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Integer values of generating functions for Lucas sequences

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INTEGER VALUES OF GENERATING FUNCTIONS FOR LUCAS SEQUENCES

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Abstract. It is known that the generating function of the Fibonacci sequence, $F(t) = \sum_{k=0}^{\infty} F_k t^k = t/(1-t-t^2)$, attains an integer value if and only if $t = F_k/F_{k+1}$ for some $k \in \mathbb{Z}$. In this article, we generalize this result for the Lucas sequences and the companion Lucas sequences associated to $(P, \pm 1)$, clarifying a role of the arithmetic of real quadratic number fields.

Introduction

The Lucas sequences, including the Fibonacci sequence, have been studied widely for a long time. There is left an enormous accumulation of research, and it seems that there remains an abundance of ore to mine.

For example, let $\{F_k\}_{k\geq 0}$ and $\{\Lambda_k\}_{k\geq 0}$ denote the Fibonacci sequence and the Lucas sequence, repectively, and put

$$F(t) = \sum_{k=0}^{\infty} F_k t^k = \frac{t}{1-t-t^2},\ G(t) = \sum_{k=0}^{\infty} \Lambda_k t^k = \frac{2-t}{1-t-t^2}.$$

It was recently that Hong [1] observed that $F(F_n/F_{n+1})$, $G(F_n/F_{n+1})$ and $G(\Lambda_n/\Lambda_{n+1})$ are integers for $n \geq 0$ and posed a question which rational number q assures $F(q) \in \mathbb{Z}$ or $G(q) \in \mathbb{Z}$. Soon after, Pongsriiam [3] answered the question, establishing the following results:
(1) Let $q \in \mathbb{Q}$. Then, $F(q)$ is an integer if and only if $q = F_n/F_{n+1}$ or $-F_{n+1}/F_n$ for some n;
(2) Let $q \in \mathbb{Q}$. Then, $G(q)$ is an integer if and only if $q = F_n/F_{n+1}$, $-F_{n+1}/F_n$, Λ_n/Λ_{n+1} or $-\Lambda_{n+1}/\Lambda_n$ for some n.

Tsuno ([6],[7]) generalized Pongsriiam's result to the generating functions for sequences given by the Pell eqautions. Their argument depends on skillful combination of various formulas for the sequences defined by recurrence relation of order 2.

In this article, we reexamine their results and generalize (1) and (2) for the Lucas sequences and the companion Lucas sequences associated to $(P, \pm 1)$.

Main Result I (=Theorem 2.3) *Let $P, Q \in \mathbb{Z}$ with $P \neq 0$, $Q = \pm 1$, $P^2 - 4Q > 0$ and $(P, Q) \neq (\pm 3, 1)$. Put $f(t) = t/(1 - Pt + Qt^2)$, the generating function of the Lucas sequnce associated to (P, Q). Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = L_n/L_{n+1}$ for some $n \in \mathbb{Z}$.*

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Main Result II (=Theorem 3.5) *Let $P, Q \in \mathbb{Z}$ with $P \neq 0$, $Q = \pm 1$ and $P^2 - 4Q > 0$. Put $f(t) = (2 - Pt)/(1 - Pt + Qt^2)$, the generating function of the companion Lucas sequnce associated to (P, Q).*
(1) *Assume $Q = -1$. Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = L_n/L_{n+1}$ or S_n/S_{n+1} for some $n \in \mathbb{Z}$.*
(2) *Assume $Q = 1$. Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = L_n/L_{n+1}$, S_n/S_{n+1}, $(L_{n+1} - L_n)/(L_{n+2} - L_{n+1})$ or $(L_n + L_{n+1})/(L_{n+2} + L_{n+1})$ for some $n \in \mathbb{Z}$.*

Now we explain the organization of the article. In the Section 1, we recall needed facts on the Lucas sequences though most of them are well known. We treat linear recurrence sequences also for negative indices, which simplifies formulas and the argument. Main Result I and Main Result II are proven in the Section 2 and in the Section 3, respectively. It should be mentioned that two main results follow from Dirichlet's unit theorem for real quadratic number fields. In the Section 4, we compare preceeding results and ours. In the Section 5, we remark upon an unlooked-for relation between our main result and the group $G_{P,Q}(\mathbb{Q})/\Theta$ investigated in [4] and [5].

Notation

For a ring R, $R^\times$ denotes the multiplicative group of invertible elements of R.

$\mathcal{L}(P, Q; \mathbb{Z})$**,** $\mathcal{L}(P, Q; \mathbb{Q})$**:** defined in 1.1
$\{L_k\}_{k \geq 0}$**:** the Lucas sequence associated to (P, Q), recalled in 1.1
$\{S_k\}_{k \geq 0}$**:** the companion Lucas sequence associated to (P, Q), recalled in 1.1
$\{F_k\}_{k \geq 0}$**:** the Fibonacci sequence
$\{\Lambda_k\}_{k \geq 0}$**:** the Lucas sequence, recalled in 1.2
(a, b)**:** the greatest common divisor of $a, b \in \mathbb{Z}$
$G_{P,Q}(\mathbb{Q})$**:** defined in 5.3
$G_{(P,Q)}(\mathbb{Q})$**:** defined in 5.3
$U_{P,Q}(\mathbb{Q})$**:** defined in 5.3
$G_{(P,Q)}(\mathbb{Q})/\Theta$**:** defined in 5.3

1. Recall: Lucas sequences

In the section, we fix $P, Q, \in \mathbb{Z}$ and put $D = P^2 - 4Q$.

Notation 1.1. For $P, Q \in \mathbb{Z}$, we put

$$\mathcal{L}(P, Q; \mathbb{Z}) = \{\{w_k\}_{k \geq 0} \in \mathbb{Z}^{\mathbb{N}} \ ; \ w_{k+2} - Pw_{k+1} + Qw_k = 0 \text{ for each } k \geq 0\}$$

and

$$\mathcal{L}(P, Q; \mathbb{Q}) = \{\{w_k\}_{k \geq 0} \in \mathbb{Q}^{\mathbb{N}} \ ; \ w_{k+2} - Pw_{k+1} + Qw_k = 0 \text{ for each } k \geq 0\}.$$

The sequence $\{L_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Z})$ defined by $(L_0, L_1) = (0,1)$ is called the *Lucas sequence* associated to (P,Q), and $\{S_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Z})$ defined by $(S_0, S_1) = (2, P)$ is called the *companion Lucas sequence* associated to (P,Q).

As is well known, for $\{w_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Q})$, we have

$$w_{n+1}^2 - Pw_{n+1}w_n + Qw_n^2 = (w_1^2 - Pw_1w_0 + Qw_0^2)Q^n.$$

Example 1.2. The Lucas sequence associated to $(P,Q) = (1,-1)$ is nothing but the Fibonacci sequence $\{F_k\}_{k\geq 0}$. On the other hand, the companion Lucas sequence associated to $(P,Q) = (1,-1)$ is traditionally called the Lucas sequence and denoted by $\{L_k\}_{k\geq 0}$. To avoid the confusion, we shall denote by $\{\Lambda_k\}_{k\geq 0}$ the Lucas sequence.

Definition 1.3. Assume that $Q \neq 0$. Let $\{w_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Q})$. Then we can define terms w_k for $k < 0$ inductively by the recurrence relation

$$w_k = \frac{P}{Q}w_{k+1} - \frac{1}{Q}w_{k+2}.$$

Hereinafter we enumerate several formulas concerning Lucas sequences.

Formulas 1.4. *Let $P, Q \in \mathbb{Z}$ with $Q \neq 0$. Then we have:*

(1) $w_{-n}w_{n+1} - Qw_{-n-1}w_n = w_0(2w_1 - Pw_0)$ *for* $\{w_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Q})$.

(2) $L_{-n} = -\dfrac{L_n}{Q^n}$, $S_{-n} = \dfrac{S_n}{Q^n}$.

(3) $\dfrac{L_{-n-1}}{L_{-n}} = \dfrac{1}{Q}\dfrac{L_{n+1}}{L_n}$, $\dfrac{S_{-n-1}}{S_{-n}} = \dfrac{1}{Q}\dfrac{S_{n+1}}{S_n}$.

Proof. We can easily verify the formulas (1) and (2) by induction on $n > 0$. The formula (3) is an immediate consequence of (2).

Formulas 1.5. *Let $P, Q \in \mathbb{Z}$ with $P^2 - 4Q \neq 0$. Let α, β denote the roots of the quadratic equation $t^2 - Pt + Q = 0$. Then we have:*

(1) $w_n = \dfrac{1}{\alpha - \beta}\{(w_1 - \beta w_0)\alpha^n - (w_1 - \alpha w_0)\beta^n\}$ for $\{w_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Q})$.

In particular,

(2) $L_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, $S_n = \alpha^n + \beta^n$.

Defintion 1.6. Let $P, Q \in \mathbb{Z}$ and $\{w_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Z})$. The generating function for $\{w_k\}_{k\geq 0}$ is defined by

$$f(t) = \sum_{k\geq 0} w_k t^k \in \mathbb{Z}[[t]].$$

As is well known, we have

$$f(t) = \frac{w_0 + (w_1 - Pw_0)t}{1 - Pt + Qt^2}.$$

For example, the generating function for the Lucas sequence $\{L_k\}_{k\geq 0}$ is given by

$$f(t) = \frac{t}{1 - Pt + Qt^2},$$

and the generating function for the companion Lucas sequence $\{S_k\}_{k\geq 0}$ is given by

$$f(t) = \frac{2-Pt}{1-Pt+Qt^2}.$$

Formulas 1.7. *Put* $f(t) = \dfrac{w_0+(w_1-Pw_0)t}{1-Pt+Qt^2}$*. Then we have:*

(1) $f\big(\dfrac{s}{r}\big) = \dfrac{r\{w_0r+(w_1-Pw_0)s\}}{r^2-Prs+Qs^2}$ for $r, s \in \mathbb{Z}$.

(2) $f\big(\dfrac{v_n}{v_{n+1}}\big) = \dfrac{v_{n+1}\{w_0v_{n+1}+(w_1-Pw_0)v_n\}}{(v_1^2-Pv_1v_0+Qv_0^2)Q^n}$ for $\{v_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Q})$.

Formulas 1.8. *Put* $f(t) = \dfrac{t}{1-Pt+Qt^2}$*. Then we have:*

(1) $f\big(\dfrac{s}{r}\big) = \dfrac{rs}{r^2-Prs+Qs^2}$ *for* $r, s \in \mathbb{Z}$.

(2) $f\big(\dfrac{v_n}{v_{n+1}}\big) = \dfrac{v_{n+1}v_n}{(v_1^2-Pv_1v_0+Qv_0^2)Q^n}$ *for* $\{v_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Q})$.

(3) $f\big(\dfrac{L_n}{L_{n+1}}\big) = \dfrac{L_{n+1}L_n}{Q^n}$.

(4) $f\big(\dfrac{L_{-n-1}}{L_{-n}}\big) = f\big(\dfrac{L_n}{L_{n+1}}\big)$.

Proof. We can easily deduce the formula (3) from (2), noting $L_1^2 - PL_1L_0 + QL_0^2 = 1$. The formula (4) follows from (3) and 1.4 (2).

Formulas 1.9. *Put* $f(t) = \dfrac{2-Pt}{1-Pt+Qt^2}$*. Then we have:*

(1) $f\big(\dfrac{s}{r}\big) = \dfrac{r(2r-Ps)}{r^2-Prs+Qs^2}$ *for* $r, s \in \mathbb{Z}$.

(2) $f\big(\dfrac{v_n}{v_{n+1}}\big) = \dfrac{v_{n+1}(2v_{n+1}-Pv_n)}{(v_1^2-Pv_1v_0+Qv_0^2)Q^n}$ *for* $\{v_k\}_{k\geq 0} \in \mathcal{L}(P,Q;\mathbb{Z})$.

(3) $f\big(\dfrac{S_n}{S_{n+1}}\big) = -\dfrac{S_{n+1}L_n}{Q^n}$.

(4) $f\big(\dfrac{L_n}{L_{n+1}}\big) = \dfrac{L_{n+1}S_n}{Q^n}$.

(5) $f\big(\dfrac{S_{-n-1}}{S_{-n}}\big) = f\big(\dfrac{L_n}{L_{n+1}}\big)$.

(6) $f\big(\dfrac{L_{-n-1}}{L_{-n}}\big) = f\big(\dfrac{S_n}{S_{n+1}}\big)$.

Proof. We can easily deduce the formulas (3) and from (2), noting

$$S_1^2 - PS_1S_0 + QS_0^2 = -P^2 + 4Q = D,\ 2S_{n+1} - PS_n = DL_n, 2L_{n+1} - L_n = S_n.$$

The formulas (5) and (6) are combinations of (3), (4) and 1.4 (2).

2. Main result I

Lemma 2.1. *Let $P, Q \in \mathbb{Z}$ with $P \neq 0$, $Q = \pm 1$ and $P^2 - 4Q > 0$. Let α be a root of the quadraic equation $t^2 - Pt + Q = 0$. Then α generates the multiplicative group $\mathbb{Z}[\alpha]^\times/\{\pm 1\}$ except for $(P, Q) = (\pm 3, 1)$.*

Proof. The multiplicative group $\mathbb{Z}[\alpha]^\times/\{\pm 1\}$ is cyclic as is well known. Assume that α does not generate the multiplicative group $\mathbb{Z}[\alpha]^\times/\{\pm 1\}$. Then there exists $\varepsilon \in \mathbb{Z}[\alpha]^\times$ such that $\alpha = \pm\varepsilon^k$ for some $k \geq 2$. Then we obtain $\mathbb{Z}[\varepsilon^k] = \mathbb{Z}[\varepsilon]$, which implies

$$\varepsilon^2 - \varepsilon - 1 = 0,\ \varepsilon^2 + \varepsilon - 1 = 0,\ \varepsilon^2 - \varepsilon + 1 = 0 \text{ or } \varepsilon^2 + \varepsilon + 1 = 0.$$

However, the latter two cases are excluded since ε is real. In the first case we have $\varepsilon = (1 \pm \sqrt{5})/2$, and in the second case we have $\varepsilon = (-1 \pm \sqrt{5})/2$. These correspond to the cases of $(P, Q) = (3, 1)$ and $(P, Q) = (-3, 1)$, respectively.

Lemma 2.2. *Let $P, Q, r, s \in \mathbb{Z}$ with $(r, Q) = 1$, $(r, s) = 1$ and $r \neq 0$. Put $f(t) = t/(1 - Pt + Qt^2)$. Then, $f(s/r)$ is an integer if and only if $r^2 - Prs + Qs^2 = \pm 1$.*

Proof. We can easily verify the assertion, noting that (a) $f(s/r) = rs/(r^2 - Prs + Qs^2)$, (b) $(r^2 - Prs + Qs^2, r) = (Qs^2, r) = 1$ and (c) $(r^2 - Prs + Qs^2, s) = (r^2, s) = 1$.

Theorem 2.3. *Let $P, Q \in \mathbb{Z}$ with $P \neq 0$, $Q = \pm 1$, $P^2 - 4Q > 0$ and $(P, Q) \neq (\pm 3, 1)$. Put $f(t) = t/(1 - Pt + Qt^2)$. Then, $f(q)$ ($q \in \mathbb{Q}$) is an integer if and only if $q = L_n/L_{n+1}$ for some $n \in \mathbb{Z}$.*

Proof. As is remarked in Formula 1.8, we have

$$f\left(\frac{L_n}{L_{n+1}}\right) = \frac{L_{n+1}L_n}{Q^n} \in \mathbb{Z}$$

for $n \in \mathbb{Z}$ ($n \neq 0$).

Conversely, assume that $f(q)$ is an integer. Put

$$D = P^2 - 4Q,\ \alpha = \frac{P + \sqrt{D}}{2},\ \beta = \frac{P - \sqrt{D}}{2}.$$

Then α is invertible in the ring $\mathbb{Z}[\alpha]$ since $\alpha\beta = Q = \pm 1$. Futhermore, α generates the multiplicative group $\mathbb{Z}[\alpha]^\times/\{\pm 1\}$ since $(P, Q) \neq (\pm 3, 1)$.

Now put

$$q = \frac{s}{r},\ r, s \in \mathbb{Z} \text{ with } (r, s) = 1.$$

Then, by Lemma 2.2, we obtain $r^2 - Prs + Qs^2 = \pm 1$, which implies that $r - \alpha s$ is invertible in $\mathbb{Z}[\alpha]$. Hence there exists $n \in \mathbb{Z}$ such that

$$r - \alpha s = \beta^n,\ r - \beta s = \alpha^n$$

or

$$r - \alpha s = -\beta^n,\ r - \beta s = -\alpha^n.$$

Hence, by Lemma 2.1, we obtain

$$(r, s) = (L_{n+1}, L_n) \text{ or } (-L_{n+1}, -L_n),$$

noting the formula $L_k = \dfrac{\alpha^k - \beta^k}{\alpha - \beta}$.

Propsition 2.4.1. (The case of $P = 3$ and $Q = 1$) *Put* $f(t) = t/(1 - 3t + t^2)$*, and let* $q \in \mathbb{Q}$*. Then,* $f(q)$ *is an integer if and only if* $q = F_n/F_{n+2}$ *for some* $n \in \mathbb{Z}$*.*

Proof. We can deduce

$$f\bigl(\frac{F_n}{F_{n+2}}\bigr) = (-1)^n F_{n+2} F_n$$

for $n \in \mathbb{Z}$ ($n \neq -2$) immediately from the equality $F_{n+2}^2 - 3F_{n+2}F_n + F_n^2 = (-1)^n$.

Conversely, put $\varepsilon = (1 + \sqrt{5})/2$. Then the roots of the quadratic equation $t^2 - 3t + 1 = 0$ are given by $\alpha = \varepsilon^2 = (3 + \sqrt{5})/2$ and $\beta = \varepsilon^{-2} = (3 - \sqrt{5})/2$. Furthermore, ε generates the multiplicative group $\mathbb{Z}[\varepsilon^2]^\times/\{\pm 1\} = \mathbb{Z}[\varepsilon]^\times/\{\pm 1\}$.

Now, let $\{L_k\}_{k \in \mathbb{Z}}$ denote the Lucas sequence associated to $(P, Q) = (3, 1)$. Then we have $L_k = F_{2k}$ for each $k \in \mathbb{Z}$. Now put

$$q = \frac{s}{r},\ r, s \in \mathbb{Z} \text{ with } (r, s) = 1.$$

Then, by Lemma 2.2, we obtain $r^2 - 3rs + s^2 = \pm 1$, which implies that $r - \alpha s$ is invertible in $\mathbb{Z}[\alpha]$. Hence there exists $n \in \mathbb{Z}$ such that

$$r - \alpha s = \varepsilon^{-n},\ r - \beta s = \varepsilon^n$$

or

$$r - \alpha s = -\varepsilon^{-n},\ r - \beta s = -\varepsilon^n.$$

Then we obtain

$$(r, s) = (F_{n+2}, F_n) \text{ or } (-F_{n+2}, -F_n),$$

noting $F_k = \dfrac{\varepsilon^k - \varepsilon^{-k}}{\varepsilon - \varepsilon^{-1}}$ and $\alpha - \beta = \varepsilon - \varepsilon^{-1}$.

Remark 2.4.2. Let $\{L_k\}_{k \geq 0}$ denote the Lucas sequence associated to $(P, Q) = (3, 1)$. Then we have $\{F_{2k}\}_{k \geq 0} = \{L_k\}_{k \geq 0}$ and $\{F_{2k+1}\}_{k \geq 0} = \{L_{k+1} - L_k\}_{k \geq 0}$.

Propsition 2.5.1. (The case of $P = -3$ and $Q = 1$) *Put* $f(t) = t/(1 + 3t + t^2)$*, and let* $q \in \mathbb{Q}$*. Then* $f(q)$ *is an integer if and only if* $q = -F_n/F_{n+2}$ *for some* $n \in \mathbb{Z}$*.*

Proof. We can verify

$$f\bigl(-\frac{F_n}{F_{n+2}}\bigr) = (-1)^{n-1} F_{n+2} F_n$$

for $n \in \mathbb{Z}$ ($n \neq -2$) and prove the assertion as in Propsition 2.4.1.

Remark 2.5.2. Let $\{L_k\}_{k \geq 0}$ denote the Lucas sequence associated to $(P, Q) = (-3, 1)$. Then we have $\{(-1)^{k-1} F_{2k}\}_{k \geq 0} = \{L_k\}_{k \geq 0}$ and $\{(-1)^k F_{2k+1}\}_{k \geq 0} = \{L_{k+1} + L_k\}_{k \geq 0}$.

3. Main result II

Lemma 3.1. *Let $P, Q \in \mathbb{Z}$, and put*

$$D = P^2 - 4Q,\ \alpha = \frac{P+\sqrt{D}}{2},\ \beta = \frac{P-\sqrt{D}}{2}.$$

Assume that D is not a square. Let $r, s, r', s' \in \mathbb{Q}$. Then, $(r-s\alpha)/(r-s\beta) = (r'-s'\alpha)/(r'-s'\beta)$ if and only if $(r:s) = (r':s')$.

Proof. We obtain the conclusion immediately, simplifying $(r-s\alpha)(r'-s'\beta) = (r-s\beta)(r'-s'\alpha)$ and noting that α and β are linearly independent over $\mathbb{Q}$.

Lemma 3.2. *Let $P, Q, r, s \in \mathbb{Z}$ with $(r, Q) = 1$, $(r, s) = 1$ and $r \neq 0$, and put $f(t) = (2-Pt)/(1-Pt+Qt^2)$. Then, $f(s/r)$ is an integer if and only if $2r - Ps$ is divisible by $r^2 - Prs + Qs^2$.*

Proof. First note $f(s/r) = r(2r-Ps)/(r^2-Prs+Qs^2)$. Then, $f(s/r)$ is an integer if and only if $r(2r-Ps)$ is divisible by $r^2-Prs+Qs^2$. In this case, $2r-Ps$ is divisible by $r^2-Prs+Qs^2$ since $(r, r^2-Prs+Qs^2) = 1$.

Corollary 3.3. *Let $P, Q, r, s \in \mathbb{Z}$ with $P^2 - 4Q \neq 0$, $Q = \pm 1$, $(r, s) = 1$ and $r \neq 0$, and put $f(t) = (2-Pt)/(1-Pt+Qt^2)$, $D = P^2 - 4Q$ and $\alpha = (P+\sqrt{D})/2$. If $f(s/r)$ is an integer, then $(r-s\alpha)/(r-s\beta)$ is an invertible element of $\mathbb{Z}[\sqrt{D}]$.*

Proof. By Lemma 3.2, $2r-Ps$ is divisible by $r^2-Prs+Qs^2$. Put now $\eta = r-s\alpha$ and $\bar{\eta} = r-s\beta$. Then, we have $\mathrm{Nr}\,\eta = \mathrm{Nr}\,\bar{\eta} = r^2-Prs+Qs^2$ and $\eta+\bar{\eta} = 2r-Ps$. These imply that $\mathrm{Nr}\,\eta/\bar{\eta} = 1$ and $1/\eta + 1/\bar{\eta} \in \mathbb{Z}$, and therefore, $\eta/\bar{\eta} \in \mathbb{Z}[\eta] \subset \mathbb{Z}[\sqrt{D}]$. Hence the result.

Lemma 3.4. *Let $P, Q \in \mathbb{Z}$. Assume that $P^2 - 4Q \neq 0$. Let α and β be the roots of the quadratic equation $t^2 - Pt + Q = 0$. Then we have:*

(1) $\dfrac{L_{n+1}-\alpha L_n}{L_{n+1}-\beta L_n} = \dfrac{\beta^n}{\alpha^n} = \dfrac{\beta^{2n}}{Q^n}$,

(2) $\dfrac{S_{n+1}-\alpha S_n}{S_{n+1}-\beta S_n} = -\dfrac{\beta^n}{\alpha^n} = -\dfrac{\beta^{2n}}{Q^n}$,

(3) $\dfrac{(L_{n+2}-L_{n+1})-\alpha(L_{n+1}-L_n)}{(L_{n+2}-L_{n+1})-\beta(L_{n+1}-L_n)} = -\dfrac{\beta^{n+1}}{\alpha^n} = -\beta^{2n+1}$ if $Q = 1$,

(4) $\dfrac{(L_{n+2}+L_{n+1})-\alpha(L_{n+1}+L_n)}{(L_{n+2}+L_{n+1})-\beta(L_{n+1}+L_n)} = \dfrac{\beta^{n+1}}{\alpha^n} = \beta^{2n+1}$ if $Q = 1$.

Proof. We can readily verify (1) and (2), noting

$$(\alpha^{n+1}-\beta^{n+1})-\alpha(\alpha^n-\beta^n) = (\alpha-\beta)\beta^n,\ (\alpha^{n+1}-\beta^{n+1})-\beta(\alpha^n-\beta^n) = (\alpha-\beta)\alpha^n,$$

$$(\alpha^{n+1}+\beta^{n+1})-\alpha(\alpha^n+\beta^n) = -(\alpha-\beta)\beta^n,\ (\alpha^{n+1}+\beta^{n+1})-\beta(\alpha^n-\beta^n) = (\alpha-\beta)\alpha^n.$$

Assume now $Q = 1$. Then we obtain $\alpha\beta = 1$, and therefore,

$$(L_{n+2}-L_{n+1})-\alpha(L_{n+1}-L_n)=\frac{\beta^{n+1}-\beta^n}{\alpha-\beta}=\frac{\beta^{n+1}(1-\alpha)}{\alpha-\beta},$$
$$(L_{n+2}-L_{n+1})-\beta(L_{n+1}-L_n)=\frac{\alpha^{n+1}-\alpha^n}{\alpha-\beta}=\frac{\alpha^{n}(\alpha-1)}{\alpha-\beta},$$
$$(L_{n+2}+L_{n+1})-\alpha(L_{n+1}+L_n)=\frac{\beta^{n+1}+\beta^n}{\alpha-\beta}=\frac{\beta^{n+1}(1+\alpha)}{\alpha-\beta},$$
$$(L_{n+2}+L_{n+1})-\beta(L_{n+1}+L_n)=\frac{\alpha^{n+1}+\alpha^n}{\alpha-\beta}=\frac{\alpha^{n}(\alpha+1)}{\alpha-\beta}.$$

Theorem 3.5. Let $P, Q \in \mathbb{Z}$ with $P \neq 0$, $Q = \pm 1$ and $P^2 - 4Q > 0$. Put $f(t) = (2 - Pt)/(1 - Pt + Qt^2)$.
(1) Assume $Q = -1$. Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = L_n/L_{n+1}$ or S_n/S_{n+1} for some $n \in \mathbb{Z}$.
(2) Assume $Q = 1$. Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = L_n/L_{n+1}$, S_n/S_{n+1}, $(L_{n+1} - L_n)/(L_{n+2} - L_{n+1})$ or $(L_n + L_{n+1})/(L_{n+2} + L_{n+1})$ for some $n \in \mathbb{Z}$.

Proof. As is remarked in Formulas 1.9, we have

$$f\Big(\frac{S_n}{S_{n+1}}\Big)=-\frac{S_{n+1}L_n}{Q^n}\in\mathbb{Z},\ f\Big(\frac{L_n}{L_{n+1}}\Big)=\frac{L_{n+1}S_n}{Q^n}\in\mathbb{Z}.$$

Moreover, in the case of $Q = 1$, we can verify

$$f\Big(\frac{L_{n+1}-L_n}{L_{n+2}-L_{n+1}}\Big)=-(L_{n+2}-L_{n+1})(L_{n+1}+L_n)\in\mathbb{Z},$$
$$f\Big(\frac{L_{n+1}+L_n}{L_{n+2}+L_{n+1}}\Big)=(L_{n+2}+L_{n+1})(L_{n+1}-L_n)\in\mathbb{Z},$$

noting

$$(L_{n+2}-L_{n+1})^2-P(L_{n+2}-L_{n+1})(L_{n+1}-L_n)+(L_{n+1}-L_n)^2=2-P,$$
$$2(L_{n+2}-L_{n+1})-P(L_{n+1}-L_n)=-(2-P)(L_{n+1}+L_n),$$

and

$$(L_{n+2}+L_{n+1})^2-P(L_{n+2}+L_{n+1})(L_{n+1}+L_n)+(L_{n+1}+L_n)^2=2+P,$$
$$2(L_{n+2}+L_{n+1})-P(L_{n+1}+L_n)=(2+P)(L_{n+1}-L_n),$$

repectively.

Conversely, assume that $f(q)$ is an integer. Put

$$D=P^2-4Q,\ \alpha=\frac{P+\sqrt{D}}{2},\ \beta=\frac{P-\sqrt{D}}{2}.$$

Then α is invertible in the ring $\mathbb{Z}[\alpha]$.

Now we assume $(P, Q) \neq (\pm 3, 1)$, which implies that α generates the multiplicative group $\mathbb{Z}[\alpha]^{\times}/\{\pm 1\}$ by Lemma 2.1. Put

$$q = \frac{s}{r},\ r, s \in \mathbb{Z} \text{ with } (r, s) = 1.$$

Then, by Corollary 3.3, $(r - s\alpha)/(r - s\beta)$ is an invertible element of $\mathbb{Z}[\alpha]$.

First assume $Q = -1$. Then we have

$$\frac{r - s\alpha}{r - s\beta} = \pm(-1)^n \beta^{2n}$$

for some $n \in \mathbb{Z}$ since $\mathrm{Nr}(r - \alpha s) = \mathrm{Nr}(r - \beta s)$ and $\mathrm{Nr}\,\beta = -1$. Hence we obtain

$$(s : r) = (L_n : L_{n+1}) \text{ or } (S_n : S_{n+1})$$

by Lemma 3.1 and Lemma 3.4.

Assume now $Q = 1$. Then we have

$$\frac{r - s\alpha}{r - s\beta} = \pm\beta^{2n} \text{ or } \frac{r - s\alpha}{r - s\beta} = \mp\beta^{2n+1}$$

for some $n \in \mathbb{Z}$. Then we obtain

$$(s : r) = (L_n : L_{n+1}), (S_n : S_{n+1}), (L_{n+1} - L_n : L_{n+2} - L_{n+1}) \text{ or } (L_{n+1} + L_n : L_{n+2} + L_{n+1})$$

again by Lemma 3.1 and Lemma 3.4.

We treat the case of $(p, Q) = (\pm 3, 1)$ separately in 3.6 and 3.7.

Remark 3.5.1. In the case of $Q = 1$, we have

$$\frac{L_{-n-1} - L_{-n-2}}{L_{-n} - L_{-n-1}} = \frac{L_{n+2} - L_{n+1}}{L_{n+1} - L_n},\ \frac{L_{-n-1} + L_{-n-2}}{L_{-n} + L_{-n-1}} = \frac{L_{n+2} + L_{n+1}}{L_{n+1} + L_n}$$

and

$$f\Big(\frac{L_{n+2} - L_{n+1}}{L_{n+1} - L_n}\Big) = f\Big(\frac{L_{n+1} + L_n}{L_{n+2} + L_{n+1}}\Big),\ f\Big(\frac{L_{n+1} + L_n}{L_{n+2} + L_{n+1}}\Big) = f\Big(\frac{L_{n+1} - L_n}{L_{n+2} - L_{n+1}}\Big).$$

Propsition 3.6.1. (The case of $P = 3$ and $Q = 1$) *Put $f(t) = (2t - 3)/(1 - 3t + t^2)$, and let $q \in \mathbb{Q}$. Then $f(q)$ is an integer if and only if $q = F_n/F_{n+2}$ or Λ_n/Λ_{n+2} for some $n \in \mathbb{Z}$.*

Proof. Put $\varepsilon = (1 + \sqrt{5})/2$. Then the roots of the quadratic equation $t^2 - 3t + 1 = 0$ are given by $\alpha = \varepsilon^2 = (3 + \sqrt{5})/2$ and $\beta = \varepsilon^{-2} = (3 - \sqrt{5})/2$. Furthermore, ε generates the multiplicative group $\mathbb{Z}[\varepsilon^2]^{\times}/\{\pm 1\} = \mathbb{Z}[\varepsilon]^{\times}/\{\pm 1\}$.

Now put

$$q = \frac{s}{r},\ r, s \in \mathbb{Z} \text{ with } (r, s) = 1,$$

and assume that $f(q)$ is an integer. Then, By Corollary 3.3, there exists $n \in \mathbb{Z}$ such that

$$\frac{r - s\alpha}{r - s\beta} = \pm\varepsilon^{-2n}$$

since $\mathrm{Nr}(r - \alpha s) = \mathrm{Nr}(r - \beta s)$ and $\mathrm{Nr}\,\varepsilon = -1$. That is to say, there exists $n \in \mathbb{Z}$ such that

$$\frac{r - s\alpha}{r - s\beta} = \pm\beta^n.$$

Hence we obtain

$$(s:r)=(L_n:L_{n+1}),(S_n:S_{n+1}),(L_{n+1}-L_n:L_{n+2}-L_{n+1})\text{ or }(L_{n+1}+L_n:L_{n+2}+L_{n+1})$$

by Lemma 3.1 and Lemma 3.4. At last, we obtain the result, noting

$$L_n=F_{2n},\ S_n=\Lambda_{2n},\ L_{n+1}-L_n=F_{2n+1},\ L_{n+1}+L_n=\Lambda_{2n+1}.$$

Remark 3.6.2. Let $\{L_k\}_{k\geq 0}$ denote the Lucas sequence associated to $(P,Q)=(3,1)$. Then we have $\{F_{2k}\}_{k\geq 0}=\{L_k\}_{k\geq 0}$ and $\{F_{2k+1}\}_{k\geq 0}=\{L_{k+1}-L_k\}_{k\geq 0}$, as is remarked in 2.4.2, and $\{\Lambda_{2k}\}_{k\geq 0}=\{S_k\}_{k\geq 0}$ and $\{\Lambda_{2k+1}\}_{k\geq 0}=\{L_{k+1}+L_k\}_{k\geq 0}$.

We can similarly prove the following:

Propsition 3.7.1. (The case of $P=-3$ and $Q=1$) *Put* $f(t)=(2t+3)/(1+3t+t^2)$, *and let* $q\in\mathbb{Q}$. *Then* $f(q)$ *is an integer if and only if* $q=-F_n/F_{n+2}$ *or* $-\Lambda_n/\Lambda_{n+2}$ *for some* $n\in\mathbb{Z}$.

Remark 3.7.2. Let $\{L_k\}_{k\geq 0}$ denote the Lucas sequence associated to $(P,Q)=(-3,1)$. Then we have $\{(-1)^{k-1}F_{2k}\}_{k\geq 0}=\{L_k\}_{k\geq 0}$ and $\{(-1)^kF_{2k+1}\}_{k\geq 0}=\{L_{k+1}+L_k\}_{k\geq 0}$, as is remarked in 2.5.2, and $\{(-1)^k\Lambda_{2k}\}_{k\geq 0}=\{S_k\}_{k\geq 0}$ and $\{(-1)^k\Lambda_{2k+1}\}_{k\geq 0}=\{L_{k+1}-L_k\}_{k\geq 0}$.

4. Preceeding results

4.1. Let N be a positive integer. Assume that N is not a square. Let (a,b) denote the minimal solution of the Pell equation $x^2-Ny^2=\pm 1$. Define two integer sequences $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ by

$$U_n=\frac{(a+b\sqrt{N})^n-(a-b\sqrt{N})^n}{2\sqrt{N}}$$

and

$$V_n=\frac{(a+b\sqrt{N})^n+(a-b\sqrt{N})^n}{2}.$$

Put $P=2a$ and $Q=a^2-Nb^2=\pm 1$. Then $\{U_n\}_{n\geq 0},\{V_n\}_{n\geq 0}\in\mathcal{L}(P,Q;\mathbb{Z})$. The generating functions of $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ are given by

$$\frac{bt}{1-Pt+Qt^2}$$

and

$$\frac{1}{2}\,\frac{1-at}{1-Pt+Qt^2},$$

respectively. We have also

$$U_n=bL_n,\ V_n=\frac{1}{2}S_n$$

for each $n\in\mathbb{Z}$, where $\{L_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ denote the Lucas sequence and the companion Lucas sequence associated to (P,Q), respectively.

Tsuno [6] proves the following assertions:

(1) Put $f(t)=\dfrac{bt}{1-Pt+Qt^2}$, and let $q\in\mathbb{Q}$. Then, $f(q)$ is an integer if and only if $q=U_n/U_{n+1}$ or QU_{n+1}/U_n for some $n\geq 0$.

(2) Put $f(t) = \frac{1}{2}\frac{1-at}{1-Pt+Qt^2}$, and let $q \in \mathbb{Q}$. Then, $f(q)$ is an integer if and only if $q = U_n/U_{n+1}$, V_n/V_{n+1}, $q = QU_{n+1}/U_n$ or QV_{n+1}/V_n for some $n \geq 0$.

Noting

$$U_n/U_{n+1} = L_n/L_{n+1},\ U_{n+1}/U_n = L_{n+1}/L_n,\ V_n/V_{n+1} = S_n/S_{n+1}, V_{n+1}/V_n = S_{n+1}S_n$$

and

$$L_{n+1}/L_n = QL_{-n-1}/L_{-n},\ S_{n+1}/S_n = QS_{-n-1}/S_{-n},$$

we can restate the above assetions as follows:
(1)′ Put $f(t) = \frac{bt}{1-Pt+Qt^2}$, and let $q \in \mathbb{Q}$. Then, $f(q)$ is an integer if and only if $q = L_n/L_{n+1}$ for some $n \in \mathbb{Z}$.
(2)′ Put $f(t) = \frac{1}{2}\frac{1-at}{1-Pt+Qt^2}$, and let $q \in \mathbb{Q}$. Then, $f(q)$ is an integer if and only if $q = L_n/L_{n+1}$ or S_n/S_{n+1} for some $n \in \mathbb{Z}$.

Now we deduce these assertions from ours.

In the case of (1)′ we have

$$f\Big(\frac{L_n}{L_{n+1}}\Big) = \frac{bL_{n+1}L_n}{Q^n} \in \mathbb{Z}.$$

Conversely, put $q = s/r$ ($r, s \in \mathbb{Z}$ with $(r, s) = 1$), and assume that $f(q)$ is an integer. Then b is divisible by $r^2-Prs+Qs^2$ since brs is divisible by $r^2-Prs+Qs^2$ and $(rs, r^2-Prs+Qs^2) = 1$. Put now $\eta = (r + sa) - sb\sqrt{N}$ and $d = (r + sa, sb)$. Then $\mathrm{Nr}\,\eta = r^2 - Prs + Qs^2$. Moreover, η/d is invertible in $\mathbb{Z}[\sqrt{N}]$,

Indeed, b is divisible by d^2 and b/d^2 is divisible by $\mathrm{Nr}(\eta/d)$ since b is divisible by $\mathrm{Nr}\,\eta$. Assume now $\mathrm{Nr}(\eta/d) \neq \pm 1$, and let p be a prime divisor of $\mathrm{Nr}(\eta/d)$. Then, we could conclude that b/d and $(r + sa)/d$ are both divisible by p, noting

$$\mathrm{Nr}\,\frac{\eta}{d} = \Big(\frac{r+sa}{d}\Big)^2 - s^2\Big(\frac{b}{d}\Big)^2.$$

However, this contradicts the fact that $(r + sa)/d$ and b/d are prime to each other.

The multiplicatve group $\mathbb{Z}[\sqrt{N}]^\times/\{\pm 1\}$ is generated by $\alpha = a + b\sqrt{N}$ since (a, b) is the minimal solution of the Pell equation $x^2 - Ny^2 = \pm 1$. Hence, we obtain $(r - s\alpha)/d = \pm\beta^n$ and $(r - s\beta)/d = \pm\alpha^n$, and therefore $r/s = L_{n+1}/L_n$ for some $n \in \mathbb{Z}$.

On the other hand, in the case of (2)′ we have

$$f\Big(\frac{S_n}{S_{n+1}}\Big) = \frac{S_{n+1}L_n}{2},\ f\Big(\frac{L_n}{L_{n+1}}\Big) = -\frac{L_{n+1}S_n}{2},$$

$$f\Big(\frac{L_{n+1}-L_n}{L_{n+2}-L_{n+1}}\Big) = -\frac{1}{2}(L_{n+2}-L_{n+1})(L_{n+1}+L_n),$$

$$f\Big(\frac{L_{n+1}+L_n}{L_{n+2}+L_{n+1}}\Big) = \frac{1}{2}(L_{n+2}+L_{n+1})(L_{n+1}-L_n).$$

Hence, we can conclude

$$f\Big(\frac{S_n}{S_{n+1}}\Big), f\Big(\frac{L_n}{L_{n+1}}\Big) \in \mathbb{Z},$$

noting that S_k is even for each $k \in \mathbb{Z}$ since $S_0 = 2$ and $S_1 = 2$. Furtheremore, we can verify $L_k \equiv k \mod 2$ for each k, noting $L_0 = 0$, $L_1 = 1$ and $L_2 \equiv 0 \mod 2$. Hence we obtain

$$f\big(\frac{L_{n+1} - L_n}{L_{n+2} - L_{n+1}}\big), f\big(\frac{L_{n+1} + L_n}{L_{n+2} + L_{n+1}}\big) \notin \mathbb{Z}.$$

4.2. Let N be a positive integer. Assume that N is not a square. Let (a, b) denote the minimal solution of the Pell equation $x^2 - Ny^2 = \pm 4$. Define two integer sequences $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ by

$$U_n = \frac{(a + b\sqrt{N})^n - (a - b\sqrt{N})^n}{2^n\sqrt{N}}$$

and

$$V_n = \frac{(a + b\sqrt{N})^n + (a - b\sqrt{N})^n}{2^n}.$$

Put $P = a$ and $Q = (a^2 - Nb^2)/4 = \pm 1$. Then $\{U_n\}_{n\geq 0}, \{V_n\}_{n\geq 0} \in \mathcal{L}(P, Q\ \mathbb{Z})$. The generating functions of $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ are given by

$$\frac{bt}{1 - Pt + Qt^2}$$

and

$$\frac{2 - at}{1 - Pt + Qt^2},$$

respectively. We have also

$$U_n = bL_n,\ V_n = S_n$$

for each $n \in \mathbb{Z}$, where $\{L_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ denote the Lucas sequence and the companion Lucas sequence associated to (P, Q), respectively.

Tsuno [7] proves the following assertions, under the assumption $N \geq 5$:

(1) Put $f(t) = \dfrac{bt}{1 - Pt + Qt^2}$, and let $q \in \mathbb{Q}$. Then, $f(q)$ is an integer if and only if $q = U_n/U_{n+1}$ or QU_{n+1}/U_n for some $n \geq 0$.

(2) Put $f(t) = \dfrac{1}{2}\dfrac{1 - at}{1 - Pt + Qt^2}$, and let $q \in \mathbb{Q}$. Then:

(a) Assume $Q = -1$. Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = U_n/U_{n+1}$, $-U_{n+1}/U_n$, V_n/V_{n+1} or $-V_{n+1}/V_n$ for some $n \in \mathbb{Z}$.

(b) Assume $Q = 1$. Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = \pm 1$ or $q = U_n/U_{n+1}$, U_n/U_{n+1}, V_n/V_{n+1}, V_{n+1}/V_n, $U_{2n-1}/(U_{2n} \pm U_1$ or $U_{2n+1})/(U_{2n+1} \pm U_1)$ some $n \in \mathbb{Z}$.

We can restate (1) and (2) as follows:

(1)′ Put $f(t) = \dfrac{bt}{1 - Pt + Qt^2}$, and let $q \in \mathbb{Q}$. Then, $f(q)$ is an integer if and only if $q = L_n/L_{n+1}$ for some $n \in \mathbb{Z}$.

(2)′ Put $f(t) = \dfrac{2 - at}{1 - Pt + Qt^2}$, and let $q \in \mathbb{Q}$. Then:

(a) Assume $Q = -1$. Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = L_n/L_{n+1}$ or S_n/S_{n+1} for some $n \in \mathbb{Z}$.

(b) Assume $Q = 1$. Then, $f(q)$ $(q \in \mathbb{Q})$ is an integer if and only if $q = L_n/L_{n+1}$, S_n/S_{n+1}, $(L_n - L_{n-1})/(L_{n+1} - L_n)$ or $(L_n - L_{n-1})/(L_{n+1} - L_n)$ some $n \in \mathbb{Z}$.

Indeed, we can deduce the assertion $(1)'$ from Theorem 2.3 as in 4.1. Now we deduce the assertion $(2)'$ from Theorem 3.5. First note

$$U_n/U_{n+1} = L_n/L_{n+1},\ U_{n+1}/U_n = L_{n+1}/L_n,\ V_n/V_{n+1} = S_n/S_{n+1}, V_{n+1}/V_n = S_{n+1}/S_n$$

and

$$L_{n+1}/L_n = QL_{-n-1}/L_{-n},\ S_{n+1}/S_n = QS_{-n-1}/S_{-n},$$

Furthermore, if $Q = 1$, then we have

$$\frac{L_0 - L_{-1}}{L_1 - L_0} = 1,\ \frac{L_0 + L_{-1}}{L_1 - L_0} = -1$$

and

$$\frac{L_n - L_{n-1}}{L_{n+1} - L_n} = \frac{L_{2n-1}}{L_{2n} - 1}, \frac{L_n + L_{n-1}}{L_{n+1} + L_n} = \frac{L_{2n-1}}{L_{2n} + 1},$$
$$\frac{L_{n+1} + L_n}{L_n + L_{n-1}} = \frac{L_{2n+1}}{L_{2n} - L_1}, \frac{L_{n+1} - L_n}{L_n - L_{n-1}} = \frac{L_{2n+1}}{L_{2n} + L_1},$$

which follow from

$$(L_n - L_{n-1})(L_{2n} - 1) = L_{2n-1}(L_{n+1} - L_n),\ (L_n + L_{n-1})(L_{2n} + 1) = L_{2n-1}(L_{n+1} + L_n),$$

$$(L_{n+1} + L_n)(L_{2n} - L_1) = L_{2n+1}(L_n + L_{n-1}),\ (L_{n+1} - L_n)(L_{2n} + L_1) = L_{2n+1}(L_n - L_{n-1}),$$

respectively. We can honestly verify these equalities, using the formula

$$L_n L_m = \frac{S_{n+m} - S_{n-m}}{D}.$$

Hence, the assertion $(2)'$ is nothing but Theorem 3.5.

5. An observation

In this section, we fix $P, Q \in \mathbb{Z}$ and put $D = P^2 - 4Q$.

Notation 5.1. Let $P, Q \in \mathbb{Z}$. As is well known, the map $\{w_k\}_{k\geq 0} \mapsto (w_0, w_1)$ gives rise to a $\mathbb{Q}$-linear isomorphism $\mathcal{L}(P, Q; \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}^2$.

Now put $\tilde{R} = \mathbb{Q}[t]/(t^2 - Pt + Q)$ and $\theta = t \mod (t^2 - Pt + Q)$. We define a $\mathbb{Q}$-linear map $\omega : \tilde{R} \to \mathbb{Q}$ by $\omega(a + b\theta) = b$ $(a, b \in \mathbb{Q})$. Moreover, we define a $\mathbb{Q}$-linear map $\tilde{\omega} : \tilde{R} \to \mathbb{Q}^{\mathbb{N}}$ by $\tilde{\omega}(\eta) = \{\omega(\eta\theta^k)\}_{k\geq 0}$. For $\eta = a + b\theta \in \tilde{R}$, we have $\tilde{\omega}(\eta) = \{b, a + Pb, \dots\}$.

We can verify the following statements, paraphrasing the proofs of [4, Prop.3.2 and Cor.3.3].

(1) The $\mathbb{Q}$-linear map $\tilde{\omega} : \tilde{R} \to \mathcal{L}(P, Q; \mathbb{Q}) \subset \mathbb{Q}^{\mathbb{N}}$ is bijective.

(2) A $\mathbb{Q}$-algebra structure of $\mathcal{L}(P, Q; \mathbb{Q})$ is defined through the $\mathbb{Q}$-linear isomorphism $\tilde{\omega} : \tilde{R} \xrightarrow{\sim} \mathcal{L}(P, Q; \mathbb{Q})$. Then the Lucas sequence $\{L_k\}_{k\geq 0} = \tilde{\omega}(1)$ is the unit of the ring $\mathcal{L}(P, Q; \mathbb{Q})$.

More precisely, let $\boldsymbol{w} = \{w_k\}_{k\geq 0}, \boldsymbol{w}' = \{w'_k\}_{k\geq 0} \in \mathcal{L}(P, Q; \mathbb{Q})$. Then the product of $\boldsymbol{w}$ and $\boldsymbol{w}'$ is given by

$$(w_0 w'_1 + w_1 w'_0 - P w_0 w'_0, w_1 w'_1 - Q w_0 w'_0, \dots).$$

It is readily seen that the multiplication by θ on $\tilde{R}$ induces the shift operation $\{w_k\}_{k\geq 0} \mapsto \{w_{k+1}\}_{k\geq 0}$ on $\mathcal{L}(P, Q; \mathbb{Q})$ through the isomorphism $\tilde{\omega} : \tilde{R} \xrightarrow{\sim} \mathcal{L}(P, Q; \mathbb{Q})$.

(3) Let $\eta = a + b\theta \in \tilde{R} = \mathbb{Q}[t]/(t^2 - Pt + Q)$ $(a, b \in \mathbb{Q})$. Then $\eta \mapsto \bar{\eta}$ gives rise to a $\mathbb{Q}$-automorphism of $\tilde{R}$. Moreover, we define $\mathrm{Nr}\,\eta \in \mathbb{Q}$ by $\mathrm{Nr}\,\eta = \eta\bar{\eta} = a^2 + Pab + Qb^2$. For example, we have $\mathrm{Nr}\,\theta = Q$. Obviously, η is invertible in $\tilde{R}$ if and only if $\mathrm{Nr}\,\eta \neq 0$.

Now let $\boldsymbol{w} = \{w_k\}_{k\geq 0} \in \mathcal{L}(P, Q; \mathbb{Q})$. Define $\Delta(\boldsymbol{w}) \in \mathbb{Q}$ by $\Delta(\boldsymbol{w}) = w_1^2 - Pw_0w_1 + Qw_0^2$. If $\eta \in \tilde{R}$ and $\boldsymbol{w} = \tilde{\omega}(\eta)$, then we have $\mathrm{Nr}\,\eta = \Delta(\boldsymbol{w})$. Therefore, the sequence $\boldsymbol{w} = \{w_k\}_{k\geq 0}$ is invertible in $\mathcal{L}(P, Q; \mathbb{Q})$ if and only if $\Delta(\boldsymbol{w}) = w_1^2 - Pw_0w_1 + Qw_0^2 \neq 0$.

Notation 5.2. We put $\delta = -P + 2\theta \in \tilde{R}$. Then we have $\delta^2 = D$ and $\mathrm{Nr}\,\delta = -D$. The sequence $\tilde{\omega}(\delta)$ is nothing but the companion Lucas sequence $(S_k)_{k\geq 0}$ associated to (P, Q).

Notation 5.3. We define groups $G_{P,Q}(\mathbb{Q})$, $G_{(P,Q)}(\mathbb{Q})$ and $U_{P,Q}(\mathbb{Q})$ by

$$\begin{aligned} G_{P,Q}(\mathbb{Q}) &= (\mathbb{Q}[t]/(t^2 - Pt + Q))^\times, \\ G_{(P,Q)}(\mathbb{Q}) &= \mathrm{Coker}[i : \mathbb{Q}^\times \to (\mathbb{Q}[t]/(t^2 - Pt + Q))^\times], \\ U_{P,Q}(\mathbb{Q}) &= \mathrm{Ker}[\mathrm{Nr} : (\mathbb{Q}[t]/(t^2 - Pt + Q))^\times \to \mathbb{Q}^\times]. \end{aligned}$$

Here $i : \mathbb{Q}^\times \to (\mathbb{Q}[t]/(t^2 - Pt + Q))^\times$ denotes the inclusion map. Moreover, we define a homomorphism of groups $\gamma : G_{P,Q}(\mathbb{Q}) \to U_{P,Q}(\mathbb{Q})$ by $\gamma(\eta) = \eta/\bar{\eta} = \eta^2/\mathrm{Nr}\,\eta$. Then we have $\mathrm{Ker}[\gamma : G_{P,Q}(\mathbb{Q}) \to U_{P,Q}(\mathbb{Q})] = \mathbb{Q}^\times$, and γ is surjective by Hilbert 90. Hence γ induces an isomorphism of groups $\tilde{\gamma} : G_{(P,Q)}(\mathbb{Q}) = G_{P,Q}(\mathbb{Q})/\mathbb{Q}^\times \xrightarrow{\sim} U_{P,Q}(\mathbb{Q})$. It is readily seen:

(a) If D is a square in $\mathbb{Q}^\times$, then $U_{P,Q}(\mathbb{Q})$ is isomorphic to the multiplicative group $\mathbb{Q}^\times$;

(b) If $D = 0$, then $U_{P,Q}(\mathbb{Q})$ is isomorphic to the additive group $\mathbb{Q}$;

(c) If D is not a square in $\mathbb{Q}$, then $U_{P,Q}(\mathbb{Q})$ is isomorphic to the multiplicative group $\mathrm{Ker}[\mathrm{Nr} : \mathbb{Q}(\sqrt{D})^\times \to \mathbb{Q}^\times]$.

Hence, if $D \neq 0$, then we obtain $\gamma(\delta) = -1$, which is a unique element of order 2 of $U_{P,Q}(\mathbb{Q})$.

Assume now $Q \neq 0$. Then θ is invertible in $\tilde{R} = \mathbb{Q}[t]/(t^2 - Pt + Q)$. Let Θ denote the subgroup of $G_{(P,Q)}(\mathbb{Q})$ generated by θ. Then the group $G_{(P,Q)}(\mathbb{Q})/\Theta$ is isomorphic to the group $G(f)$ defined by Laxton [2] under the assumptions: $(P, Q) = 1$ and $D = P^2 - 4Q \neq 0$ ([4, Th.4.2] and [5, Th.4.2]). Here $f(t) = t^2 - Pt + Q$.

Remark 5.3.1. The groups $G_{P,Q}(\mathbb{Q})$, $G_{(P,Q)}(\mathbb{Q})$ and $U_{P,Q}(\mathbb{Q})$ are the $\mathbb{Q}$-rational points of the group schemes $G_{P,Q}$, $G_{(P,Q)}$ and $U_{P,Q}$, respectively. For details, we refer to [4, Section 1] and [5, Section 1].

Hereafter, we investigate the elements of order 2 of $G_{(P,Q)}(\mathbb{Q})/\Theta$.

Proposition 5.4. *Let $\eta \in G_{P,Q}(\mathbb{Q})$. Then:*

(1) *There exists $\xi \in U_{P,Q}(\mathbb{Q})$ such that $\xi^2 = \gamma(\eta)$ if and only if $\mathrm{Nr}\,\eta$ is a square in $\mathbb{Q}$. In this case, the solutions of the equation $\xi^2 = \gamma(\eta)$ in $U_{P,Q}(\mathbb{Q})$ are given by $\xi = \pm\eta/\sqrt{\mathrm{Nr}\,\eta}$.*

(2) *Assume that $\mathrm{Nr}\,\eta$ is a square in $\mathbb{Q}$, and put $\eta = u + v\delta$ $(u, v \in \mathbb{Q})$. If $Dv \neq 0$, then we have $\pm\eta/\sqrt{\mathrm{Nr}\,\eta} = \gamma(\eta \pm \sqrt{\mathrm{Nr}\,\eta})$.*

Proof. (1) Assume first that $\mathrm{Nr}\,\eta$ is a square in $\mathbb{Q}$. Then we have $\pm\eta/\sqrt{\mathrm{Nr}\,\eta} \in U_{P,Q}(\mathbb{Q})$ and $(\pm\eta/\sqrt{\mathrm{Nr}\,\eta})^2 = \eta^2/\mathrm{Nr}\,\eta = \gamma(\eta)$.

Conversely, assume that there exists $\xi \in U_{P,Q}(\mathbb{Q})$ such that $\xi^2 = \gamma(\eta)$. Taking $\tilde{\xi} \in G_{P,Q}(\mathbb{Q})$ such that $\gamma(\tilde{\xi}) = \xi$, we obtain $\eta = a\xi^2$ for some $a \in \mathbb{Q}^\times$. This implies $\mathrm{Nr}\,\eta = a^2(\mathrm{Nr}\,\xi)^2$.

(2) Put $\tilde{\xi} = \eta \pm \sqrt{\mathrm{Nr}\,\eta}$. Then we obtain $\tilde{\xi}^2 = 2(u \pm \sqrt{\eta})\eta$, and therefore $\gamma(\tilde{\xi})^2 = \gamma(\eta)$ since $\mathrm{Nr}\,\eta = u^2 - Dv^2 \neq u^2$.

Remark 5.5. Assume $D = 0$. Let $\eta = u + v\delta \in \tilde{R} = \mathbb{Q}[t]/(t^2 - Pt + Q)$ $(u, v \in \mathbb{Q})$. Then we obtain $\eta^2 = u^2 + 2uv\delta$ and $\mathrm{Nr}\,\eta = u^2$, noting $\delta^2 = D$. Hence, η is invertible in $\tilde{R}$ if and only if $u \neq 0$. In this case, we have $\gamma(\eta) = 1 + 2v\delta/u$, and the solutions of $\xi^2 = \gamma(\eta)$ in $U_{P,Q}(\mathbb{Q})$ are given by $\xi = \pm(1 + v\delta/u)$.

Corollary 5.6. *Assume that $Q \neq 0$ and $D \neq 0$. Then there exists $\xi \in U_{P,Q}(\mathbb{Q})$ such that $\xi^2 = \gamma(\theta)$ in $U_{P,Q}(\mathbb{Q})$ if and only if $Q = \mathrm{Nr}\,\theta$ is a square in $\mathbb{Q}$. In this case, the solutions of the equation $\xi^2 = \gamma(\theta)$ in $U_{P,Q}(\mathbb{Q})$ are given by $\xi = \pm\theta/\sqrt{Q} = \gamma(\theta \pm \sqrt{Q})$.*

The following assertion is a direct consequence of Corollary 5.6.

Corollary 5.7. *Assume that $Q \neq 0$ and $D \neq 0$. Then*:
(1) *If Q is a square in $\mathbb{Q}$ and $P \neq 0, \pm\sqrt{Q}$, then the kernel of the square map on $G_{(P,Q)}(\mathbb{Q})/\Theta$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*
(2) *If Q is not a square in $\mathbb{Q}$, then the kernel of the square map on $G_{(P,Q)}(\mathbb{Q})/\Theta$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Example 5.8. Assume $Q = 1$ and $P \neq 0, \pm 1, \pm 2$. Then we have

$$(\theta + 1)^2 = (P + 2)\theta,\ \mathrm{Nr}(\theta + 1) = P + 2,\ \gamma(\theta + 1) = \theta,$$

$$(\theta - 1)^2 = (P - 2)\theta,\ \mathrm{Nr}(\theta - 1) = -P + 2,\ \gamma(\theta - 1) = -\theta,$$

$$(\theta + 1)(\theta - 1) = P\theta - 2 = \theta\delta.$$

Furthermore, the kernel of the square map on $G_{(P,1)}(\mathbb{Q})/\Theta$ is given by $\{[\theta] = 1, [\theta+1], [\theta-1], [\delta]\}$.

Observation 5.9. Put $f(t) = (2 - t)/(1 - Pt + t^2)$, the generating function of the companion Lucas sequence associated to $(P, 1)$. Let $q \in \mathbb{Q}$. Theorem 3.5, Proposition 3.6.1 and Proposition 3.7.1 assert that $f(q) \in \mathbb{Z}$ if and only if $q = w_n/w_{n+1}$ for some $n \in \mathbb{Z}$, where $\{w_k\}_{k \geq 0} = \tilde{\omega}(\eta)$ and $[\eta] \in \{[\theta], [\delta], [\theta+1], [\theta-1]\} \subset G_{(P,1)}(\mathbb{Q})/\Theta$. The author is not sure whether this is a chance or an apperance of a deeper fact. However, the following examples suggest that there is hidden something to consider.

Example 5.10.1. Let $P = 0$ and $Q = 1$. Then we have

$$\{L_k\}_{k\geq 0} = \{0, 1, 0, -1, 0, 1, \dots\},\ \{S_k\}_{k\geq 0} = \{2, 0, -2, 0, 2, 0, \dots\},$$

$$\{L_{k+1} + L_k\}_{k\geq 0} = \{1, 1, -1, -1, 1, 1, \dots\},\ \{L_{k+1} - L_k\}_{k\geq 0} = \{1, -1, -1, 1, 1, -1, \dots\},$$

and the kernel of the square map on $G_{(P,1)}(\mathbb{Q})/\Theta$ is given by $\{[\theta]=[\delta],[\theta+1]=[\theta-1]\}$. Moreover, let $q\in\mathbb{Q}$.

(1) Put $f(t)=t/(1+t^2)$. Then, $f(q)\in\mathbb{Z}$ if and only if $q=0$, i.e. $q=L_n/L_{n+1}$ for some $n\in\mathbb{Z}$.

(2) Put $f(t)=2/(1+t^2)$. Then, $f(q)\in\mathbb{Z}$ if and only if $q=0,\pm1$, i.e. $q=L_n/L_{n+1}$ or $(L_{n+1}+L_n)/(L_{n+2}+L_{n+1})$ for some $n\in\mathbb{Z}$.

Example 5.10.2. Let $P=1$ and $Q=1$. Then we have

$$\{L_k\}_{k\geq 0}=\{0,1,1,0,-1,-1,0,1,\dots\},\ \{S_k\}_{k\geq 0}=\{2,1,-1,-2,-1,1,2,1,\dots\},$$
$$\{L_{k+1}+L_k\}_{k\geq 0}=\{1,2,1,-1,-2,-1,2,1,\dots\},\ \{L_{k+1}-L_k\}_{k\geq 0}=\{1,0,-1,-1,0,1,1,0,\dots\},$$

and the kernel of the square map on $G_{(P,1)}(\mathbb{Q})/\Theta$ is given by $\{[\theta]=[\theta-1],[\delta]=[\theta+1]\}$. Moreover, let $q\in\mathbb{Q}$.

(1) Put $f(t)=t/(1-t+t^2)$. Then, $f(q)\in\mathbb{Z}$ if and only if $q=0,1$, i.e. $q=L_n/L_{n+1}$ for some $n\in\mathbb{Z}$.

(2) Put $f(t)=(2-t)/(1-t+t^2)$. Then, $f(q)\in\mathbb{Z}$ if and only if $q=0,1,2,1/2$, i.e. $q=L_n/L_{n+1}$ or S_n/S_{n+1} for some $n\in\mathbb{Z}$.

Example 5.10.3. Let $P=-1$ and $Q=1$. Then we have

$$\{L_k\}_{k\geq 0}=\{0,1,-1,0,1,\dots\},\ \{S_k\}_{k\geq 0}=\{2,-1,-1,2,-1,\dots\},$$
$$\{L_{k+1}+L_k\}_{k\geq 0}=\{1,0,-1,1,0,\dots\},\ \{L_{k+1}-L_k\}_{k\geq 0}=\{1,-2,1,1,-2,\dots\},$$

and the kernel of the square map on $G_{(P,1)}(\mathbb{Q})/\Theta$ is given by $\{[\theta]=[\theta+1],[\delta]=[\theta-1]\}$. Moreover, let $q\in\mathbb{Q}$.

(1) Put $f(t)=t/(1+t+t^2)$. Then, $f(q)\in\mathbb{Z}$ if and only if $q=0,-1$, i.e. $q=L_n/L_{n+1}$ for some $n\in\mathbb{Z}$.

(2) Put $f(t)=(2+t)/(1+t+t^2)$. Then, $f(q)\in\mathbb{Z}$ if and only if $q=0,-1,-2,-1/2$, i.e. $q=L_n/L_{n+1}$ or S_n/S_{n+1} for some $n\in\mathbb{Z}$.

Example 5.10.4. Let $P=2$ and $Q=1$. Then we have

$$L_k=k1^{k-1},\ S_k=2\cdot 1^k,\ L_{k+1}+L_k=2k1^{k-1}+1^k,\ L_{k+1}-L_k=1^k$$

and the kernel of the square map on $G_{(P,1)}(\mathbb{Q})/\Theta$ is given by $\{[\theta],[\theta+1]\}$. Moreover, let $q\in\mathbb{Q}$.

(1) Put $f(t)=t/(1-t)^2$. Then, $f(q)\in\mathbb{Z}$ if and only if $q=n/(n+1)=L_n/L_{n+1}$ for some $n\in\mathbb{Z}$.

(2) Put $f(t)=(2-2t)/(1-t)^2=2/(1-t)$. Then, $f(q)\in\mathbb{Z}$ if and only if $q=n/(n+1)=L_n/L_{n+1}$ or $q=(2n+1)/(2n+3)=(L_{n+1}+L_n)/(L_{n+2}+L_{n+1})$ for some $n\in\mathbb{Z}$.

Example 5.10.5. Let $P=-2$ and $Q=1$. Then we have

$$L_k=k(-1)^{k-1},\ S_k=2\cdot(-1)^k,\ L_{k+1}+L_k=(-1)^k,\ L_{k+1}-L_k=-2k(-1)^{k-1}+(-1)^k$$

and the kernel of the square map on $G_{(P,1)}(\mathbb{Q})/\Theta$ is given by $\{[\theta], [\theta - 1]\}$. Moreover, let $q \in \mathbb{Q}$.

(1) Put $f(t) = t/(1+t)^2$. Then, $f(q) \in \mathbb{Z}$ if and only if $q = -n/(n+1) = L_n/L_{n+1}$ for some $n \in \mathbb{Z}$.

(2) Put $f(t) = (2+2t)/(1+t)^2 = 2/(1+t)$. Then, $f(q) \in \mathbb{Z}$ if and only if $q = -n/(n+1) = L_n/L_{n+1}$ or $q = -(2n+1)/(2n+3) = (L_{n+1} + L_n)/(L_{n+2} + L_{n+1})$ for some $n \in \mathbb{Z}$.

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