On volume functions of special flow polytopes associated to the root system of type A and On equivariant index of a generalized Bott manifold

Yuki Sugiyama

Introduction

In the thesis, we treat the two themes.

In Part I, we consider the volume of a special kind of flow polytope. We show that its volume satisfies a certain system of differential equations, and conversely, the solution of the system of differential equations is unique up to a constant multiple. In addition, we give an inductive formula for the volume with respect to the rank of the root system of type A.

In Part II, we consider the equivariant index of a generalized Bott manifold. We show the multiplicity function of the equivariant index is given by the density function of a generalized twisted cube. In addition, we give a Demazure-type character formula of this representation.

1 On volume functions of special flow polytopes associated to the root system of type A

The number of lattice points and the volume of a convex polytope are important and interesting objects and have been studied from various points of view (see, e.g., [1]). For example, the number of lattice points of a convex polytope associated to a root system is called the Kostant partition function, and it plays an important role in representation theory of Lie groups (see, e.g., [4]).

We consider a flow polytope associated to the root system of type A. In [2], a number of theoretical results related to the Kostant partition function and the volume function of a flow polytope can be found. In particular, it is shown that these functions for the nice chamber are written as iterated residues ([2, Lemma 3]).

The purpose of this part is to characterize the volume function of a flow polytope for the nice chamber in terms of a system of differential equations, based on a result in [2]. In order to state the main results, we give some notation. Let $e_1, \dots, e_{r+1}$ be the standard basis of $\mathbb{R}^{r+1}$ and let

$$A_r^+ = \{e_i - e_j \,|\, 1 \le i < j \le r+1\}$$

be the positive root system of type A with rank r. We assign a positive integer $m_{i,j}$ to each i and j with $1 \le i < j \le r+1$. Let us set $m = (m_{i,j})$ and $M = \sum_{1 \le i < j \le r+1} m_{i,j}$. For $a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in \mathbb{R}^{r+1}$, where $a_i \in \mathbb{R}_{\ge 0}$ $(i = 1, \dots, r)$, the following polytope $P_{A_r^+,m}(a)$ is called the flow polytope associated to the root system of type A:

$$P_{A_r^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \;\middle|\; \begin{array}{l} 1 \le i < j \le r+1\,,\ 1 \le k \le m_{i,j}\,,\ y_{i,j,k} \ge 0\,, \\ \sum_{1 \le i < j \le r+1} \sum_{1 \le k \le m_{i,j}} y_{i,j,k}(e_i - e_j) = a \end{array} \right\}.$$

Note that the flow polytopes in [2] include the case that some of $m_{i,j}$'s are zero, whereas we exclude such cases in this part. We denote the volume of $P_{A_r^+,m}(a)$ by $v_{A_r^+,m}(a)$.

The open set

$$\mathfrak{c}_{\mathrm{nice}} := \{a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in \mathbb{R}^{r+1} \,|\, a_i > 0, i = 1, \dots, r\}$$

in $\mathbb{R}^{r+1}$ is called the nice chamber. We are interested in the volume $v_{A_r^+,m}(a)$ when a is in the closure of the nice chamber, and then it is written by $v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}$. It is a homogeneous polynomial of degree $M - r$. The first result of this part is the following.

Theorem 1.1 *Let $a = \sum_{i=1}^r a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}_{\mathrm{nice}}}$, and let $v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}(a)$ be the volume of $P_{A_r^+,m}(a)$. Then $v = v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}(a)$ satisfies the system of differential equations as follows:*

$$\begin{cases} \partial_r^{m_{r,r+1}} v = 0 \\ (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} v = 0 \\ \qquad \vdots \\ (\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} v = 0, \end{cases}$$

where $\partial_i = \frac{\partial}{\partial a_i}$ for $i = 1, \dots, r$. Conversely, the polynomial $v = v(a)$ of degree $M - r$ satisfying the above equations is equal to a constant multiple of $v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}(a)$.

In addition, we show the volume $v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}(a)$ is written by a linear combination of $v_{A_{r-1}^+,m',\mathfrak{c}'_{\mathrm{nice}}}(a')$ and its partial derivatives, where $m' = (m_{i,j})_{2 \le i < j \le r+1}$, $\mathfrak{c}'_{\mathrm{nice}}$ is the nice chamber of A_{r-1}^+, and $a' = \sum_{i=2}^r a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}'_{\mathrm{nice}}}$.

2 On equivariant index of a generalized Bott manifold

A *Bott tower* of height n is a sequence:

$$M_n \stackrel{\pi_n}{\to} M_{n-1} \stackrel{\pi_{n-1}}{\to} \cdots \stackrel{\pi_2}{\to} M_1 \stackrel{\pi_1}{\to} M_0 = \{\text{a point}\}$$

of complex manifolds $M_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j)$, where $\underline{\mathbb{C}}$ is the trivial line bundle over M_{j-1}, E_j is a holomorphic line bundle over M_{j-1}, $\mathbb{P}(\cdot)$ denotes the projectivization, and $\pi_j : M_j \to M_{j-1}$ is the projection of the $\mathbb{C}P^1$-bundle. We call M_j a *j-stage Bott manifold*. The notion of a Bott tower was introduced by Grossberg and Karshon ([3]).

A *generalized Bott tower* is a generalization of a Bott tower. A generalized Bott tower of height m is a sequence:

$$B_m \stackrel{\pi_m}{\to} B_{m-1} \stackrel{\pi_{m-1}}{\to} \cdots \stackrel{\pi_2}{\to} B_1 \stackrel{\pi_1}{\to} B_0 = \{\text{a point}\},$$

of complex manifolds $B_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j^{(1)} \oplus \cdots \oplus E_j^{(n_j)})$, where $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}, $E_j^{(k)}$ is a holomorphic line bundle over B_{j-1} for $k = 1, \ldots, n_j$. We call B_j a *j-stage generalized Bott manifold*. Generalized Bott manifolds are a certain class of toric manifolds, so it is interesting to investigate the specific properties of generalized Bott towers.

In [3], Grossberg and Karshon showed the multiplicity function of the *equivariant index* for a holomorphic line bundle over a Bott manifold is given by the density function of a *twisted cube*, which is determined by the structure of the Bott manifold and the line bundle over it. From this, they derived a Demazure-type character formula.

The purpose of this part is to generalize the results in [3] to generalized Bott manifolds. We generalize the twisted cube, and we call it the *generalized twisted cube*. We show the multiplicity function of the equivariant index for a holomorphic line bundle over the generalized Bott manifold is given by the density function of the associated generalized twisted cube. From this, we derive a Demazure-type character formula. In order to state the main results, we give some notation. Let $\mathbf{L}$ be a holomorphic line bundle over a generalized Bott manifold B_m, which is constructed from integers $\{\ell_i\}$ and $\{c_{i,j}^{(k)}\}$. Let $N = \sum_{j=1}^m n_j$, and let $T^N = S^1 \times \cdots \times S^1$. We consider the action of T^N on B_m as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m] = [\mathbf{t}_1\mathbf{z}_1, \ldots, \mathbf{t}_m\mathbf{z}_m],$$

where $\mathbf{t}_i = (t_{i,1}, \ldots, t_{i,n_i}), \mathbf{z}_i = (z_{i,0}, \ldots, z_{i,n_i}), \mathbf{t}_i\mathbf{z}_i = (z_{i,0}, t_{i,1}z_{i,1}, \ldots, t_{i,n_i}z_{i,n_i})$ for $i = 1, \ldots, m$. Also we consider the action of $T = T^N \times S^1$ on $\mathbf{L}$ as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m, t_{m+1}) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m, v] = [\mathbf{t}_1\mathbf{z}_1, \ldots, \mathbf{t}_m\mathbf{z}_m, t_{m+1}v]. \tag{2.1}$$

We define the generalized twisted cube as follows. It is defined to be the set of $x = (x_{1,1}, \ldots, x_{m,n_m}) \in \mathbb{R}^N$ which satisfies

$$A_i(x) \leq \sum_{k=1}^{n_i} x_{i,k} \leq 0,\ x_{i,k} \leq 0 \ \ (1 \leq k \leq n_i)$$

$$\text{or } 0 < \sum_{k=1}^{n_i} x_{i,k} < A_i(x),\ x_{i,k} > 0 \ \ (1 \leq k \leq n_i),$$

for $1 \leq i \leq m$, where

$$A_i(x) = \begin{cases} -\ell_m & (i = m) \\ -(\ell_i + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k}) & (1 \leq i \leq m-1). \end{cases}$$

We denote the generalized twisted cube by C. We also define $\mathrm{sgn}(x_{i,k}) = 1$ for $x_{i,k} > 0$ and $\mathrm{sgn}(x_{i,k}) = -1$ for $x_{i,k} \leq 0$. The *density function* of the generalized twisted cube is defined to be $\rho(x) = (-1)^N \prod_{1 \leq i \leq m, 1 \leq k \leq n_i} \mathrm{sgn}(x_{i,k})$ when $x \in C$ and 0 elsewhere.

Let $\mathfrak{t}$ be the Lie algebra of T and let $\mathfrak{t}^*$ be its dual space. Let $\ell^* \subset i\mathfrak{t}^*$ be the integral weight lattice and let $\mathrm{mult} : \ell^* \to \mathbb{Z}$ be the multiplicity function of the equivariant index. The first main result of this part is the following:

Theorem 2.1 *Fix integers $\{c_{i,j}^{(k)}\}$ and $\{\ell_j\}$. Let $\mathbf{L} \to B_m$ be the corresponding line bundle over a generalized Bott manifold. Let $\rho : \mathbb{R}^N \to \{-1, 0, 1\}$ be the density function of the generalized twisted cube C which is determined by these integers. Consider the torus action of $T = T^N \times S^1$ as in (2.1). Then the multiplicity function for $\ell^* \cong \mathbb{Z}^N \times \mathbb{Z}$ is given by*

$$\mathrm{mult}(x, k) = \begin{cases} \rho(x) & (k = 1) \\ 0 & (k \neq 1). \end{cases}$$

In addition, from Theorem 2.1, we obtain a Demazure-type character formula.

References

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