Extension of Olsen's inequality to Morrey-Lorentz spaces

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Abstract

Morrey-Lorentz spaces, which are an extension of Morrey and Lorentz spaces, were introduced by Ragusa [17] in 2012. Morrey spaces were introduced by Morrey [14] to investigate the solutions of second-order elliptic partial differential equations. Lorentz [13] defined Lorentz spaces and compared them with Lebesgue and Morrey spaces (see [13, Theorem 3]). In particular, according to [13, Theorem 2], Lorentz spaces are separable, but Morrey spaces are not. Lorentz spaces can be constructed by the real interpolation spaces of Lebesgue spaces (see, e.g., [1]). Morrey spaces have weak Lebesgue spaces as proper subspaces. In this study, by showing the proper embedding for Morrey-Lorentz spaces, we explained the Morrey-Lorentz scale enjoys diversity. In addition, Morrey spaces are used to obtain the Fefferman-Phong inequality (see (1)).

Olsen's inequality represents the weighted boundedness of fractional integral operators on Morrey spaces (see [16]). Taking the gradient of functions, we see that this inequality is an extension of the Fefferman-Phong inequality; for a potential $V \geq 0$,

$$\int_{\mathbb{R}^n} |u(x)|^2 V(x)\,\mathrm{d}x \leq C_V \int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x. \tag{1}$$

According to [2, p. 143], this inequality is a necessary condition for the positivity of the Schrödinger operator $-\Delta - V$. This is such an important problem that one considers the optimality of the constant C_V appearing in the above estimate. As a result, when V belongs to some Morrey spaces, Olsen proved the above estimates. Since then, many authors have investigated generalizations for Olsen's inequality, including generalized Morrey spaces [19], Orlicz-Morrey spaces [18, 3] of various types, and mixed Morrey spaces [15]. In particular, according to [20, Proposition 4.1], we can no longer relax the condition on the local integrability (see [19]).

Olsen's inequality cannot simply be proved by a mere combination of the Hölder inequality and the boundedness of the fractional integral operator on Morrey spaces. Seemingly, Olsen's inequality can be obtained by combining boundedness of the Riesz potential and Hölder's inequality; however this is not the case. To this end, the proof of this inequality is very difficult, and many authors have given alternative proofs. Tanaka [21] used the Calderón-Zygmund decomposition for the family of dyadic cubes to additionally give the vector-valued extension. In addition, Iida et al. [12] provided the atomic decomposition for Morrey spaces, and as an application, they proved Olsen's inequality. In [5], the author applied Tanaka's method to the generalization for its inequality. In this thesis, we refer to the ideas from the paper by Iida et al. to obtain an extension to its inequality for Morrey-Lorentz spaces. We gave the main theorem as follows:

Theorem 1. *Let* $0 < \alpha < n$, $1 < p_1 \le p_0 < \infty$, $1 < q_1 \le q_0 < \infty$, $1 < r_1 \le r_0 < \infty$, *and* $0 < p_2, r_2 \le \infty$. *Assume that*

$$r_1 < q_1, \quad \frac{1}{q_0} \le \frac{\alpha}{n} < \frac{1}{p_0}, \quad \frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}.$$

If we suppose either of the following;

(1) $0 < r_2, p_2 < \infty$ *and* $\frac{r_0}{p_0} = \frac{r_1}{p_1} = \frac{r_2}{p_2}$,

(2) $r_2 = p_2 = \infty$ *and* $\frac{r_0}{p_0} = \frac{r_1}{p_1}$,

then we have

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \|g\|_{\mathrm{W}\mathcal{M}^{q_0}_{q_1}} \|f\|_{\mathcal{M}^{p_0}_{p_1,p_2}}$$

for any non-negative $f \in \mathcal{M}^{p_0}_{p_1,p_2}(\mathbb{R}^n)$ *and any* $g \in \mathrm{W}\mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n)$.

The Taylor and Fourier expansions are classically well known as decompositions of functions classically. Decomposing functions yields approximations of functions. In this thesis, we employ our "atomic decomposition" as a method for decomposition. The Taylor and Fourier expansions use some power and trigonometric functions, respectively, while atomic decomposition uses some functions with compact support that are orthogonal to polynomials up to a fixed order. The origin of atomic decomposition goes back to the investigation of Hardy spaces. We provided the atomic decompositions for Hardy-Morrey-Lorentz spaces as follows:

Theorem 2. *Suppose that the parameters* p, q, r, s, t, v *satisfy*

$$0 < q \le p < \infty, \quad 0 < r \le \infty, \quad 1 < t \le s < \infty, \quad 0 < v \le 1,$$

$$q < t, \quad p < s, \quad v < \min(q, r).$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n) \cap \mathcal{P}_{d_v}(\mathbb{R}^n)^\perp$ *and* $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ *fulfill*

$$\|a_j\|_{\mathrm{W}\mathcal{M}^s_t} \le |Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} < \infty.$$

Then, $f = \sum_{j=1}^\infty \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{H\mathcal{M}^p_{q,r}} \lesssim_{p,q,r,s,t} \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}}. \tag{2}$$

Theorem 3. *Suppose that the real parameters* p, q, r, *and* K *satisfy*

$$0 < q \le p < \infty, \quad 0 < r \le \infty, \quad K \in \mathbb{N}_0 \cap \left(\frac{n}{q_0} - n - 1, \infty \right),$$

where $q_0 := \min(1, q)$. *Let* $f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$. *Then, there exists a triplet*

$$\{\lambda_j\}_{j=1}^\infty \subset [0,\infty), \quad \{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n), \quad \text{and} \quad \{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K^\perp(\mathbb{R}^n)$$

such that $f = \sum_{j=1}^\infty \lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and that for all* $v > 0$,

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} \lesssim_v \|f\|_{H\mathcal{M}^p_{q,r}}.$$

This paper presents the author's achievements, systematically combining [5, 9].

Additionally, the present author investigated many kinds of operators, including the boundedness of bilinear fractional integral operators of Grafakos type [4, 11], universality of neural networks with ReLU activations [6], boundedness of composition operators on Morrey and weak Morrey spaces [7], predual spaces of weak Orlicz spaces [8], and pointwise multiplier spaces from Besov spaces to Banach lattices [10].

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