Ricci flow with bounded curvature integrals
and
Decompositions of the space of Riemannian metrics on a compact manifold with boundary

中央大学大学院, 理工学研究科, 数学専攻, 濱中 翔太
Department of Mathematics, Graduate School of Science and Engineering, Chuo University, Shota Hamanaka

Abstract

In this thesis, we treat two themes on differential geometry.

In the Part I, we study the Ricci flow on a closed manifold and finite time interval $[0, T)$ $(T < \infty)$ on which certain integral curvature energies are finite. A Ricci flow on a manifold M is given by a smooth family $g(t)$ $(t \in [0, T))$, of Riemannian metrics satisfying the evolution equation

$$\partial_t g(t) = -2 \operatorname{Ric}_{g(t)}.$$

The Ricci flow equation was introduced by Hamilton [4] in 1982. In the same paper, he stated the existence and uniqueness of the Ricci flow on a closed manifold. Moreover, he proved that a Ricci flow on a closed manifold develops a singularity at a finite time T (i.e., T is the maximal existence time of the flow) if and only if the maximum of the norm of the Riemannian curvature tensor blows up at T. On the other hand, Wang [7] characterized the maximal existence time of the flow by certain geometric energies which consist of integral bounds rather than point-wise ones. Later, Di Matteo [3] generalized Wang's results using mixed integral norms which are parametrized by $\alpha, \beta \in (1, \infty)$ with $\alpha \geq \frac{n}{2}\frac{\beta}{\beta-1}$.

Theorem 0.1 ([3, Theorem 1.2]). *Let $(M^n, g(t))_{t\in[0,T)}$ $(T < +\infty)$ be a smooth Ricci flow such that $(M, g(t))$ is complete and has bounded curvature for every t in $[0, T)$. Assume that the initial time-slice $(M, g(0))$ satisfies $\operatorname{inj}(M, g(0)) > 0$ where $\operatorname{inj}(M, g(0))$ denotes the injectivity radius of $(M, g(0))$. Assume also that*

$$\big|\big|\,||\operatorname{Rm}(\cdot, t)||_{L^\alpha(M)}\big|\big|_{L^\beta([0,T))} < +\infty$$

for some pair $(\alpha, \beta) \in (1, +\infty) \times (1, +\infty) \subset \mathbb{R}^2$ with

$$\alpha \geq \frac{n}{2}\frac{\beta}{\beta - 1}.$$

Then this flow can be extended smoothly over T.

Theorem 0.2 ([3, Theorem 1.3]). *Let $(M^n, g(t))_{t\in[0,T)}$ $(T < +\infty)$ be a smooth Ricci flow such that $(M, g(t))$ is complete and has bounded curvature for every t in $[0, T)$. Assume that the initial time-slice $(M, g(0))$ satisfies $\operatorname{inj}(M, g(0)) > 0$. Assume also that the following conditions hold:*

$$(1)\ \operatorname{Ric}(x, t) \geq -A \cdot g(x, t) \quad \text{for all } (x, t) \in M \times [0, T),$$

$$(2)\ \big|\big|\,||R(\cdot, t)||_{L^\alpha(M)}\big|\big|_{L^\beta([0,T))} < +\infty$$

for some $A \in \mathbb{R}$ and pair $(\alpha, \beta) \in (1, +\infty) \times (1, +\infty) \subset \mathbb{R}^2$ with

$$\alpha \geq \frac{n}{2}\frac{\beta}{\beta - 1}.$$

Then this flow can be extended smoothly over T.

In the Part I, we study the case that $(\alpha, \beta) = (n/2, \infty)$ and $(\infty, 1)$. Under some stronger assumptions, we prove that in dimension four, such flow converges to a smooth Riemannian manifold except for finitely many orbifold singularities.

Main Theorem 1 (cf. [1, Corollary 1.11]). *Let $(M^4, g(t))_{t\in[0,T)}$ $(T < \infty)$ be a 4-dimensional closed (i.e., M is smooth compact and connected manifold without boundary) Ricci flow satisfying*

$$(*) \quad || \sup_M |R_{g(t)}| ||_{L^1([0,T))} \leq C < +\infty$$

for some positive constant C, where $R_{g(t)}$ denotes the scalar curvature of $g(t)$. Then there exists a positive constant $\varepsilon = \varepsilon(M, g(0), T)$ such that the following holds : $(*)_{p_0,\varepsilon}$ For fixed $p_0 > 2$, assume that there exists $r > 0$ such that

$$\sup_{t\in[0,T)} ||R_{g(t)}||_{L^{p_0}(B(x,r,t))} \leq \varepsilon$$

for all $x \in M$, where $B(x, r, t)$ denotes the geodesic open ball centered at x of radius r with respect to $g(t)$. Then $(M, g(t))$ converges to an orbifold in the smooth Cheeger-Gromov sense. More specifically, we can find a decomposition $M = M^{\mathrm{reg}} \bigcup M^{\mathrm{sing}}$ with the following properties:
(1) M^{reg} is open and connected in M,
(2) M^{sing} is a zero set with respect to the Riemannian volume measure $dvol_{g(t)}$ for all $t \in [0, T)$,
(3) $g(t)$ smoothly converges to a Riemannian metric g_T on M^{reg} as $t \to T$.
(4) (M^{reg}, g_T) can be compactified to a metric space $(\bar{M}^{\mathrm{reg}}, \bar{d})$ by adding finitely many points and the differentiable structure on M^{reg} can be extended to a smooth orbifold structure on $\bar{M}^{\mathrm{reg}}$ such that the orbifold singularities are of cone type,
(5) Around every orbifold singularity of $(\bar{M}^{\mathrm{reg}}, \bar{d})$ the metric g_T satisfies

$$|\nabla^m \mathrm{Rm}| = o(\rho^{-2-m}) \text{ and } |\nabla^m \mathrm{Ric}| = O(\rho^{-1-m-\frac{2}{p_0}}) \text{ as } \rho \to 0, \quad \text{for all } m \geq 0$$

where ρ denotes the distance to the singularity. Furtheremore, for every $\varepsilon > 0$ we can find a smooth orbifold metric $\bar{g}_\varepsilon$ on $\bar{M}^{\mathrm{reg}}$ such that the following holds:

$$||g_T - \bar{g}_\varepsilon||_{C^0(M^{\mathrm{reg}}, \bar{g}_\varepsilon)} + ||g_T - \bar{g}_\varepsilon||_{W^{2,2}(M^{\mathrm{reg}}, \bar{g}_\varepsilon)} < \varepsilon.$$

Here, the C^0 and $W^{2,2}$-norms are taken with respect to $\bar{g}_\varepsilon$.

We also show that in higher dimensions, the same assertions hold for a closed Ricci flow satisfying another conditions of integral curvature bounds.

Main Theorem 2 (cf. [8, Theorem A]). *Let $(M^n, g(t))_{t\in[0,T)}$ $(T < \infty)$ be a n-dimensional $(n \geq 5)$ closed Ricci flow satisfying $(*)$ in Main Theorem 1. Then there exists a positive constsnat $\varepsilon = \varepsilon(M, g(0), n, T)$ such that the following holds : Suppose that*

$$\sup_{t\in[0,T)} ||\mathrm{Rm}_{g(t)}||_{L^{n/2}(M)} < +\infty$$

and $(*)_{p_0,\varepsilon}$ for some $p_0 > n/2$ in Main Theorem 1 holds. Then the assertions (1)-(5) in Main Theorem 1 hold.

Moreover, we show that such flows can be extended over T by an orbifold Ricci flow.

In the Part II, for a compact manifold M with non-empty boundary ∂M, we give a Koiso-type decomposition theorem, as well as an Ebin-type slice theorem, for the space of all Riemannian metrics on M endowed with a fixed conformal class on ∂M. In the case that $\partial M = \emptyset$, Ebin [2] particularly has proved a slice theorem for the pullback action of the diffeomorphism group on the space $\mathscr{M}$, of all Riemannian metrics on M. In [5], Koiso has extended it to an Inverse Limit Hilbert (ILH for brevity)-version. Moreover, he has also studied the conformal action on $\mathscr{M}$, and consequently has proved the following decomposition theorem for $\mathscr{M}$.

Theorem 0.3 (Koiso's decomposition theorem [6, Corollary 2.9]). *Let M^n be a closed n-manifold ($n \geq 3$), $\mathscr{M}$ the space of all Riemannian metrics on M and $\mathrm{Diff}(M)$ the diffeomorphism group of M. Set also*

$$C^\infty_+(M) := \{f \in C^\infty(M) \mid f > 0 \text{ on } M\},$$

$$\check{\mathfrak{S}} := \left\{ g \in \mathscr{M} \,\middle|\, \mathrm{Vol}(M, g) = 1,\ R_g = \mathrm{const},\ \frac{R_g}{n-1} \notin \mathrm{Spec}(-\Delta_g) \right\},$$

where $\mathrm{Vol}(M, g)$, R_g and $\mathrm{Spec}(-\Delta_g)$ denote respectively the volume of (M, g), the scalar curvature of g and the set of all non-zero eigenvalues of the (non-negative) Laplacian $-\Delta_g$ of g. Note that these four spaces become naturally ILH-manifolds. For any $g = f\bar{g}$ ($f \in C^\infty_+$, $\bar{g} \in \check{\mathfrak{S}}$) and any smooth deformation $\{g(t)\}_{t\in(-\varepsilon,\varepsilon)}$ of g for sufficiently small $\varepsilon > 0$, then there exist uniquely smooth deformations $\{f(t)\}_{t\in(-\varepsilon,\varepsilon)} (\subset C^\infty_+(M))$ of f, $\{\phi(t)\}_{t\in(-\varepsilon,\varepsilon)} (\subset \mathrm{Diff}(M)\ )$ of the identity id_M and $\{g(t)\}_{t\in(-\varepsilon,\varepsilon)} (\subset \check{\mathfrak{S}}\ )$ of $\bar{g}$ with $\delta_g(\bar{g}'(0)) = 0$ such that

$$g(t) = f(t)\phi(t)^*\bar{g}(t).$$

Here, $\delta_g(\bar{g}'(0))$ denotes the divergence $-\ \nabla^i_g(\bar{g}'(0))_i$ with respect to g.

We generalize these results to the case that $\partial M \neq \emptyset$ with some suitable boundary conditions.

Main Theorem 3. *For any $g = f\bar{g}$ ($f \in C^\infty_+(M)_N$, $\bar{g} \in \check{\mathfrak{S}}_{C^1_0}$) and any smooth deformation $\{g(t)\}_{t\in(-\varepsilon,\varepsilon)} (\subset \mathscr{M}_{C^1_0})$ of g for sufficiently small $\varepsilon > 0$, there exist smooth deformations $\{f(t)\}_{t\in(-\varepsilon,\varepsilon)} (\subset C^\infty_+(M)_N)$ of f, $\{\phi(t)\}_{t\in(-\varepsilon,\varepsilon)} (\subset \mathrm{Diff}_{C_0})$ of id_M and $\{\bar{g}(t)\}_{t\in(-\varepsilon,\varepsilon)} (\subset \check{\mathfrak{S}}_{C^1_0})$ of $\bar{g}$ with $\delta_g(\bar{g}'(0)) = 0$ such that*

$$g(t) = f(t)\phi(t)^*\bar{g}(t).$$

The spaces in Main Theorem 3 are defined as follows, respectively.

Fix a Riemannian metric g_0 on M with $H_{g_0} = 0$ along ∂M and set its conformal class $C := [g_0]$ on M. ν_{g_0} denotes the outer unit normal vector field along ∂M with respect to g_0. When two metrics g and $\tilde{g}$ on M have the same 1-jets $j^1_x g = j^1_x \tilde{g}$ for all $x \in \partial M$, we write it as $j^1_{\partial M} g = j^1_{\partial M} \tilde{g}$. Set also

$$C^\infty_+(M)_N := \{f \in C^\infty_+(M) \mid \nu_{g_0}(f)|_{\partial M} = 0\},$$

$$\mathscr{M}_{C_0} := \{g \in \mathscr{M} \mid g = fg_0 \text{ on } \partial M \text{ for some } f \in C^\infty_+(M),\ H_g = 0 \text{ on } \partial M\},$$

$$\mathscr{M}_{C^1_0} := \{g \in \mathscr{M} \mid j^1_{\partial M} g = j^1_{\partial M}(fg_0) \text{ for some } f \in C^\infty_+(M)_N\},$$

$$\mathfrak{S}_{C^{(1)}_0} := \{g \in \mathscr{M}_{C^{(1)}_0} \mid \mathrm{Vol}(M, g) = 1,\ R_g = \mathrm{const}\},$$

$$\check{\mathfrak{S}}_{C_0^{(1)}} := \left\{ g \in \mathfrak{S}_{C_0^{(1)}} \;\middle|\; \frac{R_g}{n-1} \notin \mathrm{Spec}(-\Delta_g; \mathrm{Neumann}) \right\},$$

$$\mathrm{Diff}_{C_0} := \left\{ \phi \in \mathrm{Diff}(M) \;\middle|\; j^1_{\partial M}(\phi^* g_0) = j^1_{\partial M}(f g_0) \text{ on } \partial M \text{ for some } f \in C^\infty_+(M)_N \right\},$$

where $\mathrm{Spec}(-\Delta_g; \mathrm{Neumann})$ denotes the set of all non-zero eigenvalues of $-\Delta_g$ with the Neumann boundary condition. As a corollary, we give a characterization of relative Einstein metrics. Moreover, we also give the following sufficient condition for a positive constant scalar curvature metric on a manifold with boundary to be a relative Yamabe metric, which is a natural relative version of the classical Yamabe metric.

Theorem 0.4. *Let g be a relative Yamabe metric on a compact connected smooth manifold M of dimension $n \geq 3$ with non-empty smooth boundary ∂M with $R_g > 0$ on M. Assume that h is a relative metric on M with constant scalar curvature and that φ is a diffeomorphism of M such that $dv_{\varphi^* h} = \gamma dv_g$ for some positive constant γ. If*

$$R_h h \leq R_g g, \tag{1}$$

then h is also a relative Yamabe metric. Moreover, if

$$R_h h < R_g g, \tag{2}$$

then h is a unique relative Yamabe metric (up to positive constant) in the relative confomal class $[h]_0$ of h. Here, $[h]_0 := \left\{ g \in [h] \;\middle|\; H_g = 0 \text{ on } \partial M \right\} = \left\{ u^{\frac{4}{n-2}} \cdot h \;\middle|\; u \in C^\infty_+(M),\ \nu_h(u) = 0 \text{ on } \partial M \right\}$, where ν_h denotes the inward unit normal vector field of ∂M with respect to h on M.

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