_\lambda(\mathbf{X})). \end{gathered}$$

It seems, however, that the analytical application of the inequality of this Corollary is tough in general. Below we present examples to illustrate these observations. To do so, we introduce a function

$$f(t) = \Big(1-\frac{\varphi'(t)}{\varphi'(\lambda)}\Big)(1-\lambda+\lambda\log\lambda) - \Big(1-\frac{\lambda}{t}\Big)\Big(1-\lambda+\frac{\varphi(\lambda)}{\varphi'(\lambda)}\Big)$$

for $\lambda \le t \le 1$. It is easy to see that Corollary means that $f \ge 0$ on $\lambda \le t \le 1$ (resp. $f \le 0$ on $\lambda \le t \le 1$) is equivalent to $\mathrm{CCVaR}_\lambda(\mathbf{X}) \ge \mathrm{MCVaR}_\lambda(\mathbf{X})$ (resp. $\mathrm{CCVaR}_\lambda(\mathbf{X}) \le \mathrm{MCVaR}_\lambda(\mathbf{X})$)).

By virtue that we will consider the value of λ near to 1, indeed, we take $\lambda = 0.95$ and/or $\lambda = 0.99$ in our computation, the sign of $f(1)$, by the continuity of f, will be a dominant factor in the evaluation of (4.10). We are therefore content merely with the expansion of $f(1)$ in terms of $1-\lambda$, which seems sufficient in view of our numerical results in the next chapter. In addition, as a matter of fact, the evaluation of $f'(t)$ on $\lambda \le t \le 1$ requires too much task.

Example 6. *Let the generator be $\varphi(t) = \log(t^{-1}(1-\theta(1-t)))$ for $\theta \in [-1,1)$; that is, we consider the Ali-Mikhail-Haq family, which yields*

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}.$$

We then see that the corresponding $CCVaR^{AMH}$ becomes

$$\mathrm{CCVaR}_\lambda^{AMH}(\mathbf{X}) = \frac{\int_\lambda^1 (\omega F_{X_1}^{(-1)}(t) + (1-\omega)F_{X_2}^{(-1)}(t))\Big(1-\frac{\lambda(1-\theta(1-\lambda))}{t(1-\theta(1-t))}\Big)dt}{1-\lambda+\frac{\lambda(1-\theta(1-\lambda))}{1-\theta}\log\frac{\lambda}{1-\theta(1-\lambda)}}$$

Now we infer that

$$\begin{aligned}
\frac{f(1)}{\lambda} &= 1-\lambda+\log\lambda+\frac{1-\theta(1-\lambda)}{1-\theta}\Big(-(1-\theta)(1-\lambda+\lambda\log\lambda)+(1-\lambda)\log\frac{1-\theta(1-\lambda)}{\lambda}\Big)\\
&= -\frac{1}{2}(1-\lambda)^2-\frac{1}{3}(1-\lambda)^3-\frac{1}{4}(1-\lambda)^4-\cdots\\
&\quad -(1-\theta(1-\lambda))\Big((1-\lambda)^2-\frac{\lambda}{2}(1-\lambda)^2-\frac{\lambda}{3}(1-\lambda)^3-\frac{\lambda}{4}(1-\lambda)^4-\cdots\Big)\\
&\quad +\frac{1-\theta(1-\lambda)}{1-\theta}(1-\lambda)\Big((1-\theta)(1-\lambda)+\frac{1-\theta^2}{2}(1-\lambda)^2+\frac{1-\theta^3}{3}(1-\lambda)^3+\cdots\Big)\\
&= -\frac{\theta^2}{6}(1-\lambda)^4+O((1-\lambda)^5),
\end{aligned}$$

taking into account that, by the Taylor expansion,

$$\begin{aligned}
\log\lambda = \log(1-(1-\lambda)) &= -(1-\lambda)-\frac{1}{2}(1-\lambda)^2-\cdots\\
\log(1-\theta(1-\lambda)) &= -\theta(1-\lambda)-\frac{\theta^2}{2}(1-\lambda)^2-\cdots
\end{aligned}$$

Thus it is expected that

$$\mathrm{CCVaR}_\lambda^{AMH}(\mathbf{X}) \le \mathrm{MCVaR}_\lambda(\mathbf{X})$$

Example 7. *Let the generator be $\varphi(t)=\log(1-\theta\log t)$ for $0<\theta\le 1$; that is, we consider the Gumbel-Barnett family, which yields*

$$C(u,v)=uv\exp(-\theta\log u\log v).$$

We then see that the corresponding CCVaR^{GB} becomes

$$\mathrm{CCVaR}_\lambda^{GB}(\mathbf{X})=\frac{\int_\lambda^1(\omega F_{X_1}^{(-1)}(t)+(1-\omega)F_{X_2}^{(-1)}(t))\Big(1-\frac{\lambda}{t}\frac{(1-\theta\log\lambda)}{(1-\theta\log t)}\Big)dt}{1-\lambda-\frac{\lambda}{\theta}(1-\theta\log\lambda)\log(1-\theta\log\lambda)}.$$

Now we infer that

$$\frac{f(1)}{\lambda}=1-\lambda+\log\lambda+\frac{1-\theta\log\lambda}{\theta}\Big(-\theta(1-\lambda+\lambda\log\lambda)+(1-\lambda)\log(1-\theta\log\lambda)\Big).$$

Applying the Taylor expansion

$$\begin{aligned}
\log(1-\theta\log\lambda) &= -\theta\log\lambda-\frac{\theta^2}{2}(\log\lambda)^2-\frac{\theta^3}{3}(\log\lambda)^3-\cdots\\
&= \theta(1-\lambda)+\frac{\theta}{2}(1-\lambda)^2+\frac{\theta}{3}(1-\lambda)^3-\frac{\theta^2}{2}\Big(-(1-\lambda)-\frac{1}{2}(1-\lambda)^2-\cdots\Big)^2\\
&\quad +\frac{\theta^3}{3}(1-\lambda)^3+\cdots,
\end{aligned}$$

we further compute that

$$
\begin{aligned}
\frac{f(1)}{\lambda} &= -\frac{1}{2}(1-\lambda)^2 - \frac{1}{3}(1-\lambda)^3 - \frac{1}{4}(1-\lambda)^4 - \cdots \\
&\quad - (1-\theta\log\lambda)\Big((1-\lambda)^2 - \frac{\lambda}{2}(1-\lambda)^2 - \frac{\lambda}{3}(1-\lambda)^3 - \frac{\lambda}{4}(1-\lambda)^4 - \cdots\Big) \\
&\quad + (1-\theta\log\lambda)(1-\lambda)\Big((1-\lambda) + \frac{1}{2}(1-\lambda)^2 + \frac{1}{3}(1-\lambda)^3 + \cdots \\
&\qquad - \frac{\theta}{2}(1-\lambda)^2 - \theta(1-\lambda)^3 + \frac{\theta^2}{3}(1-\lambda)^3 + \cdots\Big) \\
&= -\Big(\Big(1 - \frac{7}{12}\lambda\Big)\theta + \frac{\theta^2}{6}\Big)(1-\lambda)^4 + O((1-\lambda)^5)
\end{aligned}
$$

which is negative for $\theta << 1$. Thus it is expected, in this case, that

$$
\mathrm{CCVaR}^{GB}_{\lambda}(\mathbf{X}) \le \mathrm{MCVaR}_{\lambda}(\mathbf{X}),
$$

which agrees with our numerical computation.

Example 8. *Let the generator be $\varphi(t) = (-\log t)^{\theta}$ for $1 \le \theta < \infty$; that is, we consider the Gumbel family, which yields*

$$
C(u,v) = \exp(-((-\log u)^{\theta} + (-\log v)^{\theta})^{1/\theta}).
$$

We then see that the corresponding CCVaR^G becomes

$$
\mathrm{CCVaR}^{G}_{\lambda}(\mathbf{X}) = \frac{\int_{\lambda}^{1}(\omega F^{(-1)}_{X_1}(t) + (1-\omega)F^{(-1)}_{X_2}(t))\Big(1 - \frac{\lambda(-\log t)^{\theta-1}}{t(-\log\lambda)^{\theta-1}}\Big)dt}{1 - \lambda + \frac{\lambda}{\theta}\log\lambda}.
$$

In this case, we have that

$$
\begin{aligned}
\frac{f(1)}{\lambda} &= 1 - \lambda + \log\lambda - \frac{1}{\theta}(1-\lambda)\log\lambda \\
&= \frac{2-\theta}{2\theta}(1-\lambda)^2 + O((1-\lambda)^3)
\end{aligned}
$$

which is positive for $\theta < 2$ and it is expected, in this case, that

$$
\mathrm{CCVaR}^{G}_{\lambda}(\mathbf{X}) \ge \mathrm{MCVaR}_{\lambda}(\mathbf{X}),
$$

which agrees with our numerical computation in the next chapter.

Chapter 5

Empirical study

In this chapter, the numerical studies applied to estimation of VaR and CVaR are presented. The main idea is to show that this thesis approach, which consists in merging the copula modeling and normal mixture gaussian innovations, results in better estimates for VaR in the sense of backtesting accepting. We compare this thesis approach with classical VaR methodologies, and copula with normal and Student-t distributed standardized residual innovations already implemented in Palaro and Hotta (2006) and the plain VaR with Gaussian mixture ARMA-GARCH model described on Lee and Lee (2011). We present three cases, the first using copulas proposed by Palaro and Hotta (2006). Secondly, a case with Archimedian copulas in order to use the new determination formula. And finally, a case for CCVaR is presented and compared to MCVaR of Lee and Prékopa (2013).

5.1 A case study for VaR with copulas

In this example, we want to show that VaR estimation with this thesis method, consisting in merging copula with Gaussian mixture ARMA-GARCH models, results in better estimates than plain Gaussian mixture ARMA-GARCH Value at Risk by Lee and Lee (2011). These results are published in Molina Barreto, Ishimura, and Yoshizawa (2019).

5.1.1 Descriptive Statistics

The data base used for our empirical analysis consists of daily geometric return obtained from closing prices for the NASDAQ and Nikkei 225 from 22 August 2013 to 21 August 2018 with a total of 1188 trading days. The data is taken from Yahoo Finance. Table (5.1) contains descriptive statistics and Figure (5.1) presents plots of both series. The implementation is performed with MATLAB.

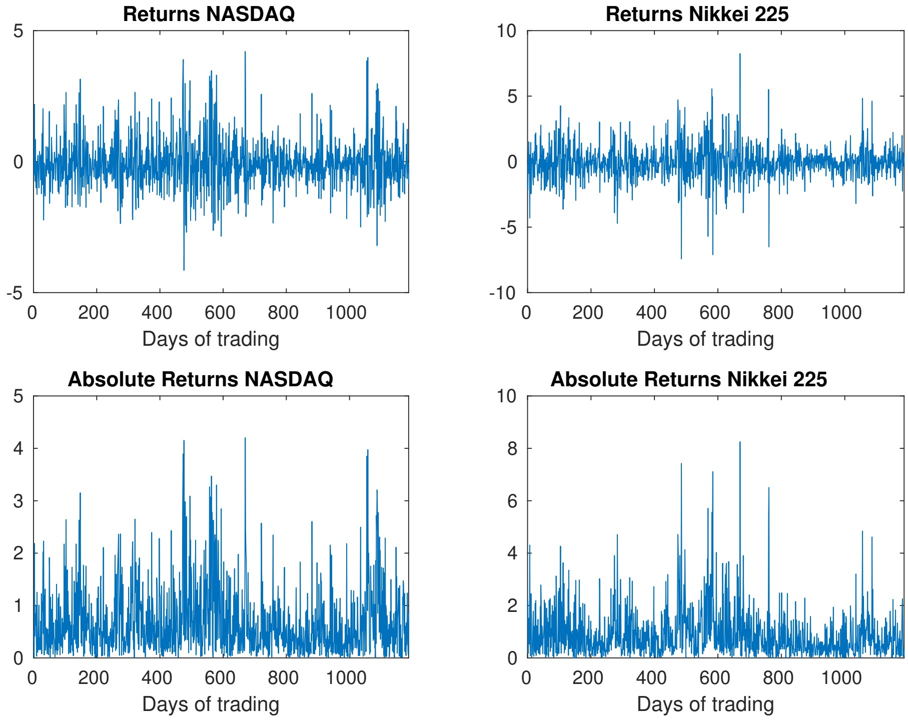

FIGURE 5.1: Daily and absolute returns of NASDAQ and Nikkei 225

Statistics	NASDAQ	Nikkei 225
Mean	−0.0652	−0.0428
Standard Deviation	0.9546	1.3048
Minimum	−4.1520	−7.4262
Median	−0.1105	−7.4262
Maximum	4.2023	8.2529
Kurtosis	5.2800	7.8141
Asymmetry	0.6011	0.1742

TABLE 5.1: Descriptive Statistics of daily log-returns of NASDAQ and Nikkei 225

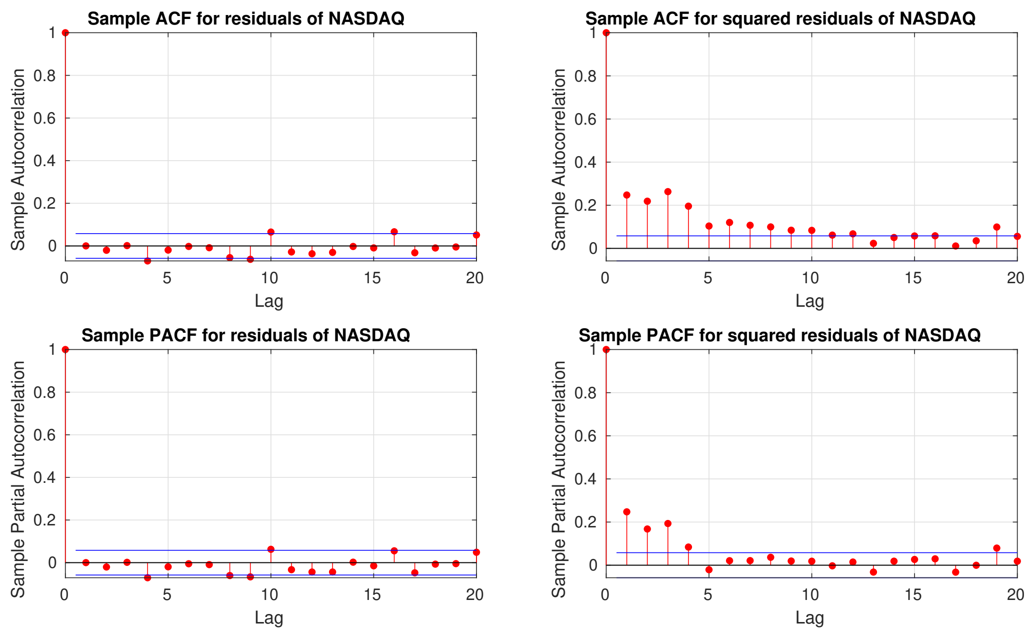

FIGURE 5.2: ACF and PACF plots for NASDAQ

5.1.2 Margin modeling

For specifying a model for both series, we consider an ARMA(p, q)-GARCH(r, s) for $t = 1, \ldots, 1187$. Distribution for standardized residuals is set to a Gaussian mixture model. In order to choose adequate orders for ARMA-GARCH model, we consult methodology proposed by Francq and Zakoian (2019) in Chapter 5. It states that select minimal orders (r, s) for GARCH with sample auto-correlations estimates must be inside significance bands $\pm 1.96/\sqrt{T}$. We can also consider Ljung-Box Q test (see (McLeod and Li, 1983)) in which the null hypothesis that a series of residuals exhibits no auto-correlation for a fixed number of lags L, against the alternative that some auto-correlation coefficient $\rho(k)$, $k = 1, ..., L$ is nonzero. In figure 5.2 and 5.3 we can observe auto-correlation function (ACF) and partial auto-correlation function (PACF) for losses in NASDAQ and Nikkei series, respectively.

In fact, we fitted two AR(1)-GARCH(1,1) for both series as initial models with a two component Gaussian mixture. This selection was considered to see there was no autocorrelation nor squared auto-correlation in the residuals. Also it is usual to consider two or three components for the mixture of normals (here we only report the case of two). We also performed a Ljung Box test to infer the null hypothesis is not rejected from lag 1 to 5. We report these values in table (5.2). Finally, we report values for Kolmogorov-Smirnov (KS), Chi-square goodness-of-fit test (CSG) and Anderson-Darling test used for uniformity test for the standardized residuals.

Parameter	NASDAQ	Nikkei225
a_0	-0.0966	-0.0894
	(0.6188)	(0.6350)
a_1	-0.0339	-0.0331
	(0.8024)	(0.8366)
c_0	0.1288	0.0906
	(0.5923)	(0.5279)
c_1	0.1820	0.1600
	(0.7934)	(0.5687)
d_1	0.6748	0.7964
	(1.1545)	(0.6589)
π	0.7545	0.6791
	(2.1270)	(2.4874)
μ_1	-0.1858	-0.1247
	(1.2318)	(1.0617)
μ_2	0.5710	0.2639
	(6.5601)	(3.9875)
σ_1^2	0.5163	0.4324
	(1.9407)	(2.2565)
σ_2^2	2.0541	2.0988
	(8.5172)	(10.6249)
AIC	15.7468	14.5864
$Q^2(1)$	0.9840	0.8117
$Q^2(5)$	0.8655	0.6028
KS	0.0449	0.0221
χ^2	0.2043	0.0262
AD	0.0562	0.0218

TABLE 5.2: Model parameters of univariate Gaussian mixture ARMA-GARCH. Standard errors between brackets. Last values correspond to p-values for each test.

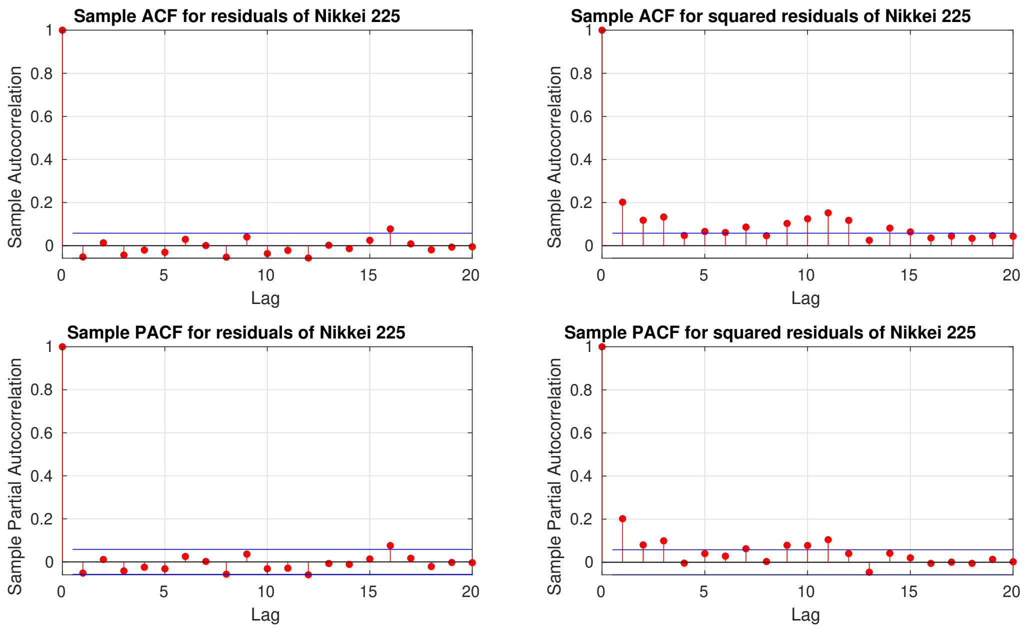

FIGURE 5.3: ACF and PACF plots for Nikkei 225

A good alternative to choose between sereval models is to use Akaike's information criterion (AIC) defined as:

$$AIC(M) = -2\text{log-likelihood}(\theta_t) + 2M$$

where M is the number of parameters in the maximum likelihood estimates. AIC's coefficient penalties in the case of more parameters, so better model will have a smaller AIC value. (Akaike, 1973) We can consider a good fit for estimates of the margins if transform given by (3.1) is close to uniform cumulative distribution. In this example, we can see this fit in figure (5.5)

5.1.3 VaR estimation

We again consider the portfolio of equal weight. First we estimate the parameters using the data from $t = 1$ to $t = 487$ as initial window and update the parameters each day for both the marginal distributions and for the copula. Our target is to find the solution of (**??**) for VaR at the level $\lambda = 0.95$ and $\lambda = 0.99$ concerning the data from $t = 489$ to $t = 1188$ (699 days). At table (5.3), we can look at the proportion of observations where the loss exceeded the confidence level. We then compare the forecast VaR with the actual return of the portfolio. However, the computation is highly demanding and a Monte-Carlo simulation is preferred. By looking at the number of failures, we could infer the copula models have fewer number of failures for this portfolio with respect to other methods. But if we consider the effect

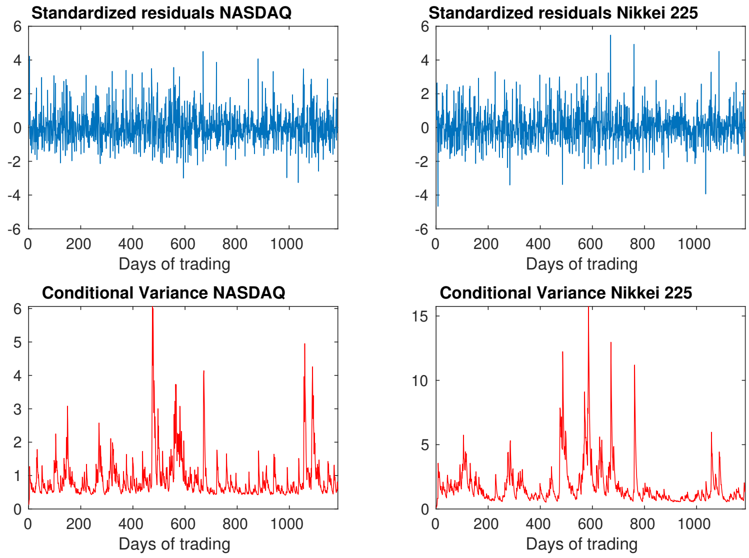

FIGURE 5.4: Empirical distribution of transformed series

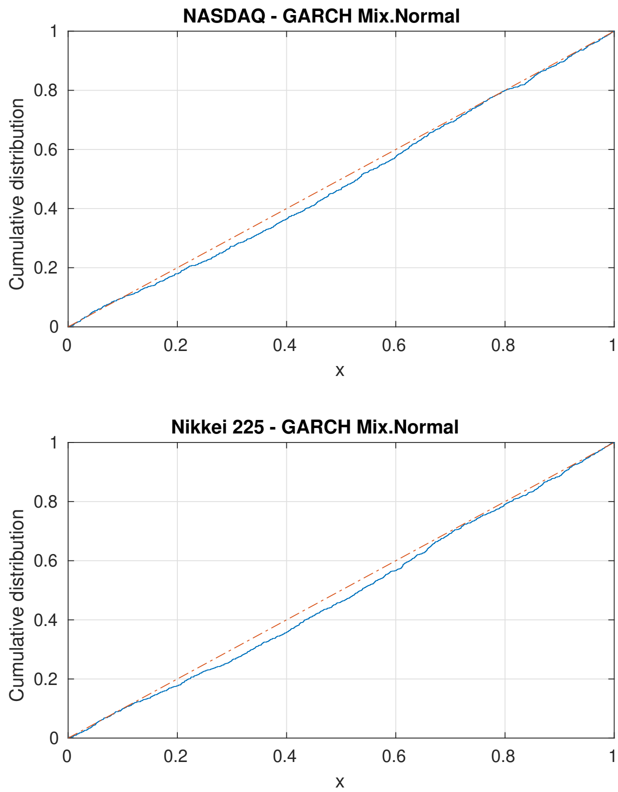

FIGURE 5.5: Empirical distribution of transformed series

of nonlinear dependence given by the copula and the improvement of implementing it to the computation of VaR, we can see it outperform in higher levels of confidence. We also compared with benchmark models like Variance Covariance and Historical VaR. In all cases, the model with Symmetrized Joe Clayton copula gives the best results. Data is exhibited in Figure (5.6) and (5.7).

5.1.4 Backtesting

To ascertain the outcome of computation, several back-testings are considered.Here we appeal to Binomial test (Bin), Kupiec's POF test (POF), and Christoffersen's test (CCI), respectively. Results are shown in table (5.3).

We can infer the proposed models with copulas results in better estimations than plain ARMA-GARCH Normal mixture models. Thanks to the property of copula, we can explain a better non-linear correlation between the two indexes studied here. In effect, for extreme losses, the copulas give better estimates and pass all the back-tests.

5.2 A case study for VaR with Archimedean copulas

In this case, we study the VaR estimation with some Archimedean copula namely, Clayton, Gumbel and Frank. Estimation can be done numerically from formula (3). Results are published in Molina Barreto and Ishimura, 2020.

5.2.1 Descriptive Statistics

We consider a portfolio composed of two assets: the S&P 500 and Jakarta Stock Exchange Composite Index (JCI). The data contains 2377 daily closing prices from December 7 2009 to December 6 2019, and we compute the daily log-returns and ignore the entries that are not available at the same time in any of both markets. The data period excludes the direct effect of the United States subprime mortgage crisis that started in 2007. The data is taken from Yahoo Finance, and implementation is performed with MATLAB. Statistics are displayed in table (5.5) and Figure (5.8) shows the plot for both log-return series. We remark the excess of kurtosis and the significance of negative asymmetry in this case.

5.2.2 Margin modeling

For each marginal series, we consider a general AR(1)-GARCH(1,1) model with innovations with a two compounded Gaussian mixture distribution. This idea seems accurate due to the characteristics, such as asymmetry and excess of kurtosis, that can be seen on the series. We

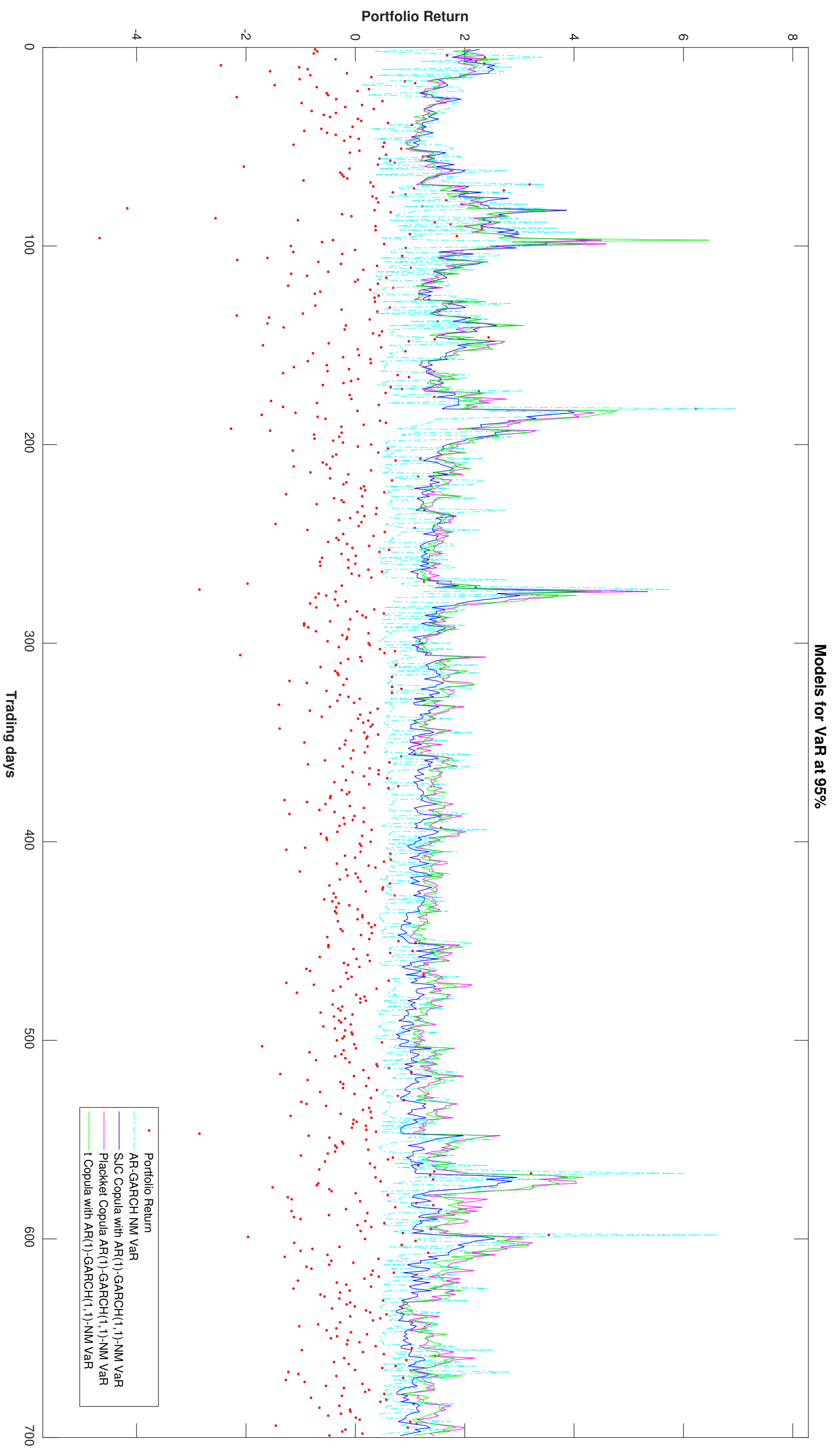

FIGURE 5.6: One day ahead forecasts of VaR at $\lambda = 0.95$ for portfolio of NASDAQ and Nikkei 225 with Gaussian mixture margins and various copulas

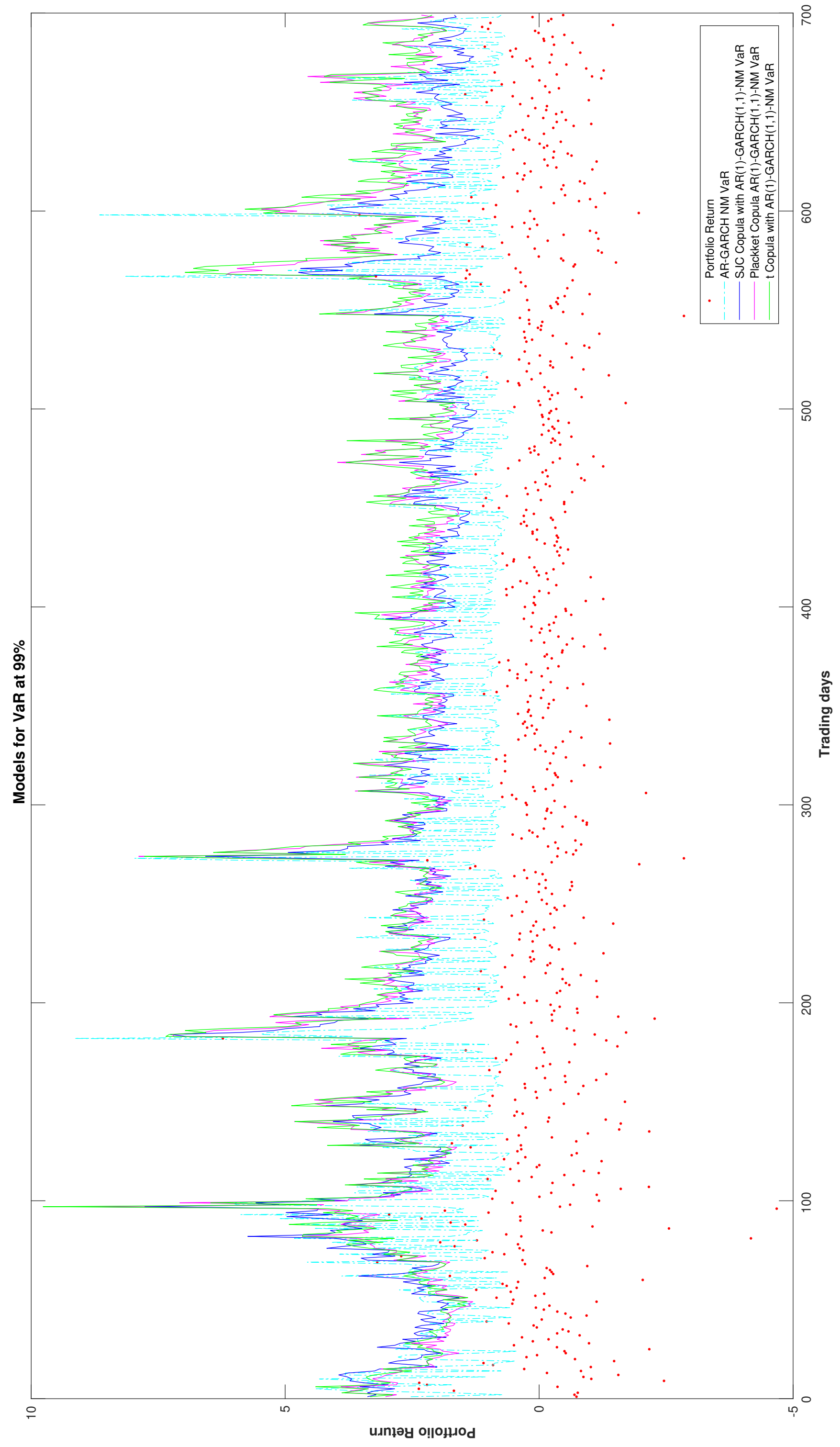

FIGURE 5.7: One day ahead forecasts of VaR at $\lambda = 0.99$ for portfolio of NASDAQ and Nikkei 225 with Gaussian mixture margins and various copulas

Model	Bin	z_α	p_{bin}	Failures	Prop. Fails	POF	Lik. ratio	p_{POF}	CCI	Lik. ratio	p_{CCI}
											$\lambda = 0.95$
ARMA-GARCH-NM	accept	0.8764	0.3808	40	0.0572	accept	0.7354	0.3911	accept	1.2003	0.2733
SJC - NM	accept	-0.8491	0.3903	30	0.0429	accept	0.7735	0.3791	reject	4.1760	0.0410
Plackett - NM	reject	-2.5945	0.0095	20	0.0286	reject	7.9066	0.0049	accept	0.2788	0.5975
t-Student - NM	reject	-2.4210	0.0155	21	0.0300	reject	6.7964	0.0091	accept	0.1956	0.6583
Historical	accept	-1.3797	0.1677	27	0.0386	accept	2.0584	0.1514	reject	9.0449	0.0026
Variance - Covariance	accept	1.2235	0.2211	42	0.0601	accept	1.4102	0.2350	reject	6.4041	0.0114
											$\lambda = 0.99$
ARMA-GARCH-NM	reject	2.2846	0.0223	13	0.0186	reject	4.1645	0.0413	reject	5.4431	0.0196
SJC - NM	accept	-0.7565	0.4494	5	0.0072	accept	0.6353	0.4254	accept	0.0722	0.7882
Plackett - NM	accept	-0.7564	0.4494	5	0.0072	accept	0.6353	0.4254	accept	0.0722	0.7882
t-Student - NM	accept	-1.5168	0.1293	3	0.0043	accept	2.9278	0.0871	accept	0.0259	0.8721
Historical	accept	-0.3763	0.7067	6	0.0086	accept	0.1488	0.6997	accept	0.1040	0.7470
Variance - Covariance	reject	3.4251	0.0006	16	0.0229	reject	8.5973	0.0034	accept	0.7922	0.3734

TABLE 5.3: Backtesting for estimated VaR models with Copulas

Model	Uncond. Normal	p_{UN}	Test. Stat.	Critical Value	Uncond. T	p_{UT}	Test. Stat.	Critical Value
								$\lambda = 0.95$
AR-GARCH - NM	accept	0.1316	-0.1973	-0.2864	accept	0.1652	-0.1973	-0.3410
SJC - NM	accept	0.5000	0.1102	-0.2864	accept	0.5000	0.1102	-0.3410
PLA - NM	accept	0.5000	0.3843	-0.2864	accept	0.5000	0.3843	-0.3410
t-Student - NM	accept	0.5000	0.3956	-0.2864	accept	0.5000	0.3956	-0.3410
Historical	accept	0.5000	0.2159	-0.2864	accept	0.5000	0.2159	-0.3410
Variance - Covariance	reject	0.0038	-0.4918	-0.2864	reject	0.0151	-0.4918	-0.3410
								$\lambda = 0.99$
AR-GARCH - NM	reject	0.0204	-0.8703	-0.6696	reject	0.0387	-0.8703	-0.7762
SJC - NM	accept	0.5000	0.0524	-0.6696	accept	0.5000	0.0524	-0.7762
PLA - NM	accept	0.5000	0.1406	-0.6696	accept	0.5000	0.1406	-0.7762
t-Student - NM	accept	0.5000	0.4192	-0.6696	accept	0.5000	0.4192	-0.7762
Historical	accept	0.5000	0.0644	-0.6696	accept	0.5000	0.0644	-0.7762
Variance - Covariance	reject	0.0001	-2.0199	-0.6696	reject	0.0019	-2.0199	-0.7762

TABLE 5.4: Backtesting for estimated CVaR models with Copulas

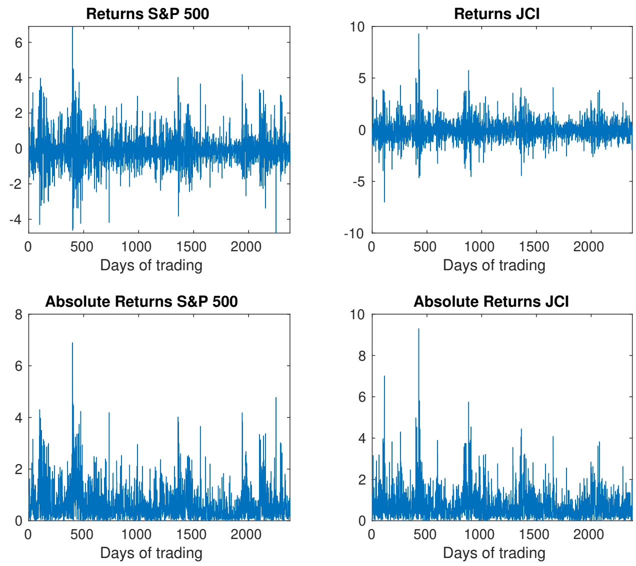

FIGURE 5.8: Daily and absolute returns of S&P 500 and JCI

Statistics	S&P	JCI
Mean	−0.0441	−0.0384
Standard Deviation	0.9582	1.0504
Minimum	−4.7775	−7.0136
Median	−0.0611	−0.1002
Maximum	6.8958	9.2997
Kurtosis	7.3810	9.1121
Asymmetry	0.4615	0.5916

TABLE 5.5: Descriptive statistics of daily log-returns of S&P 500 and JCI stock indices

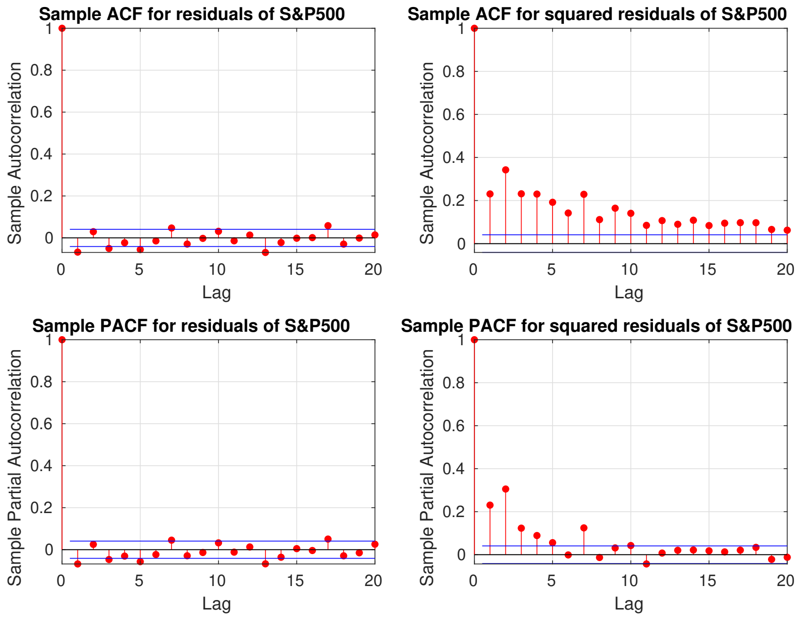

FIGURE 5.9: ACF and PACF plots for S&P500

have observed the ARMA-GARCH with normal mixture distributed innovation models fit this kind of series better than plain ARMA-GARCH with normal or t distributions.

This selection was considered to see there was no auto-correlation nor squared auto-correlation in the residuals. We also performed Ljung Box test to infer the null hypothesis is not rejected from lag 1 to 5. Values for Kolmogorov-Smirnov (KS), Chi-square goodness-of-t test (CSG) and Anderson-Darling test used for uniformity test for the standardized residuals are all accepted for significance level of 5%. In figure (5.11) are plotted Conditional variance and Standardized Residuals for both series.

5.2.3 VaR estimation

We again consider portfolio of equal weights. First, we estimate the parameters using data from $t = 1$ to $t = 1376$ as the initial window and update the parameters each day as for the marginal distributions as for the copula. Our target is to find the solution formula for VaR at the level and concerning the data from $t = 1377$ to $t = 2376$ (1000 days). We then compare the forecast VaR with the actual return of the portfolio.

By looking at the value of failures, we could infer that performance of the proposed model over classical estimations of VaR as Historical or Variance-Covariance is better thanks to the effect of nonlinear dependence given by the copula and the improvement of implementing it to the computation of VaR. We also compared with benchmark models like Variance-Covariance. In all cases, the model with Clayton-Normal mixture gives the best results. Data is exhibited in Figure (5.13) and Figure (5.14).

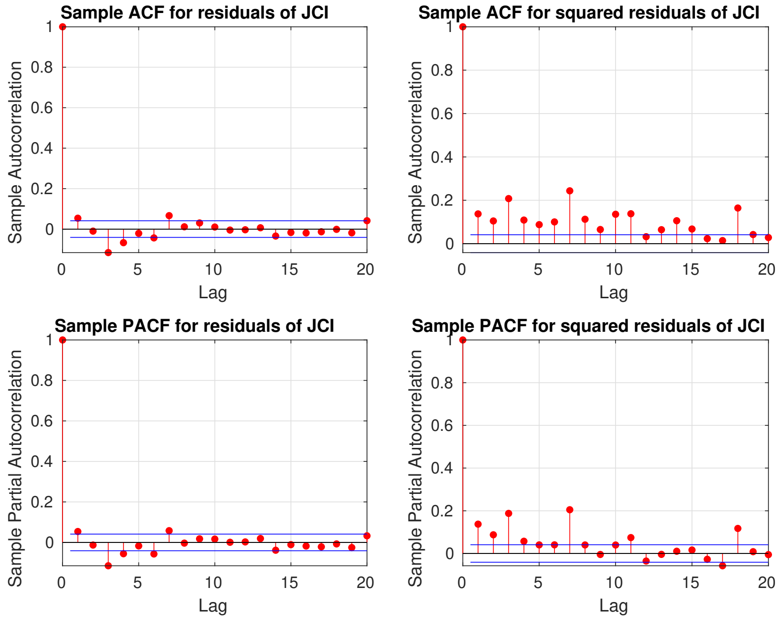

FIGURE 5.10: ACF and PACF plots for JCI

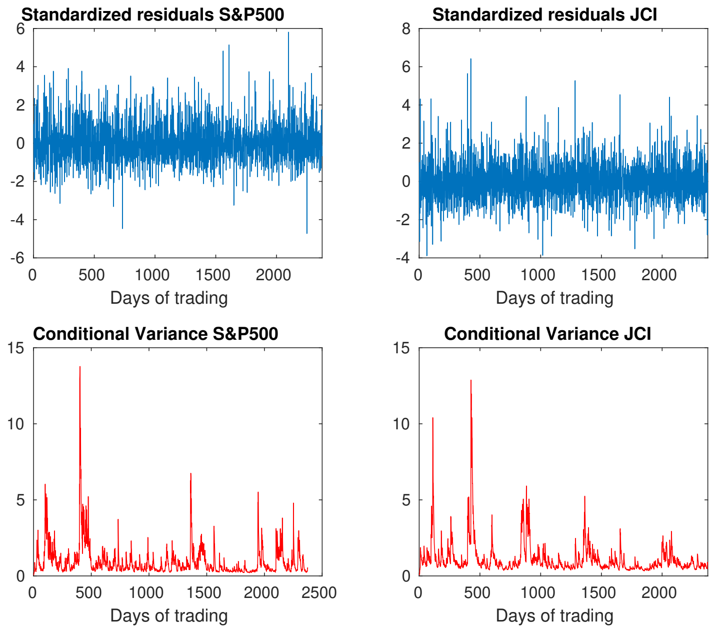

FIGURE 5.11: Conditional variance and Standardized Residuals for log returns series of S&P 500 and JCI stock indices

Parameter	S&P500	JCI
a_0	-0.0928	-0.0511
	(0.5625)	(0.5811)
a_1	-0.0705	0.0394
	(0.7274)	(0.6402)
c_0	0.0440	0.0474
	(0.2219)	(0.3465)
c_1	0.1895	0.1284
	(0.6909)	(0.6524)
d_1	0.7708	0.8318
	(0.7003)	(0.8021)
π	0.6707	0.5811
	(2.7581)	(2.6726)
μ_1	-0.1499	-0.0959
	(1.1543)	(1.1897)
μ_2	0.3054	0.4083
	(4.3484)	(7.5223)
σ_1^2	0.4483	0.5995
	(2.4498)	(5995)
σ_2^2	1.9846	2.4983
	(9.1468)	(15.5862)
AIC	16.0043	15.4625
$Q^2(1)$	0.4284	0.7436
$Q^2(5)$	0.9054	0.8887
KS	0	0.2559
χ^2	0	0.2293
AD	0.0001	0.3000

TABLE 5.6: Model parameters of univariate Gaussian mixture ARMA-GARCH. Standard errors between brackets. Last values correspond to p-values for each test.

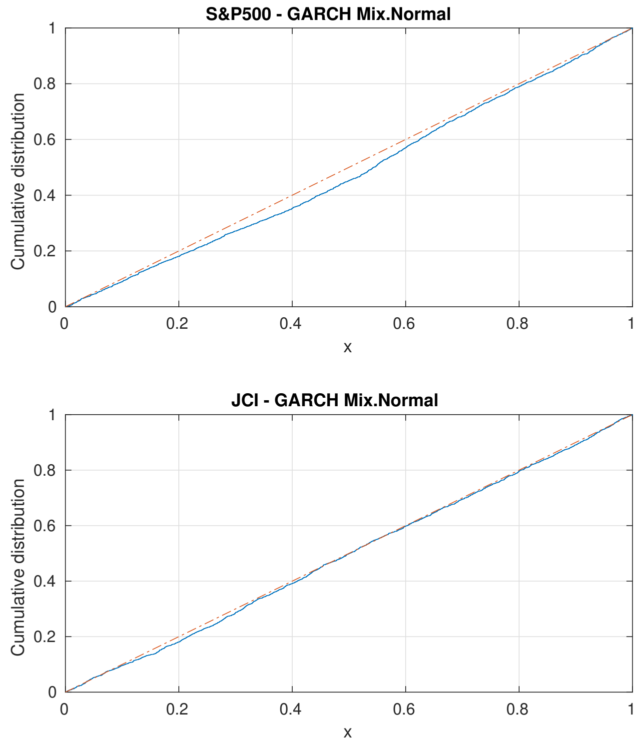

FIGURE 5.12: Empirical distribution of transformed series

5.2.4 Backtesting

We again have used the Binomial test (Bin), Kupiec's POF test (POF), and Christoffersen's test (CCI), respectively. The result is shown in figure (5.7). Our empirical analysis has shown that the proposed models with copulas results in better estimations than models such as Historical or Var-Covariance methods. Thanks to the property of copula, we can explain a better non-linear correlation between the two indexes studied here. In effect, for extreme losses, the copulas give better estimates and pass all back-tests. We can also observe a similar behavior for the three copulas.

5.3 A case study for CCVaR and MCVaR

We now turn our attention to empirical analysis of estimating CCVaR.

5.3.1 Description of data

The data base used for our empirical analysis consists of daily geometric return obtained from closing prices for the S&P 500 and Nikkei 225 from September 9 2010 to September 3 2020 with a total of 2362 trading days. The data is taken from Yahoo Finance. Table (5.9)

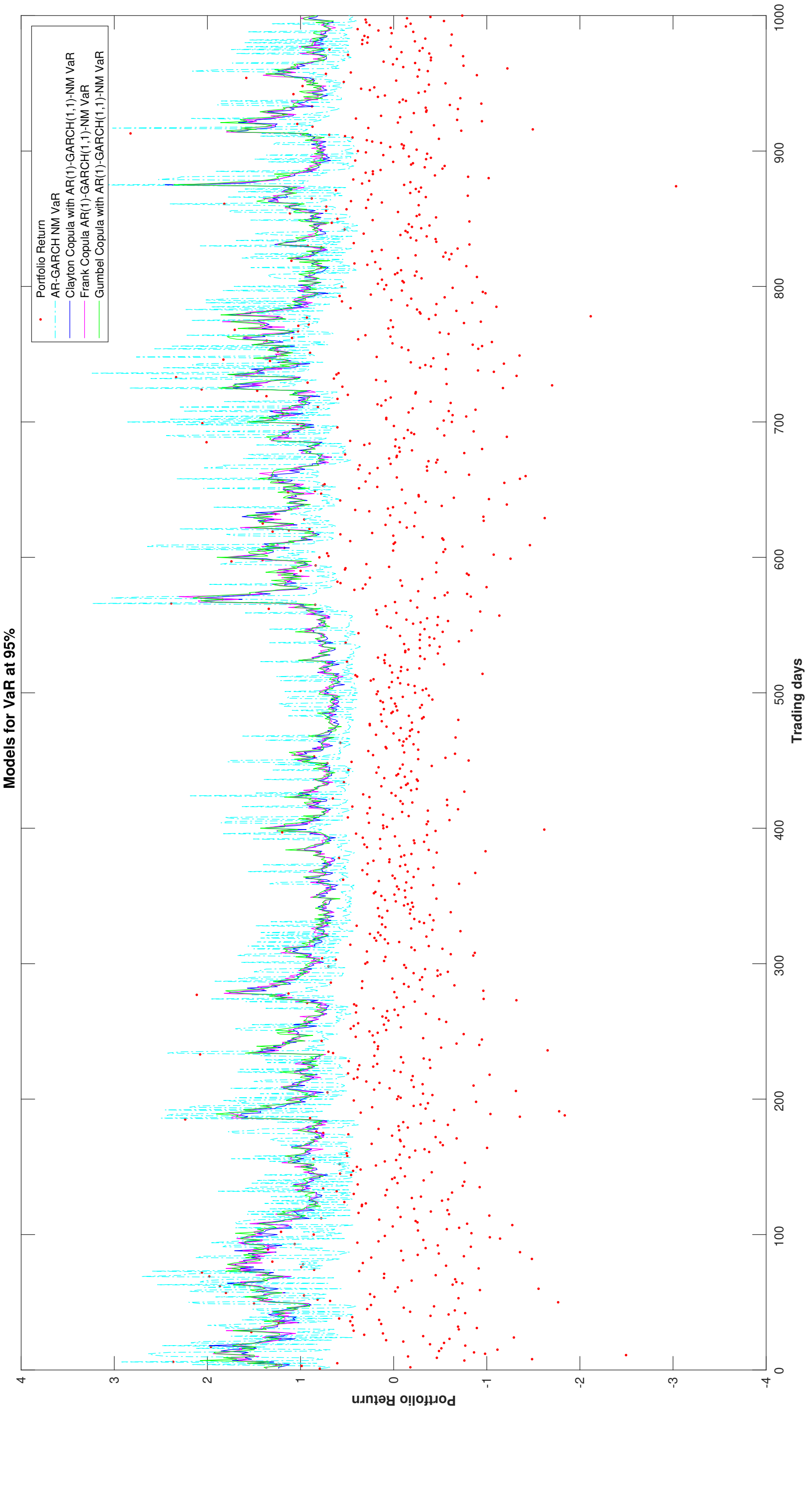

FIGURE 5.13: One day ahead forecasts of VaR at $\lambda = 0.95$ for portfolio of S&P500 and JCI with Gaussian mixture margins and various copulas

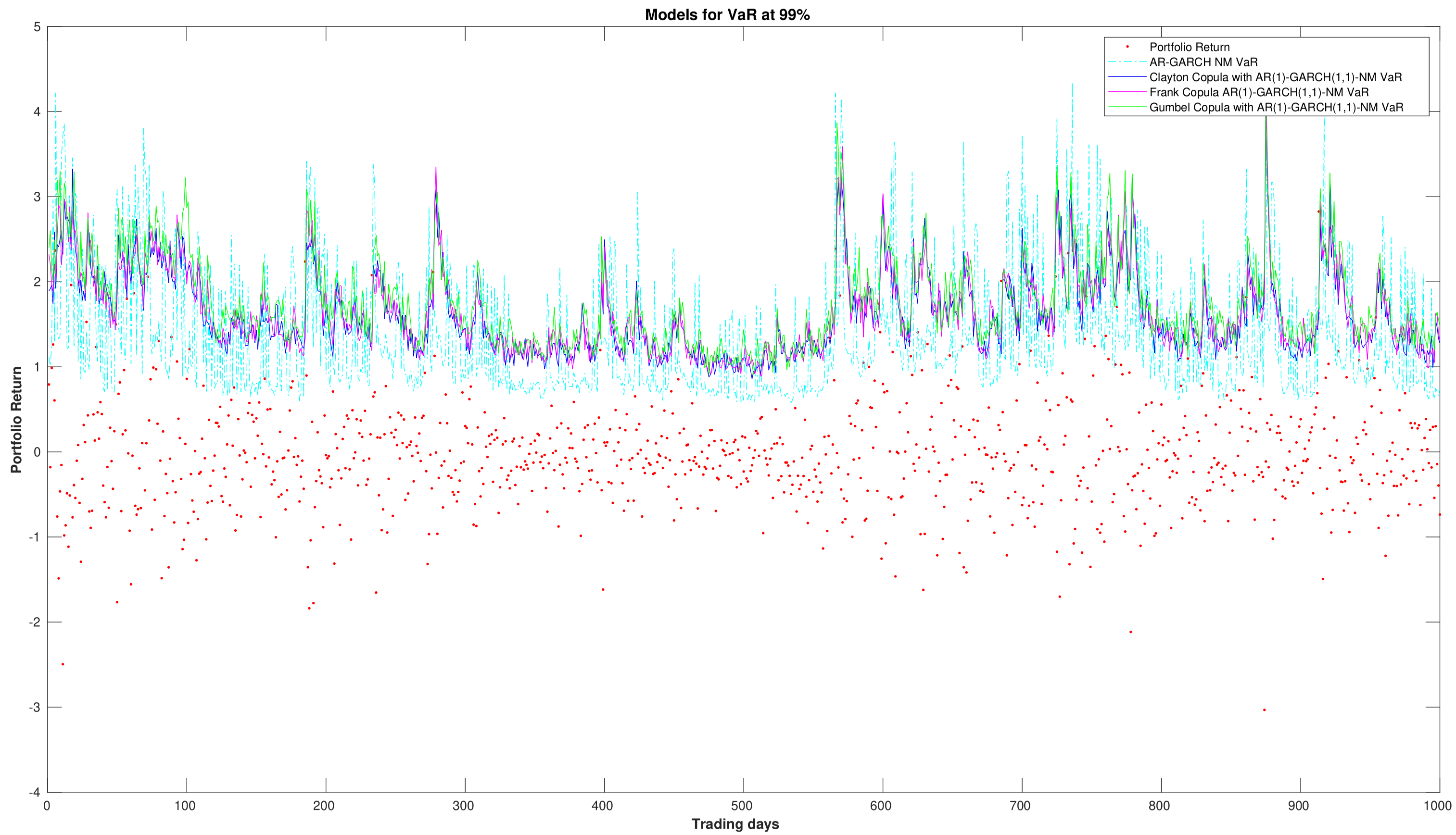

FIGURE 5.14: One day ahead forecasts of VaR at $\lambda = 0.99$ for portfolio of S&P500 and JCI with Gaussian mixture margins and various copulas

Model	Bin	z_α	p_{bin}	Failures	Prop. Fails	POF	Lik. ratio	p_{POF}	CCI	Lik. ratio	p_{CCI}
											$\lambda = 0.95$
ARMA-GARCH-NM	reject	2.3215	0.0203	66	0.0660	reject	4.9184	0.0266	accept	1.6805	0.1949
Clayton - NM	accept	0.0000	1.0000	50	0.0500	accept	0.0000	1.0000	accept	2.2008	0.1379
Frank - NM	accept	-0.1451	0.8846	49	0.0490	accept	0.0212	0.8843	accept	2.4347	0.1187
Gumbel - NM	accept	-0.2902	0.7717	48	0.0480	accept	0.0853	0.7702	accept	2.6826	0.1015
Historical	accept	-1.5960	0.1105	39	0.0390	accept	2.7469	0.0974	accept	3.1141	0.0776
Variance - Covariance	accept	1.7411	0.0817	62	0.0620	accept	2.8260	0.0927	accept	2.4326	0.1188
											$\lambda = 0.99$
ARMA-GARCH-NM	reject	2.2247	0.0261	17	0.0170	reject	4.0910	0.0431	accept	0.5886	0.4430
Clayton - NM	accept	1.9069	0.0565	16	0.0160	accept	3.0766	0.0794	accept	0.5209	0.4705
Frank - NM	accept	1.2713	0.2036	14	0.0140	accept	1.4374	0.2306	accept	0.3980	0.5281
Gumbel - NM	accept	0.6356	0.5250	12	0.0120	accept	0.3798	0.5377	accept	0.2918	0.5891
Historical	accept	-1.5891	0.1120	5	0.0050	accept	3.0937	0.0786	accept	0.0503	0.8225
Variance - Covariance	reject	4.4495	0.0000	24	0.0240	reject	14.2214	0.0002	accept	1.1817	0.2770

TABLE 5.7: Backtesting for estimated VaR models with Copulas

Model	Uncond. Normal	p_{UN}	Test. Stat.	Critical Value	Uncond. T	p_{UT}	Test. Stat.	Critical Value
								$\lambda = 0.95$
AR-GARCH - NM	reject	0.0097	-0.3426	-0.2359	reject	0.0262	-0.3426	-0.2806
Clayton - NM	accept	0.2284	-0.1054	-0.2359	accept	0.2463	-0.1054	-0.2806
Frank - NM	accept	0.3215	-0.0653	-0.2359	accept	0.3339	-0.0653	-0.2806
Gumbel - NM	accept	0.4723	-0.0074	-0.2359	accept	0.4635	-0.0074	-0.2806
Historical	accept	0.5000	0.2600	-0.2359	accept	0.5000	0.2600	-0.2806
Variance - Covariance	reject	0.0016	-0.4457	-0.2359	reject	0.0086	-0.4457	-0.2806
								$\lambda = 0.99$
AR-GARCH - NM	reject	0.0235	-0.6803	-0.5485	reject	0.0433	-0.6803	-0.6362
Clayton - NM	reject	0.0139	-0.7695	-0.5485	reject	0.0296	-0.7695	-0.6362
Frank - NM	reject	0.0499	-0.5490	-0.5485	accept	0.0758	-0.5490	-0.6362
Gumbel - NM	accept	0.2114	-0.2582	-0.5485	accept	0.2238	-0.2582	-0.6362
Historical	accept	0.5000	0.5443	-0.5485	accept	0.5000	0.5443	-0.6362
Variance - Covariance	reject	0.0001	-1.8376	-0.5485	reject	0.0008	-1.8376	-0.6362

TABLE 5.8: Backtesting for estimated CVaR models with Archimedean Copulas

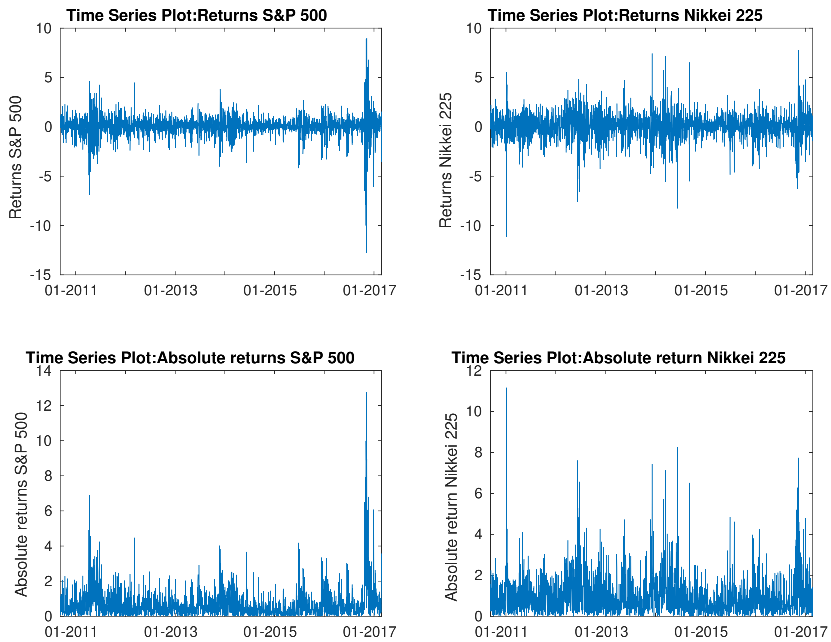

FIGURE 5.15: Daily and absolute returns of S&P 500 and Nikkei 225

Statistics	S&P 500	Nikkei 225
Mean	0.0485	0.0405
Standard	1.1267	1.3707
Minimum	-12.7652	-11.1534
Median	0.0715	0.0687
Maximum	8.9683	7.7314
Kurtosis	21.4719	8.6458
Asymmetry	-1.0968	-0.4276

TABLE 5.9: Descriptive Statistics of daily log-returns of S&P 500 and Nikkei 225

contains descriptive statistics and Figure (5.15) presents plots of both series. The implementation is performed with MATLAB and R.

Both series present asymmetry and have large kurtosis. In both cases, we can observe the negative value of asymmetry for both series, indicating the likeliness of negative returns, and excess of kurtosis shows fatter tails than the normal distribution. We can also observe the effects of volatility clustering. It would be a good idea to consider different models to Normal or t-distributed innovations for each series.

5.3.2 Margins modeling

For this study, we consider three different distributions: Standard Normal, Student-t and Hansen's Skewed-t. We recall Hansen's definition (Hansen, 1994): Let y_t be a random variable which follows a conditional Skewed- t distribution with density function $f(\cdot; \nu_t, \lambda_t)$ and mean zero and variance one by construction, in order to be a suitable model for the standardized residuals of a conditional mean and variance model. The conditional parameters ν_t, λ_t control the kurtosis and skewness of the variable, respectively, while the density function is given by:

$$f(y_t; \nu_t, \lambda_t \mid \mathcal{F}_{t-1}) = \begin{cases} bc\left(1 + \frac{1}{\nu_t - 2}\left(\frac{by_t + a}{1 - \lambda_t}\right)^2\right)^{-(\nu_t+1)/2} & \text{for } y_t \leq -\frac{a}{b} \\ bc\left(1 + \frac{1}{\nu_t - 2}\left(\frac{by_t + a}{1 + \lambda_t}\right)^2\right)^{-(\nu_t+1)/2} & \text{for } y_t > -\frac{a}{b} \end{cases} \tag{5.1}$$

This density is defined for $2 < \nu_t < \infty$ and $-1 < \lambda_t < 1$ The constants a,b and c are given by

$$a = 4\lambda_t c\left(\frac{\nu_t - 2}{\nu_t - 1}\right)$$

$$b = 1 + 3\lambda_t^2 - a^2$$

$$c = \frac{\Gamma\left(\frac{\nu_t+1}{2}\right)}{\sqrt{\pi(\nu_t - 2)}\Gamma\left(\frac{\nu_t}{2}\right)}$$

The results for marginal models are shown in Table (5.10). In fact, we fitted two AR(1)-GARCH(1,1) for both series as initial models with three specified distribution for the standardized residuals. This selection was considered to see there was no auto-correlation nor squared auto-correlation in the residuals. We also performed a Ljung-Box test to infer the null hypothesis is not rejected from lag 1 to 10. We also report values for Kolmogorov-Smirnov (KS), Chi-square goodness-of-fit test (CSG) and Anderson-Darling test used for uniformity test for standardized residuals series. We can consider a good fit for the model if the transformed series are closed to uniform distribution. In figure 4, we can observe the comparison for each specified model.

5.3.3 Copula parameter estimation

In table (5.11),we present maximum likelihood estimates for three copulas and margins modeled by different distribution. We report standard errors, and the value of the log-likelihood function evaluated on the optimum, Akaike's information criterion (AIC). So when selecting between several models, the best fit will be given for the model with the lowest AIC. We can also consider de quadratic distance between the estimate and the empirical copula in a finite

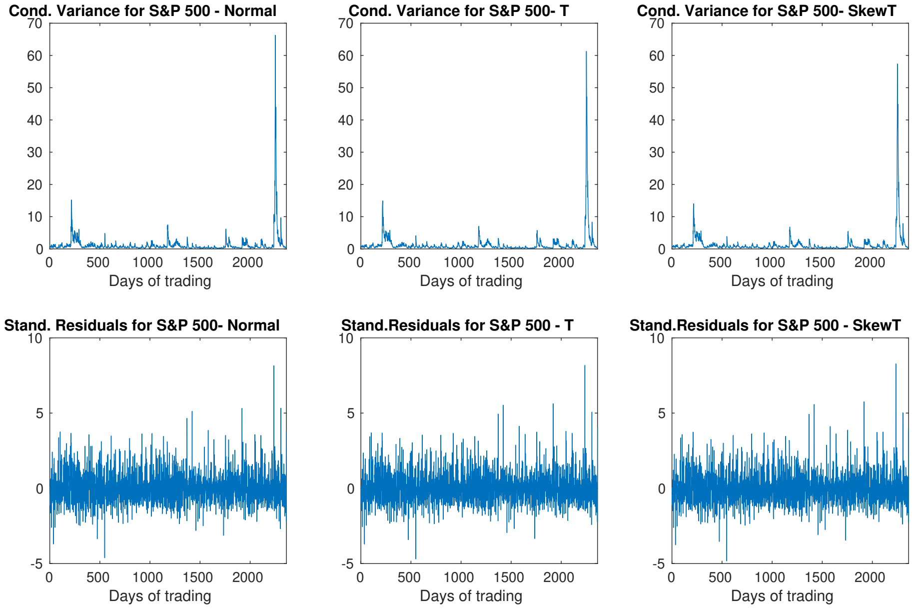

FIGURE 5.16: Conditional Variance and standardized residuals of S&P 500 with Normal, t and Skewed-t innovations

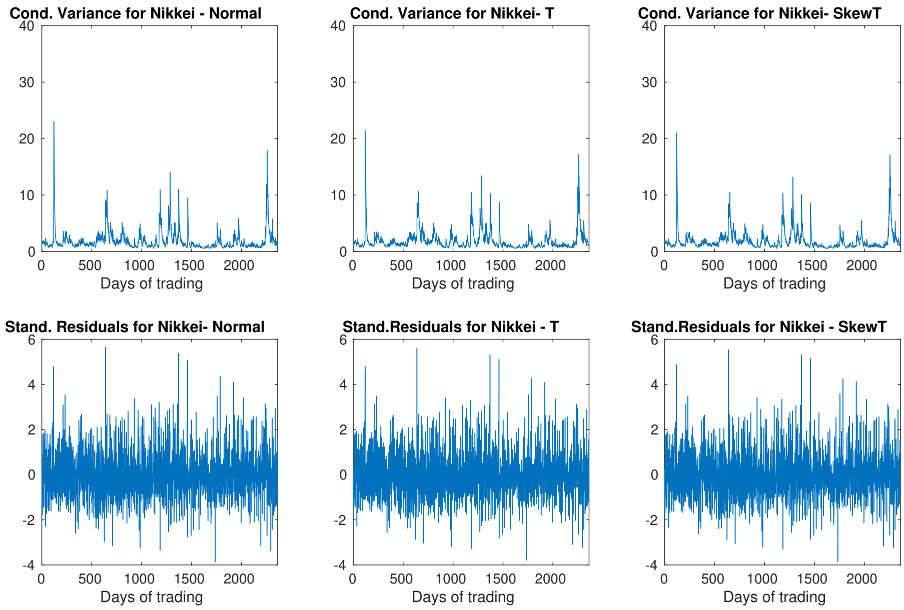

FIGURE 5.17: Conditional Variance and standardized residuals of Nikkei 225 with Normal, t and Skewed-t innovations

			S&P 500			Nikkei 225
	Normal	Student-t	Skewed-t	Normal	Student-t	Skewed-t
a_0	-0.0960	-0.1025	-0.0744	-0.0808	-0.0972	-0.0715
	(0.0139)	(0.0132)	(0.0143)	(0.0229)	(0.0217)	(0.0281)
b_0	-0.0660	-0.0664	-0.0836	-0.0312	-0.0374	-0.0445
	(0.0242)	(0.2198)	(0.0205)	(0.0229)	(0.0215)	(0.0472)
c_0	0.0500	0.0274	0.0253	0.06907	0.0606	0.0574
	(0.0049)	(0.0066)	(0.0060)	0.0104	(0.0157)	(0.0288)
c_1	0.2239	0.1839	0.1726	0.1230	0.1184	0.1158
	(0.0155)	(0.0256)	(0.0230)	(0.0099)	(0.0170)	(0.0228)
d_1	0.7418	0.8103	0.8193	0.8368	0.8542	0.8573
	(0.0170)	(0.0230)	(0.0210)	0.0124	(0.0197)	(0.0425)
λ			0.1465			0.0854
			(0.0287)			(0.0295)
ν		4.4574	4.8783		5.4431	5.7832
		(0.4381)	(0.4893)		(0.6580)	(2.0810)
AIC	5.9218×10^3	5.6802×10^3	5.6540×10^3	7.6931×10^3	7.5569×10^3	7.5470×10^3
$Q^2(1)$	0.8565	0.6884	0.6375	0.0325	0.0057	0.0033
$Q^2(10)$	0.7684	0.7988	0.8016	0.1097	0.0564	0.0416
KS	6.6143×10^{-5}	0^*	0.0222	1.0033×10^{-4}	3.7565×10^{-9}	0.0536
χ^2	0^*	0^*	0.0040	0^*	0^*	0.0074
AD	2.5413×10^{-7}	2.5413×10^{-7}	0.0798	6.0298×10^{-6}	2.5413×10^{-7}	0.1278

TABLE 5.10: Parameter estimates of AR(1)-GARCH(1-1) model for S&P 500 and Nikkei 225

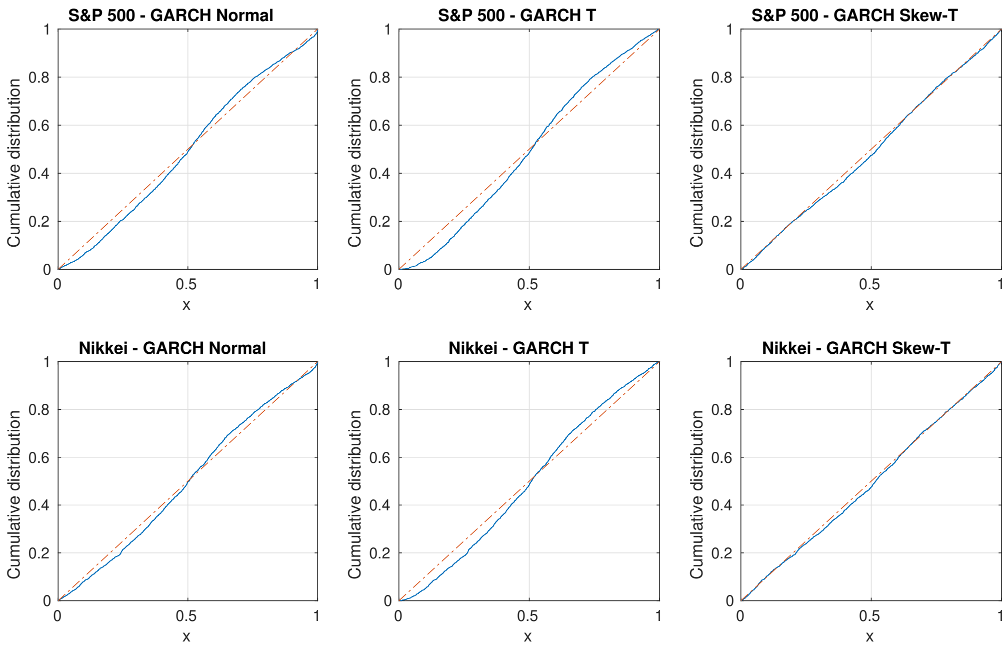

FIGURE 5.18: Empirical distribution of the transformed series for both risk assets

set of bivariate points. For instance, we are concerned with points whose value of empirical copula is larger than level λ

5.3.4 Computation of MCVaR and CCVaR

We have consired a portfolio with equal weights in the two risk assets ($\omega = 0.5$) and estimated MCVaR and CCVaR for levels $\lambda = 0.95$ and $\lambda = 0.99$. With the estimates for copula and quantile function, we compute equation (4.10) with numerical methods implemented on MATLAB. Results are reported in Table (5.12).

We can observe that for Ali-Mikhail copula with parameter positive, inequality in the example 5 is not satisfied, so in this case we have

$$\mathrm{CCVaR}_{\lambda}^{AMH}(\mathbf{X}) \leq \mathrm{MCVaR}_{\lambda}(\mathbf{X}).$$

for both $\lambda = 0.95$ and $\lambda = 0.99$. In Gumbel-Barnet copula, parameter θ is very close to zero, so close values are expected for CCVaR to MCVaR. We also observed that estimates for MCVaR and CCVaR with Student-t innovations are greater than those with Normal or Skewed-t distributions. Further research is needed to establish the relationship between these estimates.

5.3.5 Behavior through time of MCVaR and CCVaR

We also estimate CCVaR for a period of several days in which parameters for margins and copula are estimated each new trading day. Then we plot values for MCVaR and CCVaR to compare both models and infer that volatility clustering has effectively been captured by the margins and the proportion of portfolio. Figure (5.19) and (5.20) represents MCVaR and CCVaR for $\lambda = 0.95$ and $\lambda = 0.99$, respectively. In all cases, we have taken Skewed-t distribution for comparison.

It seems Ali-Mikhail-Haq and Gumbel-Barnett copulas give estimates for CCVaR very close to MCVaR corresponding to independence copula. Gumbel copula seems to overestimate CCVaR for both levels. We hope we can establish this observed behavior in future research.

Copula	Margins	θ	Stand. Error	Log-Lik	AIC $\times 10^4$	$d_{0.95}$	$d_{0.99}$
	Normal	0.6321	(0.0359)	59.5836	1.3498	16.6739	17.5587
Ali-Mikhail-Haq	Student-t	0.8414	(0.0585)	75.9893	1.3087	16.6716	17.5386
	Skewed-t	0.5407	(0.0115)	56.1783	1.3091	17.2131	18.5005
	Normal	1.0006×10^{-5}	(0.0254)	-0.0046	1.3617	17.0295	17.9464
Gumbel-Barnett	Student-t	1.0008×10^{-5}	(0.2723)	-0.0030	1.3239	17.2949	18.1247
	Skewed-t	1.0005×10^{-5}	(0.0229)	-0.0047	1.3203	17.4967	18.7582
	Normal	1.1014	(0.0274)	61.9544	1.3493	16.8856	17.8225
Gumbel	Student-t	1.2639	(0.0221)	87.1683	1.3065	17.0033	17.8698
	Skewed-t	1.1604	(0.0035)	77.0485	1.3049	17.3456	18.6411

TABLE 5.11: Parameter estimates and standard errors for Ali-Mikhail-Haq, Gumbel-Barnett and Gumbel copula

		$\lambda = 0.95$		$\lambda = 0.99$	
Copula	Margins	MCVaR	CCVaR	MCVaR	CCVaR
Ali-Mikhail-Haq	Normal	2.1021	2.1011	2.6450	2.6448
	Student-t	3.0944	3.0901	4.7003	4.6992
	Skewed-t	2.5763	2.5744	3.8874	3.8865
Gumbel-Barnett	Normal	2.1021	2.1020	2.6450	2.6449
	Student-t	3.0944	3.0943	4.7003	4.7002
	Skewed-t	2.5763	2.5760	3.8874	3.8869
Gumbel	Normal	2.1021	2.2062	2.6450	2.7545
	Student-t	3.0944	3.3462	4.7003	5.0718
	Skewed-t	2.5763	2.8216	3.8874	4.2546

TABLE 5.12: Values of MCVaR and CCVaR for Ali-Mikhail-Haq, Gumbel-Barnett and Gumbel copula with different margins

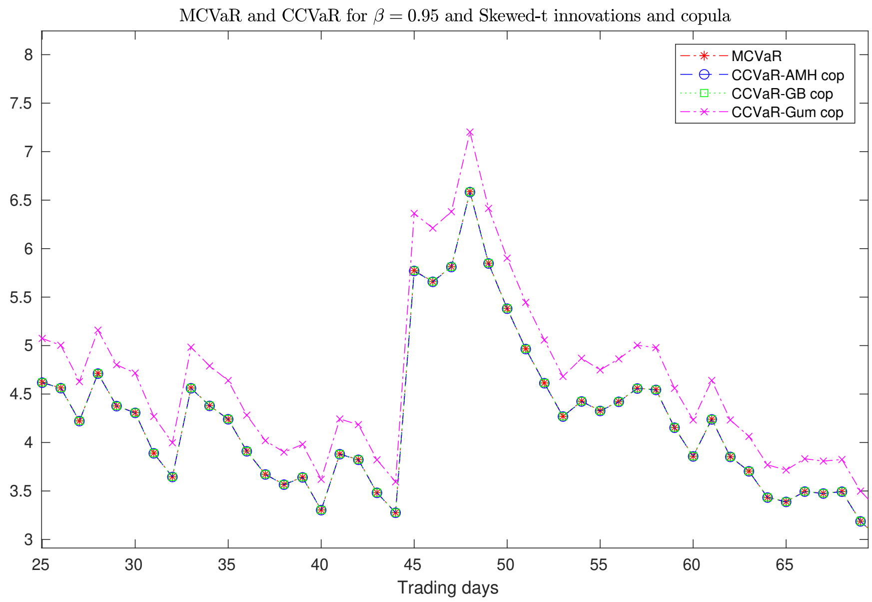

FIGURE 5.19: MCVaR and CCVaR for $\beta = 0.95$ and Skewed-t innovations and copula

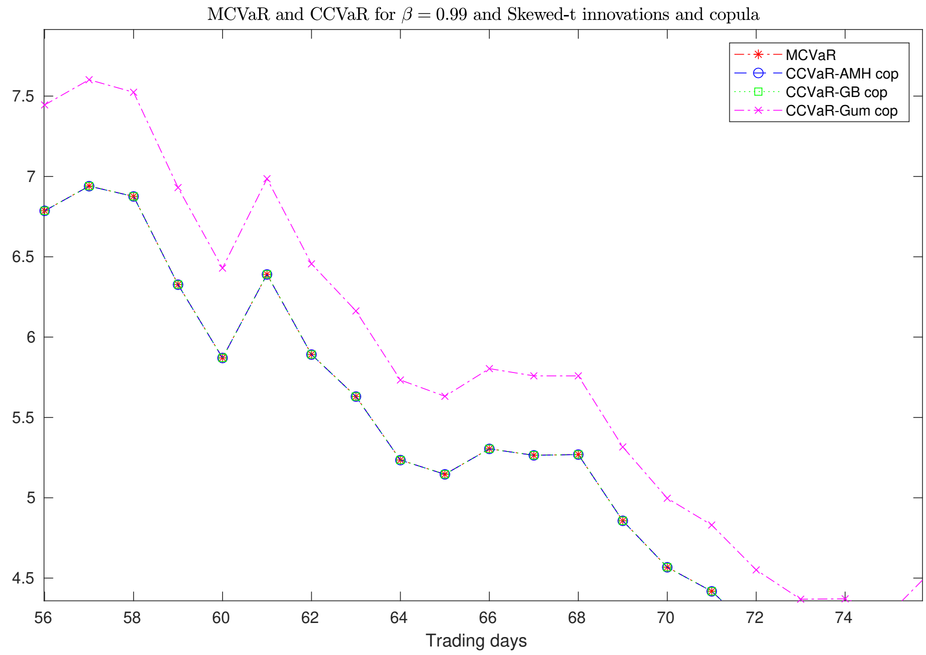

FIGURE 5.20: MCVaR and CCVaR for $\beta = 0.99$ and Skewed-t innovations and copula

Chapter 6

Conclusions

VaR is a fundamental tool in the context of portfolio and risk management. One limitation of VaR measure is that it gives little importance to the most extreme losses since the skewness and kurtosis of the distribution are not adequately reflected. CVaR is a function of VaR defined as the conditional expected value of losses that exceed VaR. This measure possesses desirable properties as convexity and sub-additivity regardless of the functional form of the lost profit distribution is. In this doctoral thesis we have explored specific models for the estimation of VaR and CVaR for a portfolio of two assets which are not necessarily independent but possibly non-linearly related. I also have considered that the dynamics of risk assets present asymmetry and heavy tails. In this context, we decided to model asset losses along with the direct effects of volatility employing an ARMA-GARCH model with residual innovations distributed with a gaussian mixture of standard normal. And not only that, but I can also take advantage of algorithms like Expectation maximization (EM) for rapid estimation of mixture parameters. It is meaningful to measure the robustness of proposed methods for estimating VaR and CVaR.

In addition, I also wish to adequately take into account the non-linear dependence between risk assets. This can be performed by considering copula as a tool that models the dependence between various random variables. Copulas can be also considered as a tool to generate multivariate cumulative distribution functions which margins are uniformly distributed. Archimedean copulas were considered for their easiness to implement in numerical and analytical scenarios. For the particular case of Archimedean copulas, I have developed a formula to estimate the VaR whose solution can be obtained numerically. VaR can be computed implementing a ordinary Monte Carlo simulation by taking sample random returns from conditional distribution and reevaluating portfolio at a specified time. Consequently Value at Risk can be determined by taking the empirical quantile at λ of the simulated loss portfolio.

VaR is defined for a single random variable, and there has been much effort such that the definition is extended to involve multivariate random vectors. Here, I also present an alternative definition for multivariate copula-based CVaR. I could consider this as a possible generalization of MVaR (Lee and Prékopa, 2013), whose expression can be obtained by taking the independence copula. This definition seems to be more accessible and easier to estimate than CCVaR by Krzemienowski and Szymczyk (2016). It can even be easily calculated in the case of Archimedean copulas of which I equally found a relatively ready-to-calculate expression and established some relationships with MVaR for some examples of copulas.

In the numerical examples, I operated portfolios consisting of risky assets with several periods of high volatility and and negative skewness. Empirical analysis has shown that choosing a good specification for the distribution of residual innovations along with copulation can result in adequate estimates for VaR and thus CVaR. In particular, mixtures of Gaussian distributions seem to be suitable for assets with strong asymmetry and leptokurtosis. We have compared classical methods for VaR estimation such as variance-covariance method and historical estimation, with our copula approach with ARMA-GARCH mixed Gaussian distributions. Indeed, I observe the latter reacts to the effects of volatility over time. Thanks to the property of copula, I can explain a more significant non-linear correlation between risk factors studied here. In effect, the copulas produce better estimates for VaR than classical ones. In fact, utilizing the backtesting methodology, I have seen that for more extreme losses, the effect of copula improves the outcome of the tests. I have also developed a numerical implementation to our definition of CCVaR, which also confirms the critical importance of choosing the precise specification for the margins.

Intended research should focus on working with more than two risk assets. The task is arduous for dimensions greater than two. For the multivariate copula, I can consider again the Archimedean families. Part of this modelling was already performed in Savu (2010), but he also stated that unfortunately these copulas suffer from a significantly limited dependence structure, since all k-dimensional marginal distributions are identical $(k < d)$. Actual problems come with the specification of the parameters as the estimation uses depends on difficult expressions and Monte Carlo simulation also looks very inefficient. Another approach is the use of vine copulas. This tool allows labeling constraints in high order dimensional distributions. As Cooke (2010) stated; vine copulas owe their increasing popularity to the fact that they leverage from bivariate copula and enable extension to arbitrary dimensions. I intend to use these tools for modeling VaR for the portfolio problem in higher dimensions, but their implementation seems complicated as the number of variables arise. For future research, I also want to apply the theory of vine copulas and Archimedean copulas for the estimation of the Value at Risk. It must be considered that this development must be easy and fast to implement numerically. Possibly this task is partly solved in the case of

Archimedean copulas because they enjoy well algebraic properties and let the estimation of their parameters and simulation to be almost straightforward. I should be capable to exploit their properties for a fast calculation of the Value at Risk (Hofert, Mächler, and Mcneil, 2012). However, these copulas suffer from a severely limited dependence structure. Subsequent research should intend to use more general multivariate copulas such as Vine or Hierarchical Archimedean (Savu and Trede, 2010).

Appendix A

Proofs

A.1 Proof of Theorem 3

For simplicity, we assume that $F_Z(z)$ is continuous and strictly monotone. General cases are treated with obvious modifications. First we see that $\mathrm{VaR}_\lambda(Z)$ is determined by the equation

$$\begin{aligned}
\lambda &= P(Z \le z) = P(\omega X + (1-\omega)Y \le z) \\
&= \int_0^z ds \int_0^s \frac{\partial^2 C_\varphi}{\partial u \partial v}\left(F_X\left(\frac{t}{\omega}\right), F_Y\left(\frac{s-t}{1-\omega}\right)\right)\frac{1}{\omega} f_X\left(\frac{t}{\omega}\right)\frac{1}{1-\omega} f_Y\left(\frac{s-t}{1-\omega}\right) dt \\
&= \int_0^{\frac{z}{1-\omega}} \frac{\partial C_\varphi}{\partial v}\left(F_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right), F_Y(y)\right) f_Y(y) dy \\
&= \int_0^{\frac{z}{1-\omega}} \left[\frac{\partial}{\partial y}\left(c_\varphi\left(F_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right), F_Y(y)\right)\right)\right. \\
&\quad \left. + \frac{1-\omega}{\omega}\frac{\partial C_\varphi}{\partial u}\left(F_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right), F_Y(y)\right) f_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right)\right] dy
\end{aligned}$$

Now, thanks to the assumptions that C_φ is Archimedean, we derive

$$\begin{aligned}
&\mathrm{P}\left(\omega X + (1-\omega) Y \le z\right) \\
&\quad = \int_0^{\frac{z}{1-\omega}} \frac{1-\omega}{\omega} \frac{\varphi'\left(F_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right)\right)}{\varphi'\left(\varphi^{(-1)}\left(\varphi\left(F_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right)\right) + \varphi\left(F_Y(y)\right)\right)\right)} f_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right) dy \\
&\quad = (1-\omega)\int_0^{\frac{z}{1-\omega}} \frac{d}{dz}\varphi^{(-1)}\left(\varphi\left(F_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right)\right) + \varphi\left(F_Y(y)\right)\right) dy \\
&\quad = (1-\omega)\frac{d}{dz}\int_0^{\frac{z}{1-\omega}} C_\varphi\left(F_X\left(\frac{z}{\omega} - \frac{1-\omega}{\omega} y\right), F_Y(y)\right) dy \\
&\quad = \frac{d}{dx}\int_0^x C_\varphi\left(F_X\left(\frac{1}{\omega}(x-y)\right), F_Y\left(\frac{y}{1-\omega}\right)\right) dy
\end{aligned}$$

This completes the proof. In above computation, we note that the boundary conditions of copulas must be taken into account.

A.2 Proof of Theorem 4

The proof is implemented in an elementary fashion. By the standard approximation argument, we may assume that φ is C^2-class. If the copula C is Archimedean of the above form (3.7), we learn that

$$dC(u,v) = \frac{-\varphi''(t)}{(\varphi'(t))^3}\varphi'(u)\varphi'(v)dudv,$$

where we have put $t = \varphi^{(-1)}(\varphi(u)+\varphi(v))$. Taking into account of the symmetry of u, v, we have

$$\begin{aligned}
&\mathrm{CCVaR}_\lambda(\mathbf{X}) \\
&= \frac{1}{\iint_{\{\varphi(u)+\varphi(v)\le\varphi(\lambda)\}} \frac{-\varphi''(t)}{(\varphi'(t))^3}\varphi'(u)\varphi'(v)dudv} \\
&\quad \cdot \Big\{ \iint_{\{\varphi(u)+\varphi(v)\le\varphi(\lambda)\}} (\omega F_{X_1}^{(-1)}(u) + (1-\omega)F_{X_2}^{(-1)}(u)) \frac{-\varphi''(t)}{(\varphi'(t))^3}\varphi'(u)\varphi'(v)dudv \Big\}.
\end{aligned}$$

Now applying the change of variables

$$(u,v) \to (u,t) \quad \text{where} \quad t = \varphi^{(-1)}(\varphi(u)+\varphi(v)),$$

we infer that

$$\begin{aligned}
&\mathrm{CCVaR}_\lambda(\mathbf{X}) \\
&= \frac{1}{\iint_{\{\lambda\le t\le u\}} \frac{-\varphi''(t)}{(\varphi'(t))^2}\varphi'(u)dudt} \\
&\quad \cdot \Big\{ \iint_{\{\lambda\le t\le u\}} (\omega F_{X_1}^{(-1)}(u) + (1-\omega)F_{X_2}^{(-1)}(u)) \frac{-\varphi''(t)}{(\varphi'(t))^2}\varphi'(u)dudt \Big\} \\
&= \frac{\int_\lambda^1 \Big((\omega F_{X_1}^{(-1)}(u) + (1-\omega)F_{X_2}^{(-1)}(u))\varphi'(u) \int_\lambda^u \frac{-\varphi''(t)}{(\varphi'(t))^2}dt \Big) du}{\int_\lambda^1 \varphi'(u)du \int_\lambda^u \frac{-\varphi''(t)}{(\varphi'(t))^2}dt} \\
&= \frac{\int_\lambda^1 (\omega F_{X_1}^{(-1)}(u) + (1-\omega)F_{X_2}^{(-1)}(u))\varphi'(u) \Big[\frac{1}{\varphi'(t)}\Big]_\lambda^u du}{\int_\lambda^1 \varphi'(u) \Big[\frac{1}{\varphi'(t)}\Big]_\lambda^u du} \\
&= \frac{\int_\lambda^1 (\omega F_{X_1}^{(-1)}(u) + (1-\omega)F_{X_2}^{(-1)}(u)) \Big(1 - \frac{\varphi'(u)}{\varphi'(\lambda)}\Big) du}{1 - \lambda + \frac{\varphi(\lambda)}{\varphi'(\lambda)}},
\end{aligned}$$

which implies the theorem.

We remark that a similar calculation for the denominator is already employed in the literature (see for instance Theorem 4.3.4 in Nelsen (2007)).

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