Extension of Olsen's inequality to Morrey-Lorentz spaces

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Preface

Morrey-Lorentz spaces, which are an extension of Morrey and Lorentz spaces, were introduced by Ragusa [44] in 2012. Morrey spaces were introduced by Morrey [38] to investigate the solutions of second-order elliptic partial differential equations. Lorentz [37] defined Lorentz spaces and compared them with Lebesgue and Morrey spaces (see [37, Theorem 3]). In particular, according to [37, Theorem 2], Lorentz spaces are separable, but Morrey spaces are not. Lorentz spaces can be constructed from the real interpolation spaces of Lebesgue spaces (see, e.g., [4]). Morrey spaces have weak Lebesgue spaces as proper subspaces (see Proposition 2.16 (3)). In addition, Morrey spaces are used to obtain the Fefferman-Phong inequality (see (1)).

Olsen's inequality represents the weighted boundedness of fractional integral operators on Morrey spaces (see [42]). Taking the gradient of functions, we see that this inequality is an extension of the Fefferman-Phong inequality; for a potential $V \geq 0$,

$$\int_{\mathbb{R}^n} |u(x)|^2 V(x)\,\mathrm{d}x \leq C_V \int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x. \tag{1}$$

According to [12, p. 143], this inequality is a necessary condition for the positivity of the Schrödinger operator $-\Delta - V$. This is such an important problem that one considers the optimality of the constant C_V appearing in the above estimate. As a result, when V belongs to some Morrey spaces, Olsen proved the above estimates. Since then, many authors have investigated generalizations for Olsen's inequality, including generalized Morrey spaces [51], Orlicz-Morrey spaces [18, 50] of various types, and mixed Morrey spaces [41]. In particular, according to [52, Proposition 4.1], we can no longer relax the condition on the local integrability in Theorem 1.7 (see [51]).

Olsen's inequality cannot simply be proved by a mere combination of the Hölder inequality and the boundedness of the fractional integral operator on Morrey spaces (see Section 5.2 in detail). Seemingly, Olsen's inequality can be obtained by combining boundedness of the Riesz potential and Hölder's inequality; however this is not the case. For this reason, the proof of this inequality is very difficult, and many authors have given alternative proofs. Tanaka [57] used the Calderón-Zygmund decomposition for the family of dyadic cubes to additionally give the vector-valued extension. Iida et al. [33] provided the atomic decomposition for Morrey spaces, and as an application, they proved Olsen's inequality. In [22], the author applied Tanaka's method to the generalization for its inequality. In this thesis, we refer to the ideas from the paper by Iida et al. to obtain an extension to its inequality for Morrey-Lorentz spaces.

The Taylor and Fourier expansions are classically well known as decompositions of functions. Decomposing functions yields approximations of functions. In this thesis, we employ our "atomic decomposition" as a method for the decomposition. The Taylor and Fourier expansions use power and trigonometric functions, respectively, while our atomic decomposition uses some functions with compact support that are orthogonal to polynomials up to a fixed order. The origin of atomic decomposition goes back to the investigation of Hardy spaces.

This thesis presents the author's achievements, systematically combining and refining [22, 26].

Additionally, the author investigated many kinds of operators, including the boundedness of bilinear fractional integral operators of Grafakos type [21, 28], universality of neural networks with ReLU activations [23], boundedness of composition operators on Morrey and weak Morrey spaces [24], predual spaces of weak Orlicz spaces [25], and pointwise multiplier spaces from Besov spaces to Banach lattices [27].

Chapter 1

Introduction

The goal of this chapter is to present a brief overview of basic concepts and our results.

This chapter is organized as follows: In Section 1.1, we introduce the notation used in this thesis. In Section 1.2, we explain the theory of function spaces and give some examples as an extension of Lebesgue spaces. In Section 1.3, we introduce the investigation of Hardy spaces and their atomic decomposition. In Section 1.4, we present the main theorem. In Section 1.5, as an application to the Schrödinger operator, we give the Fefferman-Phong inequality.

1.1 Notation

Throughout this thesis, we use the following notation:

1. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In this thesis, $n \in \mathbb{N}$ stands for a dimension.

2. $d_v := [n(1/v - 1)]$ for $v \in (0, 1]$.

3. For $0 < p \le \infty$, the conjugate number p' of p is defined by $1/p + 1/p' = 1$. Here, when $0 < p < 1$, we understand $p' = \infty$.

4. Denote by $\mathcal{Q}(\mathbb{R}^n)$ the set of all cubes in $\mathbb{R}^n$ that are parallel to the coordinate axes.

5. For $Q \in \mathcal{Q}(\mathbb{R}^n)$, $\ell(Q)$ and $c(Q)$ represent the side-length and center of Q, respectively. In addition, we denote by $Q(x, r)$ the cube of radius $r > 0$ centered at $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as follows:

$$Q(x, r) := [x_1 - r, x_1 + r) \times \cdots \times [x_n - r, x_n + r).$$

   For simplicity, we write $Q(r)$ instead of $Q(0, r)$.

6. Given $Q \in \mathcal{Q}(\mathbb{R}^n)$ and $\alpha > 0$, αQ represents the cube concentric to Q with sidelength $\alpha \ell(Q)$:
$$\alpha Q(x, r) := Q(x, \alpha r)$$
for $x \in \mathbb{R}^n$ and $r > 0$.

7. The closure of $Q \in \mathcal{Q}(\mathbb{R}^n)$ is denoted by $\overline{Q}$:
$$\overline{Q(x, r)} := [x_1 - r, x_1 + r] \times \cdots \times [x_n - r, x_n + r]$$
for $x \in \mathbb{R}^n$ and $r > 0$.

8. For $j \in \mathbb{Z}$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, we define
$$Q_{jm} := \left[\frac{m_1}{2^j}, \frac{m_1 + 1}{2^j}\right) \times \cdots \times \left[\frac{m_n}{2^j}, \frac{m_n + 1}{2^j}\right),$$
and Q_{jm} is called a dyadic cube. Denote by $\mathcal{D}_j(\mathbb{R}^n)$ the set of all such cubes with side length 2^{-j}, and set
$$\mathcal{D}(\mathbb{R}^n) := \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j(\mathbb{R}^n).$$
.

9. We denote by $B(x, r)$ the ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$:
$$B(x, r) := \{y \in \mathbb{R}^n \, : \, |x - y| < r\}.$$
We write $B(r)$ instead of $B(0, r)$. The symbol $\mathcal{B}(\mathbb{R}^n)$ represents the set of all balls $B(x, r)$ for $x \in \mathbb{R}^n$ and $r > 0$.

10. We use C to denote a positive constant that may vary from one occurrence to another. If $A \le CB$, then we write $A \lesssim B$ or $B \gtrsim A$, and if $A \lesssim B \lesssim A$, we write $A \sim B$. In particular, when we want to emphasize that the constant C depends on the parameters α, β, γ, etc., we write $A \lesssim_{\alpha,\beta,\gamma,\dots} B$ and $A \sim_{\alpha,\beta,\gamma,\dots} B$ instead of $A \lesssim B$ and $A \sim B$, respectively.

11. Let E be a measurable set in $\mathbb{R}^n$. Then, χ_E denotes the indicator function for E.

12. We define $L^0(\mathbb{R}^n)$ as the space of all measurable functions on $\mathbb{R}^n$.

13. Denote by $\mathcal{P}_K(\mathbb{R}^n)$ the set of all polynomial functions with degree less than or equal to K. The set $\mathcal{P}_K(\mathbb{R}^n)^\perp$ denotes the set of $f \in L^0(\mathbb{R}^n)$ for which
$$\langle \cdot \rangle^K f \in L^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} x^\alpha f(x) \, \mathrm{d}x = 0$$
for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le K$, where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. Such a function f is said to satisfy the *moment condition of order K*.

14. For a space $E(\mathbb{R}^n)$ with quasi-norm $\|\cdot\|_E$, set

$$E_{\rm loc}(\mathbb{R}^n) := \{f \in L^0(\mathbb{R}^n) \,:\, \|f\chi_K\|_E < \infty, \text{ for all compact sets } K \text{ in } \mathbb{R}^n\}.$$

15. For a measurable set E with $|E| \neq 0$ and $f \in L^0(\mathbb{R}^n)$,

$$m_E(f) := \frac{1}{|E|}\int_E f(x)\,{\rm d}x,$$

and the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} \chi_Q(x) m_Q(|f|), \quad x \in \mathbb{R}^n$$

for $f \in L^0(\mathbb{R}^n)$. More generally, for $\eta \in (0,\infty)$, we define its powered version by $M^{(\eta)}f := (M[|f|^\eta])^{1/\eta}$ for $f \in L^0(\mathbb{R}^n)$.

16. Let $0 < \alpha < n$. We define the fractional integral operator I_α by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,{\rm d}y, \quad x \in \mathbb{R}^n$$

for $f \in L^0(\mathbb{R}^n)$. Note that the integral defining $I_\alpha f$ converges in many cases as we will show.

17. The symbol $\mathcal{S}(\mathbb{R}^n)$ represents the Schwartz space, and its continuous linear functional space is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

18. When X and Y are sets, $X \subset Y$ represents the inclusion of sets. In addition, if both X and Y are quasi-normed spaces endowed with $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and if the natural embedding mapping $X \to Y$ is bounded, we write $X \hookrightarrow Y$. Moreover, when $X \hookrightarrow Y$ and $Y \hookrightarrow X$, we write $X \cong Y$.

19. For $0 < u \le \infty$, $\ell^u(\mathbb{N})$ denotes the set of all sequences $\{a_j\}_{j=1}^\infty$ with finite quasi-norm

$$\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^u} := \begin{cases} \left(\displaystyle\sum_{j=1}^\infty |a_j|^u\right)^{\frac{1}{u}}, & 0 < u < \infty, \\ \displaystyle\sup_{j\in\mathbb{N}} |a_j|, & u = \infty. \end{cases}$$

1.2 Function space theory and Lebesgue spaces

The theory of function spaces is of intersect in harmonic analysis. By a "function space," we mean a linear subspace of the space of all functions on a set X. In this thesis, we work in the setting of the Euclidean space $X = \mathbb{R}^n$. In harmonic analysis, various operators are used, and their continuity on some function spaces endowed with (quasi-)norms, which is called boundedness, is investigated. For this reason, the investigation of function spaces is fundamental.

Here, we recall the Lebesgue space $L^p(\mathbb{R}^n)$, $0 < p \le \infty$, which is a fundamental example of a function space. Let $0 < p \le \infty$. Define the Lebesgue space $L^p(\mathbb{R}^n)$ to be the linear space of all $f \in L^0(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{L^p} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|, & p = \infty. \end{cases}$$

Lebesgue spaces are quasi-Banach spaces. In particular, if $p \ge 1$, the Lebesgue space $L^p(\mathbb{R}^n)$ is a Banach space. When $0 < p < \infty$, the dual space of $L^p(\mathbb{R}^n)$ is equivalent to $L^{p'}(\mathbb{R}^n)$. In particular, if $1 < p < \infty$, the Lebesgue space $L^p(\mathbb{R}^n)$ is reflexive.

It is well known that Hardy and Littlewood [19] and Sobolev [54] proved the boundedness of a fractional integral operator on Lebesgue spaces, which is called the Hardy-Littlewood-Sobolev inequality: if $0 < \alpha < n$ and $1 < p < s < \infty$ satisfies $1/s = 1/p - \alpha/n$, then

$$\|I_\alpha f\|_{L^s} \lesssim \|f\|_{L^p} \tag{1.1}$$

for all $f \in L^p(\mathbb{R}^n)$ (see, e.g., [16, Theorem 1.2.3]). Hereafter, (1.1) is extended to boundedness on Morrey spaces, which is called Adams' theorem (see [2, 7]).

To discuss the boundedness property more precisely, many function spaces that are extensions of Lebesgue spaces have been introduced and investigated. In this study, the author considers Lorentz and Morrey spaces. Here, we recall the following spaces.

Definition 1.1. For $t > 0$ and $f \in L^0(\mathbb{R}^n)$, its distribution function $m_f(t)$ and rearrangement function $f^*(t)$ are defined by

$$m_f(t) := |\{x \in \mathbb{R}^n : |f(x)| > t\}|$$

and

$$f^*(t) := \inf\{\alpha > 0 : m_f(\alpha) \le t\},$$

respectively. Here, it is assumed that $\inf \emptyset = \infty$.

Let $0 < p, q \le \infty$. We define the Lorentz space $L^{p,q}(\mathbb{R}^n)$ to be the linear space of all $f \in L^0(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{L^{p,q}} := \begin{cases} \left(\displaystyle\int_0^\infty \left[t^{\frac{1}{p}} f^*(t) \right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}, & 0 < p, q < \infty, \\ \sup_{t > 0} t^{\frac{1}{p}} f^*(t), & 0 < p \le q = \infty. \end{cases}$$

In particular, $L^{p,\infty}(\mathbb{R}^n)$ is isomorphic with coincidence of norms to the weak Lebesgue space $\mathrm{W}L^p(\mathbb{R}^n)$ (see [15, Proposition 1.4.5 (16)]), whose weak Lebesgue quasi-norm $\|\cdot\|_{\mathrm{W}L^p}$ is defined by

$$\|f\|_{\mathrm{W}L^p} := \sup_{\lambda>0} \lambda m_f(\lambda)^{\frac{1}{p}} = \sup_{\lambda>0} \lambda \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\|_{L^p}.$$

We do not consider the space $L^{\infty,q}(\mathbb{R}^n)$ for $0 < q < \infty$.

Remark 1.2. Let $0 < q < \infty$. According to [15, Example 1.4.8], the only function with finite quasi-norm $\|\cdot\|_{L^{\infty,q}}$ is zero, i.e., $L^{\infty,q}(\mathbb{R}^n) = \{0\}$.

Definition 1.3. Let $0 < q \le p < \infty$. We define the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ as the space of all $f \in L^0(\mathbb{R}^n)$ with the finite quasi-norm

$$\|f\|_{\mathcal{M}^p_q} := \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}.$$

In addition, the weak Morrey space $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$ is defined as the space of all $f \in L^0(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{\mathrm{W}\mathcal{M}^p_q} := \sup_{\lambda>0} \lambda \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\|_{\mathcal{M}^p_q}.$$

The fundamental properties of these function spaces are discussed in Chapter 2. In this thesis, we employ Morrey-Lorentz spaces introduced by Ragusa [44].

Definition 1.4. Let $0 < q \le p < \infty$ and $0 < r \le \infty$. We define the Morrey-Lorentz space $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ as the space of all $f \in L^0(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{\mathcal{M}^p_{q,r}} := \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L^{q,r}}.$$

These function spaces are extensions of Lorentz and Morrey spaces as follows.

Proposition 1.5. *Let $0 < q \le p < \infty$ and $0 < r \le \infty$. Then,*

$$\mathcal{M}^p_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad \mathcal{M}^p_{p,\infty}(\mathbb{R}^n) = \mathrm{W}L^p(\mathbb{R}^n), \quad \mathcal{M}^p_{p,r}(\mathbb{R}^n) = L^{p,r}(\mathbb{R}^n),$$

$$\mathcal{M}^p_{q,q}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n), \quad \mathcal{M}^p_{q,\infty}(\mathbb{R}^n) = \mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$$

with coincidence quasi-norms.

The proof of each equality is straightforward, and we omit the proofs.

1.3 Hardy spaces and their atomic decomposition

To prove the extension of the Olsen inequality to Morrey-Lorentz spaces, we use atomic decomposition. The origin of the investigation of atomic decomposition goes back to the theory of Hardy spaces (see [35]).

Recall that for $0 < p < \infty$, the Hardy space $H^p(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm $\|f\|_{H^p} := \left\|\sup_{t>0} |e^{t\Delta} f|\right\|_{L^p}$ is finite, where $e^{t\Delta} f$ represents the heat expansion of f for $t > 0$:

$$e^{t\Delta} f(x) = \left\langle \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x-\cdot|^2}{4t}\right), f \right\rangle, \quad x \in \mathbb{R}^n.$$

For later use, we recall the following two fundamental notions (see [8]):

(1) Topologize $\mathcal{S}(\mathbb{R}^n)$ by the norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) := \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$$

for each $N \in \mathbb{N}$. Define $\mathcal{F}_N := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}$.

(2) Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The grand maximal function $\mathcal{M}f$ is given by

$$\mathcal{M}f(x) := \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| : t > 0,\, \psi \in \mathcal{F}_N\}, \quad x \in \mathbb{R}^n,$$

where we choose and fix a large integer N.

We remark that

$$\|f\|_{H^p} \sim \|\mathcal{M}f\|_{L^p}$$

for all $f \in H^p(\mathbb{R}^n)$ (see, e.g., [55, Chapter 3]).

To date, many authors have investigated atomic decomposition for extended Hardy spaces, including Hardy-Lorentz spaces [1,36,43], Orlicz-Hardy spaces [40], Hardy spaces with variable exponents [9,39,46], Hardy-Morrey spaces [33,34], generalized Hardy-Morrey spaces [3], Hardy-Orlicz-Morrey spaces [18,50] of various types, and mixed Hardy-Morrey spaces [41]. Here, we consider Hardy-Morrey-Lorentz spaces.

In particular, Strönberg and Tochinsky established the theory of atomic decomposition for weighted Hardy spaces. Let w be a locally integrable function, and recall that w is an A_1-weight when $Mw \lesssim w$. We define the weighted L^1-space $L^1(\mathbb{R}^n, w)$ by the space of all $f \in L^0(\mathbb{R}^n)$ with finite norm

$$\|f\|_{L^1(w)} := \int_{\mathbb{R}^n} |f(x)| w(x)\, \mathrm{d}x,$$

and we set

$$H^1(\mathbb{R}^n, w) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \,:\, \|f\|_{H^1(w)} := \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{L^1(w)} < \infty \right\}.$$

We use the following atomic decomposition for $H^1(\mathbb{R}^n, w)$ with A_1-weight w.

Theorem 1.6 ([40, 56])**.** *Let w be an A_1-weight, and let $f \in H^1(\mathbb{R}^n, w)$. Then, there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that*

$$|a_j| \le \chi_{Q_j}, \quad \int_{\mathbb{R}^n} a_j(x)\,\mathrm{d}x = 0, \quad \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{L^1(w)} \lesssim \|f\|_{H^1(w)}.$$

In particular, $H^1(\mathbb{R}^n, w)$ is embedded in $L^1(\mathbb{R}^n, w)$.

1.4 Main theorem

For Morrey spaces, sharp Olsen's inequality was given by Sawano, Sugano, and Tanaka as follows.

Theorem 1.7 ([52, Proposition 1.8])**.** *Let $0 < \alpha < n$, $1 < p_1 \le p_0 < \infty$, $1 < q_1 \le q_0 < \infty$, and $1 < r_1 \le r_0 < \infty$. Assume that*

$$r_1 < q_1, \quad \frac{1}{q_0} \le \frac{\alpha}{n} < \frac{1}{p_0}, \quad \frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}, \quad \frac{r_0}{p_0} = \frac{r_1}{p_1}.$$

Then,

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_{r_1}} \lesssim \|g\|_{\mathcal{M}^{q_0}_{q_1}} \|f\|_{\mathcal{M}^{p_0}_{p_1}}$$

for any $f \in \mathcal{M}^{p_0}_{p_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n)$.

Remark 1.8. We compare Theorem 1.7 with the original version of Olsen's inequality from [42, Theorem 2], where Olsen assumed that

$$\frac{1}{r_1} = \frac{1}{q_1} + \frac{1}{p_1} - \frac{\alpha}{n}$$

instead of the condition $r_0/p_0 = r_1/p_1$ in Theorem 1.7.

The goal of this paper is to prove the following Olsen inequality for Morrey-Lorentz spaces.

Theorem 1.9. *Let $0 < \alpha < n$, $1 < p_1 \le p_0 < \infty$, $1 < q_1 \le q_0 < \infty$, $1 < r_1 \le r_0 < \infty$, and $0 < p_2, r_2 \le \infty$. Assume that*

$$r_1 < q_1, \quad \frac{1}{q_0} \le \frac{\alpha}{n} < \frac{1}{p_0}, \quad \frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}.$$

If we suppose either of the following;

(1) $0 < r_2, p_2 < \infty$ *and* $\dfrac{r_0}{p_0} = \dfrac{r_1}{p_1} = \dfrac{r_2}{p_2}$,

(2) $r_2 = p_2 = \infty$ *and* $\dfrac{r_0}{p_0} = \dfrac{r_1}{p_1}$,

then we have

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \|g\|_{\mathrm{W}\mathcal{M}^{q_0}_{q_1}} \|f\|_{\mathcal{M}^{p_0}_{p_1,p_2}}$$

for any $f \in \mathcal{M}^{p_0}_{p_1,p_2}(\mathbb{R}^n)$ *and any* $g \in \mathrm{W}\mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n)$.

Comparing the sharp Olsen inequality [52, Proposition 1.8], we learn that Theorem 1.9 is improved in that the condition $g \in \mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n)$ in Theorem 1.7 is replaced by $g \in \mathrm{W}\mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n)$. With this result, we remark that the embedding $\mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n) \hookrightarrow \mathrm{W}\mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n)$ is proper (see [17] and Theorem 3.7 to follow).

In particular, we can rewrite Case (2) in Theorem 1.9 in terms of weak Morrey spaces as follows.

Theorem 1.10. *Let* $0 < \alpha < n$, $1 < p_1 \le p_0 < \infty$, $1 < q_1 \le q_0 < \infty$, *and* $1 < r_1 \le r_0 < \infty$. *Assume that*

$$r_1 < q_1, \quad \frac{1}{q_0} \le \frac{\alpha}{n} < \frac{1}{p_0}, \quad \frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}, \quad \frac{r_0}{p_0} = \frac{r_1}{p_1}.$$

Then we have

$$\|g \cdot I_\alpha f\|_{\mathrm{W}\mathcal{M}^{r_0}_{r_1}} \lesssim \|g\|_{\mathrm{W}\mathcal{M}^{q_0}_{q_1}} \|f\|_{\mathrm{W}\mathcal{M}^{p_0}_{p_1}}$$

for any $f \in \mathrm{W}\mathcal{M}^{p_0}_{p_1}(\mathbb{R}^n)$ *and* $g \in \mathrm{W}\mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n)$.

1.5 Fefferman-Phong inequality and Schrödinger operator

Olsen's inequality generalizes the Fefferman-Phong inequality. To verify this, we provide the extension of the Fefferman-Phong inequality as an application of Theorem 1.9 in this section.

Let $n \ge 3$. The Fefferman-Phong inequality reads

$$\int_{\mathbb{R}^n} |u(x)|^2 V(x)\,\mathrm{d}x \le C_V \int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x \tag{1.2}$$

for a potential $V \ge 0$. This inequality yields the positivity of the Schrödinger operator $L := -\Delta - V$. In fact, when $0 < C_V \le 1$, using integration by parts, we have

$$(Lu, u)_{L^2} = \int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x - \int_{\mathbb{R}^n} |u(x)|^2 V(x)\,\mathrm{d}x \ge (1 - C_V) \int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x \ge 0.$$

In particular, Fefferman [12, p. 143] proved (1.2) with $C_V = K\|V\|_{L^{n/2}}$ for some $K > 0$. Namely, $V \in L^{n/2}(\mathbb{R}^n)$ with $\|V\|_{L^{n/2}} \le K^{-1}$ implies that the operator $-\Delta - V$ is positive.

Replacing f by $|\nabla u|$ in Olsen's inequality of Theorem 1.7, one obtains the Fefferman-Phong inequality (1.2) with $C_V = K\|V\|_{\mathcal{M}_q^{n/2}}$ for some $K > 0$. Thus, Olsen's inequality extends the condition $V \in L^{n/2}(\mathbb{R}^n)$ to $V \in \mathcal{M}_q^{n/2}(\mathbb{R}^n)$. Similarly, we can transform Theorem 1.9 into the following assertion.

Theorem 1.11. *Let $n \ge 3$ and $1 < q \le n/2 < \infty$. Then, there exists a constant $K > 0$ such that*

$$\int_{\mathbb{R}^n} |u(x)|^2 V(x)\,\mathrm{d}x \le K\|V\|_{\mathrm{W}\mathcal{M}_q^{\frac{n}{2}}} \int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x$$

for all $u \in L^0(\mathbb{R}^n)$ such that $\nabla u \in (L^2(\mathbb{R}^n))^n$ and non-negative functions $V \in \mathrm{W}\mathcal{M}_q^{\frac{n}{2}}(\mathbb{R}^n)$.

To prove Theorem 1.11, we use the following pointwise estimate to connect I_1 and ∇.

Theorem 1.12. *Let $n \ge 2$. Then, $|f| \lesssim I_1[|\nabla f|]$ for all $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$.*

Proof. Fix $x \in \mathbb{R}^n$. We suppose $n \ge 3$; the case of $n = 2$ can be handled similarly except that we must handle the kernel $\log|x-y|$. We omit the proof for the case of $n = 2$. Thanks to [16, Section 1.2.1], we have $f(x) \simeq_n I_2[\Delta f](x)$. Because $n \ge 3$, we can perform integration by parts to obtain

$$f(x) \simeq_n \sum_{k=1}^n \int_{\mathbb{R}^n} \frac{x_k - y_k}{|x-y|^n} \partial_k f(y)\mathrm{d}y.$$

Then, by the triangle inequality for integrals, we have the desired result. □

Proof of Theorem 1.11. We assume that Theorem 1.9 holds to give the proof of Theorem 1.11. It is well known that the homogeneous Sobolev space

$$\dot{H}^1(\mathbb{R}^n) := \{f \in L^0(\mathbb{R}^n) \,:\, \|f\|_{\dot{H}^1} := \||\nabla f|\|_{L^2} < \infty\}$$

has a dense subset $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ (see, e.g., [60, Proposition 1.22]). Then, we may assume that $u \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$.

Combining with Theorems 1.9 and 1.12, we have

$$\int_{\mathbb{R}^n} |u(x)|^2 V(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^n} I_1[|\nabla u|](x)^2 V(x)\,\mathrm{d}x \lesssim K\|V\|_{\mathrm{W}\mathcal{M}_q^{\frac{n}{2}}} \int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x,$$

as desired. □

Chapter 2

Classical function spaces

Morrey-Lorentz spaces are built upon Morrey spaces and Lorentz spaces. Thus, we recall their fundamental properties.

Throughout this chapter, we introduce some classical function spaces and their fundamental properties. We introduce Lorentz spaces in Section 2.1 and Morrey spaces in Section 2.2.

2.1 Lorentz spaces

On the basis of the definition of Lorentz spaces in Definition 1.1, we introduce their well-known properties.

First, we give $\|\chi_E\|_{L^{p,q}}$.

Lemma 2.1 ([15, Example 1.4.8]). *Let $0 < p < \infty$ and $0 < q \leq \infty$, and let E be a measurable set in $\mathbb{R}^n$. Then,*

$$\|\chi_E\|_{L^{p,q}} = \left(\frac{p}{q}\right)^{\frac{1}{q}} |E|^{\frac{1}{p}},$$

where we assume that $(p/q)^{1/q} = 1$ for $q = \infty$.

We recall the dilation property for Lorentz quasi-norms.

Lemma 2.2 ([15, Remark 1.4.7]). *Let $0 < p, q \leq \infty$ and $t \in (0, \infty)$. Then*

$$\|f(t\cdot)\|_{L^{p,q}} = t^{-\frac{n}{p}} \|f\|_{L^{p,q}}$$

for all $f \in L^{p,q}(\mathbb{R}^n)$.

Importantly, $L^{p,q}(\mathbb{R}^n)$ is normable as the following proposition shows.

Proposition 2.3 ([6, Section 6.2]). *If $p > 1$ and $q \geq 1$, then the space $L^{p,q}(\mathbb{R}^n)$ is normable. In particular, if we set*

$$\|f\|^{\dagger}_{L^{p,q}} := \begin{cases} \left(\int_0^\infty \left[t^{\frac{1}{p}} f^{**}(t) \right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}}, & 1 < p < \infty, 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^{**}(t), & 1 < p \leq q = \infty, \end{cases}$$

for $f \in L^0(\mathbb{R}^n)$, where $f^{**}(t) := t^{-1} \int_0^t f^*(s) \, \mathrm{d}s$ for $t > 0$, then we have

$$\|f\|^{\dagger}_{L^{p,q}} \sim \|f\|_{L^{p,q}},$$

and $L^{p,q}(\mathbb{R}^n)$ is a Banach space with the norm $\| \cdot \|^{\dagger}_{L^{p,q}}$.

We have the following inclusion relation.

Proposition 2.4 ([15, Proposition 1.4.5 (15) and Proposition 1.4.10]). *Let $0 < p$, $q_1, q_2 \leq \infty$. The following assertions hold*:

(1) $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

(2) *If $0 < q_1 \leq q_2 \leq \infty$, then*

$$L^{p,q_1}(\mathbb{R}^n) \hookrightarrow L^{p,q_2}(\mathbb{R}^n) \hookrightarrow L^{p,\infty}(\mathbb{R}^n).$$

Here, we present examples of Lorentz functions, demonstrating the diversity of Lorentz spaces.

Example 2.5 ([15, Exercise 1.4.8]). If $0 < p < \infty$ and $0 < q_1 < q_2 < \infty$, then

$$f(x) := (1 + |x|)^{-\frac{n}{p}} (\log(2 + |x|))^{-\frac{1}{q_1}} \in L^{p,q_2}(\mathbb{R}^n) \setminus L^{p,q_1}(\mathbb{R}^n).$$

We discretize Example 2.5, working in $\mathbb{R}$.

Example 2.6 ([5, p. 56]). Let $0 < p < \infty$, $0 < q_1 < q_2 < \infty$, and let $J = J(p, q_1) \gg 1$ be such that the sequence $\{2^{nj/p}/j^{1/q_1}\}_{j=J}^\infty$ is increasing for $j \in \mathbb{N} \cap [J, \infty)$. Taking

$$f := \sum_{j=J}^\infty \frac{2^{\frac{nj}{p}}}{j^{\frac{1}{q_1}}} \chi_{B(2^{-j}) \setminus B(2^{-j-1})},$$

one has

$$f \in L^{p,q_2}(\mathbb{R}^n) \setminus L^{p,q_1}(\mathbb{R}^n). \tag{2.1}$$

In fact, using the fact that

$$f^* = \sum_{j=J}^\infty \frac{2^{\frac{nj}{p}}}{j^{\frac{1}{q_1}}} \chi_{\left[\frac{\nu_n}{2^{n(j+1)}}, \frac{\nu_n}{2^{nj}}\right)},$$

where ν_n is the volume of a unit ball, one has

$$\|f\|_{L^{p,q}} = \begin{cases} \infty, & q = q_1, \\ \nu_n^{\frac{1}{p}} \left\{ \frac{p}{q} \left(1 - \frac{1}{2^{\frac{nq}{p}}} \right) \right\}^{\frac{1}{q}} \left(\sum_{j=J}^{\infty} \frac{1}{j^{\frac{q}{q_1}}} \right)^{\frac{1}{q}}, & q \in (q_1, \infty). \end{cases}$$

This proves (2.1).

As before, we prove the Fatou property for Lorentz quasi-norms.

Lemma 2.7 ([15, Exercise 1.4.11 (a)]). *Let $0 < p, q \le \infty$, and let $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$ be a nonnegative collection such that $f = \lim_{j\to\infty} f_j$ exists a.e. Then, we have*

$$\|f\|_{L^{p,q}} \le \liminf_{j\to\infty} \|f_j\|_{L^{p,q}}.$$

We can extend Hölder's inequality.

Lemma 2.8 ([32, Theorem 4.5]). *Assume that $0 < p, p_1, p_2 < \infty$ and $0 < q, q_1, q_2 \le \infty$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then,

$$\|f \cdot g\|_{L^{p,q}} \le \frac{p_1^{\frac{1}{p_1}} p_2^{\frac{1}{p_2}}}{p^{\frac{1}{p}}} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \tag{2.2}$$

for all $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,q_2}(\mathbb{R}^n)$. In particular,

$$\|f \cdot g\|_{L^{p,q}} \le 2^{\frac{1}{p}} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \tag{2.3}$$

for all $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,q_2}(\mathbb{R}^n)$.

Proof. We suppose $q, q_1, q_2 < \infty$. According to [15, Proposition 1.4.5 (7)],

$$(f \cdot g)^*(t) \le f^*(\alpha t) g^*((1-\alpha)t)$$

for all $t > 0$ and $0 < \alpha < 1$. Then using Hölder's inequality, we have

$$\begin{aligned} \|f \cdot g\|_{L^{p,q}} &\le \left(\int_0^\infty \left[t^{\frac{1}{p}} f^*(\alpha t) g^*((1-\alpha)t) \right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\ &\le \left(\int_0^\infty \left[t^{\frac{1}{p_1}} f^*(\alpha t) \right]^{q_1} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q_1}} \cdot \left(\int_0^\infty \left[t^{\frac{1}{p_2}} g^*((1-\alpha)t) \right]^{q_2} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q_2}} \\ &= \frac{1}{\alpha^{\frac{1}{p_1}}} \|f\|_{L^{p_1,q_1}} \cdot \frac{1}{(1-\alpha)^{\frac{1}{p_2}}} \|g\|_{L^{p_2,q_2}}. \end{aligned}$$

Optimizing the most right-hand side in α, we obtain (2.2). We omit the proofs of the cases of $q = \infty$, $q_1 = \infty$, and $q_2 = \infty$ due to their similarity. In addition, because $(p_1/p)^{p/p_1} (p_2/p)^{p/p_2} \le 2$, we obtain (2.3). We finish the proof of Lemma 2.8. □

We now state the maximal inequality.

Proposition 2.9. *Let* $0 < p, q \leq \infty$ *and* $0 < \eta < \infty$. *If* $1 < p \leq \infty$, *then for all* $f \in L^{p,q}(\mathbb{R}^n)$,

$$\|Mf\|_{L^{p,q}} \lesssim \|f\|_{L^{p,q}}. \tag{2.4}$$

More generally, if $0 < \eta < p \leq \infty$, *then for all* $f \in L^{p,q}(\mathbb{R}^n)$,

$$\|M^{(\eta)}f\|_{L^{p,q}} \lesssim \|f\|_{L^{p,q}}. \tag{2.5}$$

Proof. We outline the proof here, we extend Proposition 2.9 in Theorem 2.10, where we give the detailed proof. The equation (2.4) is equivalent to

$$\|M[|f|^\eta]\|_{L^{\tilde{p},\tilde{q}}} \lesssim \||f|^\eta\|_{L^{\tilde{p},\tilde{q}}},$$

where $\tilde{p} := p/\eta > 1$ and $\tilde{q} := q/\eta$. Then, it suffices to prove (2.4).

It is well known that

$$\|Mf\|_{\mathrm{W}L^{p_0}} \lesssim \|f\|_{L^{p_0}}$$

for any $p_0 \in [1, \infty]$ (see, e.g., [15, Theorem 2.1.6]). Applying Hunt's interpolation theorem [15, Theorem 1.4.19], we conclude (2.4). Also, refer to Lemma 2.11 below for more details. □

We extend Proposition 2.9 to the vector-valued setting.

Theorem 2.10. *Let* $1 < p, u \leq \infty$ *and* $0 < q \leq \infty$. *Then, for all sequences* $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$,

$$\left\| \left(\sum_{j=1}^\infty M f_j{}^u \right)^{\frac{1}{u}} \right\|_{L^{p,q}} \lesssim \left\| \left(\sum_{j=1}^\infty |f_j|^u \right)^{\frac{1}{u}} \right\|_{L^{p,q}}.$$

According to [10], the case of $1 < q \leq \infty$ in Theorem 2.10 is obtained. However, the case of $0 < q \leq 1$ is not well understood. Thus, what is new in Theorem 2.10 is the case of $0 < q \leq 1$.

To prove Theorem 2.10, we invoke a result from the textbook of Bergh and Löfström [4]. We denote by $L^{p,q}(\ell^u, \mathbb{R}^n)$ the set of all sequences $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$ for which

$$\|\{f_j\}_{j=1}^\infty\|_{L^{p,q}(\ell^u)} := \left\| \left(\sum_{j=1}^\infty |f_j|^u \right)^{\frac{1}{u}} \right\|_{L^{p,q}} < \infty.$$

The space $L^p(\ell^u, \mathbb{R}^n)$ represents $L^{p,p}(\ell^u, \mathbb{R}^n)$.

To prove Theorem 2.10, we will use the real interpolation technique.

Lemma 2.11 ([4, Theorem 5.3.1])**.** *Let p_0, p_1, q, $u \in (0,\infty]$ and $0<\eta<1$ satisfy $p_0 \neq p_1$. Define $p \in (0,\infty]$ by*

$$\frac{1}{p}=\frac{1-\eta}{p_0}+\frac{\eta}{p_1}. \tag{2.6}$$

Then,

$$(L^{p_0}(\ell^u,\mathbb{R}^n),L^{p_1}(\ell^u,\mathbb{R}^n))_{\eta,q}\cong L^{p,q}(\ell^u,\mathbb{R}^n)$$

with equivalence of norms.

Proof of Theorem 2.10. We resort to a technique from [14]. Fix $f \in L^0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ for a while. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we have

$$Mf(x)=\sup_{\substack{y\in\mathbb{Q}^n,\\ r\in\mathbb{Q}\cap(0,\infty)}}\chi_{Q(y,r)}(x)m_{Q(y,r)}(|f|).$$

Let $r_1,r_2,\dots$ be an enumeration of $\mathbb{Q}\cap(0,\infty)$, and let $y_1,y_2,\dots$ be that of $\mathbb{Q}^n$. Then,

$$Mf(x)=\lim_{J\to\infty}\max_{k,l\in\{1,2,\dots,J\}}\chi_{Q(y_k,r_l)}(x)m_{Q(y_k,r_l)}(|f|).$$

Here and below, we fix such enumerations and write

$$M_Jf(x):=\max_{k,l\in\{1,2,\dots,J\}}\chi_{Q(y_k,r_l)}(x)m_{Q(y_k,r_l)}(|f|)$$

for each $J \in \mathbb{N}$. Using Fatou's lemma and the Fatou property for the Lorentz quasi-norm $\|\cdot\|_{L^{p,q}}$ (see Lemma 2.7), we need only show that

$$\|\{M_Jf_j\}_{j=1}^{\infty}\|_{L^{p,q}(\ell^u)}\le C\|\{f_j\}_{j=1}^{\infty}\|_{L^{p,q}(\ell^u)} \tag{2.7}$$

with constant C independent of J.

By the definition of the operator M_J, we can find $k(x),l(x)\in\{1,2,\dots,J\}$ such that

$$M_Jf(x)=\chi_{Q(y_{k(x)},r_{l(x)})}(x)m_{Q(y_{k(x)},r_{l(x)})}(|f|). \tag{2.8}$$

We may assume that such $(k(x),l(x))$ is the smallest couple in the lexicographic order of $\{1,2,\dots,J\}^2$ among (k,l) satisfying (2.8), so the mapping $x\mapsto(k(x),l(x))$ is measurable. Write

$$E_{k,l}(f):=\{x\in\mathbb{R}^n\ :\ k(x)=k,\ l(x)=l\}\quad((k,l)\in\{1,2,\dots,J\}^2).$$

Then by the definition of $E_{k,l}(f)$, we have

$$M_Jf(x)=\sum_{k,l=1}^{J}\chi_{E_{k,l}(f)\cap Q(y_k,r_l)}(x)m_{Q(y_k,r_l)}(|f|).$$

We fix parameters $p_0 \in (1,p)$, $p_1 \in (p,\infty)$, and $\eta \in (0,1)$ satisfying (2.6). Write

$$\Phi(\{h_j\}_{j=1}^\infty) = \{\Phi_j(h_j)\}_{j=1}^\infty := \left\{ \sum_{k,l=1}^{J} \chi_{E_{k,l}(f_j)\cap Q(y_k,r_l)} m_{Q(y_k,r_l)}(h_j) \right\}_{j=1}^\infty$$

for $\{h_j\}_{j=1}^\infty \subset L^1_{\rm loc}(\mathbb{R}^n)$. Because Φ is a linear operator and $|\Phi_j(h_j)| \le Mh_j$, Φ is bounded on $L^{p_0}(\ell^u,\mathbb{R}^n)$ and $L^{p_1}(\ell^u,\mathbb{R}^n)$. Consequently, thanks to Lemma 2.11, Φ is bounded on $L^{p,q}(\ell^u,\mathbb{R}^n)$, that is, (2.7) holds. The proof of Theorem 2.10 is therefore complete. □

Let $0<\eta<\infty$ and $0<\theta\le\infty$. For a measurable function f defined on $\mathbb{R}^n$, define a function $M^{(\eta,\theta)}f$ by

$$M^{(\eta,\theta)}f(x) := \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} \chi_Q(x)\frac{\|f\chi_Q\|_{L^{\eta,\theta}}}{\|\chi_Q\|_{L^{\eta,\theta}}}, \quad x\in\mathbb{R}^n.$$

When $\theta=\eta$, we have $M^{(\eta,\eta)} = M^{(\eta)}$. The boundedness of the operator $M^{(\eta,\theta)}$ acting on Lorentz spaces is used in the proof of Theorem 3.17 below.

Proposition 2.12. [22, Proposition 2] *Let $0<p,q,\theta\le\infty$ and $0<\eta<\infty$. If $\eta<p$, then $M^{(\eta,\theta)}$ is bounded on $L^{p,q}(\mathbb{R}^n)$.*

Proof. Due to the Hölder inequality for Lorentz quasi-norms (see Lemma 2.8), for $\tilde\eta\in(\eta,p)$, we have

$$M^{(\eta,\theta)}f \lesssim M^{(\tilde\eta)}f.$$

Therefore, by the $L^{p,q}(\mathbb{R}^n)$-boundedness of $M^{(\tilde\eta)}$ (see Proposition 2.9), we obtain the result. □

2.2 Morrey spaces

We verify the $\mathcal{M}^p_q(\mathbb{R}^n)$-norm of the special indicator functions of subsets related to the Cantor dust.

Example 2.13 ([49, Example 11])**.** Let $0<q<p<\infty$, and let $R>1$ satisfy $(1+R)^{n/p-n/q}2^{n/q}=1$. We define

$$F_j := \begin{cases} [0,1]^n, & j=0, \\ \left\{ y + \sum_{k=1}^{j} R(1+R)^{k-1}a_k \,:\, \{a_k\}_{k=1}^j \subset \{0,1\}^n,\, y\in[0,1]^n \right\}, & j\in\mathbb{N} \end{cases}$$

for each $j\in\mathbb{N}_0$, and we set

$$F := \bigcup_{j\in\mathbb{N}_0} F_j. \tag{2.9}$$

Then,

$$F_j = F\cap[0,(1+R)^j]^n, \quad \|\chi_F\|_{\mathcal{M}^p_q} \sim 1. \tag{2.10}$$

Similar to Lemma 2.2, it is known that the dilation property for Morrey quasi-norms holds.

Lemma 2.14 ([49, Theorem 17]). *Let $0 < q \le p < \infty$ and $t \in (0, \infty)$. Then*

$$\|f(t\cdot)\|_{\mathcal{M}^p_q} = t^{-\frac{n}{p}} \|f\|_{\mathcal{M}^p_q}$$

for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

The following proposition is fundamental.

Proposition 2.15. *Let $0 < q \le p < \infty$. If $q \ge 1$, then $\mathcal{M}^p_q(\mathbb{R}^n)$ is a Banach space. Meanwhile, if $q < 1$, then $\mathcal{M}^p_q(\mathbb{R}^n)$ is a quasi-Banach space.*

We recall the inclusion property.

Proposition 2.16. *The following assertions hold:*

(1) *If $0 < p < \infty$, then*

$$\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

(2) *If $0 < q_2 < q_1 \le p < \infty$, then the embedding*

$$\mathcal{M}^p_{q_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2}(\mathbb{R}^n)$$

holds and is proper.

(3) [59, p. 136] *If $0 < q < p < \infty$, then the embedding*

$$\mathrm{W}L^p(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_q(\mathbb{R}^n)$$

holds and is proper.

(4) *If $1 \le p < \infty$, then there exists a sufficiently large number $N \in \mathbb{N}$ such that*

$$|\langle f, \varphi \rangle| \lesssim \|f\|_{\mathcal{M}^p_1} \cdot p_N(\varphi) \tag{2.11}$$

for all $f \in \mathcal{M}^p_1(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. In particular, the embedding

$$\mathcal{M}^p_1(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

holds.

Proof. The proofs of (1) and (2) follow easily from Definition 1.3. In addition, refer to [59] and Proposition 3.5 later for a detailed proof of (3).

For convenience, we give the proof of (4) have. We estimate

$$
\begin{aligned}
|\langle f,\varphi\rangle| &\le \int_{\mathbb{R}^n} |f(x)\varphi(x)|\,\mathrm{d}x \le \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^n} p_n(\varphi)\,\mathrm{d}x \\
&\le \left(\int_{[-1,1]^n} |f(x)|\,\mathrm{d}x + \sum_{j=1}^{\infty} \frac{1}{(\sqrt{n}2^{j-1})^n} \int_{[-2^j,2^j]^n} |f(x)|\,\mathrm{d}x \right) p_n(\varphi) \\
&\lesssim \left(\|f\|_{\mathcal{M}^p_1} + \frac{2^{\left(2-\frac{1}{p}\right)n}}{\sqrt{n}^n} \sum_{j=1}^{\infty} 2^{-\frac{jn}{p}} \|f\|_{\mathcal{M}^p_1} \right) p_n(\varphi) \\
&= \left(1 + \frac{2^{\left(2-\frac{1}{p}\right)n}}{\sqrt{n}^n} \frac{2^{-\frac{n}{p}}}{1-2^{-\frac{n}{p}}} \right) \|f\|_{\mathcal{M}^p_1} \cdot p_n(\varphi).
\end{aligned}
$$

Then, (2.11) is proved. □

As the following lemma shows, Morrey spaces can be embedded into weighted Lebesgue spaces. Lemma 2.17 is a starting point for us to consider weighted Hardy spaces in Section 1.3.

Lemma 2.17. *Let* $1 \le p < \tau < \infty$*. Then,*

$$
\mathcal{M}^p_1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n, (M\chi_{[-1,1]^n})^{\frac{1}{\tau}}).
$$

Proof. The proof is similar to [49, Proposition 285]. □

The Hardy-Littlewood maximal operator is bounded, as the following proposition shows.

Proposition 2.18 ([7])**.** *Let* $1 \le q \le p < \infty$*. Then, the following assertions hold:*

(1) *For all* $f \in \mathcal{M}^p_1(\mathbb{R}^n)$,

$$
\|Mf\|_{\mathrm{W}\mathcal{M}^p_1} \lesssim \|f\|_{\mathcal{M}^p_1}.
$$

(2) *If* $1 < q \le p < \infty$*, for all* $f \in \mathcal{M}^p_q(\mathbb{R}^n)$,

$$
\|Mf\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}.
$$

We can extend Proposition 2.18 to the vector-valued setting.

Theorem 2.19 ([53, Theorem 2.4] and [58, Lemma 2.5])**.** *Let* $1 < q \le p < \infty$ *and* $1 < u \le \infty$*. Then,*

$$
\left\| \left(\sum_{j=1}^{\infty} M f_j{}^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q}
$$

for all sequences $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$.

Chapter 3

Morrey-Lorentz spaces

Having introduced fundamental facts regarding Lorentz spaces and Morrey spaces, we now consider Morrey-Lorentz spaces according to Definition 1.4.

Chapter 3 contains the characterizations of Morrey-Lorentz spaces under the theory of function spaces, which has been investigated by many authors. We introduce fundamental properties in Section 3.1, predual spaces in Section 3.2, the boundedness of the Hardy-Littlewood maximal operator and its vector-valued extension in Section 3.3, the boundedness of the fractional integral operator and maximal operators in Section 3.4, and the atomic decomposition in Section 3.5. For convenience, we provide the proofs of all statements in Section 3.1, the boundedness of the Hardy-Littlewood maximal operator given in Theorem 3.13 in Section 3.3 and fractional operators given in Proposition 3.15 and 3.16 in Section 3.4. The results on predual spaces of Morrey-Lorentz spaces given in Section 3.2 are already known. The Fefferman-Stein inequality given in Theorem 3.13 in Section 3.3 and atomic decompositions for Morrey-Lorentz spaces given in Theorems 3.17 and 3.20 in Section 3.5 are our new results.

3.1 Fundamental properties

Although we cannot calculate $\|\chi_E\|_{\mathcal{M}^p_{q,r}}$ for all measurable subsets E, we can do so for any cube E.

Proposition 3.1. *Let $0 < q \le p < \infty$ and $0 < r \le \infty$, and let $Q \in \mathcal{Q}(\mathbb{R}^n)$. Then*

$$\|\chi_Q\|_{\mathcal{M}^p_{q,r}} = \left(\frac{q}{r}\right)^{\frac{1}{r}} |Q|^{\frac{1}{p}}.$$

Proof. By Lemma 2.1,

$$\|\chi_Q\|_{\mathcal{M}^p_{q,r}} = \left(\frac{q}{r}\right)^{\frac{1}{r}} \sup_{R\in\mathcal{Q}(\mathbb{R}^n)} |R|^{\frac{1}{p}-\frac{1}{q}}|Q\cap R|^{\frac{1}{q}}.$$

Then, combining the estimates

$$\sup_{R\in\mathcal{Q}(\mathbb{R}^n)} |R|^{\frac{1}{p}-\frac{1}{q}}|Q\cap R|^{\frac{1}{q}} \le \sup_{R\in\mathcal{Q}(\mathbb{R}^n)} |Q\cap R|^{\frac{1}{p}-\frac{1}{q}}|Q\cap R|^{\frac{1}{q}} = |Q|^{\frac{1}{p}}$$

and

$$\sup_{R\in\mathcal{Q}(\mathbb{R}^n)} |R|^{\frac{1}{p}-\frac{1}{q}}|Q\cap R|^{\frac{1}{q}} \ge \sup_{R\in\mathcal{Q}(\mathbb{R}^n),\, R\subset Q} |R|^{\frac{1}{p}-\frac{1}{q}}|R|^{\frac{1}{q}} = |Q|^{\frac{1}{p}},$$

we obtain the desired result. □

Similar to Lemma 2.2, the dilation property is obtained for Morrey-Lorentz quasi-norms.

Lemma 3.2. *Let $0 < q \le p < \infty$, $0 < r \le \infty$, and $t \in (0,\infty)$. Then,*

$$\|f(t\cdot)\|_{\mathcal{M}^p_{q,r}} = t^{-\frac{n}{p}}\|f\|_{\mathcal{M}^p_{q,r}}$$

for all $f \in \mathcal{M}^p_{q,r}(\mathbb{R}^n)$.

Proof. Fix $Q = Q(x_0, r_0) \in \mathcal{Q}(\mathbb{R}^n)$. Observe that

$$Q = \{x \in \mathbb{R}^n : tx \in Q(tx_0, tr_0)\}.$$

Then using Lemma 2.2, we have

$$\begin{aligned} |Q|^{\frac{1}{p}-\frac{1}{q}}\|f(t\cdot)\chi_Q\|_{L^{q,r}} &= |Q|^{\frac{1}{p}-\frac{1}{q}} \cdot t^{-\frac{n}{q}}\|f\chi_{Q(tx_0,tr_0)}\|_{L^{q,r}} \\ &= t^{-\frac{n}{p}} \cdot |Q(tx_0, tr_o)|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{Q(tx_0,tr_0)}\|_{L^{q,r}}, \end{aligned}$$

as required. □

We discuss equivalence of norms obtained by the geometry of the underlying spaces.

Proposition 3.3. *Let $0 < q \le p < \infty$ and $0 < r \le \infty$. Then*

$$\|f\|_{\mathcal{M}^p_{q,r}} \sim \|f\|^{\rm dyadic}_{\mathcal{M}^p_{q,r}} \sim \|f\|^{\rm ball}_{\mathcal{M}^p_{q,r}}$$

for all $f \in \mathcal{M}^p_{q,r}(\mathbb{R}^n)$, where

$$\|f\|^{\rm dyadic}_{\mathcal{M}^p_{q,r}} := \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_Q\|_{L^{q,r}}, \quad \|f\|^{\rm ball}_{\mathcal{M}^p_{q,r}} := \sup_{B\in\mathcal{B}(\mathbb{R}^n)} |B|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_B\|_{L^{q,r}}.$$

We omit the proof of Proposition 3.3. In particular, the case of $q = r$ in Proposition 3.3 is discussed in [49, Remark 1 (1)].

Importantly, Morrey-Lorentz spaces are normable as the following proposition shows; the space $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ inherits its normability from $L^{q,r}(\mathbb{R}^n)$ (see Proposition 2.3).

Proposition 3.4. *If $1 < q \le p < \infty$ and $1 \le r \le \infty$, then $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ is normable. More precisely, if we set*

$$\|f\|^{\dagger}_{\mathcal{M}^p_{q,r}} := \sup_{Q \in \mathcal{Q}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|^{\dagger}_{L^{q,r}}$$

for $f \in L^0(\mathbb{R}^n)$, then we have

$$\|f\|^{\dagger}_{\mathcal{M}^p_{q,r}} \sim \|f\|_{\mathcal{M}^p_{q,r}},$$

and $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ is a Banach space under the norm $\|\cdot\|^{\dagger}_{\mathcal{M}^p_{q,r}}$.

We now prove some fundamental embedding relations.

Proposition 3.5 ([44, Theorem 3.1])**.** *The following assertions hold:*

(1) *If $0 < q \le p < \infty$ and $0 < r_1 \le r_2 \le \infty$, then*

$$\mathcal{M}^p_{q,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q,r_2}(\mathbb{R}^n).$$

(2) *If $0 < q_2 < q_1 \le p < \infty$ and $0 < r_1, r_2 \le \infty$, then*

$$\mathcal{M}^p_{q_1,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2,r_2}(\mathbb{R}^n).$$

Proof. (1) is trivial from Proposition 2.4. To prove (2), by (1) and Lemmas 2.1 and 2.8, it suffices to show that

$$\mathcal{M}^p_{q_1,\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{\tilde{q}_2}(\mathbb{R}^n)$$

for $\tilde{q}_2 \in (q_2, q_1)$.

Fix $Q \in \mathcal{Q}(\mathbb{R}^n)$. By the Layer-Cake formula (see, e.g., [15, Proposition 1.1.4]),

$$\begin{aligned} \|f\chi_Q\|^{\tilde{q}_2}_{L^{\tilde{q}_2}} &= \tilde{q}_2 \int_0^\infty \lambda^{\tilde{q}_2-1} |\{x \in Q \,:\, |f(x)| > \lambda\}| \,\mathrm{d}\lambda \\ &\le \tilde{q}_2 \int_0^\infty \lambda^{\tilde{q}_2-1} \min(|Q|, \lambda^{-q_1} \|f\chi_Q\|^{q_1}_{L^{q_1,\infty}}) \,\mathrm{d}\lambda \\ &= \frac{q_1}{q_1-\tilde{q}_2} |Q|^{1-\frac{\tilde{q}_2}{q_1}} \|f\chi_Q\|^{\tilde{q}_2}_{L^{q_1,\infty}}. \end{aligned}$$

We conclude that

$$|Q|^{\frac{1}{p}-\frac{1}{\tilde{q}_2}} \|f\chi_Q\|_{L^{\tilde{q}_2}} \le \left(\frac{q_1}{q_1-\tilde{q}_2}\right)^{\frac{1}{\tilde{q}_2}} |Q|^{\frac{1}{p}-\frac{1}{q_1}} \|f\chi_Q\|_{L^{q_1,\infty}} \le \left(\frac{q_1}{q_1-\tilde{q}_2}\right)^{\frac{1}{\tilde{q}_2}} \|f\|_{\mathcal{M}^p_{q_1,\infty}}.$$

□

The proper embedding $\mathcal{M}^p_{q_1,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2,r_2}(\mathbb{R}^n)$ in Proposition 3.5 (2) is already known.

Remark 3.6. When $0 < q_2 < q_1 < \infty$ and $0 < r_1, r_2 \le \infty$, by

$$\mathcal{M}^p_{q_1,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{\frac{2q_1+q_2}{3}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{\frac{q_1+2q_2}{3}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2,r_2}(\mathbb{R}^n),$$

the embedding $\mathcal{M}^p_{q_1,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2,r_2}(\mathbb{R}^n)$ is proper.

Embeddings as in Proposition 3.5 (1) are proper.

Theorem 3.7. *Let $0 < q < p < \infty$ and $0 < r < \tilde{r} \le \infty$. Then, the embedding*

$$\mathcal{M}^p_{q,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q,\tilde{r}}(\mathbb{R}^n)$$

is proper.

A direct consequence of Theorem 3.7 is that the Morrey-Lorentz scale enjoys diversity.

To prove Theorem 3.7, we use the following lemma:

Lemma 3.8. *Let $0 < q < p < \infty$. Set F as in Example 2.13 (2.9) above and*

$$V_k := \begin{cases} \emptyset, & k = 0, \\ \{x \in \mathbb{R}^n \,:\, (1+R)^{-k}x \in F\}, & k \in \mathbb{N}, \end{cases} \tag{3.1}$$

and define

$$f := \sum_{k=1}^{\infty} a_k \chi_{V_k \setminus V_{k-1}},$$

where $\{a_k\}_{k=1}^{\infty}$ is a non-increasing sequence. Then for any $r_0 \in (0, \infty]$,

$$\|f\|_{\mathcal{M}^p_{q,r_0}} \sim_{p,q} \sup_{j \in \mathbb{N}_0} |[0,(1+R)^j]^n|^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}},$$

where the implicit constant in "$\sim_{p,q}$" is independent of r_0.

Proof. It is clear that

$$\|f\|_{\mathcal{M}^p_{q,r_0}} \ge \sup_{j \in \mathbb{N}_0} |[0,(1+R)^j]^n|^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}}.$$

If $Q \in \mathcal{Q}(\mathbb{R}^n)$ satisfies $|Q| \le 1$, then by the monotonicity of $\{a_k\}_{k=1}^{\infty}$,

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L^{q,r_0}} \le |[0,1]^n|^{\frac{1}{p}-\frac{1}{q}} \|f\chi_{[0,1]^n}\|_{L^{q,r_0}}. \tag{3.2}$$

Meanwhile, we fix $j \in \mathbb{N}$ and suppose that $Q \in \mathcal{Q}(\mathbb{R}^n)$ satisfies $(1+R)^{(j-1)n} < |Q| \le (1+R)^{jn}$. Note that for each $k \in \mathbb{N}$,

$$V_k = \bigcup_{l=1}^{\infty} \bigcup_{\{c_l\}_{l=1}^{\infty}} (R^l c_l + V_k^j),$$

where we take the above union, $\bigcup_{\{c_l\}_{l=1}^\infty}$, over all sequences $\{c_l\}_{l=1}^\infty$ satisfying that $c_l \in \{0,(1+R)^j\}^n$ for all $l \in \mathbb{N}$, and $\{c_l\}_{l=1}^\infty \in (\ell^1(\mathbb{R}^n))^n$. Thus, by simple geometric observation, we may consider $Q \in \mathcal{Q}(\mathbb{R}^n)$ such that

$$Q \cap (R^{l_0}c_{l_0} + V_k^j) \neq \emptyset$$

for some unique $l_0 \in \mathbb{N}$. Choosing $Q' \in \mathcal{Q}(\mathbb{R}^n)$ as the smallest cube containing $\bigcup_{c\in\{0,(1+R)^j\}^n}(R^{l_0}c + V_k^j)$, we have

$$Q \subset Q', \quad |Q| \sim |Q'| = |[0,(1+R)^{j+1}]^n|.$$

Here, we remark that Q' is independent of $k \in \mathbb{N}$. Then

$$\|f\chi_Q\|_{L^{q,r_0}} \le \|f\chi_{Q'}\|_{L^{q,r_0}}.$$

Meanwhile, by the monotonicity of $\{k^{-1/r}(1+R)^{-nk/p}\}_{k=1}^\infty$,

$$\|f\chi_{Q'}\|_{L^{q,r_0}} \le \|f\chi_{[0,(1+R)^{j+1}]^n}\|_{L^{q,r_0}}.$$

It follows that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_Q\|_{L^{q,r_0}} \le |[0,(1+R)^{j-1}]^n|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{[0,(1+R)^{j+1}]^n}\|_{L^{q,r_0}}. \tag{3.3}$$

Combining the estimates of (3.2) and (3.3), we conclude that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_Q\|_{L^{q,r_0}} \le (1+R)^{-\frac{2n}{p}+\frac{2n}{q}} \sup_{j\in\mathbb{N}_0} |[0,(1+R)^j]^n|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}}$$

for all $Q \in \mathcal{Q}(\mathbb{R}^n)$. □

Proof of Theorem 3.7. Set F and $\{V_k\}_{k=0}^\infty$ as in Example 2.13 (2.9) and in Lemma 3.8 (3.1) above, respectively. Then by (2.10), we verify that the function

$$f := \begin{cases} \displaystyle\sum_{k=1}^\infty \frac{1}{k^{\frac{1}{r}}(1+R)^{\frac{nk}{p}}}\chi_{V_k\setminus V_{k-1}}, & \tilde{r} < \infty, \\ \displaystyle\sum_{k=1}^\infty \frac{1}{(1+R)^{\frac{nk}{p}}}\chi_{V_k\setminus V_{k-1}} \sim \sup_{k\in\mathbb{N}} \frac{\chi_F((1+R)^{-k}\cdot)}{\|\chi_F((1+R)^{-k}\cdot)\|_{\mathcal{M}^p_q}}, & \tilde{r} = \infty \end{cases}$$

belongs to $\mathcal{M}^p_{q,\tilde{r}}(\mathbb{R}^n) \setminus \mathcal{M}^p_{q,r}(\mathbb{R}^n)$.

First, we prove the case of $\tilde{r} < \infty$. By the definitions of Morrey-Lorentz quasi-norms and the function f,

$$\sup_{j\in\mathbb{N}} |[0,(1+R)^j]^n|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}} = \|f\|_{\mathcal{M}^p_{q,r_0}}$$

for all $r_0 \in (0,\infty)$. Here, setting

$$V_k^j := V_k \cap [0,(1+R)^j]^n, \quad j,k \in \mathbb{N},$$

we see that

$$V_k^j = \emptyset$$

for all $k \in \mathbb{N} \cap (j,\infty)$, and

$$|V_k^j| = \|\chi_F((1+R)^{-k}\cdot)\chi_{[0,(1+R)^j)]^n}\|_{L^1} = (1+R)^{nk}\|\chi_{F_{j-k}}\|_{L^1} = (1+R)^{nk}2^{(j-k)n}$$

for all $k \in \mathbb{N} \cap [1,j]$. According to [15, Example 1.4.2],

$$\left(f\chi_{[0,(1+R)^j]^n}\right)^* = \left(\sum_{k=1}^{j} \frac{1}{k^{\frac{1}{r}}(1+R)^{\frac{nk}{p}}}\chi_{V_k^j \setminus V_{k-1}^j}\right)^* = \sum_{k=1}^{j} \frac{1}{k^{\frac{1}{r}}(1+R)^{\frac{nk}{p}}}\chi_{[|V_{k-1}^j|,|V_k^j|)}.$$

for $j \in \mathbb{N}$. It follows that

$$\begin{aligned}
\|f\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}}^{r_0} &= \int_0^\infty \left[t^{\frac{1}{q}}\sum_{k=1}^{j} \frac{1}{k^{\frac{1}{r}}(1+R)^{\frac{nk}{p}}}\chi_{[|V_{k-1}^j|,|V_k^j|)}\right]^{r_0} \frac{\mathrm{d}t}{t} \\
&\sim \frac{\{(1+R)^n 2^{(j-1)n}\}^{\frac{r_0}{q}}}{(1+R)^{\frac{nr_0}{p}}} + \sum_{k=2}^{j} \frac{\{(1+R)^{nk}2^{(j-k)n}\}^{\frac{r_0}{q}} - \{(1+R)^{n(k-1)}2^{(j-k+1)n}\}^{\frac{r_0}{q}}}{k^{\frac{r_0}{r}}(1+R)^{\frac{nkr_0}{p}}} \\
&\sim \sum_{k=1}^{j} \frac{2^{\frac{jnr_0}{q}}}{k^{\frac{r_0}{r}}}
\end{aligned}$$

for all $j \in \mathbb{N} \setminus \{1\}$ and $r_0 \in (r,\infty)$. Hence,

$$\|f\|_{\mathcal{M}^p_{q,r_0}} \sim \left(\sum_{k=1}^{\infty} \frac{1}{k^{\frac{r_0}{r}}}\right)^{\frac{1}{r_0}} = \begin{cases} \infty, & r_0 = r, \\ \zeta\left(\dfrac{r_0}{r}\right)^{\frac{1}{r_0}}, & r_0 \in (r,\infty), \end{cases}$$

where $\zeta(s)$, $s > 1$, is the Riemann zeta function. This proves that

$$f \in \mathcal{M}^p_{q,\tilde{r}}(\mathbb{R}^n) \setminus \mathcal{M}^p_{q,r}(\mathbb{R}^n).$$

Next, we prove the case of $\tilde{r} = \infty$. Similar to the approach for the case of $\tilde{r} < \infty$, we have

$$\|f\chi_{[0,(1+R)^j]^n}\|_{L^{q,r}}^r \sim \sum_{k=1}^{j} 2^{\frac{jnr}{q}} = j2^{\frac{jnr}{q}}$$

for all $j \in \mathbb{N}$, and hence,

$$\|f\|_{\mathcal{M}^p_{q,r}} = \infty.$$

Meanwhile, as mentioned in [49, Example 17], we see that

$$\|f\|_{\mathcal{M}^p_{q,\infty}} = 1.$$

We finish the proof of Theorem 3.7. □

With Theorem 3.7, we can characterize the condition under which $\mathcal{M}^{p_0}_{q_0,r_0}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^{p_1}_{q_1,r_1}(\mathbb{R}^n)$ holds.

Theorem 3.9. *For $0 < q_i \le p_i < \infty$, $0 < r_i \le \infty$ and $i = 0, 1$,*

$$\mathcal{M}^{p_0}_{q_0,r_0}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^{p_1}_{q_1,r_1}(\mathbb{R}^n) \tag{3.4}$$

if and only if any one of the following conditions holds:

(1) $p_0 = p_1$ *and* $q_0 > q_1$ *or*

(2) $p_0 = p_1$, $q_0 = q_1$, *and* $r_0 \le r_1$.

Proof. The "only if" part follows from Proposition 3.5. Thus, we need only verify the "if" part.

Assuming (3.4), we have

$$\|f(t\cdot)\|_{\mathcal{M}^{p_0}_{q_0,r_0}} \gtrsim \|f(t\cdot)\|_{\mathcal{M}^{p_1}_{q_1,r_1}}$$

for all $f \in \mathcal{M}^{p_0}_{q_0,r_0}(\mathbb{R}^n)$ and $t \in (0, \infty)$, where the implicit constant appearing in "$\gtrsim$" is independent of f and t. By the dilation property for $\|\cdot\|_{\mathcal{M}^{p_0}_{q_0,r_0}}$ and $\|\cdot\|_{\mathcal{M}^{p_1}_{q_1,r_1}}$ (see Lemma 3.2),

$$\|f\|_{\mathcal{M}^{p_0}_{q_0,r_0}} \gtrsim t^{\frac{n}{p_0}-\frac{n}{p_1}} \|f\|_{\mathcal{M}^{p_1}_{q_1,r_1}}.$$

When we take the limits $t \to 0$ and ∞, $t^{\frac{n}{p_0}-\frac{n}{p_1}}$ must remain bounded. Thus, $p_0 = p_1$.

Next, we suppose that $p := p_0 = p_1$ and (3.4). Then, by the strict monotonicity for the embedding of Morrey-Lorentz spaces, $q_0 \ge q_1$ (see Remark 3.6 for details).

Finally, we assume that $p := p_0 = p_1$, $q := q_0 = q_1$, and (3.4). The strict monotonicity for the embedding of Morrey-Lorentz spaces (see Theorem 3.7) gives $r_0 \le r_1$ again. □

Although the norm of $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ is not absolutely continuous, we still have its Fatou property.

Lemma 3.10 (Fatou property for Morrey-Lorentz space)**.** *Let $0 < q \le p < \infty$ and $0 < r \le \infty$, and let $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$ be a nonnegative collection such that $f = \lim_{j\to\infty} f_j$ exists a.e. Then, we have*

$$\|f\|_{\mathcal{M}^p_{q,r}} \le \liminf_{j\to\infty} \|f_j\|_{\mathcal{M}^p_{q,r}}.$$

Proof. For each $Q \in \mathcal{Q}(\mathbb{R}^n)$,

$$f_j \chi_Q \to f \chi_Q \quad \text{a.e.},$$

and therefore, by Lemma 2.7,

$$\|f\chi_Q\|_{L^{q,r}} \le \liminf_{j\to\infty} \|f_j\chi_Q\|_{L^{q,r}}.$$

Consequently,

$$\begin{aligned}\|f\|_{\mathcal{M}^p_{q,r}} &\le \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \liminf_{j\to\infty} \|f_j\chi_Q\|_{L^{q,r}} \le \liminf_{j\to\infty} \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f_j\chi_Q\|_{L^{q,r}} \\ &= \liminf_{j\to\infty} \|f_j\|_{\mathcal{M}^p_{q,r}}.\end{aligned}$$

□

The Hölder inequality for Morrey-Lorentz quasi-norms can be obtained from that for Lorentz spaces.

Lemma 3.11. *Assume that* $0 < p, p_1, p_2, q, q_1, q_2, r, r_1, r_2 \le \infty$ *satisfies*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Then

$$\|fg\|_{\mathcal{M}^p_{q,r}} \le \frac{q_1^{\frac{1}{q_1}} q_2^{\frac{1}{q_2}}}{q^{\frac{1}{q}}} \|f\|_{\mathcal{M}^{p_1}_{q_1,r_1}} \|g\|_{\mathcal{M}^{p_2}_{q_2,r_2}}$$

for all $f \in \mathcal{M}^{p_1}_{q_1,r_1}(\mathbb{R}^n)$ *and* $g \in \mathcal{M}^{p_2}_{q_2,r_2}(\mathbb{R}^n)$. *In particular,*

$$\|fg\|_{\mathcal{M}^p_{q,r}} \le 2^{\frac{1}{q}} \|f\|_{\mathcal{M}^{p_1}_{q_1,r_1}} \|g\|_{\mathcal{M}^{p_2}_{q_2,r_2}} \tag{3.5}$$

for all $f \in \mathcal{M}^{p_1}_{q_1,r_1}(\mathbb{R}^n)$ *and* $g \in \mathcal{M}^{p_2}_{q_2,r_2}(\mathbb{R}^n)$.

Proof. Fix $Q \in \mathcal{Q}(\mathbb{R}^n)$. Using the Hölder inequality for Lorentz quasi-norms (see Lemma 2.8), we have

$$\begin{aligned}|Q|^{\frac{1}{p}-\frac{1}{q}} \|fg\chi_Q\|_{L^{q,r}} &\le \frac{q_1^{\frac{1}{q_1}} q_2^{\frac{1}{q_2}}}{q^{\frac{1}{q}}} \left(|Q|^{\frac{1}{p_1}-\frac{1}{q_1}} \|f\chi_Q\|_{L^{q_1,r_1}}\right) \left(|Q|^{\frac{1}{p_2}-\frac{1}{q_2}} \|g\chi_Q\|_{L^{q_2,r_2}}\right) \\ &\le \frac{q_1^{\frac{1}{q_1}} q_2^{\frac{1}{q_2}}}{q^{\frac{1}{q}}} \|f\|_{\mathcal{M}^{p_1}_{q_1,r_1}} \|g\|_{\mathcal{M}^{p_2}_{q_2,r_2}}.\end{aligned}$$

In addition, because $(q_1/q)^{q/q_1}(q_2/q)^{q/q_2} \le 2$, as before, we obtain (3.5). We finish the proof of Lemma 3.11. □

3.2 Predual spaces

Ferreira [13, Lemma 3.1] and Ho [31, Theorem 3.5] obtained a description of a predual space of Morrey-Lorentz spaces. Let $1 < q \le p < \infty$ and $1 < r \le \infty$. Then the predual space $\mathcal{H}^{p'}_{q',r'}(\mathbb{R}^n)$ of the Morrey-Lorentz space $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ is given by

$$\mathcal{H}^{p'}_{q',r'}(\mathbb{R}^n) = \left\{ g = \sum_{j=1}^{\infty} \mu_j b_j \ : \ \{\mu_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}), \text{ each } b_j \text{ is a } (p', q', r')\text{-block} \right\}.$$

Here, by a (p', q', r')-block, we mean an $L^{q',r'}(\mathbb{R}^n)$-function supported on a cube $Q \in \mathcal{Q}(\mathbb{R}^n)$ with $L^{q',r'}(\mathbb{R}^n)$-norm less than or equal to $|Q|^{\frac{1}{q'}-\frac{1}{p'}}$. The norm of $\mathcal{H}^{p'}_{q',r'}(\mathbb{R}^n)$ is defined by

$$\|g\|_{\mathcal{H}^{p'}_{q',r'}} = \inf \sum_{j=1}^{\infty} |\mu_j|,$$

where inf is over all admissible expressions above. As in Theorem 3.12 below, the norm equivalence

$$\|f\|_{\mathcal{M}^p_{q,r}} \sim \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| \, \mathrm{d}x \ : \ \|g\|_{\mathcal{H}^{p'}_{q',r'}} = 1 \right\}$$

is obtained. In particular, many authors have obtained the predual space of the Lorentz space $L^{p,r}(\mathbb{R}^n)$, Morrey space $\mathcal{M}^p_q(\mathbb{R}^n) = \mathcal{M}^p_{q,q}(\mathbb{R}^n)$, and weak Morrey space $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n) = \mathcal{M}^p_{q,\infty}(\mathbb{R}^n)$ as $L^{p',r'}(\mathbb{R}^n) = \mathcal{H}^{p'}_{p',r'}(\mathbb{R}^n)$ (Hunt [32, (2.7)]), and $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n) = \mathcal{H}^{p'}_{q',q'}(\mathbb{R}^n)$ (Zorko [61, Proposition 5]) and $\mathcal{H}^{p'}_{q',1}(\mathbb{R}^n)$ (Ho [30, Theorem 3.6] and Sawano and El-Shabrawy [48, Theorem 2.5]), respectively.

Theorem 3.12 ([13,31])**.** *Let $1 < q \le p < \infty$ and $1 < r \le \infty$. Then, the following assertions hold*:

(1) *Any $f \in \mathcal{M}^p_{q,r}(\mathbb{R}^n)$ defines a continuous functional L_f by*

$$L_f : \mathcal{H}^{p'}_{q',r'}(\mathbb{R}^n) \ni g \longmapsto \int_{\mathbb{R}^n} f(x)g(x) \, \mathrm{d}x \in \mathbb{C}$$

on $\mathcal{H}^{p'}_{q',r'}(\mathbb{R}^n)$, and

$$\|L_f\|_{\left(\mathcal{H}^{p'}_{q',r'}\right)^*} \lesssim \|f\|_{\mathcal{M}^p_{q,r}}$$

holds.

(2) *Conversely, every continuous functional L on $\mathcal{H}^{p'}_{q',r'}(\mathbb{R}^n)$ can be realized as $L = L_f$ with some $f \in \mathcal{M}^p_{q,r}(\mathbb{R}^n)$, and*

$$\|f\|_{\mathcal{M}^p_{q,r}} \lesssim \|L\|_{\left(\mathcal{H}^{p'}_{q',r'}\right)^*}$$

holds.

(3) *The correspondence*

$$\tau : \mathcal{M}^p_{q,r}(\mathbb{R}^n) \ni f \longmapsto L_f \in \left(\mathcal{H}^{p'}_{q',r'}(\mathbb{R}^n)\right)^*$$

is an isomorphism.

3.3 Boundedness of the Hardy-Littlewood maximal operator and its vector-valued extension

We extend Proposition 2.18 to Morrey-Lorentz spaces.

Theorem 3.13. *Let $1 < q \le p < \infty$ and $0 < r \le \infty$. Then, for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$,*

$$\|Mf\|_{\mathcal{M}^p_{q,r}} \lesssim \|f\|_{\mathcal{M}^p_{q,r}}.$$

The proof of this hinges on the local/global strategy.

Proof of Theorem 3.13. Fix $Q \in \mathcal{Q}(\mathbb{R}^n)$. We decompose

$$f = f\chi_{2Q} + f\chi_{(2Q)^c} =: f_1 + f_2,$$

and using the subadditivity of M, we have

$$Mf(x) \le Mf_1(x) + Mf_2(x).$$

First, by the boundedness of M on $L^{q,r}(\mathbb{R}^n)$ (see Proposition 2.9),

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\|(Mf_1)\chi_Q\|_{L^{q,r}} \lesssim |Q|^{\frac{1}{p}-\frac{1}{q}}\|f_1\|_{L^{q,r}} \lesssim \|f\|_{\mathcal{M}^p_{q,r}}. \tag{3.6}$$

Second, a simple geometric observation shows that

$$Mf_2(x) \lesssim \sup_{R\in\mathcal{Q}(\mathbb{R}^n),\, R\supset Q} m_R(|f|) \le |Q|^{-\frac{1}{p}}\|f\|_{\mathcal{M}^p_1} \tag{3.7}$$

for all $x \in Q$, and hence, it follows from the embedding $\mathcal{M}^p_{q,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{1,1}(\mathbb{R}^n) = \mathcal{M}^p_1(\mathbb{R}^n)$ (see Proposition 3.5) and Proposition 3.1 that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\|(Mf_2)\chi_Q\|_{L^{q,r}} \lesssim \|f\|_{\mathcal{M}^p_{q,r}}. \tag{3.8}$$

Combining the two estimates of (3.6) and (3.8), we obtain the result. □

We can also extend Theorem 2.19 to Morrey-Lorentz spaces.

Theorem 3.14. *Let $1 < q \le p < \infty$, $0 < r \le \infty$, and $1 < u \le \infty$. Then, for all sequences $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$,*

$$\left\|\left(\sum_{j=1}^\infty Mf_j{}^u\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_{q,r}} \lesssim \left\|\left(\sum_{j=1}^\infty |f_j|^u\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_{q,r}}.$$

Proof. The case of $u = \infty$ can be dealt with by the use of the following pointwise inequality:

$$Mf_k(x) \le M\left[\sup_{j\in\mathbb{N}} |f_j|\right](x), \quad x \in \mathbb{R}^n$$

for each $k \in \mathbb{N}$ and Theorem 3.13. We may therefore assume that $u < \infty$. Fix $Q \in \mathcal{Q}(\mathbb{R}^n)$. We decompose

$$f_j = f_j\chi_{2Q} + f_j\chi_{(2Q)^c} =: f_{j,1} + f_{j,2}$$

for each $j \in \mathbb{N}$. Then by the subadditivity of M, it suffices to show that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left\|\left(\sum_{j=1}^{\infty} Mf_{j,\nu}{}^u\right)^{\frac{1}{u}}\chi_Q\right\|_{L^{q,r}} \lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^u\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_{q,r}} \tag{3.9}$$

for each $\nu = 1, 2$.

First, we estimate the part $\nu = 1$. By Theorem 2.10, we have

$$\begin{aligned} |Q|^{\frac{1}{p}-\frac{1}{q}}\left\|\left(\sum_{j=1}^{\infty} Mf_{j,1}{}^u\right)^{\frac{1}{u}}\chi_Q\right\|_{L^{q,r}} &\lesssim |Q|^{\frac{1}{p}-\frac{1}{q}}\left\|\left(\sum_{j=1}^{\infty}|f_{j,1}|^u\right)^{\frac{1}{u}}\right\|_{L^{q,r}} \\ &\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^u\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_{q,r}}. \end{aligned}$$

Then, (3.9) is obtained for $\nu = 1$.

Second, we estimate the part $\nu = 2$. Fix $x \in Q$. The same idea as in (3.7) gives us that

$$Mf_{j,2}(x) \lesssim \sup_{R\in\mathcal{Q}(\mathbb{R}^n),\, R\supset Q} m_R(|f_j|) \lesssim \sum_{k=1}^{\infty} m_{2^{k-1}Q}(|f_j|).$$

By Minkowski's inequality, we have

$$\left(\sum_{j=1}^{\infty} Mf_{j,2}(x)^u\right)^{\frac{1}{u}} \lesssim \sum_{k=1}^{\infty} m_{2^{k-1}Q}\left(\left(\sum_{j=1}^{\infty}|f_j|^u\right)^{\frac{1}{u}}\right) \lesssim |Q|^{-\frac{1}{p}}\left\|\left(\sum_{j=1}^{\infty}|f_j|^u\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_1}.$$

Hence, from the embedding $\mathcal{M}^p_{q,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{1,1}(\mathbb{R}^n) = \mathcal{M}^p_1(\mathbb{R}^n)$ (see Proposition 3.5) and Proposition 3.1, we conclude (3.9) for $\nu = 2$. □

3.4 Boundedness of the fractional integral and maximal operators

We prove the boundedness of I_α.

Proposition 3.15. [22, Proposition 3] *Let* $0 < \alpha < n$, $1 < q \le p < \infty$, $1 < t \le s < \infty$, *and* $0 < r, u \le \infty$. *Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.$$

If we suppose either

(1) $0 < r, u < \infty$ *and* $\dfrac{s}{p} = \dfrac{t}{q} = \dfrac{u}{r}$ *or*

(2) $r = u = \infty$ *and* $\dfrac{s}{p} = \dfrac{t}{q}$,

then we have

$$\|I_\alpha f\|_{\mathcal{M}^s_{t,u}} \lesssim \|f\|_{\mathcal{M}^p_{q,r}}$$

for all $f \in \mathcal{M}^p_{q,r}(\mathbb{R}^n)$.

Proof. To prove this proposition, we employ Hedberg's idea from [29]. Fix $x \in \mathbb{R}^n$ and $\rho > 0$. We decompose

$$f = f\chi_{B(x,\rho)} + f\chi_{B(x,\rho)^c} =: f_1 + f_2.$$

We estimate

$$\begin{aligned} |I_\alpha f_1(x)| &\le \sum_{j=1}^{\infty} \int_{2^{-j}\rho \le |x-y| < 2^{-j+1}\rho} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, \mathrm{d}y \\ &\le \sum_{j=1}^{\infty} \frac{1}{(2^{-j}\rho)^{n-\alpha}} \int_{|x-y|<2^{-j+1}\rho} |f(y)| \, \mathrm{d}y \lesssim \rho^\alpha M f(x) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |I_\alpha f_2(x)| &\le \sum_{j=1}^{\infty} \int_{2^{j-1}\rho \le |x-y| < 2^{j}\rho} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, \mathrm{d}y \\ &\le \sum_{j=1}^{\infty} \frac{1}{(2^{j-1}\rho)^{n-\alpha}} \int_{|x-y|<2^{j}\rho} |f(y)| \, \mathrm{d}y \lesssim \sum_{j=1}^{\infty} (2^{j-1}\rho)^{\alpha - \frac{n}{p}} \|f\|_{\mathcal{M}^p_1} \\ &\sim \rho^{-\frac{n}{s}} \|f\|_{\mathcal{M}^p_1}. \end{aligned} \tag{3.11}$$

Combining the estimates of (3.10) and (3.11), we have

$$|I_\alpha f(x)| \lesssim \rho^\alpha M f(x) + \rho^{-\frac{n}{s}} \|f\|_{\mathcal{M}^p_1}.$$

Because this estimate holds for all $\rho > 0$, it follows that

$$|I_\alpha f(x)| \lesssim Mf(x)^{1-\frac{p\alpha}{n}} \|f\|_{\mathcal{M}_1^p}^{\frac{p\alpha}{n}}.$$

Consequently, using the boundedness of M on $\mathcal{M}_{q,r}^p(\mathbb{R}^n)$ (see Theorem 3.13) and the embedding $\mathcal{M}_{q,r}^p(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{1,1}^p(\mathbb{R}^n) = \mathcal{M}_1^p(\mathbb{R}^n)$ (see Proposition 3.5), we conclude that

$$\|I_\alpha f\|_{\mathcal{M}_{t,u}^s} \lesssim \|Mf\|_{\mathcal{M}_{q,r}^p}^{1-\frac{p\alpha}{n}} \|f\|_{\mathcal{M}_1^p}^{\frac{p\alpha}{n}} \lesssim \|f\|_{\mathcal{M}_{q,r}^p}.$$

□

Let $0 \leq \alpha < n$. We define the fractional maximal operator M_α by

$$M_\alpha f(x) := \sup_{Q \in \mathcal{Q}(\mathbb{R}^n)} \frac{\chi_Q(x)}{\ell(Q)^{n-\alpha}} \int_Q |f(y)| \, \mathrm{d}y, \quad x \in \mathbb{R}^n$$

for $f \in L^0(\mathbb{R}^n)$.

In a way similar to the proof of Proposition 3.15, we can prove the boundedness of M_α.

Proposition 3.16. [22, Proposition 3] *Let $0 \leq \alpha < n$, $1 < q \leq p < \infty$, $1 < t \leq s < \infty$ and $0 < r, u \leq \infty$. Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.$$

If we suppose either

(1) $0 < r, u < \infty$ *and* $\dfrac{s}{p} = \dfrac{t}{q} = \dfrac{u}{r}$ *or*

(2) $r = u = \infty$ *and* $\dfrac{s}{p} = \dfrac{t}{q}$,

then we have

$$\|M_\alpha f\|_{\mathcal{M}_{t,u}^s} \lesssim \|f\|_{\mathcal{M}_{q,r}^p}$$

for all $f \in \mathcal{M}_{q,r}^p(\mathbb{R}^n)$.

Proof. The case of $\alpha = 0$ is equivalent to Theorem 3.13. In addition, combining the pointwise estimate $M_\alpha f \lesssim I_\alpha[|f|]$ and Proposition 3.15, we obtain the case of $0 < \alpha < n$. □

3.5 Atomic decomposition

The goal of this section is to prove the following synthesis result.

Theorem 3.17. *Suppose that the parameters p, q, r, s, t, v satisfy*

$$0 < q \le p < \infty, \quad 0 < r \le \infty, \quad 0 < t \le s < \infty, \quad 0 < v \le 1,$$

$$q < t, \quad p < s, \quad v < \min(q, r).$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n)$, and $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ fulfill

$$\|a_j\|_{\mathrm{W}\mathcal{M}^s_t} \le |Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} < \infty.$$

Then, $f = \sum_{j=1}^\infty \lambda_j a_j$ converges a.e. and satisfies

$$\|f\|_{\mathcal{M}^p_{q,r}} \lesssim_{p,q,r,s,t,v} \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}}. \tag{3.12}$$

In particular, $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in $L^{q,r}_{\mathrm{loc}}(\mathbb{R}^n)$ if $r < \infty$ and in $L^{p,r}(\mathbb{R}^n)$ if $p = q$ and $r < \infty$.

In this theorem, we can take the atoms $\{a_j\}_{j=1}^\infty$ from a larger space, that is, the weak Morrey space $\mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n)$. We can choose the parameter v freely.

It is possible to transplant Theorem 3.17 to Lorentz spaces and weak Morrey spaces, as follows:

Corollary 3.18. *Suppose that the parameters p, r, s, t, v satisfy*

$$0 < p < t \le s < \infty, \quad 0 < r \le \infty, \quad 0 < v \le 1, \quad v < \min(p, r).$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n)$, and $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ fulfill

$$\|a_j\|_{\mathrm{W}\mathcal{M}^s_t} \le |Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{L^{p,r}} < \infty.$$

If $v < \min(p, r)$, then $f = \sum_{j=1}^\infty \lambda_j a_j$ converges a.e. and satisfies

$$\|f\|_{L^{p,r}} \lesssim_{p,r,s,t,v} \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{L^{p,r}}.$$

In particular, $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in $L^{p,r}(\mathbb{R}^n)$ if $r < \infty$.

Corollary 3.19. *Suppose that the parameters p, q, s, t, v satisfy*

$$0 < q \le p < \infty, \quad 0 < t \le s < \infty, \quad 0 < v \le 1, \quad p < s, \quad v < q < t.$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill

$$\|a_j\|_{\mathrm{W}\mathcal{M}^s_t} \le |Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathrm{W}\mathcal{M}^p_q} < \infty.$$

Then $f = \sum_{j=1}^\infty \lambda_j a_j$ converges a.e. and satisfies

$$\|f\|_{\mathrm{W}\mathcal{M}^p_q} \lesssim_{p,q,s,t,v} \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathrm{W}\mathcal{M}^p_q}.$$

Proof of Theorem 3.17. We employ the argument from the proof of Theorem 1.1 in [33].

By the decomposition of Q_j, we may assume that each Q_j is a dyadic cube. We may assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \ge N$. In addition, let us assume that the a_j's are non-negative. By the embedding $\ell^v(\mathbb{N}) \hookrightarrow \ell^1(\mathbb{N})$ and the duality argument, we note that

$$\|f\|_{\mathcal{M}^p_{q,r}}^v \le \left\| \sum_{j=1}^\infty |\lambda_j a_j|^v \right\|_{\mathcal{M}^{\tilde{p}}_{\tilde{q},\tilde{r}}} \sim \sup \left\{ \int_{\mathbb{R}^n} \sum_{j=1}^\infty |\lambda_j a_j(x)|^v |g(x)| \, \mathrm{d}x : \|g\|_{\mathcal{H}^{\tilde{p}'}_{\tilde{q}',\tilde{r}'}} = 1 \right\},$$

where we set $\tilde{p} := p/v$, $\tilde{q} := q/v$ and $\tilde{r} := r/v$. Then, we may assume that the a_j's are non-negative and g is a non-negative $(\tilde{p}', \tilde{q}', \tilde{r}')$-block with associated dyadic cube Q. Then, we show that

$$\int_{\mathbb{R}^n} \sum_{j=1}^\infty (\lambda_j a_j(x))^v g(x) \, \mathrm{d}x \lesssim_{p,q,r,s,t,v} \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}}^v. \tag{3.13}$$

Assume first that each Q_j contains Q as a proper subset. If we group the j's such that all Q_j are identical, we can assume that each Q_j is a j-th parent of Q for each $j \in \mathbb{N}$. Then, by the Hölder inequality for Lorentz spaces (see Lemma 2.8),

$$\begin{aligned}
\int_{\mathbb{R}^n} \sum_{j=1}^\infty (\lambda_j a_j(x))^v g(x) \, \mathrm{d}x &= \sum_{j=1}^\infty \lambda_j^v \int_Q a_j(x)^v g(x) \, \mathrm{d}x \\
&\lesssim \sum_{j=1}^\infty \lambda_j^v \|a_j \chi_Q\|_{L^{t,\infty}}^v \|g\|_{L^{\tilde{q}',\tilde{r}'}} |Q|^{\frac{v}{q} - \frac{v}{t}} \\
&\le \sum_{j=1}^\infty \lambda_j^v \|a_j\|_{\mathrm{W}\mathcal{M}^s_t}^v |Q|^{-\frac{v}{s} + \frac{v}{t}} |Q|^{\frac{1}{\tilde{q}'} - \frac{1}{\tilde{p}'}} |Q|^{\frac{v}{q} - \frac{v}{t}} \\
&= \sum_{j=1}^\infty \lambda_j^v |Q_j|^{\frac{v}{s}} |Q|^{\frac{v}{p} - \frac{v}{s}}.
\end{aligned}$$

Note that by Proposition 3.1, for each $J \in \mathbb{N}$,

$$\left\| \left(\sum_{k=1}^{\infty} (\lambda_k \chi_{Q_k})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} \geq \lambda_J \|\chi_{Q_J}\|_{\mathcal{M}^p_{q,r}} \sim \lambda_J |Q_J|^{\frac{1}{p}}.$$

Consequently, it follows from the condition $p < s$ that

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} (\lambda_j a_j(x))^v g(x)\,\mathrm{d}x &\lesssim \sum_{j=1}^{\infty} |Q_j|^{\frac{v}{s}-\frac{v}{p}} |Q|^{\frac{v}{p}-\frac{v}{s}} \cdot \left\| \left(\sum_{k=1}^{\infty} (\lambda_k \chi_{Q_k})^v \right)^{\frac{1}{v}} \right\|^v_{\mathcal{M}^p_{q,r}} \\ &\lesssim \left\| \left(\sum_{k=1}^{\infty} (\lambda_k \chi_{Q_k})^v \right)^{\frac{1}{v}} \right\|^v_{\mathcal{M}^p_{q,r}}.\end{aligned}$$

Conversely, assume that Q contains each Q_j. Then, again by the Hölder inequality (see Lemma 2.8),

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} (\lambda_j a_j(x))^v g(x)\,\mathrm{d}x = \sum_{j=1}^{\infty} \lambda_j^v \int_{Q_j} a_j(x)^v g(x)\,\mathrm{d}x &\lesssim \sum_{j=1}^{\infty} \lambda_j^v \|a_j\|^v_{L^{t,\infty}} \|g\chi_{Q_j}\|_{L^{\tilde{t}',1}} \\ &\leq \sum_{j=1}^{\infty} \lambda_j \|a_j\|^v_{\mathrm{W}\mathcal{M}^s_t} |Q_j|^{-\frac{v}{s}+\frac{v}{t}} \|g\chi_{Q_j}\|_{L^{\tilde{t}',1}} \\ &\leq \sum_{j=1}^{\infty} \lambda_j^v |Q_j|^{\frac{v}{t}} \|g\chi_{Q_j}\|_{L^{\tilde{t}',1}},\end{aligned}$$

where $\tilde{t} := t/v$. Thus, in terms of the maximal operator $M^{(\tilde{t}',1)}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} (\lambda_j a_j(x))^v g(x)\,\mathrm{d}x &\leq \sum_{j=1}^{\infty} \lambda_j^v |Q_j| \cdot \inf_{y \in Q_j} M^{(\tilde{t}',1)} g(y) \\ &\leq \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j}(y))^v \right) \chi_Q(y) M^{(\tilde{t}',1)} g(y)\,\mathrm{d}y.\end{aligned}$$

Hence, we obtain (3.13) by the Hölder inequality (see Lemma 2.8) and the $L^{\tilde{q}',1}(\mathbb{R}^n)$-boundedness of the maximal operator $M^{(\tilde{t}',1)}$ (see Proposition 2.12).

It remains to check the convergence of the sum. Here, when $r < \infty$, by the estimate of (3.12), the Lebesgue convergence theorem yields

$$\left\| \left(\sum_{j=1}^{\infty} \lambda_j a_j - \sum_{j=1}^{J} \lambda_j a_j \right) \chi_R \right\|_{L^{q,r}} \leq \left\| \left(\sum_{j=J+1}^{\infty} |\lambda_j a_j| \right) \chi_R \right\|_{L^{q,r}} \to 0$$

as $J \to \infty$ for each $R \in \mathcal{Q}(\mathbb{R}^n)$. Namely, $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $L^{q,r}_{\mathrm{loc}}(\mathbb{R}^n)$. The case of $p = q$ and $r < \infty$ can be also dealt with by a similar approach. □

The next assertion concerns the decomposition of functions in $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$.

Theorem 3.20. *Let $1 < q \le p < \infty$, $0 < r \le \infty$, $K \in \mathbb{N}_0$, and $f \in \mathcal{M}^p_{q,r}(\mathbb{R}^n)$. Then, there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K^\perp(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that, for all $v > 0$,*

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} \lesssim_v \|f\|_{\mathcal{M}^p_{q,r}}. \tag{3.14}$$

Theorem 3.20 is proved by combining Proposition 4.1 and Theorem 4.6 below.

As special cases of Theorem 3.20, we obtain decomposition theorems for Lorentz spaces and weak Morrey spaces.

Corollary 3.21. *Let $1 < p < \infty$, $0 < r \le \infty$, $K \in \mathbb{N}_0$, and $f \in L^{p,r}(\mathbb{R}^n)$. Then, there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K^\perp(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that for all $v > 0$,*

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{L^{p,r}} \lesssim_v \|f\|_{L^{p,r}}.$$

Corollary 3.22. *Let $1 < q \le p < \infty$, $K \in \mathbb{N}_0$, and $f \in \mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$. Then there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K^\perp(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that for all $v > 0$,*

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathrm{W}\mathcal{M}^p_q} \lesssim_v \|f\|_{\mathrm{W}\mathcal{M}^p_q}.$$

Except for the topology of the convergence of the sums in Theorem 3.17, Theorems 3.17 and 3.20 are special cases of Theorems 4.4 and 4.6 later, respectively, which concerns the decomposition of Hardy-Morrey-Lorentz spaces. In fact, thanks to Proposition 4.1 later, $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ and $H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ are isomorphic for $1 < q \le p < \infty$ and $0 < r \le \infty$. Thus, we can apply Theorems 4.4 and 4.6 to Morrey-Lorentz spaces.

Chapter 4

Atomic decomposition for Hardy-Morrey-Lorentz spaces

In Chapter 3, we introduced our results on decompositions of the functions in Morrey-Lorentz spaces. To prove the results in Chapter 3, we address a wider framework of consider Hardy-Morrey-Lorentz spaces.

This chapter is organized as follows: In Section 4.1, we introduce Hardy-Morrey-Lorentz spaces and compare them with Morrey-Lorentz spaces. In Section 4.2, we give the atomic decomposition for Hardy-Morrey-Lorentz spaces. In Section 4.3, we provide characterizations using the grand maximal functions defined in Section 1.3 for Hardy-Morrey-Lorentz spaces. We give the proof of convergence of the atomic decompositions and the norm estimates in Sections 4.5 and 4.6, respectively. In Section 4.7, we prove Theorem 4.6.

4.1 Hardy-Morrey-Lorentz spaces

Recall that for $0 < q \leq p < \infty$ and $0 < r \leq \infty$, the *Hardy-Morrey-Lorentz space* $H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm $\|f\|_{H\mathcal{M}^p_{q,r}} := \left\|\sup_{t>0} |e^{t\Delta} f|\right\|_{\mathcal{M}^p_{q,r}}$ is finite. In addition, $H\mathcal{M}^p_{q,\infty}(\mathbb{R}^n)$ coincides with the Hardy-weak Morrey space $H\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$ introduced by Ho in [30].

Concerning $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ and $H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$, we have the following assertion:

Proposition 4.1. *Let $1 \leq q \leq p < \infty$ and $0 < r \leq \infty$.*

(1) *If $f \in \mathcal{M}^p_{q,r}(\mathbb{R}^n)$ and $q > 1$, then $f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ and $\|f\|_{H\mathcal{M}^p_{q,r}} \lesssim \|f\|_{\mathcal{M}^p_{q,r}}$.*

(2) *Assume that $q > 1$ or $q = 1 \geq r$. If $f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$, then f can be represented by a locally integrable function belonging to $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{M}^p_{q,r}} \lesssim \|f\|_{H\mathcal{M}^p_{q,r}}$.*

We compare Proposition 4.1 with existing results.

Remark 4.2. (1) It is noteworthy that the case of $q = 1 \geq r$ in this proposition covers a result of [20] as a special case of $r = 1$.

(2) It remains an open problem to determine whether $H\mathcal{M}^p_{1,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{1,r}(\mathbb{R}^n)$ and $\mathcal{M}^1_{1,r}(\mathbb{R}^n) = H^{1,r}(\mathbb{R}^n) \hookrightarrow L^{1,r}(\mathbb{R}^n)$ for $r > 1$. This is because the embedding

$$H\mathcal{M}^p_{1,r}(\mathbb{R}^n) \hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

fails. In fact, examples can be seen in $L^{1,r}(\mathbb{R}^n) \setminus L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (see Example 2.6).

To prove Proposition 4.1, we use the following lemma.

Lemma 4.3. (1) *For all* $f \in \mathcal{S}(\mathbb{R}^n)$,

$$e^{t\Delta} f(x) \to f(x) \quad as \quad t \downarrow 0. \tag{4.1}$$

(2) *Let* $1 \leq p < \infty$, *and let* $f \in L^1(\mathbb{R}^n, (M\chi_{[-1,1]^n})^{1/(p+1)})$. *Then for each* $x \in \mathbb{R}^n$ *and* $t > 0$,

$$\int_{\mathbb{R}^n} \left| \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x-y|^2}{4t} \right) f(y) \right| \mathrm{d}y < \infty$$

holds. Moreover, we have

$$|f| \leq \sup_{t>0} |e^{t\Delta} f|, \quad a.e. \tag{4.2}$$

(3) *For all* $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$e^{t\Delta} f \to f \quad in \quad \mathcal{S}'(\mathbb{R}^n).$$

Proof. (1) Because

$$|\exp(-|z|^2) f(x - 2\sqrt{t} z)| \leq \exp(-|z|^2) \|f\|_{L^\infty} \in L^1(\mathbb{R}^n),$$

by the Lebesgue convergence theorem, we have

$$e^{t\Delta} f(x) = \frac{1}{\sqrt{\pi^n}} \int_{\mathbb{R}^n} \exp(-|z|^2) f(x - 2\sqrt{t} z)\, \mathrm{d}z \to \frac{1}{\sqrt{\pi^n}} \int_{\mathbb{R}^n} \exp(-|z|^2) f(x)\, \mathrm{d}z = f(x)$$

as $t \downarrow 0$. This proves (4.1).

(2) To prove (4.2), it suffices to show that the set

$$E_k := \left\{ x \in \mathbb{R}^n \,:\, \limsup_{t \downarrow 0} |e^{t\Delta} f(x) - f(x)| > \frac{1}{k} \right\}$$

is a null set for all $k \in \mathbb{N}$ and $f \in L^1(\mathbb{R}^n, (M\chi_{[-1,1]^n})^{1/(p+1)})$. Here, set

$$w := \left(M\chi_{[-1,1]^n}\right)^{\frac{1}{p+1}},$$

fix $\varepsilon > 0$, and take $g \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\|f-g\|_{L^1(w)} < \varepsilon.$$

Because the function

$$(0,\infty) \ni \lambda \longmapsto \varphi(\lambda) := \exp(-\lambda) \in (0,\infty)$$

is positive and decreasing on $(0,\infty)$, we deduce from [11, Proposition 2.7] that

$$\sup_{t>0} |e^{t\Delta} f| = \sup_{t>0} \left| \frac{1}{\sqrt{(4\pi t)^n}} \varphi\left(-\frac{|\cdot|^2}{4t}\right) * f \right| \lesssim \frac{1}{\sqrt{\pi^n}} \|\varphi(|\cdot|^2)\|_{L^1} Mf.$$

Thus, by (4.1), we estimate

$$\begin{aligned} w(E_k) &\le w\left(\left\{\limsup_{t\downarrow 0} |e^{t\Delta}[f-g]| > \frac{1}{2k}\right\}\right) + w\left(\left\{|f-g| > \frac{1}{2k}\right\}\right) \\ &\lesssim w\left(\left\{M[f-g] > \frac{1}{2k}\right\}\right) + w\left(\left\{|f-g| > \frac{1}{2k}\right\}\right). \end{aligned}$$

Applying the weak-type boundedness of M on $L^1(\mathbb{R}^n, w)$ (see, e.g., [15, Theorem 7.1.9]) and Chebyshev's inequality, we conclude that

$$w(E_k) \lesssim_{[w]_{A_1}} 4k\|f-g\|_{L^1(w)} < 4k\varepsilon.$$

We finish the proof of Lemma 4.3 because $\varepsilon > 0$ and $w(x)\,\mathrm{d}x$ and $\mathrm{d}x$ are mutually absolutely continuous.

(3) We omit the proof of this statement. See [47, Theorem 1.35] for the discrete case. A minor modification suffices for the continuous case. The same argument applies to the Gaussian, although the Gaussian is not compactly supported. □

Proof of Proposition 4.1. (1) By Propositions 3.5 and 2.16, we have

$$f \in \mathcal{M}^p_{q,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_1(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

As described in [11, Proposition 2.7], we have a pointwise estimate $|e^{t\Delta}f| \lesssim Mf$. Because M has been shown to be bounded on Morrey-Lorentz spaces $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ (Theorem 3.13), we have $f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$.

(2) First, we assume that $q, r > 1$. Let $f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$. Then $\{e^{t\Delta}f\}_{t>0}$ is a bounded set of $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$, which admits a separable predual as we have seen in Theorem 3.12. Therefore, there exists a sequence $\{t_j\}_{j=1}^\infty$ decreasing to 0 such that

$\{e^{t_j\Delta}f\}_{j=1}^\infty$ converges to a function g in the weak-* topology of $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$. Because the weak-* topology of $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ is stronger than the topology of $\mathcal{S}'(\mathbb{R}^n)$, it follows from Lemma 4.3 (3) that $f = g \in \mathcal{M}^p_{q,r}(\mathbb{R}^n)$.

Next, we assume that $q > 1$ and $r \le 1$. Let $f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$. By the embedding $\mathcal{M}^p_{q,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q,q}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$ (see Proposition 3.5) and the fact proved immediately above, we can identify $f = g \in \mathcal{M}^p_q(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n, (M\chi_{[-1,1]^n})^{1/(p+1)})$ by Proposition 2.16 and Lemma 2.17. Then, employing Lemma 4.3 (2), we obtain

$$\|f\|_{\mathcal{M}^p_{q,r}} \le \left\| \sup_{t>0} |e^{t\Delta}f| \right\|_{\mathcal{M}^p_{q,r}} = \|f\|_{H\mathcal{M}^p_{q,r}} < \infty.$$

Finally, we verify the case of $q = 1 \ge r$. Using Lemma 2.17 and Proposition 3.5, we have

$$\mathcal{M}^p_{1,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{1,1}(\mathbb{R}^n) = \mathcal{M}^p_1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n, (M\chi_{[-1,1]^n})^{\frac{1}{p+1}}).$$

Because $(M\chi_{[-1,1]^n})^{1/(p+1)}$ is an A_1-weight (see [11, Theorem 7.7]), by Theorem 1.6,

$$H\mathcal{M}^p_{1,r}(\mathbb{R}^n) \hookrightarrow H^1(\mathbb{R}^n, (M\chi_{[-1,1]^n})^{\frac{1}{p+1}}) \hookrightarrow L^1(\mathbb{R}^n, (M\chi_{[-1,1]^n})^{\frac{1}{p+1}}). \tag{4.3}$$

Consequently, it follows from Lemma 4.3 (2) that

$$H\mathcal{M}^p_{1,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{1,r}(\mathbb{R}^n).$$

□

4.2 Atomic decomposition for Hardy-Morrey-Lorentz spaces

We generalize Theorem 3.17 to Hardy-Morrey-Lorentz spaces.

Theorem 4.4. *Suppose that the parameters p, q, r, s, t, v satisfy*

$$0 < q \le p < \infty, \quad 0 < r \le \infty, \quad 1 < t \le s < \infty, \quad 0 < v \le 1,$$

$$q < t, \quad p < s, \quad v < \min(q, r).$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n) \cap \mathcal{P}_{d_v}(\mathbb{R}^n)^\perp$ and $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ fulfill

$$\|a_j\|_{\mathrm{W}\mathcal{M}^s_t} \le |Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} < \infty.$$

Then, $f=\sum_{j=1}^{\infty}\lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{H\mathcal{M}^p_{q,r}} \lesssim_{p,q,r,s,t} \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}}. \tag{4.4}$$

If $q<1$, then we can refine Theorem 4.4 as follows.

Theorem 4.5. *Suppose that the parameters* p,q,r,s,v *satisfy*

$$0<q\le p<\infty, \quad 0<r\le\infty, \quad 1\le s<\infty, \quad 0<v\le 1,$$

$$q<1, \quad p<s, \quad v<\min(q,r).$$

Assume that $\{Q_j\}_{j=1}^{\infty}\subset\mathcal{Q}(\mathbb{R}^n)$*,* $\{a_j\}_{j=1}^{\infty}\subset\mathcal{M}^s_1(\mathbb{R}^n)\cap\mathcal{P}_{d_v}(\mathbb{R}^n)^{\perp}$*, and* $\{\lambda_j\}_{j=1}^{\infty}\subset[0,\infty)$ *fulfill*

$$\|a_j\|_{\mathcal{M}^s_1}\le|Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j)\subset Q_j, \quad \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} <\infty.$$

Then, $f=\sum_{j=1}^{\infty}\lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{H\mathcal{M}^p_{q,r}} \lesssim_{p,q,r,s} \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}}. \tag{4.5}$$

It is noteworthy that we may take $t=1$ in Theorem 4.4. The cost of this is that we must replace $\mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n)$ with $\mathcal{M}^s_1(\mathbb{R}^n)$.

In light of Proposition 4.1, once we prove Theorem 4.6 below, Theorem 3.20 is also proved.

Theorem 4.6. *Suppose that the real parameters* p*,* q*,* r*, and* K *satisfy*

$$0<q\le p<\infty, \quad 0<r\le\infty, \quad K\in\mathbb{N}_0\cap\left(\frac{n}{q_0}-n-1,\infty\right),$$

where $q_0:=\min(1,q)$*. Let* $f\in H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$*. Then, there exists a triplet*

$$\{\lambda_j\}_{j=1}^{\infty}\subset[0,\infty), \quad \{Q_j\}_{j=1}^{\infty}\subset\mathcal{Q}(\mathbb{R}^n), \quad \text{and} \quad \{a_j\}_{j=1}^{\infty}\subset L^{\infty}(\mathbb{R}^n)\cap\mathcal{P}_K^{\perp}(\mathbb{R}^n)$$

such that $f=\sum_{j=1}^{\infty}\lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and that for all* $v>0$*,*

$$|a_j|\le\chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} \lesssim_v \|f\|_{H\mathcal{M}^p_{q,r}}.$$

4.3 Grand maximal functions

Hardy-Morrey-Lorentz spaces admit a characterization by using the grand maximal operator introduced in Section 1.3. The Hardy-Morrey-Lorentz quasi-norm $\|\cdot\|_{H\mathcal{M}^p_{q,r}}$ is rewritten as follows.

Proposition 4.7. *Let $0 < q \le p < \infty$ and $0 < r \le \infty$. Then,*

$$\|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}} \sim \|f\|_{H\mathcal{M}^p_{q,r}}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

The proof is similar to the case of Hardy spaces with variable exponents [9, 39]. It suffices to state the two fundamental estimates of (4.7) and (4.8) below.

Suppose that we are given an integer $K \gg 1$. We write

$$M^*_{\rm heat}f(x) := \sup_{j\in\mathbb{Z}} \left(\sup_{y\in\mathbb{R}^n} \frac{|e^{2^j\Delta}f(y)|}{(1+4^j|x-y|^2)^K} \right), \quad x \in \mathbb{R}^n. \tag{4.6}$$

The next lemma stands for the pointwise estimate for $M^*_{\rm heat}$ in terms of the usual Hardy-Littlewood maximal operator M.

Lemma 4.8 ([39, Lemma 3.2], [45, §4]). *For $0 < \theta < 1$, there exists K_θ such that for all $K \ge K_\theta$, we have*

$$M^*_{\rm heat}f(x) \lesssim M\left[\sup_{k\in\mathbb{Z}} |e^{2^k\Delta}f|^\theta\right](x)^{\frac{1}{\theta}}, \quad x \in \mathbb{R}^n \tag{4.7}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$. Here, K is the constant appearing in the definition of $M^*_{\rm heat}f(x)$ (see (4.6)).

In the course of the proof of [39, Theorem 3.3], it can be shown that

$$\mathcal{M}f(x) \sim \sup_{\tau\in\mathcal{F}_N,\, j\in\mathbb{Z}} |\tau^j * f(x)| \lesssim M^*_{\rm heat}f(x) \tag{4.8}$$

once we fix integers $K \gg 1$ and $N \gg 1$.

Combining Proposition 3.13 with the fundamental pointwise estimates of (4.7) and (4.8), Proposition 4.7 can be proved with ease. Thus, we omit the details.

4.4 Lemmas for the proofs of Theorems 4.4, 4.5 and 4.6

Lemma 4.9. *Let $1 \le s < \infty$, $K \in \mathbb{N}_0$, and $Q \in \mathcal{Q}(\mathbb{R}^n)$. Assume that $a \in \mathcal{M}^s_1(\mathbb{R}^n) \cap \mathcal{P}_K(\mathbb{R}^n)^\perp$ satisfies*

$$\operatorname{supp}(a) \subset Q, \quad \|a\|_{\mathcal{M}^s_1} \le |Q|^{\frac{1}{s}}. \tag{4.9}$$

Then, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $N > 0$,

$$\left| \int_{\mathbb{R}^n} a(x)\varphi(x)\,\mathrm{d}x \right| \lesssim \ell(Q)^{n+K+1} \sup_{y \in Q} \frac{1}{1+|y|^N}.$$

Here, the implicit constant in $\lesssim$ depends on φ.

Proof. By the mean-value theorem, there exists $\theta \in (0,1)$ depending on x, K, Q, and φ such that

$$\begin{aligned}
\int_{\mathbb{R}^n} a(x)\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n} a(x) \left(\varphi(x) - \sum_{|\alpha| \le K} \frac{1}{\alpha!} \partial^\alpha \varphi(c(Q))(x - c(Q))^\alpha \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^n} a(x) \sum_{|\beta| = K+1} \frac{1}{\beta!} \partial^\beta \varphi((1-\theta)x + \theta c(Q))(x - c(Q))^\beta \,\mathrm{d}x.
\end{aligned}$$

Then, from (4.9)

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} a(x)\varphi(x)\,\mathrm{d}x \right| &\lesssim \ell(Q)^{K+1} \sup_{y \in Q} \frac{1}{1+|y|^N} \int_Q |a(x)|\,\mathrm{d}x \\
&\le \ell(Q)^{K+1} \sup_{y \in Q} \frac{1}{1+|y|^K} \|a\|_{\mathcal{M}^s_1} \cdot |Q|^{-\frac{1}{s}+1} \\
&\lesssim \ell(Q)^{n+K+1} \sup_{y \in Q} \frac{1}{1+|y|^N},
\end{aligned}$$

as desired. □

Lemma 4.10. *Let $0 < q \le p < \infty$, $1 \le s < \infty$, and $K \in \mathbb{N}_0$. Assume that $\{\lambda_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset [0,\infty)$ and $\{a_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathcal{M}^s_1(\mathbb{R}^n) \cap \mathcal{P}_K(\mathbb{R}^n)^\perp$ satisfy*

$$\operatorname{supp}(a_Q) \subset 3Q, \quad \|a_Q\|_{\mathcal{M}^s_1} \le |Q|^{\frac{1}{s}}, \quad \left\| \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q \chi_Q \right\|_{\mathcal{M}^p_q} < \infty.$$

If $q \le 1$ and $n + K + 1 > n/q$, then for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\sum_{m=1}^{\infty} \sum_{Q \in \mathcal{D}_m(\mathbb{R}^n)} \lambda_Q |\langle a_Q, \varphi \rangle| \lesssim \left\| \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q \chi_Q \right\|_{\mathcal{M}^p_q}. \tag{4.10}$$

Proof. Fix $m \geq 1$. To prove (4.10), we use the fact that for each $\tilde{m} \in \mathbb{Z}^n$,

$$\begin{aligned}
\left\| \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q \chi_Q \right\|_{\mathcal{M}^p_q} &\gtrsim \left\| \left(\sum_{Q \in \mathcal{D}_m(\mathbb{R}^n),\, |c(Q)-\tilde{m}| \leq n} \lambda_Q \chi_Q \right) \chi_{\tilde{m}+[-n,n]^n} \right\|_{L^q} \\
&= 2^{-\frac{mn}{q}} \left(\sum_{Q \in \mathcal{D}_m(\mathbb{R}^n),\, |c(Q)-\tilde{m}| \leq n} \lambda_Q^q \right)^{\frac{1}{q}} \\
&\geq 2^{-\frac{mn}{q}} \sum_{Q \in \mathcal{D}_m(\mathbb{R}^n),\, |c(Q)-\tilde{m}| \leq n} \lambda_Q.
\end{aligned}$$

In particular, for all $R \in \mathcal{D}_m(\mathbb{R}^n)$,

$$\left\| \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q \chi_Q \right\|_{\mathcal{M}^p_q} \gtrsim 2^{-\frac{mn}{q}} \lambda_R.$$

We remark that for each $m \geq 1$ and $\tilde{m} \in \mathbb{Z}^n$,

$$\sharp\{Q \in \mathcal{D}_m(\mathbb{R}^n) \,:\, \overline{3Q} \ni 0\} = 4^n.$$

It follows from Lemma 4.9 that

$$\begin{aligned}
\sum_{Q \in \mathcal{D}_m(\mathbb{R}^n),\, \overline{3Q} \ni 0} \lambda_Q |\langle a_Q, \varphi \rangle| &\lesssim 2^{-m(n+K+1)} \sum_{Q \in \mathcal{D}_m(\mathbb{R}^n),\, \overline{3Q} \ni 0} \lambda_Q \\
&\lesssim 2^{-m\left(n+K+1-\frac{n}{q}\right)} \left\| \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q \chi_Q \right\|_{\mathcal{M}^p_q}.
\end{aligned} \tag{4.11}$$

In addition, setting

$$\mathcal{D}_{m,\tilde{m}}(\mathbb{R}^n) := \{Q \in \mathcal{D}_m(\mathbb{R}^n) \,:\, |c(Q) - \tilde{m}| \leq n\}$$

for each $m \geq 1$ and $\tilde{m} \in \mathbb{Z}^n$, we have

$$\mathcal{D}_m(\mathbb{R}^n) = \bigcup_{\tilde{m} \in \mathbb{Z}^n} \mathcal{D}_{m,\tilde{m}}(\mathbb{R}^n).$$

Then, there exists a mapping $\iota_m : \mathcal{D}_m(\mathbb{R}^n) \to \mathbb{Z}^n$ such that $Q \in \mathcal{D}_{m,\iota_m(Q)}(\mathbb{R}^n)$;

therefore

$$
\begin{aligned}
&\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n),\,\overline{3Q}\not\ni 0} \lambda_Q|\langle a_Q,\varphi\rangle| \\
&\lesssim 2^{-m(n+K+1)}\sum_{\tilde{m}\in\mathbb{Z}^n}\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n),\,\iota_m(Q)=\tilde{m}} \lambda_Q\cdot\sup_{y\in 3Q}\frac{1}{1+|y|^{n+1}} \\
&\lesssim 2^{-m(n+K+1)}\sum_{\tilde{m}\in\mathbb{Z}^n}\frac{1}{1+|\tilde{m}|^{n+1}}\sum_{Q\in\mathcal{D}_{m,\tilde{m}}(\mathbb{R}^n)}\lambda_Q \\
&\lesssim 2^{-m\left(n+K+1-\frac{n}{q}\right)}\left\|\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda_Q\chi_Q\right\|_{\mathcal{M}^p_q}.
\end{aligned}
\tag{4.12}
$$

Because

$$n+K+1>\frac{n}{q},$$

then we obtain the desired result. □

4.5 Proofs of Theorems 4.4 and 4.5: convergence of $f=\sum_{j=1}^\infty\lambda_j a_j$

First, we prove the convergence of

$$f=\sum_{j=1}^\infty\lambda_j a_j \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

We start with an important reduction. For each $J\in\mathbb{N}$, we take any cube $Q(J)\in\mathcal{D}(\mathbb{R}^n)$ with minimal volume such that $Q_J\subset 3Q(J)$, and we set

$$\mathcal{E}_Q:=\{j\in\mathbb{N}\,:\,Q=Q(j)\}$$

and

$$\lambda_Q:=\sum_{j\in\mathcal{E}_Q}\lambda_j,\quad a_Q:=\begin{cases}0, & \lambda_Q=0,\\ \dfrac{1}{\lambda_Q}\displaystyle\sum_{j\in\mathcal{E}_Q}\lambda_j a_j, & \lambda_Q\neq 0.\end{cases}$$

Note that $\{\mathcal{E}_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$ is pairwise disjoint. Then, $\{a_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$ and $\{\lambda_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$ satisfy

$$\|a_Q\|_{\mathcal{M}^s_1}\le\frac{1}{\lambda_Q}\sum_{j\in\mathcal{E}_Q}\lambda_j\|a_j\|_{\mathcal{M}^s_1}\le\frac{1}{\lambda_Q}\sum_{j\in\mathcal{E}_Q}\lambda_j|Q_j|^{\frac{1}{s}}\le|3Q|^{\frac{1}{s}}.$$

Taking $\theta \in (1/v, \infty)$, by the fact that $\chi_{Q(J)} \lesssim_n M\chi_{Q_J}$ for $J \in \mathbb{N}$, we have

$$\left\| \left(\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} (\lambda_Q \chi_Q)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} \lesssim \left\| \left(\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \left(\sum_{j \in \mathcal{E}_Q} \lambda_j (M\chi_{Q_j})^\theta \right)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}}$$
$$\leq \left\| \left(\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \sum_{j \in \mathcal{E}_Q} \left(M\left[\lambda_j^{\frac{1}{\theta}} \chi_{Q_j} \right] \right)^{\theta v} \right)^{\frac{1}{\theta v}} \right\|^\theta_{\mathcal{M}^{\theta p}_{\theta q, \theta r}}.$$

Then, by Theorem 3.14,

$$\left\| \left(\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} (\lambda_Q \chi_Q)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} \lesssim \left\| \left(\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \sum_{j \in \mathcal{E}_Q} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{\theta v}} \right\|^\theta_{\mathcal{M}^{\theta p}_{\theta q, \theta r}}$$
$$= \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{\theta v}} \right\|^\theta_{\mathcal{M}^{\theta p}_{\theta q, \theta r}} < \infty.$$

Hence, we may assume

$$\{a_Q\}_{Q \in \mathcal{D}} \subset \begin{cases} \mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n) \cap \mathcal{P}_{d_v}(\mathbb{R}^n)^\perp, & 1 < t \leq s < \infty, \\ \mathcal{M}^s_1(\mathbb{R}^n) \cap \mathcal{P}_{d_v}(\mathbb{R}^n)^\perp, & 1 = t \leq s < \infty, \end{cases} \quad \{\lambda_Q\}_{Q \in \mathcal{D}} \subset [0, \infty),$$

with $\mathrm{supp}(a_Q) \subset 3Q$ for $Q \in \mathcal{D}(\mathbb{R}^n)$ instead of

$$\{a_j\}_{j=1}^\infty \subset \begin{cases} \mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n) \cap \mathcal{P}_{d_v}(\mathbb{R}^n)^\perp, & 1 < t \leq s < \infty, \\ \mathcal{M}^s_1(\mathbb{R}^n) \cap \mathcal{P}_{d_v}(\mathbb{R}^n)^\perp, & 1 = t \leq s < \infty, \end{cases} \quad \{\lambda_j\}_{j=1}^\infty \subset [0, \infty).$$

Then, it suffices to show that

$$\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q |\langle a_Q, \varphi \rangle| = \sum_{m=-\infty}^0 \sum_{Q \in \mathcal{D}_m(\mathbb{R}^n)} \lambda_Q |\langle a_Q, \varphi \rangle| + \sum_{m=1}^\infty \sum_{Q \in \mathcal{D}_m(\mathbb{R}^n)} \lambda_Q |\langle a_Q, \varphi \rangle| < \infty. \tag{4.13}$$

First, we estimate the first part of (4.13). Fix $m \leq 0$. For each $Q \in \mathcal{D}_m(\mathbb{R}^n)$, $\overline{3Q} \not\ni 0$ implies that $|y| \geq \ell(Q)$ for all $y \in 3Q$, and then

$$|\langle a_Q, \varphi \rangle| \lesssim \int_{3Q} |a_Q(x)| \, \mathrm{d}x \sup_{y \in 3Q} \frac{1}{1 + |y|^{2n+1-\frac{n}{s}}}$$
$$\lesssim \|a_Q\|_{\mathcal{M}^s_1} \cdot |Q|^{-\frac{1}{s}+1} \sup_{y \in 3Q} \frac{1}{1 + |y|^{2n+1-\frac{n}{s}}} \lesssim |Q|^{\frac{1}{s}} \sup_{y \in 3Q} \frac{1}{1 + |y|^{n+1}}.$$

It follows that

$$\begin{aligned}\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n),\,\overline{3Q}\not\ni 0}\lambda_Q|\langle a_Q,\varphi\rangle| &\lesssim \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n),\,\overline{3Q}\not\ni 0}\lambda_Q\cdot|Q|^{\frac{1}{s}}\sup_{y\in 3Q}\frac{1}{1+|y|^{n+1}}\\ &\lesssim 2^{-\frac{mn}{s}}\sum_{\tilde m\in\mathbb{Z}^n}\frac{1}{1+|\tilde m|^{n+1}}\sum_{\substack{Q\in\mathcal{D}_m(\mathbb{R}^n),\,\overline{3Q}\not\ni 0\\ |c(Q)-\tilde m|\le n}}\lambda_Q.\end{aligned}$$

Meanwhile, if $\overline{3Q}\ni 0$, by Proposition 2.16 (4),

$$|\langle a_Q,\varphi\rangle|\lesssim_\varphi \|a_Q\|_{\mathcal{M}^s_1}\lesssim|Q|^{\frac{1}{s}}.$$

Thus,

$$\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n),\,\overline{3Q}\ni 0}\lambda_Q|\langle a_Q,\varphi\rangle|\lesssim\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n),\,\overline{3Q}\not\ni 0}\lambda_Q\cdot|Q|^{\frac{1}{s}}\le 4^n2^{-\frac{mn}{s}+\frac{mn}{p}}\sup_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda_Q|Q|^{\frac{1}{p}}.$$

Note that for each $R\in\mathcal{D}(\mathbb{R}^n)$,

$$\left\|\left(\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}(\lambda_Q\chi_Q)^v\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_{q,r}}\ge\|\lambda_R\chi_R\|_{\mathcal{M}^p_{q,r}}=\left(\frac{q}{r}\right)^{\frac{1}{r}}\lambda_R|R|^{\frac{1}{p}}$$

by Proposition 3.1. We conclude that

$$\begin{aligned}\sum_{m=-\infty}^{0}\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)}\lambda_Q|\langle a_Q,\varphi\rangle| &\lesssim\sum_{m=-\infty}^{0}2^{-\frac{mn}{s}+\frac{mn}{p}}\left\|\left(\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}(\lambda_Q\chi_Q)^v\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_{q,r}}\\ &\lesssim\left\|\left(\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}(\lambda_Q\chi_Q)^v\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_{q,r}}.\end{aligned}\tag{4.14}$$

Note that

$$n+d_v+1-\frac{n}{q_0}>n+\left(\frac{n}{v}-n-1\right)+1-\frac{n}{q_0}=\frac{n}{v}-\frac{n}{q_0}\ge 0.$$

Thus, there exists $\varepsilon\in(0,q_0)$ such that

$$n+d_v+1-\frac{n}{q_0-\varepsilon}>0,$$

where $q_0:=\min(1,q)$. Hence, by Lemma 4.10, the second part of (4.13) can be estimated as follows:

$$\sum_{m=1}^{\infty}\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)}\lambda_Q|\langle a_Q,\varphi\rangle|\lesssim\left\|\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda_Q\chi_Q\right\|_{\mathcal{M}^p_{q_0-\varepsilon_1}}\lesssim\left\|\left(\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}(\lambda_Q\chi_Q)^v\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_{q,r}},\tag{4.15}$$

where in the last inequality, we used the embedding $\mathcal{M}^p_{q,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_0-\varepsilon,q_0-\varepsilon}(\mathbb{R}^n) = \mathcal{M}^p_{q_0-\varepsilon}(\mathbb{R}^n)$ (see Proposition 3.5). Combining these estimates (4.14) and (4.15), we finish the proof of (4.13).

4.6 Proofs of Theorems 4.4 and 4.5: (4.4) and (4.5)

To prove the estimates of (4.4) and (4.5) in Theorems 4.4 and 4.5, respectively, we use the following lemma, whose proof is similar to that of Lemma 4.9.

Lemma 4.11 ([39, (5.2)]). *If $\{a_j\}_{j=1}^\infty$ satisfies the same assumptions as in Theorems* 4.4 *and* 4.5*, then*

$$\mathcal{M}a_j(x) \lesssim \chi_{3Q_j}(x) M a_j(x) + M\chi_{Q_j}(x)^{\frac{n+d_v+1}{n}}, \quad x \in \mathbb{R}^n.$$

Let us show Theorem 4.4. Using Proposition 4.7 and Lemma 4.11, we have

$$\begin{aligned}
\|f\|_{H\mathcal{M}^p_{q,r}} \sim \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}} &\leq \left\| \sum_{j=1}^\infty \lambda_j \mathcal{M} a_j \right\|_{\mathcal{M}^p_{q,r}} \\
&\lesssim \left\| \sum_{j=1}^\infty \lambda_j \left(\chi_{3Q_j} M a_j + (M\chi_{Q_j})^{\frac{n+d_v+1}{n}} \right) \right\|_{\mathcal{M}^p_{q,r}} \\
&\lesssim \left\| \sum_{j=1}^\infty \lambda_j \chi_{3Q_j} M a_j \right\|_{\mathcal{M}^p_{q,r}} + \left\| \sum_{j=1}^\infty \lambda_j (M\chi_{Q_j})^{\frac{n+d_v+1}{n}} \right\|_{\mathcal{M}^p_{q,r}} =: I_1 + I_2.
\end{aligned}$$

First, we consider I_1. We note that for each $j \in \mathbb{N}$, owing to the $\mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n) = \mathcal{M}^s_{t,\infty}(\mathbb{R}^n)$-boundedness of M (see Theorem 3.13), by applying Theorems 3.14 and 3.17 and using the fact that $\chi_{3Q_j} \leq 3^n M\chi_{Q_j}$ for each $j \in \mathbb{N}$, we have

$$\begin{aligned}
I_1 &\lesssim \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{3Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} \lesssim \left\| \left(\sum_{j=1}^\infty \lambda_j^v (M\chi_{Q_j})^2 \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} \\
&\lesssim \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}}.
\end{aligned}$$

Next, we consider I_2. Set

$$P := \frac{n+d_v+1}{n} p, \quad Q := \frac{n+d_v+1}{n} q, \quad \text{and} \quad R := \frac{n+d_v+1}{n} r.$$

Then, by Theorem 3.14 and the embedding $\ell^v(\mathbb{N}) \hookrightarrow \ell^1(\mathbb{N})$, we obtain

$$I_2 = \left\| \left[\sum_{j=1}^{\infty} \lambda_j (M\chi_{Q_j})^{\frac{n+d_v+1}{n}} \right]^{\frac{n}{n+d_v+1}} \right\|_{\mathcal{M}^P_{Q,R}}^{\frac{n+d_v+1}{n}} \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{q,r}}$$
$$\le \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{q,r}} .$$

Thus, we obtain the desired result.

Similarly, because M satisfies the weak-type boundedness on $\mathcal{M}^s_1(\mathbb{R}^n)$ (see Proposition 2.18), we can prove Theorem 4.5.

4.7 Proof of Theorem 4.6

To prove Theorem 4.6, we use a new approach provided in [41, Subsection 4.3] and the following lemma, as given in [47, Exercise 3.34].

Lemma 4.12. *Let $0 < q \le p < \infty$, $0 < r \le \infty$, $K \in \mathbb{N}$, and $0 < v < \infty$, and let $f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then, we can find $\{a_j\}_{j=1}^{\infty} \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K^\perp(\mathbb{R}^n)$ and a sequence $\{Q_j\}_{j=1}^{\infty}$ of cubes such that*

(1) $\mathrm{supp}(a_j) \subset Q_j$,

(2) $f = \sum_{j=1}^{\infty} a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$, *and*

(3) $\left(\sum_{j=1}^{\infty} (\|a_j\|_{L^\infty} \chi_{Q_j})^v \right)^{\frac{1}{v}} \lesssim \mathcal{M}f.$

Proof of Theorem 4.6. It suffices to prove the case of $v = 1$; the case of $v > 0$ can be proved similarly. Let $f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n)$. Fix $t > 0$. Because $\mathcal{D}(\mathbb{R}^n)$ is a countable set, applying Lemma 4.12 to $e^{t\Delta} f \in H\mathcal{M}^p_{q,r}(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for

$$\{3Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}, \quad \{\lambda^t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}, \quad \{\lambda_Q a^t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$$

instead of

$$\{Q_j\}_{j=1}^{\infty}, \quad \{\|a_j\|_{L^\infty}\}_{j=1}^{\infty}, \quad \{a_j\}_{j=1}^{\infty},$$

respectively, we can consider the decomposition $e^{t\Delta} f = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda^t_Q a^t_Q$ in the topology of $\mathcal{S}'(\mathbb{R}^n)$, where $a^t_Q \in \mathcal{P}_K^\perp(\mathbb{R}^n)$, $\lambda^t_Q \ge 0$, and

$$|a^t_Q| \le \chi_{3Q}, \quad \left\| \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda^t_Q \chi_{3Q} \right\|_{\mathcal{M}^p_{q,r}} \lesssim \|\mathcal{M}[e^{t\Delta} f]\|_{\mathcal{M}^p_{q,r}} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}}.$$

By the weak-* compactness of the unit ball of $L^\infty(\mathbb{R}^n)$, there exists a sequence $\{t_l\}_{l=1}^\infty$ that converges to 0 such that both $\lambda_Q = \lim_{l\to\infty} \lambda_Q^{t_l}$ and $a_Q = \lim_{l\to\infty} a_Q^{t_l}$ exist for all $Q \in \mathcal{D}$ in the sense that

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} a_Q(x)\varphi(x)\,\mathrm{d}x$$

for all $\varphi \in L^1(\mathbb{R}^n)$. We claim that $f = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_Q a_Q$ in the topology of $\mathcal{S}'(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a test function. Then, by Lemma 4.3 (3), we have

$$\langle f, \varphi\rangle = \lim_{l\to\infty} \langle e^{t_l\Delta} f, \varphi\rangle = \lim_{l\to\infty} \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_Q^{t_l} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\,\mathrm{d}x$$

from the definition of convergence in $\mathcal{S}'(\mathbb{R}^n)$. Once we fix m, we have

$$\lambda_Q^{t_l} \lesssim 2^{\frac{mn}{p}} \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}} \quad \text{and} \quad \left|\int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\,\mathrm{d}x\right| \le \int_{3Q} |\varphi(x)|\,\mathrm{d}x.$$

Additionally, by the equation

$$\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} 2^{\frac{mn}{p}} \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}} \int_{3Q} |\varphi(x)|\,\mathrm{d}x = 3^n 2^{\frac{mn}{p}} \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}} \|\varphi\|_{L^1} < \infty,$$

we can use Fubini's theorem to obtain

$$\sum_{m\in\mathbb{Z}} \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \lambda_Q^{t_l} a_Q^{t_l}(x) \right) \varphi(x)\,\mathrm{d}x = \sum_{m\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \lambda_Q^{t_l} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\,\mathrm{d}x.$$

Hereinafter, we use also abbreviation

$$a_{m,l} := \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \lambda_Q^{t_l} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\,\mathrm{d}x,$$

and we fix $0 < \varepsilon \ll 1$.

When $m \in \mathbb{Z} \cap (-\infty, 0]$, we see that

$$|a_{m,l}| \lesssim \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} 2^{\frac{mn}{p}} \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}} \int_{3Q} |\varphi(x)|\,\mathrm{d}x \lesssim_\varphi 2^{\frac{mn}{p}} \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}}$$

by the previous argument, and therefore

$$\sum_{m=-\infty}^{0} |a_{m,l}| \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}}. \tag{4.16}$$

In addition, taking $0 < \varepsilon \ll 1$ by $K + 1 > n(1/(q_0 - \varepsilon) - 1) > 0$, namely,

$$n + K + 1 > \frac{n}{q_0 - \varepsilon},$$

by Lemma 4.10, we obtain

$$\sum_{m=1}^{\infty}|a_{m,l}| \lesssim \left\| \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_Q^t \chi_{3Q} \right\|_{\mathcal{M}^p_{q,r}} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}}. \tag{4.17}$$

Thus, by (4.16) and (4.17), we obtain

$$\sum_{m=-\infty}^{\infty}|a_{m,l}| \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{q,r}}.$$

As a consequence, applying the Lebesgue convergence theorem, we obtain

$$\lim_{l\to\infty}\sum_{m=-\infty}^{\infty} a_{m,l} = \sum_{m=-\infty}^{\infty}\lim_{l\to\infty} a_{m,l}.$$

Hence, using Fubini's theorem again, we have

$$\begin{aligned}
\langle f,\varphi\rangle &= \sum_{m=-\infty}^{\infty}\left(\lim_{l\to\infty}\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \lambda_Q^{t_l}\int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\,\mathrm{d}x\right)\\
&= \sum_{m=-\infty}^{\infty}\left(\lim_{l\to\infty}\int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \lambda_Q^{t_l}a_Q^{t_l}(x)\right)\varphi(x)\,\mathrm{d}x\right)\\
&= \sum_{m=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)}\lim_{l\to\infty}\left(\int_{\mathbb{R}^n}\lambda_Q^{t_l}a_Q^{t_l}(x)\varphi(x)\,\mathrm{d}x\right)\\
&= \sum_{m=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)}\int_{\mathbb{R}^n}\lambda_Q a_Q(x)\varphi(x)\,\mathrm{d}x = \left\langle \sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda_Q a_Q, \varphi\right\rangle.
\end{aligned}$$

Consequently, we obtain the desired result. □

Chapter 5

Olsen-type inequality on Morrey-Lorentz spaces: the proof of Theorem 1.9

Having clarified the structure of Morrey-Lorentz spaces, we are now ready for the proof of the Olsen inequality in Morrey-Lorentz spaces. We prove Theorem 1.9 in Section 5.1 and make a brief remark on it in Section 5.2.

5.1 Proof of Theorem 1.9

With Theorems 3.17 and 3.20, we prove Theorem 1.9. We invoke two lemmas.

Lemma 5.1 ([33, Lemma 4.1]). *For every $Q \in \mathcal{Q}(\mathbb{R}^n)$,*

$$I_\alpha \chi_Q(x) \gtrsim \ell(Q)^\alpha \chi_Q(x)$$

for all $x \in \mathbb{R}^n$.

Lemma 5.2 ([33, Lemma 4.2]). *Let $K = 0, 1, 2, \ldots$. Suppose that A is an $L^\infty(\mathbb{R}^n)$ $\cap \mathcal{P}_K(\mathbb{R}^n)^\perp$-function supported on some $Q \in \mathcal{Q}(\mathbb{R}^n)$. Then,*

$$|I_\alpha A(x)| \le C_{\alpha,K} \|A\|_{L^\infty} \ell(Q)^\alpha \sum_{k=1}^{\infty} \frac{1}{2^{k(n+K+1-\alpha)}} \chi_{2^k Q}(x), \quad x \in \mathbb{R}^n.$$

We remark that Lemma 5.2 is proved by a method akin to Lemma 4.9.

Now, we begin the proof of Theorem 1.9. Suppose that $f \ge 0$. Note that by Fatou's lemma and the Fatou property for Morrey-Lorentz spaces (see Lemma 3.10),

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \le \left\| g \cdot \liminf_{m\to\infty} I_\alpha f_m \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \le \liminf_{m\to\infty} \|g \cdot I_\alpha f_m\|_{\mathcal{M}^{r_0}_{r_1,r_2}}$$

for all $f \in \mathcal{M}^{p_0}_{p_1,p_2}(\mathbb{R}^n)$ and $g \in \mathcal{M}^{q_0}_{q_1,q_2}(\mathbb{R}^n)$, where

$$f_m := f\chi_{B(m)}\chi_{[0,m)}(|f|) \in L^\infty_{\rm c}(\mathbb{R}^n), \quad m \in \mathbb{N}.$$

Then, we may assume that $f \in L^\infty_{\rm c}(\mathbb{R}^n)$. We decompose f according to Theorem 3.20 with sufficiently large $K \gg 1$; $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in $L^w(\mathbb{R}^n)$ for all $w \in (1,\infty)$, where $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K(\mathbb{R}^n)^\perp$, and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill (3.14).

Here, we claim that

$$I_\alpha f(x) = \sum_{j=1}^\infty \lambda_j I_\alpha a_j(x), \quad \text{a.e. } x \in \mathbb{R}^n. \tag{5.1}$$

In fact, fixing $w \in (1, n/\alpha)$ and then choosing $w^* \in (1,\infty)$ satisfying $1/w^* = 1/w - \alpha/n$, we have

$$\left\| I_\alpha f - \sum_{j=1}^N \lambda_j I_\alpha a_j \right\|_{L^{w^*}} = \left\| I_\alpha \left[f - \sum_{j=1}^N \lambda_j a_j \right] \right\|_{L^{w^*}} \lesssim \left\| f - \sum_{j=1}^N \lambda_j a_j \right\|_{L^w} \to 0$$

as $N \to \infty$ from the Hardy-Littlewood-Sobolev inequality (see (1.1)). We finish the proof of (5.1).

Then, by Lemma 5.2, we obtain

$$|g(x) I_\alpha f(x)| \lesssim \sum_{j,k=1}^\infty \frac{\lambda_j}{2^{k(n+K+1-\alpha)}} \ell(Q_j)^\alpha |g(x)| \chi_{2^k Q_j}(x).$$

Therefore, we conclude that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \|g\|_{{\rm W}\mathcal{M}^{q_0}_{q_1}} \left\| \sum_{j,k=1}^\infty \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{q_0}}}{2^{k(n+L+1)}} \cdot \frac{\ell(2^k Q_j)^{\frac{n}{q_0}}}{\|g\|_{{\rm W}\mathcal{M}^{q_0}_{q_1}}} |g| \chi_{2^k Q_j} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}}.$$

For each $(j,k) \in \mathbb{N} \times \mathbb{N}$, write

$$\kappa_{jk} := \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{q_0}}}{2^{k(n+L+1)}}, \quad b_{jk} := \frac{\ell(2^k Q_j)^{\frac{n}{q_0}}}{\|g\|_{{\rm W}\mathcal{M}^{q_0}_{q_1}}} |g| \chi_{2^k Q_j}.$$

Then,

$$\sum_{j,k=1}^\infty \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{q_0}}}{2^{k(n+K+1)}} \cdot \frac{\ell(2^k Q_j)^{\frac{n}{q_0}}}{\|g\|_{{\rm W}\mathcal{M}^{q_0}_{q_1}}} |g| \chi_{2^k Q_j} = \sum_{j,k=1}^\infty \kappa_{jk} b_{jk},$$

each b_{jk} is supported on a cube $2^k Q_j$ and

$$\|b_{jk}\|_{{\rm W}\mathcal{M}^{q_0}_{q_1}} \le |2^k Q_j|^{\frac{1}{q_0}}.$$

Observe that $\chi_{2^kQ_j} \leq 2^{kn} M\chi_{Q_j}$. Hence, if we choose $v, \theta \in \mathbb{R}$ such that

$$K > \alpha - \frac{n}{q_0} - 1 + \theta n - n, \quad \theta > \frac{1}{v} \geq \frac{1}{\min(r_1, r_2)}, \quad 0 < v \leq 1,$$

then we have

$$\begin{aligned}\left\| \left(\sum_{j,k=1}^{\infty} (\kappa_{jk}\chi_{2^kQ_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} &= \left\| \left(\sum_{j,k=1}^{\infty} \left(\frac{\lambda_j \ell(2^kQ_j)^{\alpha - \frac{n}{q_0}}}{2^{k(n+K+1)}} \chi_{2^kQ_j} \right)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \\ &\lesssim \left\| \left(\sum_{j=1}^{\infty} \left(\lambda_j \ell(Q_j)^{\alpha - \frac{n}{q_0}} (M\chi_{Q_j})^{\theta} \right)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \\ &= \left\| \left\{ \sum_{j=1}^{\infty} \left(M\left[\left(\lambda_j \ell(Q_j)^{\alpha - \frac{n}{q_0}} \chi_{Q_j} \right)^{\frac{1}{\theta}} \right] \right)^{\theta v} \right\}^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}}.\end{aligned}$$

By virtue of Theorem 3.14, the Fefferman-Stein inequality for Morrey-Lorentz spaces, alongside $f_j = (\lambda_j \ell(Q_j)^{\alpha - n/q_0} \chi_{Q_j})^{1/\theta}$, we can remove the maximal operator and obtain

$$\left\| \left(\sum_{j,k=1}^{\infty} (\kappa_{jk}\chi_{2^kQ_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \left\| \left(\sum_{j=1}^{\infty} \left(\lambda_j \ell(Q_j)^{\alpha - \frac{n}{q_0}} \chi_{Q_j} \right)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}}.$$

We distinguish two cases here:

(1) If $\alpha = n/q_0$, then $p_0 = r_0$, $p_1 = r_1$, and $p_2 = r_2$. Thus, we can use (3.14).

(2) If $\alpha > n/q_0$, then by Proposition 3.15 and Lemma 5.1, we obtain

$$\begin{aligned}&\left\| \left(\sum_{j=1}^{\infty} \left(\lambda_j \ell(Q_j)^{\alpha - \frac{n}{q_0}} \chi_{Q_j} \right)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \\ &\lesssim \left\| \left(I_{\left(\alpha - \frac{n}{q_0}\right)v} \left[\sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j} \right)^v \right] \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{p_0}_{p_1,p_2}}.\end{aligned}$$

Thus, we can still use (3.14).

Consequently, we obtain

$$\left\| \left(\sum_{j,k=1}^{\infty} (\kappa_{jk}\chi_{2^kQ_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \|f\|_{\mathcal{M}^{p_0}_{p_1,p_2}} < \infty. \tag{5.2}$$

Observe also that $q_0 > r_0$ and $q_1 > r_1$. Thus, by Theorem 3.17 and (5.2), it follows that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \|g\|_{\mathrm{W}\mathcal{M}^{q_0}_{q_1}} \left\| \left(\sum_{j,k=1}^{\infty} (\kappa_{jk}\chi_{2^kQ_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \|g\|_{\mathrm{W}\mathcal{M}^{q_0}_{q_1}} \|f\|_{\mathcal{M}^{p_0}_{p_1,p_2}}.$$

5.2 Remark on Theorem 1.9

As mentioned [51, Remark 1.9], we see that one cannot simply prove Theorem 1.9 by naively combining Proposition 3.15 and the Hölder inequality for Morrey-Lorentz quasi-norms (see Lemma 3.11). For more details, the proof of Theorem 1.9 is fundamental provided that $p_1q_0/p_0 \le q_1 \le q_0$. In fact, by virtue of Theorem 1.9,

$$\|I_\alpha f\|_{\mathcal{M}^{\tilde{p}_0}_{\tilde{p}_1,\tilde{p}_2}} \lesssim \|f\|_{\mathcal{M}^{p_0}_{p_1,p_2}}, \quad \frac{\tilde{p}_0}{p_0} = \frac{\tilde{p}_1}{p_1} = \frac{\tilde{p}_2}{p_2}, \ \frac{1}{\tilde{p}_0} = \frac{1}{p_0} - \frac{\alpha}{n}.$$

The conditions $r_0/p_0 = r_1/p_1 = r_2/p_2$ and $1/r_0 = 1/q_0 + 1/p_0 - \alpha/n$ give

$$\frac{1}{r_1} = \frac{p_0}{p_1}\left(\frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}\right) = \frac{p_0}{p_1q_0} + \frac{1}{\tilde{p}_1}.$$

This yields

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_{r_1,r_2}} \lesssim \|g\|_{\mathrm{W}\mathcal{M}^{q_0}_{q_1}} \|f\|_{\mathcal{M}^{p_0}_{p_1,p_2}}$$

by the embeddings $\mathrm{W}\mathcal{M}^{q_0}_{q_1}(\mathbb{R}^n) \hookrightarrow \mathrm{W}\mathcal{M}^{q_0}_{p_1q_0/p_0}(\mathbb{R}^n)$ and $\mathcal{M}^{\tilde{p}_0}_{\tilde{p}_1,\tilde{p}_2}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^{\tilde{p}_0}_{\tilde{p}_1,r_2}(\mathbb{R}^n)$ if $p_1q_0/p_0 \le q_1$ and $\tilde{p}_2 \le r_2$. Also observe that $1/r_0 = 1/q_0 + 1/p_0 - \alpha/n > 1/q_0$, or equivalently, $q_0 > r_0$. Thus, by $q_1 > r_1$, Theorem 1.9 is significant only when $p_1r_0/p_0 < q_1 < p_1q_0/p_0$.

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