On volume functions of special flow polytopes associated to the root system of type A and On equivariant index of a generalized Bott manifold

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Preface

In this thesis, we treat two themes.

In Part I, we consider a flow polytope associated to the root system of type A. The cone spanned by the positive roots is divided into several polyhedral cones called chambers. There is a specific chamber called the nice chamber. In this part, we call a flow polytope for the nice chamber a special flow polytope. Baldoni and Vergne showed the volume function of a special flow polytope is written as an iterated residue. In this case, we show that the volume function satisfies a certain system of differential equations, and conversely, the solution of the system of differential equations is unique up to a constant multiple. In addition, we give an inductive formula for the volume with respect to the rank of the root system of type A.

In Part II, we consider the equivariant index of a generalized Bott manifold. Grossberg and Karshon showed the multiplicity function of the equivariant index for a holomorphic line bundle over a Bott manifold is given by the density function of a twisted cube, which is determined by the structure of the Bott manifold and the line bundle over it. From this, they derived a Demazure-type character formula. In this part, we generalize the above results to generalized Bott manifolds. We show the multiplicity function of the equivariant index is given by the density function of a generalized twisted cube. In addition, we give a Demazure-type character formula of this representation.

Part I

On volume functions of special flow polytopes associated to the root system of type A

Chapter 1

Introduction

The number of lattice points and the volume of a convex polytope are important and interesting objects and have been studied from various points of view (see, e.g., [4]). For example, the number of lattice points of a convex polytope associated to a root system is called the Kostant partition function, and it plays an important role in representation theory of Lie groups (see, e.g., [9]).

We consider a flow polytope associated to the root system of type A. As explained in [2, 3], the cone spanned by the positive roots is divided into several polyhedral cones called *chambers*, and the combinatorial property of a flow polytope depends on a chamber. Moreover, there is a specific chamber called the *nice chamber*, which plays a significant role in [11]. In this part, we call a flow polytope for the nice chamber a *special flow polytope*. Also in [2, 3], a number of theoretical results related to the Kostant partition function and the volume function of a flow polytope can be found. In particular, it is shown that these functions for the nice chamber are written as iterated residues ([3, Lemma 21]). We also refer to [1] for similar formulas for other chambers in more general settings. Moreover, we mention that a generalization of the Lidskii formula is shown in [3, Theorem 38], there is a geometric proof of the Lidskii formula in [12], and combinatorial applications of this formula are given in [5, 7].

The purpose of this part is to characterize the volume function of a flow polytope for the nice chamber in terms of a system of differential equations, based on a result in [3]. In order to state the main results, we give some notation. Let $e_1, \ldots, e_{r+1}$ be the standard basis of $\mathbb{R}^{r+1}$ and let

$$A_r^+ = \{e_i - e_j \,|\, 1 \leq i < j \leq r+1\}$$

be the positive root system of type A with rank r. We assign a positive integer $m_{i,j}$ to each i and j with $1 \leq i < j \leq r+1$. Let us set $m = (m_{i,j})$ and $M = \sum_{1 \leq i < j \leq r+1} m_{i,j}$. For $a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in \mathbb{R}^{r+1}$, where $a_1 + \cdots + a_i \geq 0$ $(i = 1, \ldots, r)$, the following polytope $P_{A_r^+,m}(a)$ is called the flow polytope associated to the root system of type A:

$$P_{A_r^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \;\middle|\; \begin{array}{l} 1 \leq i < j \leq r+1\,,\ 1 \leq k \leq m_{i,j}\,,\ y_{i,j,k} \geq 0\,, \\ \sum_{1 \leq i < j \leq r+1} \sum_{1 \leq k \leq m_{i,j}} y_{i,j,k}(e_i - e_j) = a \end{array} \right\}.$$

Note that the flow polytopes in [3] include the case that some of $m_{i,j}$'s are zero, whereas we exclude such cases in this part. We denote the volume of $P_{A_r^+,m}(a)$ by $v_{A_r^+,m}(a)$.

The open set

$$\mathfrak{c}_{\text{nice}} := \{a = a_1e_1 + \cdots + a_re_r - (a_1 + \cdots + a_r)e_{r+1} \in \mathbb{R}^{r+1} \mid a_i > 0, i = 1, \ldots, r\}$$

in $\mathbb{R}^{r+1}$ is called the nice chamber. We are interested in the volume $v_{A_r^+,m}(a)$ when a is in the closure of the nice chamber, and then it is written by $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}$. It is a homogeneous polynomial of degree $M - r$. The first result of this part is the following.

Theorem 1.0.1 *Let* $a = \sum_{i=1}^r a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}_{\text{nice}}}$, *and let* $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ *be the volume of* $P_{A_r^+,m}(a)$. *Then* $v = v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ *satisfies the system of differential equations as follows:*

$$\begin{cases} \partial_r^{m_{r,r+1}} v = 0 \\ (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} v = 0 \\ \quad \vdots \\ (\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} v = 0, \end{cases}$$

where $\partial_i = \frac{\partial}{\partial a_i}$ *for* $i = 1, \ldots, r$. *Conversely, the polynomial* $v = v(a)$ *of degree* $M - r$ *satisfying the above equations is equal to a constant multiple of* $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$.

We remark that it is known that the volume function $v_{A_r^+,m}(a)$ of $P_{A_r^+,m}(a)$, as a distribution on $\mathbb{R}^r$, satisfies the differential equation

$$Lv_{A_r^+,m}(a) = \delta(a)$$

in general, where $L = \prod_{i<j}(\partial_i - \partial_j)^{m_{i,j}}$ and $\delta(a)$ is the Dirac delta function on $\mathbb{R}^r$ ([8, 11]). Note that ∂_{r+1} in the definition of L is supposed to be zero. The above theorem characterizes the function $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ on $\overline{\mathfrak{c}_{\text{nice}}}$ more explicitly. It might be interesting to see what kind of properties of the volume can be derived from Theorem 1.0.1.

In addition, in Theorem 3.0.6, we show the volume $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ is written by a linear combination of $v_{A_{r-1}^+,m',\mathfrak{c}'_{\text{nice}}}(a')$ and its partial derivatives, where $m' = (m_{i,j})_{2\le i<j\le r+1}$, $\mathfrak{c}'_{\text{nice}}$ is the nice chamber of A_{r-1}^+, and $a' = \sum_{i=2}^r a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}'_{\text{nice}}}$. It might be interesting to ask whether there is a relaton between this theorem and the inductive formulas of Schmidt–Bincer [14, (4.1), (4.24)].

This part is organized as follows. In Chapter 2, we recall the iterated residue, the Jeffrey-Kirwan residue, and the nice chamber based on [2], [3], [6] and [10]. Also, we give some examples of $P_{A_r^+,m}(a)$ and the calculations of the volume $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$. In Chapter 3, we prove the main theorems.

Chapter 2

Preliminaries

In this chapter, we set up the tools to prove the main theorems based on [2], [3], [6] and [10].

2.1 Flow polytopes and its volumes

Let $e_1, \ldots, e_{r+1}$ be the standard basis of $\mathbb{R}^{r+1}$, and let

$$V = \left\{ a = \sum_{i=1}^{r+1} a_i e_i \in \mathbb{R}^{r+1} \,\middle|\, \sum_{i=1}^{r+1} a_i = 0 \right\}.$$

We consider the positive root system of type A with rank r as follows:

$$A_r^+ = \{e_i - e_j \,|\, 1 \leq i < j \leq r+1\}.$$

Let $C(A_r^+)$ be the convex cone generated by A_r^+:

$$C(A_r^+) = \{a \in V \,|\, a_1 + \cdots + a_i \geq 0 \text{ for all } i, 1 \leq i \leq r\}.$$

We assign a positive integer $m_{i,j}$ to each i and j with $1 \leq i < j \leq r+1$, and it is called a multiplicity. Let us set $m = (m_{i,j})$ and $M = \sum_{1 \leq i < j \leq r+1} m_{i,j}$.

Definition 2.1.1 Let $a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in C(A_r^+)$. We consider the following polytope:

$$P_{A_r^+, m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \,\middle|\, \begin{array}{l} 1 \leq i < j \leq r+1\,,\ 1 \leq k \leq m_{i,j}\,,\ y_{i,j,k} \geq 0\,, \\ \sum_{1 \leq i < j \leq r+1} \sum_{1 \leq k \leq m_{i,j}} y_{i,j,k}(e_i - e_j) = a \end{array} \right\},$$

which is called the *flow polytope* associated to the root system of type A.

Remark 2.1.2 The flow polytopes in [3] include the case that $m_{i,j} = 0$ for some i and j.

The elements of A_r^+ generate a lattice $V_{\mathbb{Z}}$ in V. The lattice $V_{\mathbb{Z}}$ determines a measure da on V.

Let du be the Lebesgue measure on $\mathbb{R}^M$. Let $[\alpha_1, \ldots, \alpha_M]$ be a sequence of elements of A_r^+ with multiplicity $m_{i,j}$, and let φ be the surjective linear map from $\mathbb{R}^M$ to V defined by $\varphi(e_k) = \alpha_k$. The vector space $\ker(\varphi) = \varphi^{-1}(0)$ is of dimension $d = M - r$ and it is equipped with the quotient Lebesgue measure du/da. For $a \in V$, the affine space $\varphi^{-1}(a)$ is parallel to $\ker(\varphi)$, and thus also equipped with the Lebesgue measure du/da. Volumes of subsets of $\varphi^{-1}(a)$ are computed for this measure. In particular, we can consider the volume $v_{A_r^+,m}(a)$ of the polytope $P_{A_r^+,m}(a)$.

2.2 Total residue and iterated residue

Let $A_r = A_r^+ \cup (-A_r^+)$, and let U be the dual vector space of V. We denote by R_{A_r} the ring of rational functions $f(x_1, \ldots, x_r)$ on the complexification $U_{\mathbb{C}}$ of U with poles on the hyperplanes $x_i - x_j = 0$ $(1 \leq i < j \leq r+1)$ or $x_i = 0$ $(1 \leq i \leq r)$. A subset σ of A_r is called a *basis* of A_r if the elements $\alpha \in \sigma$ form a basis of V. In this case, we set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(x)}$$

and call such an element a *simple fraction*. We denote by S_{A_r} the linear subspace of R_{A_r} spanned by simple fractions. The space U acts on R_{A_r} by differentiation: $(\partial(u)f)(x) = (\frac{d}{d\varepsilon})f(x + \varepsilon u)|_{\varepsilon=0}$. We denote by $\partial(U)R_{A_r}$ the space spanned by derivatives of functions in R_{A_r}. It is shown in [6, Proposition 7] that $R_{A_r} = \partial(U)R_{A_r} \oplus S_{A_r}$. The projection map $\mathrm{Tres}_{A_r} : R_{A_r} \to S_{A_r}$ with respect to this decomposition is called the *total residue map*.

We extend the definition of the total residue to the space $\hat{R}_{A_r}$ consisting of functions P/Q where Q is a finite product of powers of the linear forms $\alpha \in A_r$ and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series with P_k of degree k. As the total residue vanishes outside the homogeneous component of degree $-r$ of A_r, we can define $\mathrm{Tres}_{A_r}(P/Q) = \mathrm{Tres}_{A_r}(P_{q-r}/Q)$, where q is degree of Q. For $a \in V$ and multiplicities $m = (m_{i,j}) \in (\mathbb{Z}_{\geq 0})^M$ of elements of A_r^+, the function

$$F := \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{\prod_{i=1}^r x_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}}$$

is in $\hat{R}_{A_r}$. We define $J_{A_r^+,m}(a) \in S_{A_r}$ by

$$J_{A_r^+,m}(a) = \mathrm{Tres}_{A_r} F.$$

Next, we describe the iterated residue.

Definition 2.2.1 For $f \in R_{A_r}$, we define the *iterated residue* by

$$\mathrm{Ires}_{x=0} f = \mathrm{Res}_{x_1=0} \mathrm{Res}_{x_2=0} \cdots \mathrm{Res}_{x_r=0} f(x_1, \ldots, x_r).$$

Since the iterated residue $\mathrm{Ires}_{x=0} f$ vanishes on the space $\partial(U)R_{A_r}$ as in [3], we have

$$\mathrm{Ires}_{x=0} J_{A_r^+,m}(a) = \mathrm{Ires}_{x=0} F. \tag{2.2.1}$$

2.3 Chambers and Jeffrey–Kirwan residue

Definition 2.3.1 Let $C(\nu)$ be the closed cone generated by ν for any subset ν of A_r^+ and let $C(A_r^+)_{\text{sing}}$ be the union of the cones $C(\nu)$ where ν is any subset of A_r^+ of cardinal strictly less than $r = \dim V$. By definition, the set $C(A_r^+)_{\text{reg}}$ of A_r^+-regular elements is the complement of $C(A_r^+)_{\text{sing}}$. A connected component of $C(A_r^+)_{\text{reg}}$ is called a *chamber*.

The Jeffrey–Kirwan residue [10] associated to a chamber $\mathfrak{c}$ of $C(A_r^+)$ is a linear form $f \mapsto \langle\langle \mathfrak{c}, f \rangle\rangle$ on the vector space S_{A_r} of simple fractions. Any function f in S_{A_r} can be written as a linear combination of functions f_σ, with a basis σ of A_r contained in A_r^+. To determine the linear map $f \mapsto \langle\langle \mathfrak{c}, f \rangle\rangle$, it is enough to determine it on this set of functions f_σ. So we assume that σ is a basis of A_r contained in A_r^+.

Definition 2.3.2 For a chamber $\mathfrak{c}$ and $f_\sigma \in S_{A_r}$, we define the Jeffrey–Kirwan residue $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle$ associated to a chamber $\mathfrak{c}$ as follows:

- If $\mathfrak{c} \subset C(\sigma)$, then $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle = 1$.
- If $\mathfrak{c} \cap C(\sigma) = \emptyset$, then $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle = 0$,

where $C(\sigma)$ is the convex cone generated by σ.

Remark 2.3.3 More generally, as in [3, Definition 11], the Jeffrey–Kirwan residue $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle$ is defined to be $\frac{1}{\mathrm{vol}(\sigma)}$ if $\mathfrak{c} \subset C(\sigma)$, where $\mathrm{vol}(\sigma)$ is the volume of the parallelepiped $\oplus_{\alpha\in\sigma}[0,1]\alpha$, relative to our Lebesgue measure da. In our case, the volume $\mathrm{vol}(\sigma)$ is equal to 1 since A_r is unimodular.

The volume $v_{A_r^+,m}(a)$ of the flow polytope $P_{A_r^+,m}(a)$ is written by the function $J_{A_r^+,m}(a)$ and the Jeffrey–Kirwan residue in the following.

Theorem 2.3.4 (Baldoni–Vergne [3]) *Let $\mathfrak{c}$ be a chamber of $C(A_r^+)$. Then, for $a \in \bar{\mathfrak{c}}$, the volume $v_{A_r^+,m}(a)$ of $P_{A_r^+,m}(a)$ is given by*

$$v_{A_r^+,m}(a) = \langle\langle \mathfrak{c}, J_{A_r^+,m}(a) \rangle\rangle.$$

We denote by $v_{A_r^+,m,\mathfrak{c}}(a)$ the polynomial function of a coinciding with $v_{A_r^+,m}(a)$ when $a \in \bar{\mathfrak{c}}$. It is a homogeneous polynomial of degree $M - r$.

2.4 Nice chamber

Definition 2.4.1 The open subset $\mathfrak{c}_{\text{nice}}$ of $C(A_r^+)$ is defined by

$$\mathfrak{c}_{\text{nice}} = \{a \in C(A_r^+) \,|\, a_i > 0 \ (i = 1, \ldots, r)\}.$$

The set $\mathfrak{c}_{\text{nice}}$ is in fact a chamber for the root system A_r^+ ([3]). The chamber $\mathfrak{c}_{\text{nice}}$ is called the *nice chamber*.

Lemma 2.4.2 (Baldoni–Vergne [3]) *For the nice chamber $\mathfrak{c}_{\text{nice}}$ of A_r^+ and $f \in S_{A_r}$, we have*

$$\langle\langle \mathfrak{c}_{\text{nice}}, f \rangle\rangle = \mathrm{Ires}_{x=0} f.$$

From Theorem 2.3.4, Lemma 2.4.2 and (2.2.1), we have the following corollary.

Corollary 2.4.3 (Lidskii formula [3]) *Let $a \in \overline{\mathfrak{c}_{\text{nice}}}$. Then the volume function $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ is given by*

$$v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a) = \mathrm{Ires}_{x=0} F.$$

2.5 Examples

In this section, we give some examples of the flow polytopes for A_1, A_2, and A_3, and calculate their volumes.

Example 2.5.1 When $r = 1$, the nice chamber of A_1^+ is $\mathfrak{c}_{\text{nice}} = \{a = a_1(e_1 - e_2) \mid a_1 > 0\}$. For $a = a_1(e_1 - e_2) \in \overline{\mathfrak{c}_{\text{nice}}}$,

$$P_{A_1^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^{m_{1,2}} \mid y_{i,j,k} \geq 0\,,\ y_{1,2,1} + y_{1,2,2} + \cdots + y_{1,2,m_{1,2}} = a_1 \right\}.$$

From Corollary 2.4.3, we have

$$v_{A_1^+,m,\mathfrak{c}_{\text{nice}}}(a) = \mathrm{Res}_{x_1=0} \left(\frac{e^{a_1 x_1}}{x_1^{m_{1,2}}} \right) = \frac{1}{(m_{1,2} - 1)!} a_1^{m_{1,2}-1}.$$

Example 2.5.2 When $r = 2$, there are two chambers $\mathfrak{c}_1, \mathfrak{c}_2$ of A_2^+ as below, and the nice chamber $\mathfrak{c}_{\text{nice}}$ of A_2^+ is $\mathfrak{c}_1$.

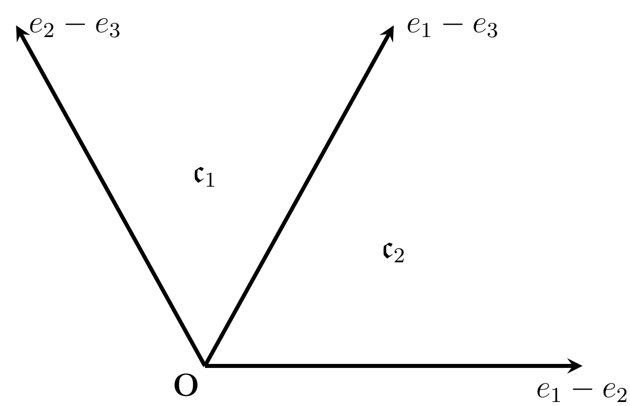

Figure 1 : The chamber of A_2^+.

For example, we set $m_{1,2} = n$ ($n \in \mathbb{Z}_{>0}$), $m_{1,3} = 1$, and $m_{2,3} = 1$. For $a = a_1e_1 + a_2e_2 - (a_1 + a_2)e_3 \in \overline{\mathfrak{c}_{\text{nice}}}$,

$$P_{A_2^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^{n+2} \;\middle|\; \begin{array}{l} y_{i,j,k} \geq 0 \\ y_{1,2,1} + y_{1,2,2} + \cdots + y_{1,2,n} + y_{1,3,1} = a_1 \\ -y_{1,2,1} - y_{1,2,2} - \cdots - y_{1,2,n} + y_{2,3,1} = a_2 \end{array} \right\}.$$

From Corollary 2.4.3, we have

$$v_{A_2^+,m,\mathfrak{c}_{\text{nice}}}(a) = \text{Ires}_{x=0} \left(\frac{e^{a_1x_1+a_2x_2}}{x_1x_2(x_1 - x_2)^n} \right) = \text{Res}_{x_1=0}\text{Res}_{x_2=0} \left(\frac{e^{a_1x_1+a_2x_2}}{x_1x_2(x_1 - x_2)^n} \right) = \frac{1}{n!}a_1^n.$$

Example 2.5.3 When $r = 3$, there are seven chambers of A_3^+ as below ([1]), and the nice chamber $\mathfrak{c}_{\text{nice}}$ of A_3^+ is $\mathfrak{c}_1$.

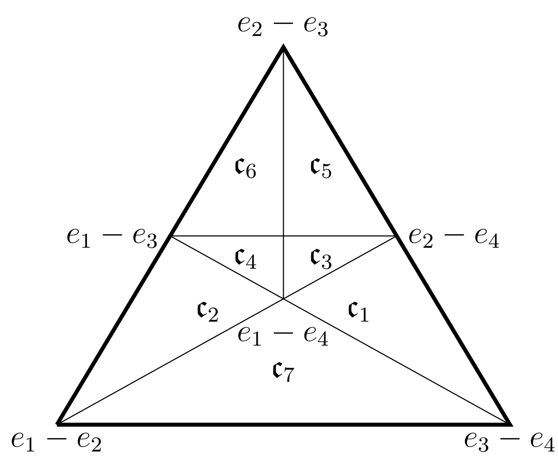

Figure 2 : The chamber of A_3^+.

For example, we set $m_{1,2} = 1$, $m_{1,3} = 1$, $m_{1,4} = 2$, $m_{2,3} = 1$, $m_{2,4} = 2$, and $m_{3,4} = 2$. For $a = \sum_{i=1}^{3} a_i(e_i - e_4) \in \overline{\mathfrak{c}_{\text{nice}}}$,

$$P_{A_3^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^{9} \;\middle|\; \begin{array}{l} y_{i,j,k} \geq 0 \\ y_{1,2,1} + y_{1,3,1} + y_{1,4,1} + y_{1,4,2} = a_1 \\ -y_{1,2,1} + y_{2,3,1} + y_{2,4,1} + y_{2,4,2} = a_2 \\ -y_{1,3,1} - y_{2,3,1} + y_{3,4,1} + y_{3,4,2} = a_3 \end{array} \right\}.$$

From Corollary 2.4.3, we have

$$\begin{aligned} v_{A_3^+,m,\mathfrak{c}_{\text{nice}}}(a) &= \text{Ires}_{x=0} \left(\frac{e^{a_1x_1+a_2x_2+a_3x_3}}{x_1^2x_2^2x_3^2(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} \right) \\ &= \frac{1}{360}a_1^3(a_1^3 + 6a_1^2a_2 + 3a_1^2a_3 + 15a_1a_2^2 + 15a_1a_2a_3 + 10a_2^3 + 30a_2^2a_3). \end{aligned}$$

Chapter 3

Main theorems of Part I

In this chapter, we prove the main theorems of this part. Let $\mathfrak{c}_{\text{nice}}$ be the nice chamber of A_r^+ and let $a = \sum_{i=1}^r a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}_{\text{nice}}}$.

Theorem 3.0.1 *For $a \in \overline{\mathfrak{c}_{\text{nice}}}$, let $P_{A_r^+,m}(a)$ be the flow polytope as in Definition 2.1.1 and let $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ be the volume of $P_{A_r^+,m}(a)$. Then $v = v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ satisfies the system of differential equations as follows:*

$$\begin{cases} \partial_r^{m_{r,r+1}} v = 0 \\ (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} v = 0 \\ \quad \vdots \\ (\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} v = 0, \end{cases}$$

where $\partial_i = \frac{\partial}{\partial a_i}$ for $i = 1, \ldots, r$.

Proof. We will prove the first two relations. Let $F = \frac{e^{a_1x_1+\cdots+a_rx_r}}{\prod_{i=1}^r x_i^{m_{i,r+1}} \prod_{1\le i<j\le r}(x_i-x_j)^{m_{i,j}}}$. It is easy to see that

$$P(\partial_1, \ldots, \partial_r)(\mathrm{Ires}_{x=0}F) = \mathrm{Ires}_{x=0}(P(\partial_1, \ldots, \partial_r)F) = \mathrm{Ires}_{x=0}(P(x_1, \ldots, x_r)F), \qquad (3.0.1)$$

where P is a polynomial. Since $\frac{e^{a_1x_1+\cdots+a_kx_k}}{\prod_{i=1}^{k-1} x_i^{m_{i,r+1}} \prod_{1\le i<j\le k}(x_i-x_j)^{m_{i,j}}}$ is holomorphic at $x_k = 0$,

$$\mathrm{Res}_{x_k=0}\left(\frac{e^{a_1x_1+\cdots+a_kx_k}}{\prod_{i=1}^{k-1} x_i^{m_{i,r+1}} \prod_{1\le i<j\le k}(x_i-x_j)^{m_{i,j}}}\right) = 0 \qquad (3.0.2)$$

for $k = 1, \ldots, r$. Therefore, from Corollary 2.4.3, (3.0.1) and (3.0.2), we obtain

$$\begin{aligned} \partial_r^{m_{r,r+1}} v &= \partial_r^{m_{r,r+1}} \mathrm{Ires}_{x=0} F = \mathrm{Ires}_{x=0} \partial_r^{m_{r,r+1}} F \\ &= \mathrm{Ires}_{x=0}\left(\frac{e^{a_1x_1+\cdots+a_rx_r}}{\prod_{i=1}^{r-1} x_i^{m_{i,r+1}} \prod_{1\le i<j\le r}(x_i-x_j)^{m_{i,j}}}\right) = 0, \end{aligned}$$

and

$$
\begin{aligned}
&(\partial_{r-1}-\partial_r)^{m_{r-1,r}}\partial_{r-1}^{m_{r-1,r+1}}v \\
&= \mathrm{Ires}_{x=0}(\partial_{r-1}-\partial_r)^{m_{r-1,r}}\partial_{r-1}^{m_{r-1,r+1}}F \\
&= \mathrm{Ires}_{x=0}(\partial_{r-1}-\partial_r)^{m_{r-1,r}}\left(\frac{e^{a_1x_1+\cdots+a_rx_r}}{x_r^{m_{r,r+1}}\prod_{i=1}^{r-2}x_i^{m_{i,r+1}}\prod_{1\le i<j\le r}(x_i-x_j)^{m_{i,j}}}\right) \\
&= \mathrm{Ires}_{x=0}\left(\frac{e^{a_1x_1+\cdots+a_rx_r}}{x_r^{m_{r,r+1}}\prod_{i=1}^{r-2}x_i^{m_{i,r+1}}\prod_{1\le i<j\le r,(i,j)\neq(r-1,r)}(x_i-x_j)^{m_{i,j}}}\right) \\
&= \mathrm{Res}_{x_1=0}\cdots\left(\mathrm{Res}_{x_{r-1}=0}\left(\frac{e^{a_1x_1+\cdots+a_{r-1}x_{r-1}}}{\prod_{i=1}^{r-2}x_i^{m_{i,r+1}}\prod_{1\le i<j\le r-1}(x_i-x_j)^{m_{i,j}}}\right.\right. \\
&\quad \left.\left.\times\mathrm{Res}_{x_r=0}\left(\frac{e^{a_rx_r}}{x_r^{m_{r,r+1}}\prod_{i=1}^{r-2}(x_i-x_r)^{m_{i,r}}}\right)\right)\right)=0.
\end{aligned}
$$

Similarly, we can verify the remaining expressions. $\square$

Remark 3.0.2 In general, it is known that the volume function $v_{A_r^+,m}(a)$ of $P_{A_r^+,m}(a)$, as a distribution on V, satisfies the differential equation

$$Lv_{A_r^+,m}(a)=\delta(a),$$

where $L=\prod_{i<j}(\partial_i-\partial_j)^{m_{i,j}}$ and $\delta(a)$ is the Dirac delta function on V ([8, 11]). Note that ∂_{r+1} in the definition of L is supposed to be zero. Theorem 3.0.1 above, together with Proposition 3.0.3 and Theorem 3.0.4 as below, characterizes the function $v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}(a)$ on $\overline{\mathfrak{c}_{\mathrm{nice}}}$ more explicitly.

Let $M_\ell=\sum_{i=\ell+1}^{r+1}m_{\ell,i}$ for $\ell=1,\dots,r$. Then we have the following proposition.

Proposition 3.0.3 *The coefficient of* $a_1^{M_1-1}a_2^{M_2-1}\cdots a_{r-1}^{M_{r-1}-1}a_r^{M_r-1}$ *in the volume function* $v_{A_r^+,m}(a)$ *is given by*

$$\frac{1}{(M_1-1)!(M_2-1)!\cdots(M_{r-1}-1)!(M_r-1)!}.$$

Proof. From the Lidskii formula in Corollary 2.4.3, we have

$$v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}(a)=\sum_{|i|=\ell-r}\frac{a_1^{i_1}}{i_1!}\cdots\frac{a_r^{i_r}}{i_r!}\mathrm{Ires}_{x=0}\left(\frac{x_1^{i_1}\cdots x_r^{i_r}}{\prod_{i=1}^{r}x_i^{m_{i,r+1}}\prod_{1\le i<j\le r}(x_i-x_j)^{m_{i,j}}}\right),$$

where $|i| = i_1 + \cdots + i_r$. When $i_\ell = M_\ell - 1$ for $\ell = 1, \ldots, r$,

$$\begin{aligned}
&\mathrm{Ires}_{x=0}\left(\frac{x_1^{M_1-1}\cdots x_r^{M_r-1}}{\prod_{i=1}^r x_i^{m_{i,r+1}}\prod_{1\le i<j\le r}(x_i-x_j)^{m_{i,j}}}\right)\\
&=\mathrm{Res}_{x_1=0}\cdots\mathrm{Res}_{x_{r-1}=0}\mathrm{Res}_{x_r=0}\left(\frac{x_1^{(\sum_{i=2}^r m_{1,i})-1}\cdots x_{r-1}^{m_{r-1,r}-1}}{x_r\prod_{1\le i<j\le r}(x_i-x_j)^{m_{i,j}}}\right)\\
&=\mathrm{Res}_{x_1=0}\cdots\mathrm{Res}_{x_{r-1}=0}\left(\frac{x_1^{(\sum_{i=2}^{r-1} m_{1,i})-1}\cdots x_{r-2}^{m_{r-2,r-1}-1}}{x_{r-1}\prod_{1\le i<j\le r-1}(x_i-x_j)^{m_{i,j}}}\right)\\
&=\mathrm{Res}_{x_1=0}\frac{1}{x_1}=1.
\end{aligned}$$

Thus we obtain the proposition. □

Theorem 3.0.4 *Let $\phi_r = \phi(a_1, \ldots, a_r)$ be a homogeneous polynomial of $a_1, \ldots, a_r$ with degree d and let $M = \sum_{1\le i<j\le r+1} m_{i,j}$. Suppose ϕ_r satisfies the system of differential equations as follows:*

$$\begin{cases}\partial_r^{m_{r,r+1}}\phi_r = 0\\ (\partial_{r-1}-\partial_r)^{m_{r-1,r}}\partial_{r-1}^{m_{r-1,r+1}}\phi_r = 0\\ \quad\vdots\\ (\partial_1-\partial_2)^{m_{1,2}}(\partial_1-\partial_3)^{m_{1,3}}\cdots(\partial_1-\partial_r)^{m_{1,r}}\partial_1^{m_{1,r+1}}\phi_r = 0.\end{cases}\tag{3.0.3}$$

(i) *If $M - r < d$, then $\phi_r = 0$.*

(ii) *If $0 \le d \le M - r$, then there is a non trivial homogeneous polynomial ϕ_r satisfying* (3.0.3).

(iii) *If $d = M - r$ in particular, ϕ_r is equal to a constant multiple of $v = v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}(a)$.*

Proof. We argue by induction on r. In the case that $r = 1$, we write

$$\phi_1 = \phi(a_1) = pa_1^d,$$

where p is a constant. If $m_{1,2} - 1 < d$ and ϕ_1 satisfies the differential equation $\partial_1^{m_{1,2}}\phi_1 = 0$, then $p = 0$ and hence $\phi_1 = 0$. If $0 \le d \le m_{1,2} - 1$, then for any $p \ne 0$, $\partial_1^{m_{1,2}}\phi_1 = 0$. Also, if $d = m_{1,2} - 1$, in particular, then $\phi_1 = pa_1^{m_{1,2}-1}$, while $v = \frac{1}{(m_{1,2}-1)!}a_1^{m_{1,2}-1}$ as in Example 2.5.1. Hence ϕ_1 is equal to a constant multiple of v.

We assume that the statement of this theorem holds for $r - 1$. We write ϕ_r as

$$\phi_r = \phi(a_1, \ldots, a_r) = g_d(a_2, \ldots, a_r) + a_1 g_{d-1}(a_2, \ldots, a_r) + \cdots + a_1^d g_0(a_2, \ldots, a_r),$$

where g_k is a homogeneous polynomial of $a_2, \dots, a_r$ with degree k for $k = 0, 1, \dots, d$. Then for $k = 0, 1, \dots, d$, g_k satisfies the differential equations as follows:

$$\begin{cases} \partial_r^{m_{r,r+1}} g_k = 0 \\ (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} g_k = 0 \\ \quad \vdots \\ (\partial_2 - \partial_3)^{m_{2,3}} (\partial_2 - \partial_4)^{m_{2,4}} \cdots (\partial_2 - \partial_r)^{m_{2,r}} \partial_2^{m_{2,r+1}} g_k = 0. \end{cases} \tag{3.0.4}$$

We set $h = (\sum_{2 \le i < j \le r+1} m_{i,j}) - (r-1)$. From the inductive assumption, if $0 \le k \le h$, then g_k is a homogeneous polynomial. On the other hand, if $h+1 \le k \le d$, then $g_k = 0$, namely,

$$g_d(a_2, \dots, a_r) = g_{d-1}(a_2, \dots, a_r) = \cdots = g_{h+1}(a_2, \dots, a_r) = 0. \tag{3.0.5}$$

(i) We consider the case of $M - r < d$. Let $M_1 = \sum_{i=2}^{r+1} m_{1,i}$. Now we compare the coefficients of $a_1^{d-h-M_1+n}$ in $(\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} \phi_r$ for $n = 0, \dots, h$. For $q = 1, \dots, M_1 - m_{1,r+1}$, we define

$$D_q = \sum_{2 \le i_1 \le r} \binom{m_{1,i_1}}{q} \partial_{i_1}^q + \cdots + \sum_{\substack{p_1 + \cdots + p_k = q \\ 2 \le i_1 < \cdots < i_k \le r}} \left(\prod_{1 \le l \le k} \binom{m_{1,i_l}}{p_l} \right) \partial_{i_1}^{p_1} \partial_{i_2}^{p_2} \cdots \partial_{i_k}^{p_k}$$
$$+ \cdots + \sum_{2 \le i_1 < \cdots < i_q \le r} \left(\prod_{1 \le l \le q} \binom{m_{1,i_l}}{1} \right) \partial_{i_1} \partial_{i_2} \cdots \partial_{i_q}.$$

Then we have the following equation:

$$\begin{aligned} &\frac{(d-h+n)!}{(d-h-M_1+n)!} g_{h-n}(a_2, \dots, a_r) - \frac{(d-h+n-1)!}{(d-h-M_1+n)!} D_1 g_{h-n+1}(a_2, \dots, a_r) \\ &+ \cdots + (-1)^j \frac{(d-h+n-j)!}{(d-h-M_1+n)!} D_j g_{h-n+j}(a_2, \dots, a_r) \\ &+ \cdots + (-1)^{M_1 - m_{1,r+1}} \frac{(d-h+n-(M_1 - m_{1,r+1}))!}{(d-h-M_1+n)!} D_{M_1 - m_{1,r+1}} g_{h-n+(M_1 - m_{1,r+1})}(a_2, \dots, a_r) \\ &= 0. \end{aligned} \tag{3.0.6}$$

When $n = 0$, from (3.0.5) and (3.0.6), we have

$$g_h(a_2, \dots, a_r) = 0.$$

When $n = 1$, we have

$$\frac{(d-h+1)!}{(d-h-M_1+1)!} g_{h-1}(a_2, \dots, a_r) - \frac{(d-h)!}{(d-h-M_1+1)!} D_1 g_h(a_2, \dots, a_r) = 0.$$

Thus we have

$$g_{h-1}(a_2, \dots, a_r) = 0.$$

Similarly, we have

$$g_{h-2}(a_2,\dots,a_r) = g_{h-3}(a_2,\dots,a_r) = \cdots = g_0(a_2,\dots,a_r) = 0$$

and hence $\phi_r = 0$.

(ii) We consider the case of $0 \le d \le M - r$. By the inductive assumption, there is a non trivial homogeneous polynomial g_{h-n_1+i} satisfying (3.0.4) for $i = 1,\dots,n_1$, where $n_1 = M - r - d + 1$. We can take

$$g_{h-n_1+i}(a_2,\dots,a_r) \neq 0.$$

When $n = n_1$, from (3.0.5) and (3.0.6),

$$\begin{aligned} g_{h-n_1}(a_2,\dots,a_r) =& \frac{(d-h+n_1-1)!}{(d-h+n_1)!} D_1 g_{h-n_1+1}(a_2,\dots,a_r) \\ &- \frac{(d-h+n_1-2)!}{(d-h+n_1)!} D_2 g_{h-n_1+2}(a_2,\dots,a_r) \\ &+ \cdots + (-1)^{n_1-1} \frac{(d-h)!}{(d-h+n_1)!} D_{n_1} g_h(a_2,\dots,a_r). \end{aligned}$$

When $n = n_1 + 1$,

$$\begin{aligned} g_{h-(n_1+1)}(a_2,\dots,a_r) =& \frac{(d-h+n_1)!}{(d-h+n_1+1)!} D_1 g_{h-n_1}(a_2,\dots,a_r) \\ &- \frac{(d-h+n_1-1)!}{(d-h+n_1+1)!} D_2 g_{h-n_1+1}(a_2,\dots,a_r) \\ &+ \cdots + (-1)^{n_1} \frac{(d-h)!}{(d-h+n_1+1)!} D_{n_1+1} g_h(a_2,\dots,a_r). \end{aligned}$$

Similarly, for $n = n_1 + 2,\dots,h$, we can express $g_{h-j}(a_2,\dots,a_r)$ $(j = n_1, n_1+1,\dots,h)$ in terms of $g_{h-j+i}(a_2,\dots,a_r)$ $(i = 1,\dots,j)$ and their partial derivatives. Namely, we can express ϕ_r in terms of $g_{h-n_1+i}(a_2,\dots,a_r)$ and their partial derivatives. It follows that $\phi_r \neq 0$ when $0 \le d \le M - r$.

(iii) If $d = M - r$ in particular, then $n_1 = 1$, and g_{h-j} $(j = 1,\dots,h)$ becomes the linear combination of g_h and their partial derivatives. Therefore ϕ_r is uniquely determined by g_h. Moreover, from the inductive assumption, $g_h = C \cdot v_{A^+_{r-1},m',\mathfrak{c}'_{\text{nice}}}$, where C is a constant, $m' = (m_{i,j})_{2\le i<j\le r+1}$, and $\mathfrak{c}'_{\text{nice}}$ is a nice chamber of A^+_{r-1}. Hence the solution of (3.0.3) is unique up to a constant multiple. On the other hand, by Theorem 3.0.1, $v_{A^+_r,m,\mathfrak{c}_{\text{nice}}}$ satisfies the system of differential equations (3.0.3). Hence ϕ_r is equal to a constant multiple of $v_{A^+_r,m,\mathfrak{c}_{\text{nice}}}$. □

Recall that in the proof of Theorem 3.0.4, we have defined the operator

$$
\begin{aligned}
D_q = &\sum_{2\le i_1\le r} \binom{m_{1,i_1}}{q} \partial_{i_1}^q + \cdots + \sum_{\substack{p_1+\cdots+p_k=q\\ 2\le i_1<\cdots<i_k\le r}} \left(\prod_{1\le l\le k} \binom{m_{1,i_l}}{p_l}\right) \partial_{i_1}^{p_1}\partial_{i_2}^{p_2}\cdots\partial_{i_k}^{p_k} \\
&+\cdots+ \sum_{2\le i_1<\cdots<i_q\le r} \left(\prod_{1\le l\le q} \binom{m_{1,i_l}}{1}\right) \partial_{i_1}\partial_{i_2}\cdots\partial_{i_q} \qquad (3.0.7)
\end{aligned}
$$

for $q = 1, \ldots, M_1 - m_{1,r+1}$.

Remark 3.0.5 *Let $M_1 = \sum_{i=2}^{r+1} m_{1,i}$. When $d = M - r$, from the proof of Theorem 3.0.4 (iii), g_{h-j} $(j = 1, \ldots, h)$ is uniquely determined as follows:*

$$
\begin{cases}
g_{h-1} = \frac{(M_1-1)!}{M_1!} D_1 g_h \\
g_{h-2} = \frac{(M_1-1)!}{(M_1+1)!} (D_1^2 - D_2) g_h \\
g_{h-3} = \frac{(M_1-1)!}{(M_1+2)!} (D_1^3 - 2D_1D_2 + D_3) g_h \\
\quad\vdots \\
g_0 = \frac{(M_1-1)!}{(M-r)!} (D_1^h - (h-1)D_1^{h-2}D_2 + \cdots + (-1)^{h-1}D_h) g_h.
\end{cases}
$$

Let $m' = (m_{i,j})_{2\le i<j\le r+1}$, $\mathfrak{c}'_{\text{nice}}$ a nice chamber of A^+_{r-1} and $a' = \sum_{i=2}^{r} a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}'_{\text{nice}}}$. From Proposition 3.0.3 and Remark 3.0.5, we obtain the following theorem.

Theorem 3.0.6 *Let $h = (\sum_{2\le i<j\le r+1} m_{i,j}) - (r-1)$ and let D_q $(q = 1, \ldots, h)$ be as in (3.0.7). Then $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ is written by the linear combination of $v_{A^+_{r-1},m',\mathfrak{c}'_{\text{nice}}}(a')$ and its partial derivatives as follows:*

$$
\begin{aligned}
v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a) = &\left\{ \frac{a_1^{M_1-1}}{(M_1-1)!} + \frac{a_1^{M_1}}{M_1!} D_1 + \frac{a_1^{M_1+1}}{(M_1+1)!} (D_1^2 - D_2) \right. \\
&+ \frac{a_1^{M_1+2}}{(M_1+2)!} (D_1^3 - 2D_1D_2 + D_3) + \cdots \\
&\left. + \frac{a_1^{M-r}}{(M-r)!} (D_1^h - (h-1)D_1^{h-2}D_2 + \cdots + (-1)^{h-1}D_h) \right\} v_{A^+_{r-1},m',\mathfrak{c}'_{\text{nice}}}(a'). \qquad (3.0.8)
\end{aligned}
$$

Example 3.0.7 Let $r = 3$, let $a = \sum_{i=1}^{3} a_i(e_i - e_4) \in \overline{\mathfrak{c}_{\text{nice}}}$ and let $a' = \sum_{i=2}^{3} a_i(e_i - e_4) \in \overline{\mathfrak{c}'_{\text{nice}}}$. We set $m_{1,2} = 1, m_{1,3} = 1, m_{1,4} = 2, m_{2,3} = 1, m_{2,4} = 2$ and $m_{3,4} = 2$ as in Example 2.5.3. Then we have

$$
v_{A_3^+,m,\mathfrak{c}_{\text{nice}}}(a) = \frac{1}{360} a_1^3 (a_1^3 + 6a_1^2a_2 + 3a_1^2a_3 + 15a_1a_2^2 + 15a_1a_2a_3 + 10a_2^3 + 30a_2^2a_3).
$$

We can check that $v = v_{A_3^+,m,\mathfrak{c}_{\rm nice}}(a)$ satisfies the system of differential equations as follows:

$$\begin{cases} \partial_3^2 v = 0 \\ (\partial_2 - \partial_3)\partial_2^2 v = 0 \\ (\partial_1 - \partial_2)(\partial_1 - \partial_3)\partial_1^2 v = 0. \end{cases}$$

Also, from Proposition 3.0.3, the coefficient of the term $a_1^3 a_2^2 a_3$ is $\frac{1}{3!2!1!} = \frac{1}{12}$. When $r = 2$,

$$v_{A_2^+,m',\mathfrak{c}'_{\rm nice}}(a') = \frac{1}{6}a_2^2(a_2 + 3a_3).$$

Therefore, we have

$$\begin{aligned} &\left\{\frac{a_1^3}{6} + \frac{a_1^4}{24}D_1 + \frac{a_1^5}{120}(D_1^2 - D_2) + \frac{a_1^6}{720}(D_1^3 - 2D_1D_2 + D_3)\right\} v_{A_2^+,m',\mathfrak{c}'_{\rm nice}}(a') \\ &= \frac{a_1^3a_2^3}{36} + \frac{a_1^3a_2^2a_3}{12} + \frac{a_1^4a_2^2}{24} + \frac{a_1^4a_2a_3}{24} + \frac{a_1^5a_2}{60} + \frac{a_1^5a_3}{120} + \frac{a_1^6}{360} = v_{A_3^+,m,\mathfrak{c}_{\rm nice}}(a) \end{aligned}$$

as in (3.0.8).

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Part II

On equivariant index of a generalized Bott manifold

Chapter 4

Introduction

A *Bott tower* of height n is a sequence:

$$M_n \stackrel{\pi_n}{\to} M_{n-1} \stackrel{\pi_{n-1}}{\to} \cdots \stackrel{\pi_2}{\to} M_1 \stackrel{\pi_1}{\to} M_0 = \{\text{a point}\}$$

of complex manifolds $M_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j)$, where $\underline{\mathbb{C}}$ is the trivial line bundle over M_{j-1}, E_j is a holomorphic line bundle over M_{j-1}, $\mathbb{P}(\cdot)$ denotes the projectivization, and $\pi_j : M_j \to M_{j-1}$ is the projection of the $\mathbb{C}P^1$-bundle. We call M_j a *j-stage Bott manifold.* The notion of a Bott tower was introduced by Grossberg and Karshon ([6]).

A *generalized Bott tower* is a generalization of a Bott tower. A generalized Bott tower of height m is a sequence:

$$B_m \stackrel{\pi_m}{\to} B_{m-1} \stackrel{\pi_{m-1}}{\to} \cdots \stackrel{\pi_2}{\to} B_1 \stackrel{\pi_1}{\to} B_0 = \{\text{a point}\},$$

of complex manifolds $B_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j^{(1)} \oplus \cdots \oplus E_j^{(n_j)})$, where $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}, $E_j^{(k)}$ is a holomorphic line bundle over B_{j-1} for $k = 1, \ldots, n_j$. We call B_j a *j-stage generalized Bott manifold.* A generalized Bott tower has been studied from various points of view (see, e.g., [2, 3, 8]). Generalized Bott manifolds are a certain class of toric manifolds, so it is interesting to investigate the specific properties of generalized Bott towers.

In [6], Grossberg and Karshon showed the multiplicity function of the *equivariant index* (see §5.4) for a holomorphic line bundle over a Bott manifold is given by the density function of a *twisted cube*, which is determined by the structure of the Bott manifold and the line bundle over it. From this, they derived a Demazure-type character formula.

The purpose of this part is to generalize the results in [6] to generalized Bott manifolds. We generalize the twisted cube, and we call it the *generalized twisted cube.* It is a special case of twisted polytope introduced by Karshon and Tolman [9] for the presymplectic toric manifold, and it is a special case of multi-polytope introduced by Hattori and Masuda [7] for the torus manifold. We show the multiplicity function of the equivariant index for a holomorphic line bundle over the generalized Bott manifold is given by the density function of the associated generalized twisted cube. From this, we derive a Demazure-type character formula. In order to state the main results, we give some notation. Let $\mathbf{L}$ be a holomorphic line bundle over a generalized Bott manifold B_m, which is constructed from integers $\{\ell_i\}$ and $\{c_{i,j}^{(k)}\}$ (see §5.1). Let $N = \sum_{j=1}^{m} n_j$, and let $T^N = S^1 \times \cdots \times S^1$. We consider the action of T^N on B_m as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m] = [\mathbf{t}_1\mathbf{z}_1, \ldots, \mathbf{t}_m\mathbf{z}_m],$$

where $\mathbf{t}_i = (t_{i,1}, \ldots, t_{i,n_i}), \mathbf{z}_i = (z_{i,0}, \ldots, z_{i,n_i}), \mathbf{t}_i\mathbf{z}_i = (z_{i,0}, t_{i,1}z_{i,1}, \ldots, t_{i,n_i}z_{i,n_i})$ for $i = 1, \ldots, m$. Also we consider the action of $T = T^N \times S^1$ on $\mathbf{L}$ as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m, t_{m+1}) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m, v] = [\mathbf{t}_1\mathbf{z}_1, \ldots, \mathbf{t}_m\mathbf{z}_m, t_{m+1}v]. \tag{4.0.1}$$

We define the generalized twisted cube as follows. It is defined to be the set of $x = (x_{1,1}, \ldots, x_{m,n_m}) \in \mathbb{R}^N$ which satisfies

$$\begin{aligned} & A_i(x) \leq \sum_{k=1}^{n_i} x_{i,k} \leq 0, \ x_{i,k} \leq 0 \ \ (1 \leq k \leq n_i) \\ & \text{or } 0 < \sum_{k=1}^{n_i} x_{i,k} < A_i(x), \ x_{i,k} > 0 \ \ (1 \leq k \leq n_i), \end{aligned}$$

for $1 \leq i \leq m$, where

$$A_i(x) = \begin{cases} -\ell_m & (i = m) \\ -(\ell_i + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k}) & (1 \leq i \leq m-1). \end{cases}$$

We denote the generalized twisted cube by C. We also define $\operatorname{sgn}(x_{i,k}) = 1$ for $x_{i,k} > 0$ and $\operatorname{sgn}(x_{i,k}) = -1$ for $x_{i,k} \leq 0$. The *density function* of the generalized twisted cube is defined to be $\rho(x) = (-1)^N \prod_{1 \leq i \leq m, 1 \leq k \leq n_i} \operatorname{sgn}(x_{i,k})$ when $x \in C$ and 0 elsewhere.

Let $\mathfrak{t}$ be the Lie algebra of T and let $\mathfrak{t}^*$ be its dual space. Let $\ell^* \subset i\mathfrak{t}^*$ be the integral weight lattice and let $\operatorname{mult} : \ell^* \to \mathbb{Z}$ be the multiplicity function of the equivariant index. The first main result of this part is the following:

Theorem 4.0.1 *Fix integers $\{c_{i,j}^{(k)}\}$ and $\{\ell_j\}$. Let $\mathbf{L} \to B_m$ be the corresponding line bundle over a generalized Bott manifold. Let $\rho : \mathbb{R}^N \to \{-1, 0, 1\}$ be the density function of the generalized twisted cube C which is determined by these integers. Consider the torus action of $T = T^N \times S^1$ as in (4.0.1). Then the multiplicity function for $\ell^* \cong \mathbb{Z}^N \times \mathbb{Z}$ is given by*

$$\operatorname{mult}(x, k) = \begin{cases} \rho(x) & (k = 1) \\ 0 & (k \neq 1). \end{cases}$$

Karshon and Tolman found a toric manifold for which the multiplicities of the equivariant index are $0, -1$, or -2 ([9, Example 6.7]). A generalized Bott manifold is different from this case by Theorem 4.0.1.

Next, we give our character formula. Let $\{e_{1,1}, \dots, e_{m,n_m}, e_{m+1}\}$ be the standard basis in $\mathbb{R}^{N+1}$, $x_i = (x_{i,1}, \dots, x_{i,n_i})$, and $e_i = (e_{i,1}, \dots, e_{i,n_i})$. Let $\Delta^-_{n,r} = \{z = (z_1, \dots, z_n) \in \mathbb{Z}^n_{\leq 0} \mid z_1 + \cdots + z_n = -r\}$, and let $\Delta^+_{n,r} = \{z = (z_1, \dots, z_n) \in \mathbb{Z}^n_{>0} \mid z_1 + \cdots + z_n = r - 1\}$. Let $\langle x_i, e_i \rangle = x_{i,1}e_{i,1} + \cdots + x_{i,n_i}e_{i,n_i}$. For every integral weight $\mu \in \ell^*$ we have a homomorphism $\lambda^\mu : T \to S^1$. We denote the integral combinations of these λ^μ's by $\mathbb{Z}[T]$. Then the operators $D_i : \mathbb{Z}[T] \to \mathbb{Z}[T]$ are defined using $c^{(k)}_{i,j}$ and ℓ_j in the following way:

$$D_i(\lambda^\mu) = \begin{cases} \displaystyle\sum_{0 \leq r \leq k_i} \sum_{x_i \in \Delta^-_{n_i,r}} \lambda^{\mu + \langle x_i, e_i \rangle} & \text{if } k_i \geq 0 \\ 0 & \text{if } -n_i \leq k_i \leq -1 \\ \displaystyle\sum_{n_i+1 \leq r \leq -k_i} \sum_{x_i \in \Delta^+_{n_i,r}} (-1)^{n_i} \lambda^{\mu + \langle x_i, e_i \rangle} & \text{if } k_i \leq -n_i - 1, \end{cases}$$

where the functions k_i are defined as follows: if $\mu = e_{m+1} + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} x_{j,k}e_{j,k}$, then $k_i(\mu) = \ell_i + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} c^{(k)}_{i,j} x_{j,k}$. From Theorem 4.0.1, we obtain the following theorem:

Theorem 4.0.2 *Consider the action of the torus T on $\mathbf{L} \to B_m$ as in (4.0.1). Denote the $(N+1)$-th component of the standard basis in $\mathbb{R}^{N+1}$ by e_{m+1}. Then the character is given by the following element of $\mathbb{Z}[T]$:*

$$\chi = D_1 \cdots D_m(\lambda^{e_{m+1}}).$$

This is a Demazure-type character formula. On the other hand, the character is also given by the localization formula with respect to the action of T ([7, Corollary 7.4]). We compare our formula with the localization formula (see Remark 6.2.7).

This part is organized as follows. In Chapter 5, we recall the generalized Bott towers and the equivariant index, and we give the definition of generalized twisted cubes. In Chapter 6, we prove the main theorems.

Chapter 5

Preliminaries

In this chapter, we set up the tools to prove the main theorems.

5.1 Generalized Bott manifolds

Definition 5.1.1 ([2]) A *generalized Bott tower* of height m is a sequence:

$$B_m \stackrel{\pi_m}{\to} B_{m-1} \stackrel{\pi_{m-1}}{\to} \cdots \stackrel{\pi_2}{\to} B_1 \stackrel{\pi_1}{\to} B_0 = \{\text{a point}\},$$

of manifolds $B_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j^{(1)} \oplus \cdots \oplus E_j^{(n_j)})$, where $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}, $E_j^{(k)}$ is a holomorphic line bundle over B_{j-1} for $k = 1, \ldots n_j$, and $\mathbb{P}(\cdot)$ denotes the projectivization. We call B_j a *j-stage generalized Bott manifold.*

The construction of the generalized Bott tower is as follows. A 1-step generalized Bott tower can be written as $B_1 = \mathbb{C}P^{n_1} = (\mathbb{C}^{n_1+1})^\times/\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts diagonally. We construct a line bundle over B_1 by $E_2^{(k)} = (\mathbb{C}^{n_1+1})^\times \times_{\mathbb{C}^\times} \mathbb{C}$ for $k = 1, \ldots, n_2$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by $a : v \mapsto a^{-c_k}v$ for some integer c_k. In $E_2^{(k)}$ we have $[z_{1,0}, \ldots, z_{1,n_1}, v] = [z_{1,0}a, \ldots, z_{1,n_1}a, a^{c_k}v]$ for all $a \in \mathbb{C}^\times$. A 2-step generalized Bott tower $B_2 = \mathbb{P}(\underline{\mathbb{C}} \oplus E_2^{(1)} \oplus \cdots \oplus E_2^{(n_2)})$ can be written as $B_2 = ((\mathbb{C}^{n_1+1})^\times \times (\mathbb{C}^{n_2+1})^\times)/G$, where the right action of $G = (\mathbb{C}^\times)^2$ is given by

$$(\mathbf{z}_1, \mathbf{z}_2) \cdot (a_1, a_2) = (z_{1,0}a_1, z_{1,1}a_1, \ldots, z_{1,n_1}a_1, z_{2,0}a_2, a_1^{c_1}z_{2,1}a_2, \ldots, a_1^{c_{n_2}}z_{2,n_2}a_2),$$

where $\mathbf{z}_j = (z_{j,0}, z_{j,1}, \ldots, z_{j,n_j})$ for $j = 1, 2$.

We can construct higher generalized Bott towers in a similar way. In this way we get an m-step generalized Bott manifold $B_m = \mathbb{P}(\underline{\mathbb{C}} \oplus E_m^{(1)} \oplus \cdots \oplus E_m^{(n_m)})$ from any collection of integers $\{c_{i,j}^{(k)}\}$:

$$B_m = ((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times)/G,$$

where the right action of $G = (\mathbb{C}^\times)^m$ is given by

$$(\mathbf{z}_1, \ldots, \mathbf{z}_m) \cdot \mathbf{a} = (\mathbf{z}'_1, \mathbf{z}'_2, \ldots, \mathbf{z}'_m),$$

where $\mathbf{z}_i = (z_{i,0}, \ldots, z_{i,n_i})$ for $i = 1, \ldots, m$, $\mathbf{a} = (a_1, \ldots, a_m) \in (\mathbb{C}^\times)^m$, $\mathbf{z}_1' = (z_{1,0}a_1, z_{1,1}a_1, \ldots, z_{1,n_1}a_1)$ and $\mathbf{z}_j' = (z_{j,0}a_j, a_1^{c_{1,j}^{(1)}} \cdots a_{j-1}^{c_{j-1,j}^{(1)}} z_{j,1}a_j, \ldots, a_1^{c_{1,j}^{(n_j)}} \cdots a_{j-1}^{c_{j-1,j}^{(n_j)}} z_{j,n_j}a_j)$ for $j = 2, \ldots, m$. We can construct a line bundle over B_m from the integers $(\ell_1, \ldots, \ell_m)$ by

$$\mathbf{L} = ((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times) \times_G \mathbb{C},$$

where $G = (\mathbb{C}^\times)^m$ acts by

$$((\mathbf{z}_1, \ldots, \mathbf{z}_m), v) \cdot \mathbf{a} = (\mathbf{z}_1', \mathbf{z}_2', \ldots, \mathbf{z}_m', a_1^{\ell_1} \cdots a_m^{\ell_m} v). \tag{5.1.1}$$

5.2 Torus action on generalized Bott towers

Let $N = \sum_{j=1}^m n_j$ and let $T^N = S^1 \times \cdots \times S^1$. We consider the action of T^N on B_m as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m] = [\mathbf{t}_1 \cdot \mathbf{z}_1, \ldots, \mathbf{t}_m \cdot \mathbf{z}_m],$$

where $\mathbf{t}_i = (t_{i,1}, \ldots, t_{i,n_i})$ and $\mathbf{t}_i \cdot \mathbf{z}_i = (z_{i,0}, t_{i,1}z_{i,1}, \ldots, t_{i,n_i}z_{i,n_i})$ for $i = 1, \ldots, m$. Also we consider the action of $T = T^N \times S^1$ on $\mathbf{L}$ as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m, t_{m+1}) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m, v] = [\mathbf{t}_1 \cdot \mathbf{z}_1, \ldots, \mathbf{t}_m \cdot \mathbf{z}_m, t_{m+1}v]. \tag{5.2.1}$$

5.3 Generalized twisted cubes

Definition 5.3.1 A *generalized twisted cube* C is defined to be the set of $x = (x_{1,1}, \ldots, x_{m,n_m}) \in \mathbb{R}^N$ which satisfies

$$\begin{aligned} & A_i(x) \leq \sum_{k=1}^{n_i} x_{i,k} \leq 0, \;\; x_{i,k} \leq 0 \;\; (1 \leq k \leq n_i) \\ & \text{or } 0 < \sum_{k=1}^{n_i} x_{i,k} < A_i(x), \;\; x_{i,k} > 0 \;\; (1 \leq k \leq n_i), \end{aligned} \tag{5.3.1}$$

for all $1 \leq i \leq m$, where

$$A_i(x) = \begin{cases} -\ell_m & (i = m) \\ -(\ell_i + \sum_{j=i+1}^m \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k}) & (1 \leq i \leq m-1). \end{cases}$$

Remark 5.3.2 (i) The generalized twisted cube is a special case of multi-polytope defined in [7]. In particular, it is a special case of twisted polytope defined in [9].

(ii) When $n_i = 1$ for all $1 \leq i \leq m$, the generalized twisted cube is the twisted cube given in [6, (2.21)].

Definition 5.3.3 We define $\mathrm{sgn}(x_{i,k}) = 1$ for $x_{i,k} > 0$ and $\mathrm{sgn}(x_{i,k}) = -1$ for $x_{i,k} \leq 0$. The *density function* of the generalized twisted cube is then defined to be $\rho(x) = (-1)^N \prod_{1\leq i\leq m, 1\leq k\leq n_i} \mathrm{sgn}(x_{i,k})$ when $x \in C$ and 0 elsewhere.

Example 5.3.4 Suppose that $m = 2, n_1 = 1, n_2 = 2, \ell_1 = 1$, and $\ell_2 = 2$. We set $c^{(1)}_{1,2} = 2$ and $c^{(2)}_{1,2} = -1$. Then the generalized twisted cube is the set of $x = (x_{1,1}, x_{2,1}, x_{2,2})$ which satisfies

- $-2 \leq x_{2,1} + x_{2,2} \leq 0,\ x_{2,1}, x_{2,2} \leq 0,$
- $-1 - 2x_{2,1} + x_{2,2} \leq x_{1,1} \leq 0$ or $0 < x_{1,1} < -1 - 2x_{2,1} + x_{2,2}$.

In Figure 1, the black dots represent the lattice points of the sign +1 and the white dots represent the sign −1.

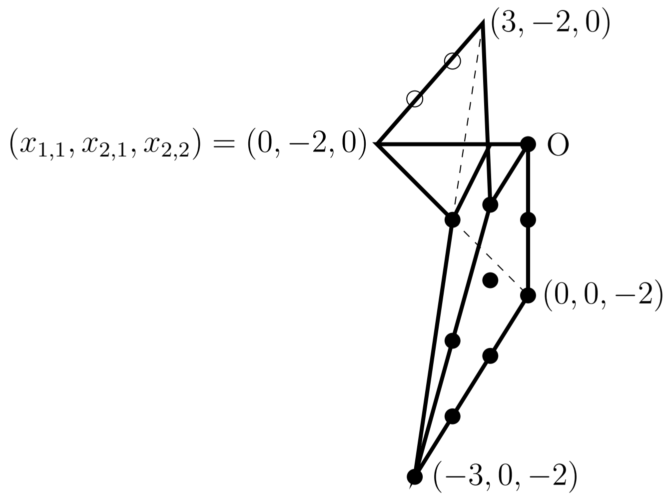

Figure 1

Example 5.3.5 Suppose that $m = 2, n_1 = 2, n_2 = 1, \ell_1 = 2$, and $\ell_2 = -6$. We set $c^{(1)}_{1,2} = -1$. Then the generalized twisted cube is the set of $x = (x_{1,1}, x_{1,2}, x_{2,1})$ which satisfies

- $0 < x_{2,1} < 6,$
- $-2 + x_{2,1} \leq x_{1,1} + x_{1,2} \leq 0,\ x_{1,1}, x_{1,2} \leq 0$ or $0 < x_{1,1} + x_{1,2} < -2 + x_{2,1},\ x_{1,1}, x_{1,2} > 0.$

In Figure 2, the white dots represent the sign −1.

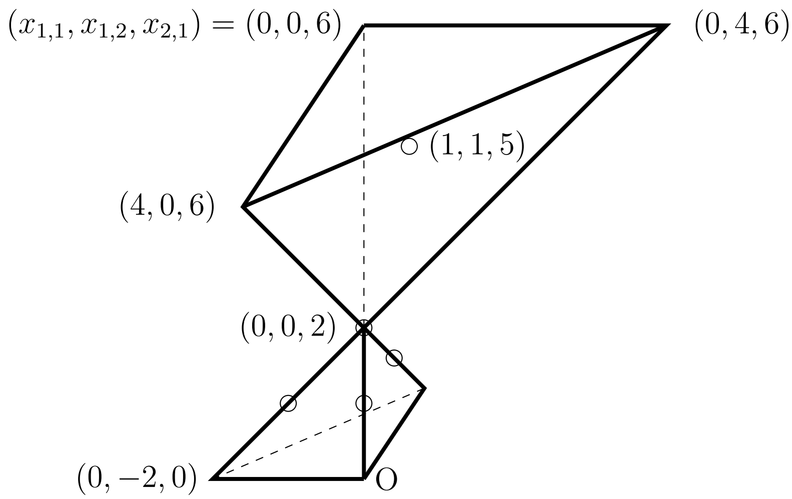

Figure 2

Example 5.3.6 Suppose that $m = 2, n_1 = n_2 = 2, \ell_1 = 1$, and $\ell_2 = 2$. We set $c^{(1)}_{1,2} = 2$ and $c^{(2)}_{1,2} = -1$. Then the generalized twisted cube is the set of $x = (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$ which satisfies

- $-2 \leq x_{2,1} + x_{2,2} \leq 0,\ x_{2,1}, x_{2,2} \leq 0,$
- $-1 - 2x_{2,1} + x_{2,2} \leq x_{1,1} + x_{1,2} \leq 0,\ x_{1,1}, x_{1,2} \leq 0$
  or $0 < x_{1,1} + x_{1,2} < -1 - 2x_{2,1} + x_{2,2},\ x_{1,1}, x_{1,2} > 0.$

The lattice points in the generalized twisted cube represent the sign -1.

5.4 Equivariant index

Let $\mathbf{L}$ be a holomorphic line bundle over a generalized Bott manifold B_m with the action of the torus T as in (5.2.1). Let $\mathcal{O}_{\mathbf{L}}$ be the sheaf of holomorphic sections. The *equivariant index* of a generalized Bott manifold is the formal sum of representation of T:

$$\mathrm{index}(B_m, \mathcal{O}_{\mathbf{L}}) = \sum (-1)^i H^i(B_m, \mathcal{O}_{\mathbf{L}})$$

The *character* of the equivariant index is the function $\chi : T \to \mathbb{C}$ which is given by $\chi = \sum (-1)^i \chi^i$ where $\chi^i(a) = \mathrm{trace}\{a : H^i(B_m, \mathcal{O}_{\mathbf{L}}) \to H^i(B_m, \mathcal{O}_{\mathbf{L}})\}$ for $a \in T$. Let $\mathfrak{t}$ be the Lie algebra of T and let $\mathfrak{t}^*$ be its dual space. Every μ in the integral weight lattice $\ell^* \subset i\mathfrak{t}^*$ defines a homomorphism $\lambda^\mu : T \to S^1$. We can write $\chi = \sum_{\mu \in \ell^*} m_\mu \lambda^\mu$. The coefficients are given by a function mult $: \ell^* \to \mathbb{Z}$, sending $\mu \mapsto m_\mu$, called the *multiplicity function* for the equivariant index.

Chapter 6

Main theorems of Part II

6.1 Multiplicity function of the equivariant index

We will show that the multiplicity function of the equivariant index of a generalized Bott manifold is given by the density function of a generalized twisted cube C. In particular, all the weights occur with a multiplicity $-1, 0$, or 1.

Theorem 6.1.1 *Fix integers $\{c_{i,j}^{(k)}\}$ and $\{\ell_j\}$. Let $\mathbf{L} \to B_m$ be the corresponding line bundle over a generalized Bott manifold. Let $\rho : \mathbb{R}^N \to \{-1, 0, 1\}$ be the density function of the generalized twisted cube C which is determined by these integers as in (5.3.1). Consider the torus action of $T = T^N \times S^1$ as in (5.2.1). Then the multiplicity function for $\ell^* \cong \mathbb{Z}^N \times \mathbb{Z}$ is given by*

$$\mathrm{mult}(x, k) = \begin{cases} \rho(x) & (k = 1) \\ 0 & (k \neq 1). \end{cases}$$

Proof ; We compute $H^*(B_m, \mathcal{O}_{\mathbf{L}})$. Take the covering $\tilde{\mathcal{U}} = \{U_{r_1} \times \cdots \times U_{r_m}\}$ of $(\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times$ for $r_1, \ldots, r_m \in \{0, 1, \ldots, n_\ell\}$ $(\ell = 1, \ldots, m)$, where $U_{r_j} = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{r_j} \times \mathbb{C}^\times \times \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n_\ell - r_j}$. This descends to the covering $\mathcal{U}$ of B_m; every intersection of sets in $\mathcal{U}$ is isomorphic to a product of $\mathbb{C}$'s and $\mathbb{C}^\times$'s. The coverings $\tilde{\mathcal{U}}$ and $\mathcal{U}$ are the Leray coverings ([5]).

Let $\mathcal{O}$ be the sheaf of holomorphic functions, and let $G = (\mathbb{C}^\times)^m$. Since holomorphic sections of $\mathcal{O}_{\mathbf{L}}$ are given by holomorphic sections of $\mathcal{O}$ which are G-invariant with respect to the action (5.1.1) ([9]), $H^*(\mathcal{U}, \mathcal{O}_{\mathbf{L}})$ is isomorphic to the G-invariant part of $H^*(\tilde{\mathcal{U}}, \mathcal{O})$. By the Leray theorem, $H^*(B_m, \mathcal{O}_{\mathbf{L}})$ is isomorphic to the G-invariant part of $H^*((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})$.

In order to compute $H^*((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})$, we compute $H^*((\mathbb{C}^{n+1})^\times, \mathcal{O})$. Let $\mathcal{U}' = \{U_0, U_1, \ldots, U_n\}$ be the covering of $(\mathbb{C}^{n+1})^\times$, let $j_0, j_1, \ldots, j_k \in \{0, 1, \ldots, n\}$ for

$k = 0, 1, \ldots, n$ and let $U_{j_0 j_1 \cdots j_k} = U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_k}$. Let $I = (i_0, i_1, \ldots, i_n) \in \mathbb{Z}^{n+1}$. The holomorphic functions on $U_{j_0 j_1 \cdots j_k}$ are given by

$$\Gamma_{\mathrm{hol}}(U_{j_0 j_1 \cdots j_k}) = \left\{ \sum_{I \in \mathbb{Z}^{n+1}, i_\ell \geq 0 (\ell \neq j_0, j_1, \ldots, j_k)} a_I z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n} \right\}.$$

Consider the Čech cochain complex

$$0 \to \check{C}^0(\mathcal{U}', \mathcal{O}) \xrightarrow{\delta^0} \check{C}^1(\mathcal{U}', \mathcal{O}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} \check{C}^n(\mathcal{U}', \mathcal{O}) \xrightarrow{\delta^n} 0,$$

where $\check{C}^i(\mathcal{U}', \mathcal{O}) = \oplus \Gamma_{\mathrm{hol}}(U_{j_0 j_1 \cdots j_i})$ $(i = 0, \ldots, n)$. The map $\delta^p : \check{C}^p(\mathcal{U}', \mathcal{O}) \to \check{C}^{p+1}(\mathcal{U}', \mathcal{O})$ is given by $\{f_{j_0 j_1 \cdots j_p}\} \mapsto \{g_{j_0 j_1 \cdots j_{p+1}}\}, g_{j_0 j_1 \cdots j_{p+1}} = \sum (-1)^k f_{j_0 j_1 \cdots \hat{j_k} \cdots j_{p+1}}$. Recall that $H^0((\mathbb{C}^{n+1})^\times, \mathcal{O}) = \operatorname{Ker} \delta^0$, and $H^n((\mathbb{C}^{n+1})^\times, \mathcal{O}) = \operatorname{Coker} \delta^{n-1}$. The torus $T^{n+1} = (S^1)^{n+1}$ acts on the holomorphic functions by $((t_0, \ldots, t_n) \cdot f)(z_0, \ldots, z_n) = f(t_0^{-1} z_0, \ldots, t_n^{-1} z_n)$. This action descends to the cohomology. The corresponding weight spaces for the weight $I \in \mathbb{Z}^{n+1}$ are

$$H^0((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = \begin{cases} \mathrm{span}(z_0^{-i_0} \cdots z_n^{-i_n}) & (I \in \mathbb{Z}^{n+1}_{\leq 0}) \\ 0 & \text{otherwise} \end{cases}$$
$$H^n((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = \begin{cases} \mathrm{span}(z_0^{-i_0} \cdots z_n^{-i_n}) & (I \in \mathbb{Z}^{n+1}_{> 0}) \\ 0 & \text{otherwise.} \end{cases}$$

We now prove $H^q((\mathbb{C}^{n+1})^\times, \mathcal{O}) = 0$ for $1 \leq q \leq n-1$. Let Δ be the fan of $(\mathbb{C}^{n+1})^\times$, and let $|\Delta| = \cup_{\sigma \in \Delta} \sigma$ be the support of Δ. Let

$$Z(I) := \{v \in |\Delta| \,;\, \langle I, v \rangle \leq \varphi(v)\},$$

where φ is the support function. From [4],

$$H^q((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = H^q(|\Delta|, |\Delta| \setminus Z(I) \,;\, \mathbb{C}).$$

Since $\mathcal{O}$ is the sheaf of holomorphic function, $\varphi(v) = 0$ for all $v \in |\Delta|$. In the case that $i_j \leq 0$ for all j, since $|\Delta|$ is contractible,

$$H^q((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = 0 \ (q \geq 1).$$

In the case that $i_j > 0$ for all j, $Z(I) = \{0\}$. Since $|\Delta| \setminus \{0\}$ is homotopic to S^{n-1},

$$H^q((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = 0 \ (q \neq n).$$

In other case, since $|\Delta| \setminus Z(I)$ is path-connected and contractible,

$$H^q((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = 0$$

for all q.

We now compute $H^*((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})$. Consider the natural action of $T^{N+m} = (S^1)^{N+m}$ on the holomorphic function. The weights are multi-indices $I' \in \mathbb{Z}^{N+m}$; we write $I' = (\mathbf{i}'_1, \ldots, \mathbf{i}'_m)$, where $\mathbf{i}'_j = (i_{j,0}, i_{j,1}, \ldots, i_{j,n_j})$ for $j = 1, \ldots, m$. From the cohomology of $(\mathbb{C}^{n+1})^\times$ that we have computed and from the Künneth formula ([1]), it follows that

$$H^q((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})_{I'} = \begin{cases} \mathrm{span}(z_{1,0}^{-i_{1,0}} z_{1,1}^{-i_{1,1}} \cdots z_{m,n_m}^{-i_{m,n_m}}) \\ 0. \end{cases}$$

The former occurs if for all ℓ we have $\mathrm{sgn}(i_{\ell,0}) = \mathrm{sgn}(i_{\ell,1}) = \cdots = \mathrm{sgn}(i_{\ell,n_\ell}) =: \varepsilon_\ell$, here $q = \sum_{\{\ell \mid \varepsilon_\ell = 1, 1 \le \ell \le m\}} n_\ell$, and $q = 0$ when $\varepsilon_\ell = -1$ for all ℓ. In particular, $(-1)^q = (-1)^N \prod_{1 \le \ell \le m, 1 \le p \le n_\ell} \mathrm{sgn}(i_{\ell,p})$.

The action (5.1.1) induces an action on functions given by

$$\begin{aligned}(a_k f)(z_{1,0}, \ldots, z_{m,n_m}) = a_k^{\ell_k} f(z_{1,0}, \ldots, z_{k-1,n_{k-1}}, &z_{k,0} a_k^{-1}, z_{k,1} a_k^{-1}, \ldots, z_{k,n_k} a_k^{-1}, \\ &\ldots, z_{\ell,0}, a_k^{-c_{k,\ell}^{(1)}} z_{\ell,1}, \ldots, a_k^{-c_{k,\ell}^{(n_\ell)}} z_{\ell,n_\ell}, \ldots).\end{aligned}$$

The monomial $z_{1,0}^{-i_{1,0}} z_{1,1}^{-i_{1,1}} \cdots z_{m,n_m}^{-i_{m,n_m}}$ is then a weight vector with a weight whose k-th coordinate is $\ell_k + i_{k,0} + \cdots + i_{k,n_k} + \sum_{\ell=k+1}^m \sum_{p=1}^{n_\ell} c_{k,\ell}^{(p)} i_{\ell,p}$. Thus the G-invariant part of $H^*((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})$ consists of those monomials $z_{1,0}^{-i_{1,0}} z_{1,1}^{-i_{1,1}} \cdots z_{m,n_m}^{-i_{m,n_m}}$ for which

$$\begin{aligned} \ell_1 + i_{1,0} + \cdots + i_{1,n_1} + \sum_{\ell=2}^m \sum_{p=1}^{n_\ell} c_{1,\ell}^{(p)} i_{\ell,p} &= 0 \\ \ell_2 + i_{2,0} + \cdots + i_{2,n_2} + \sum_{\ell=3}^m \sum_{p=1}^{n_\ell} c_{2,\ell}^{(p)} i_{\ell,p} &= 0 \\ &\vdots \\ \ell_m + i_{m,0} + \cdots + i_{m,n_m} &= 0. \end{aligned} \tag{6.1.1}$$

The action (5.2.1) induces a T action on the functions given by

$$\begin{aligned}&((t_{1,1}, \ldots, t_{m,n_m}, t_{m+1}) \cdot f)(z_{1,0}, \ldots, z_{m,n_m}) \\ &= t_{m+1} f(z_{1,0}, t_{1,1}^{-1} z_{1,1}, \ldots, z_{m,0}, t_{m,1}^{-1} z_{m,1}, \ldots, t_{m,n_m}^{-1} z_{m,n_m}).\end{aligned}$$

The weight of the monomial $z_{1,0}^{-i_{1,0}} z_{1,1}^{-i_{1,1}} \cdots z_{m,n_m}^{-i_{m,n_m}}$ with respect to this T action is $(\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_m, 1)$, where $\mathbf{i}_j = (i_{j,1}, \ldots, i_{j,n_j})$ for $j = 1, \ldots, m$. Thus the index of $(B_m, \mathcal{O}_\mathbf{L})$ is given by the set of $x = (x_{1,1}, \ldots, x_{m,n_m}, 1) = (i_{1,1}, \ldots, i_{m,n_m}, 1)$ for which there exist $(i_{1,0}, \ldots, i_{m,0})$ such that (6.1.1) is satisfied and such that $\mathrm{sgn}(i_{\ell,0}) = \mathrm{sgn}(i_{\ell,1}) = \cdots = \mathrm{sgn}(i_{\ell,n_\ell})$ for all ℓ. This is exactly the set (5.3.1). Therefore the multiplicity of the equivariant index is $(-1)^N \prod_{1 \le \ell \le m, 1 \le p \le n_\ell} \mathrm{sgn}(i_{\ell,p}) = (-1)^N \prod_{1 \le \ell \le m, 1 \le p \le n_\ell} \mathrm{sgn}(x_{\ell,p}) = \rho(x)$.
□

6.2 Character formula for the equivariant index

In the following theorem we give a formula for the character $\chi : T \to \mathbb{C}$ of the equivariant index of a generalized Bott manifold. For every integral weight $\mu \in \ell^*$ we have a homomorphism $\lambda^\mu : T \to S^1$. We denote the integral combinations of these λ^μ's by $\mathbb{Z}[T]$. Then $\chi \in \mathbb{Z}[T]$ is given by $\chi = \sum_{\mu \in \ell^*} m_\mu \lambda^\mu$ where $m_\mu = \mathrm{mult}(\mu)$.

Definition 6.2.1 Let $\{e_{1,1}, \ldots, e_{m,n_m}, e_{m+1}\}$ be the standard basis in $\mathbb{R}^{N+1}$, $x_i = (x_{i,1}, \ldots, x_{i,n_i})$ and $e_i = (e_{i,1}, \ldots, e_{i,n_i})$. Let $\Delta^-_{n,r} = \left\{z = (z_1, \ldots, z_n) \in \mathbb{Z}^n_{\leq 0} \,\middle|\, z_1 + \cdots + z_n = -r\right\}$, and let $\Delta^+_{n,r} = \{z = (z_1, \ldots, z_n) \in \mathbb{Z}^n_{>0} \,|\, z_1 + \cdots + z_n = r - 1\}$. Let $\langle x_i, e_i\rangle = x_{i,1}e_{i,1} + \cdots + x_{i,n_i}e_{i,n_i}$. Then the operators $D_i : \mathbb{Z}[T] \to \mathbb{Z}[T]$ are defined using $c^{(k)}_{i,j}$ and ℓ_j in the following way:

$$D_i(\lambda^\mu) = \begin{cases} \displaystyle\sum_{0 \leq r \leq k_i} \sum_{x_i \in \Delta^-_{n_i,r}} \lambda^{\mu + \langle x_i, e_i\rangle} & \text{if } k_i \geq 0 \\ 0 & \text{if } -n_i \leq k_i \leq -1 \\ \displaystyle\sum_{n_i+1 \leq r \leq -k_i} \sum_{x_i \in \Delta^+_{n_i,r}} (-1)^{n_i} \lambda^{\mu + \langle x_i, e_i\rangle} & \text{if } k_i \leq -n_i - 1, \end{cases}$$

where the functions k_i are defined as follows: if $\mu = e_{m+1} + \sum_{j=i+1}^m \sum_{k=1}^{n_j} x_{j,k}e_{j,k}$, then $k_i(\mu) = \ell_i + \sum_{j=i+1}^m \sum_{k=1}^{n_j} c^{(k)}_{i,j} x_{j,k}$.

From Theorem 6.1.1, we immediately obtain the following theorem.

Theorem 6.2.2 *Consider the action of the torus T on $\mathbf{L} \to B_m$ as in (5.2.1). Denote the $(N+1)$-th component of the standard basis in $\mathbb{R}^{N+1}$ by e_{m+1}. Then the character is given by the following element of $\mathbb{Z}[T]$:*

$$\chi = D_1 \cdots D_m(\lambda^{e_{m+1}}).$$

Remark 6.2.3 When $n_i = 1$ for all i, the operator D_i is given by

$$D_i(\lambda^\mu) = \begin{cases} \lambda^\mu + \lambda^{\mu - e_{i,1}} + \cdots + \lambda^{\mu - k_i e_{i,1}} & \text{if } k_i \geq 0 \\ 0 & \text{if } k_i = -1 \\ -\lambda^{\mu + e_{i,1}} - \lambda^{\mu + 2e_{i,1}} - \cdots - \lambda^{\mu - (k_i+1)e_{i,1}} & \text{if } k_i \leq -2. \end{cases}$$

We can check that this operator agrees with the one in [6, Proposition 2.32].

Example 6.2.4 Suppose that $m = 2, n_1 = 1$, and $n_2 = 2$. We set $\ell_1 = 1, \ell_2 = 2, c^{(1)}_{1,2} = 2$, and $c^{(2)}_{1,2} = -1$ as in Example 5.3.4. Then the corresponding character χ is given by

$$\begin{aligned} \chi &= D_1 D_2(\lambda^{e_3}) \\ &= D_1(\lambda^{e_3} + \lambda^{e_3 - e_{2,1}} + \lambda^{e_3 - e_{2,2}} + \lambda^{e_3 - 2e_{2,1}} + \lambda^{e_3 - e_{2,1} - e_{2,2}} + \lambda^{e_3 - 2e_{2,2}}) \\ &= \lambda^{e_3} + \lambda^{e_3 - e_{1,1}} + \lambda^{e_3 - e_{2,2}} + \lambda^{e_3 - e_{2,2} - e_{1,1}} + \lambda^{e_3 - e_{2,2} - 2e_{1,1}} - \lambda^{e_3 - 2e_{2,1} + e_{1,1}} - \lambda^{e_3 - 2e_{2,1} + 2e_{1,1}} \\ &\quad + \lambda^{e_3 - e_{2,1} - e_{2,2}} + \lambda^{e_3 - 2e_{2,2}} + \lambda^{e_3 - 2e_{2,2} - e_{1,1}} + \lambda^{e_3 - 2e_{2,2} - 2e_{1,1}} + \lambda^{e_3 - 2e_{2,2} - 3e_{1,1}}. \end{aligned}$$

Example 6.2.5 Suppose that $m = 2, n_1 = 2$, and $n_2 = 1$. We set $\ell_1 = 2, \ell_2 = -6$, and $c_{1,2}^{(1)} = -1$ as in Example 5.3.5. Then the corresponding character χ is given by

$$\begin{aligned}
\chi &= D_1 D_2(\lambda^{e_3}) \\
&= D_1(-\lambda^{e_3+e_{2,1}} - \lambda^{e_3+2e_{2,1}} - \lambda^{e_3+3e_{2,1}} - \lambda^{e_3+4e_{2,1}} - \lambda^{e_3+5e_{2,1}}) \\
&= -\lambda^{e_3+e_{2,1}} - \lambda^{e_3+e_{2,1}-e_{1,1}} - \lambda^{e_3+e_{2,1}-e_{1,2}} - \lambda^{e_3+2e_{2,1}} - \lambda^{e_3+5e_{2,1}+e_{1,1}+e_{1,2}}.
\end{aligned}$$

Example 6.2.6 Suppose that $m = 2, n_1 = 2$, and $n_2 = 2$. We set $\ell_1 = 1, \ell_2 = 2, c_{1,2}^{(1)} = 2$, and $c_{1,2}^{(2)} = -1$ as in Example 5.3.6. Then the corresponding character χ is given by

$$\begin{aligned}
\chi &= D_1 D_2(\lambda^{e_3}) \\
&= D_1(\lambda^{e_3} + \lambda^{e_3-e_{2,1}} + \lambda^{e_3-e_{2,2}} + \lambda^{e_3-2e_{2,1}} + \lambda^{e_3-e_{2,1}-e_{2,2}} + \lambda^{e_3-2e_{2,2}}) \\
&= \lambda^{e_3} + \lambda^{e_3-e_{1,1}} + \lambda^{e_3-e_{1,2}} + \lambda^{e_3-e_{2,2}} + \lambda^{e_3-e_{2,2}-e_{1,1}} + \lambda^{e_3-e_{2,2}-e_{1,2}} + \lambda^{e_3-e_{2,2}-2e_{1,1}} \\
&\quad + \lambda^{e_3-e_{2,2}-e_{1,1}-e_{1,2}} + \lambda^{e_3-e_{2,2}-2e_{1,2}} + \lambda^{e_3-2e_{2,1}+e_{1,1}+e_{1,2}} + \lambda^{e_3-e_{2,1}-e_{2,2}} + \lambda^{e_3-2e_{2,2}} \\
&\quad + \lambda^{e_3-2e_{2,2}-e_{1,1}} + \lambda^{e_3-2e_{2,2}-e_{1,2}} + \lambda^{e_3-2e_{2,2}-2e_{1,1}} + \lambda^{e_3-2e_{2,2}-e_{1,1}-e_{1,2}} + \lambda^{e_3-2e_{2,2}-2e_{1,2}} \\
&\quad + \lambda^{e_3-2e_{2,2}-3e_{1,1}} + \lambda^{e_3-2e_{2,2}-2e_{1,1}-e_{1,2}} + \lambda^{e_3-2e_{2,2}-e_{1,1}-2e_{1,2}} + \lambda^{e_3-2e_{2,2}-3e_{1,2}}.
\end{aligned}$$

Remark 6.2.7 We gave the formula for the character using the Demazure-type operators. On the other hand, the character is also given by the localization formula ([7, Corollary 7.4]). For example, when we set the parameters as in Example 6.2.4, the character is computed using the localization formula as follows:

$$\begin{aligned}
\chi = \lambda^{e_3} \Bigg(& \frac{1}{(1-\lambda^{-e_{1,1}})(1-\lambda^{-e_{2,1}})(1-\lambda^{-e_{2,2}})} + \frac{\lambda^{-2e_{2,2}}}{(1-\lambda^{-e_{1,1}})(1-\lambda^{-e_{2,1}+e_{2,2}})(1-\lambda^{e_{2,2}})} \\
&+ \frac{\lambda^{-2e_{2,1}}}{(1-\lambda^{-e_{1,1}})(1-\lambda^{e_{2,1}-e_{2,2}})(1-\lambda^{e_{2,1}})} + \frac{\lambda^{-e_{1,1}}}{(1-\lambda^{e_{1,1}})(1-\lambda^{2e_{1,1}-e_{2,1}})(1-\lambda^{-e_{1,1}-e_{2,2}})} \\
&+ \frac{\lambda^{-3e_{1,1}-2e_{2,2}}}{(1-\lambda^{e_{1,1}})(1-\lambda^{3e_{1,1}-e_{2,1}+e_{2,2}})(1-\lambda^{e_{1,1}+e_{2,2}})} \\
&+ \frac{\lambda^{3e_{1,1}-2e_{2,1}}}{(1-\lambda^{e_{1,1}})(1-\lambda^{-3e_{1,1}+e_{2,1}-e_{2,2}})(1-\lambda^{-2e_{1,1}+e_{2,1}})} \Bigg).
\end{aligned}$$

We can check that this result agrees with the result in Example 6.2.4.

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