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A remark on the atomic decomposition in Hardy spaces based on the convexification of ball Banach spaces

by

Yoshihiro Sawano and Kazuki Kobayashi

DEPARTMENT OF MATHEMATICS

CHUO UNIVERSITY

BUNKYOKU TOKYO JAPAN

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Abstract. The purpose of the present note is to slightly shorten the proof of the atomic decomposition based on the paper by Dekel et. al. The atomic decomposition in the present paper is applicable to Hardy spaces based on the convexification of ball Banach spaces. The decomposition is rather canonical although it does not depend linearly on functions. Also, this decomposition is applicable under a rather weak condition as we will see.

1. Introduction

The goal of the present paper is to consider the atomic decomposition of the Hardy space $H^p(\mathbb{R}^n)$ for $p \in (0,\infty)$. Recall that the *Hardy space* $H^p(\mathbb{R}^n)$, $0<p<\infty$, collects all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{L^p} < \infty,$$

where $\{e^{t\Delta}\}_{t>0}$ stands for the heat semigroup.

We use the following notation in the present paper: Let $\mathbb{N}_0 \equiv \{0,1,\ldots\}$. A function $f \in L^\infty(\mathbb{R}^n)$ with compact support is said to have moment of order L if

$$\int_{\mathbb{R}^n} x^\alpha f(x) \mathrm{d}x = 0$$

for all $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le L$. Let $A, B \ge 0$. Then $A \lesssim B$ means that there exists a constant $C>0$ such that $A \le CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ happen simultaneously. The index σ_p is given by $\sigma_p \equiv \frac{n}{\min(1,p)} - n$ for $0<p<\infty$.

The goal of the present note is to provide a short proof of a well-known theorem based on the paper [**?**]. To this end, we set up some notation. Let $x \in \mathbb{R}^n$ and $r>0$. We denote by $B(x,r)$ the *ball centered at x of radius r*. Namely, we write

$$B(x,r) \equiv \{y \in \mathbb{R}^n \,:\, |x-y|<r\}.$$

If $x=0$, then omit it to write $B(r)$ instead of $B(x,r)$. The set of all balls is denoted by $\mathcal{B}$.

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Theorem 1.1. *Let $0 < p \le 1$. Let $f \in H^p(\mathbb{R}^n)$ and $L \in \mathbb{Z} \cap [[\sigma_p], \infty)$. Then there exist a countable collection $\{f_j\}_{j=1}^\infty$ of $L^\infty_{\rm c}$-functions having moment of order L and a countable collection $\{B_j\}_{j=1}^\infty \subset \mathcal{B}$ such that*

$$f = \sum_{j=1}^\infty f_j \tag{1.1}$$

in $\mathcal{S}'(\mathbb{R}^n)$, that

$$\mathrm{supp}(f_j) \subset 8B_j \tag{1.2}$$

for all $j \in \mathbb{N}$ and that

$$\left(\sum_{j=1}^\infty \|f_j\|_{L^\infty}{}^p |B_j| \right)^{\frac{1}{p}} \lesssim \|f\|_{H^p}. \tag{1.3}$$

Here aB_j stands for the a-times expansion of B_j for $a > 0$. As in [**?**], the proof of Theorem **??** uses some Hilbert spaces and estimates as in Lemma **??** to control the grand maximal function. Recently Dekel, Kerkyacharian, Kyriazis and Petrushev significantly reduced this argument [**?**]. The goal of the present paper is to reexamine their proof and expand it to other Hardy spaces based on ball Banach function spaces.

In order to extend Theorem **??** to other Hardy spaces such as the one based on variable Lebesgue spaces, we slightly generalize Theorem **??**. To this end, we recall an equivalent definition of $H^p(\mathbb{R}^n)$. We will use the notation $\langle x \rangle \equiv \sqrt{1+|x|^2}$ for $x \in \mathbb{R}^n$. To simplify the notation, for $N \in \mathbb{N}_0$, we define

$$p_N(\phi) \equiv \sum_{\substack{\alpha \in {\mathbb{N}_0}^n \\ |\alpha| \le N}} \left(\sup_{x \in \mathbb{R}^n} \langle x \rangle^N |\partial^\alpha \phi(x)| \right), \quad \phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.4}$$

We define the unit ball $\mathcal{F}_N$ with respect to p_N by

$$\mathcal{F}_N \equiv \{\phi \in \mathcal{S}(\mathbb{R}^n) \, : \, p_N(\phi) \le 1\}. \tag{1.5}$$

For $j \in \mathbb{Z}$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, we write

$$\phi^j \equiv 2^{jn} \phi(2^j \cdot). \tag{1.6}$$

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We define the grand maximal operator $\mathcal{M}_N f$ by

$$\mathcal{M}_N f(x) \equiv \sup_{k \in \mathbb{Z}, \phi \in \mathcal{F}_N} \left| \phi^k * f(x) \right| \quad (x \in \mathbb{R}^n).$$

Let $0 < p \le 1$. We can say that the Hardy space $H^p(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quantity $\|f\|_{H^p} \equiv \|\mathcal{M}_N f\|_{L^p}$ is finite; this definition coincides with the one above as long as $N \gg 1$ [**?**, p. 91].

Denote by χ_E the indicator function of a set E. We refine Theorem **??** based on the spirit of Miyachi [**?**].

Theorem 1.2. *Let $0 < p \le 1$. Let $f \in H^p(\mathbb{R}^n)$ and $L \in \mathbb{Z} \cap [[\sigma_p], \infty)$. Then there exist a countable collection $\{f_j\}_{j=1}^\infty$ of $L^\infty_{\rm c}$-functions having moment of order L and a countable collection $\{B_j\}_{j=1}^\infty \subset \mathcal{B}$ satisfying* (**??**), (**??**) *and*

$$\left(\sum_{j=1}^\infty (\|f_j\|_{L^\infty} \chi_{\frac{1}{2}B_j})^u \right)^{\frac{1}{u}} \lesssim \mathcal{M}_N f \tag{1.7}$$

for all $0 < u < \infty$ *with the implicit constant depends only on* n*,* N *and* u*.*

Once Theorem **??** is proved, we can prove Theorem **??** with ease. In fact, letting $r = p \in (0,1]$, we integrate (**??**) to have (**??**). So, we concentrate on (re)proving Theorem **??** in the present note after stating some preliminary facts in Section **??**. The proof of Theorem **??** is quite akin to the one in [**?**]. Since the conclusion gets tighter as L is larger, we may assume that $L \gg 1$. However, we start the proof from scratch to clarify what is actually needed for the decomposition. We prove Theorem **??** with the spirit of [**?**]. We actually prove Theorem **??** in Section **??**. Section **??** expands what we proved in Section **??**. As the starting point, we consider weighted Hardy spaces with weights in A_1. After that, we investigate other function spaces based on weighted Hardy spaces with weights in A_1.

2. Preliminaries

A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to vanish weakly at infinity if $\psi^j * f \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $j \to -\infty$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$. Since

$$\|\psi^j * f\|_{L^\infty} = \mathrm{O}(2^{\frac{jn}{p}}\|f\|_{H^p})$$

for all $f \in H^p(\mathbb{R}^n)$, as $j \to -\infty$, any element in $H^p(\mathbb{R}^n)$ vanishes weakly at infinity.

By taking advantage of the class $\mathcal{F}_N$, we use the following observation:

Lemma 2.1. *There exists* $A > 1$ *such that*

$$\sup_{\phi \in \mathcal{F}_N} \left|\phi^k * f(x)\right| \le A \sup_{\phi \in \mathcal{F}_N} \left|\phi^k * f(y)\right| \tag{2.1}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $k \in \mathbb{Z}$ *if* $x, y \in \mathbb{R}^n$ *satisfy* $|x-y| \le 2^{2-k}$*.*

Proof. Let $\phi \in \mathcal{F}_N$. We calculate

$$\phi^k * f(x) = \left\langle f, \phi^k(x-\cdot)\right\rangle = \left\langle f, \phi^k\left((x-y)+(y-\cdot)\right)\right\rangle.$$

Let $A > 1$ be the constant in Lemma **??**. Set

$$\phi_{k,x,y}(z) \equiv \phi\left(2^k(x-y)+z\right) \quad (z \in \mathbb{R}^n).$$

Then we have $p_N(\phi_{k,x,y}) \le A p_N(\phi)$ with the constant $A > 1$ depending on N. Thus,

$$\sup_{\phi \in \mathcal{F}_N} \left|\phi^k * f(x)\right| = A \sup_{\phi \in \mathcal{F}_N} \left|A^{-1}(\phi_{k,x,y})^k * f(y)\right| \le A \sup_{\phi \in \mathcal{F}_N} \left|\phi^k * f(y)\right|,$$

proving (**??**). □

We also need the well-known Whitney covering lemma.

Lemma 2.2. *Let* Ω *be a proper open set in* $\mathbb{R}^n$*. Write* $\rho(x) \equiv \mathrm{dist}(x, \partial\Omega)$ *for* $x \in \mathbb{R}^n$*. We let* $\left\{B\left(\xi_j, \frac{\rho_j}{5}\right)\right\}_{j=1}^\infty$ *be a maximal disjoint family, where* $\rho_j \equiv \rho(\xi_j)$ *for* $j \in \mathbb{N}$*.*

(1) $\Omega = \bigcup_{j=1}^\infty B\left(\xi_j, \frac{\rho_j}{2}\right)$.

(2) *For each* $j \in \mathbb{N}$*, let*

$$\mathcal{J}_j \equiv \left\{\nu \in \mathbb{N} \cap (j,\infty) \,:\, B\left(\xi_j, \frac{3}{4}\rho_j\right) \cap B\left(\xi_\nu, \frac{3}{4}\rho_\nu\right) \neq \emptyset\right\}.$$

Then $\sharp\mathcal{J}_j \le 300^n$ *and* $7^{-1}\rho_\nu \le \rho_j \le 7\rho_\nu$ *for all* $\nu \in \mathcal{J}_j$*.*

Proof. This is essentially contained in [?]. However, the number 300 did not appear in [?]. For the sake of convenience, we clarify why this number appears. Notice that

$$\sum_{\nu\in\mathcal{J}_j}\chi_{B\left(\xi_\nu,\frac{\rho_j}{35}\right)}\le\sum_{\nu\in\mathcal{J}_j}\chi_{B\left(\xi_\nu,\frac{\rho_\nu}{5}\right)}\le\chi_{B\left(\xi_j,\frac{37}{5}\rho_j\right)},$$

since

$$\frac{3}{4}\rho_j+\frac{3}{4}\rho_\nu+\frac{1}{5}\rho_\nu\le 6\rho_j+\frac{7}{5}\rho_j=\frac{37}{5}\rho_j.$$

Thus,

$$\sharp\mathcal{J}_j\times\frac{1}{35^n}\le\frac{37^n}{5^n},$$

implying $\sharp\mathcal{J}_j\le 259^n\le 300^n$. □

3. Proof of Theorem ??

We transform Theorem **??** to the following equivalent form:

Proposition 3.1. *Let $0<p\le 1$. Let $f\in H^p(\mathbb{R}^n)$ and $L\in\mathbb{Z}\cap[[\sigma_p],\infty)$. Then there exists a countable collection $\{F_{j,r}\}_{j\in\mathbb{N},r\in\mathbb{Z}}$ of $L^\infty_{\rm c}$-functions having moment of order L with the following properties*:

(1) *In $\mathcal{S}'(\mathbb{R}^n)$,*

$$f=\sum_{(j,r)\in\mathbb{N}\times\mathbb{Z}}F_{j,r}. \tag{3.1}$$

(2) *For all $j\in\mathbb{N}$ and $r\in\mathbb{Z}$, there exist $\xi_{j,r}\in\mathbb{R}^n$ and $\rho_{j,r}>0$ such that*

$$\operatorname{supp}(F_{j,r})\subset B(\xi_{j,r},5\rho_{j,r}). \tag{3.2}$$

(3) *For all $0<u<\infty$,*

$$\left(\sum_{(j,r)\in\mathbb{N}\times\mathbb{Z}}(\|F_{j,r}\|_{L^\infty}\chi_{B(\xi_{j,r},2^{-1}\rho_{j,r})})^u\right)^{\frac{1}{u}}\lesssim\mathcal{M}_N f, \tag{3.3}$$

where the implicit constant depends on u, N and n.

Section **??** is devoted to the proof of Proposition **??** assuming that $f\neq 0$.

For each $k,r\in\mathbb{Z}$, we set

$$\Omega_r\equiv\{x\in\mathbb{R}^n\,:\,\mathcal{M}_N f(x)>2^r\}$$

and

$$V_{k,r}\equiv\{x\in\mathbb{R}^n\,:\,B(x,2^{-k+1})\subset\Omega_r\}.$$

Notice that each Ω_r is an open set and hence

$$\Omega_r=\bigcup_{k=-\infty}^{\infty}V_{k,r}.$$

If $f\in\mathcal{S}'(\mathbb{R}^n)\setminus\{0\}$, then

$$\bigcup_{r=-\infty}^{\infty}\Omega_r=\mathbb{R}^n.$$

Here is a geometric observation we need.

Lemma 3.2. *Let $l_0,l_1,k,r\in\mathbb{Z}$ and $x\in(V_{l_0+1,r}\setminus V_{l_0,r})\cap(V_{l_1+1,r+1}\setminus V_{l_1,r+1})$.*

(1) *$l_0\le l_1$.*

(2) *If* $B(x,2^{-k}) \cap (V_{k,r} \setminus V_{k,r+1}) \neq \emptyset$, *then* $l_0 \le k \le l_1+1$.
(3) *If* $l_0+2 \le k \le l_1-1$, *then* $B(x,2^{-k}) \subset V_{k,r} \setminus V_{k,r+1}$.

PROOF. We remark that $x \in (V_{l_0+1,r} \setminus V_{l_0,r}) \cap (V_{l_1+1,r+1} \setminus V_{l_1,r+1})$ if and only if $2^{-l_0} \le \mathrm{dist}(x,\partial\Omega_r) < 2^{-l_0+1}$ and $2^{-l_1} \le \mathrm{dist}(x,\partial\Omega_{r+1}) < 2^{-l_1+1}$.

(1) Since $\Omega_r \supset \Omega_{r+1}$, $\mathrm{dist}(x,\partial\Omega_{r+1}) \le \mathrm{dist}(x,\partial\Omega_r)$. Thus, in view of the above observation, the result follows immediately.
(2) Let $y \in B(x,2^{-k}) \cap (V_{k,r} \setminus V_{k,r+1})$. Since $y \in V_{k,r}$,

$$2^{-l_0+1} > \mathrm{dist}(x,\partial\Omega_r) \ge \mathrm{dist}(y,\partial\Omega_r) - |x-y| \ge 2^{1-k} - 2^{-k} = 2^{-k},$$

implying $k \ge l_0$. Likewise, since $y \notin V_{k,r+1}$,

$$2^{-l_1} \le \mathrm{dist}(x,\partial\Omega_{r+1}) \le \mathrm{dist}(y,\partial\Omega_{r+1}) + |x-y| \le 2^{1-k} + 2^{-k} < 2^{2-k}.$$

implying $k \le l_1+1$.
(3) Let $z \in B(x,2^{-k})$. Then since $x \in V_{l_0+1,r}$ and $k \ge l_0+2$,

$$\mathrm{dist}(z,\partial\Omega_r) \ge \mathrm{dist}(x,\partial\Omega_r) - |x-z| \ge 2^{-l_0} - 2^{-k} \ge 2^{1-k}.$$

Hence $B(x,2^{-k}) \subset V_{k,r}$. Likewise, since $x \notin V_{l_1,r+1}$,

$$\mathrm{dist}(z,\partial\Omega_{r+1}) \le \mathrm{dist}(x,\partial\Omega_{r+1}) + |x-z| < 2^{1-l_1} + 2^{-k} \le 2^{1-k}.$$

Hence $B(x,2^{-k}) \cap V_{k,r+1} = \emptyset$.

□

Fix an integer $L > \frac{n}{2p}$ here and below. Let $\Phi, \Psi, \Theta \in C_c^\infty(\mathbb{R}^n)$ be even functions supported in the unit ball and satisfy

$$\Psi = \Phi^1 - \Phi = \Delta^L \Theta, \quad \int_{\mathbb{R}^n} \Phi(x)\mathrm{d}x = 1. \tag{3.4}$$

The pair (Φ, Ψ, Θ) is known to exist [?]. Write $\tilde\Psi \equiv \Phi^1 + \Phi$.

Let $f \in \mathcal{S}'(\mathbb{R}^n) \setminus \{0\}$ be a distribution vanishing weakly at infinity. Also let $k, r \in \mathbb{Z}$. We set

$$f_{k,r} \equiv \Psi^k * (\chi_{V_{k,r} \setminus V_{k,r+1}} \cdot \tilde\Psi^k * f).$$

A geometric observation shows that $f_{k,r}$ is supported on Ω_r. We also need the L^∞-bound for the function of this type.

LEMMA 3.3. *Let* $\Gamma, \tilde\Gamma \in C_c^\infty(\mathbb{R}^n)$ *with* $\mathrm{supp}(\Gamma), \mathrm{supp}(\tilde\Gamma) \subset B(1)$. *Also let* $E \subset \mathbb{R}^n$ *be a measurable set. Then*

$$|\Gamma^k * (\chi_{(V_{k,r} \setminus V_{k,r+1}) \cap E} \cdot \tilde\Gamma^k * f)(x)| \lesssim 2^r$$

for all $x \in \mathbb{R}^n$.

PROOF. Since

$$\begin{aligned}
&|\Gamma^k * (\chi_{(V_{k,r} \setminus V_{k,r+1}) \cap E} \cdot \tilde\Gamma^k * f)(x)| \\
&\le \int_{V_{k,r} \setminus V_{k,r+1}} |\Gamma^k(x-y)\tilde\Gamma^k * f(y)|\mathrm{d}y \\
&\le A \int_{V_{k,r} \setminus V_{k,r+1}} |\Gamma^k(x-y)| \left(\inf_{z \in B(y,2^{2-k})} |\tilde\Gamma^k * f(z)| \right) \mathrm{d}y
\end{aligned}$$

thanks to Lemma **??**, we have

$$|\Gamma^k * (\chi_{(V_{k,r}\setminus V_{k,r+1})\cap E} \cdot \tilde{\Gamma}^k * f)(x)| \lesssim 2^r \int_{V_{k,r}\setminus V_{k,r+1}} |\Gamma^k(x-y)|\mathrm{d}y \lesssim 2^r$$

by the definition of $\mathcal{M}_N f$, $V_{k,r+1}$ and Ω_r. □

We decompose

$$f = \sum_{k=-\infty}^{\infty} \Psi^k * \tilde{\Psi}^k * f = \sum_{k=-\infty}^{\infty} \left(\sum_{r=-\infty}^{\infty} f_{k,r} \right). \tag{3.5}$$

We need to pay attention to the order of the summation in (**??**). However, if f is good enough, then we can interchange the order of the summation.

LEMMA 3.4. *Assume that $f \in H^p(\mathbb{R}^n)$ with $0 < p \le 1$ and that the integer L in (**??**) satisfies $L \in \mathbb{Z} \cap \left(\frac{n}{2p}, \infty\right)$. Then*

$$f = \sum_{k,r\in\mathbb{Z}} f_{k,r}$$

in the sense of absolute convergence in $\mathcal{S}'(\mathbb{R}^n)$. Namely,

$$\sum_{k,r\in\mathbb{Z}} |\langle f_{k,r}, \varphi \rangle| < \infty$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

PROOF. Fix $k, r \in \mathbb{Z}$. Recall that Ψ is an even function. We calculate

$$\langle f_{k,r}, \varphi \rangle = \int_{V_{k,r}\setminus V_{k,r+1}} \Psi^k * \varphi(y) \tilde{\Psi}^k * f(y)\mathrm{d}y.$$

Thanks to (**??**), by using integration by parts, we have

$$\left|\Psi^k * \varphi(y)\right| = \left|(\Delta^L \Theta)^k * \varphi(y)\right| \lesssim 2^{-\max(0,2kL)} \langle y \rangle^{-2n-1} \quad (y \in \mathbb{R}^n),$$

if $k \in \mathbb{Z}$. Meanwhile, if $y \in V_{k,r} \setminus V_{k,r+1}$, we have

$$|\tilde{\Psi}^k * f(y)| \le A p_N(\tilde{\Psi}) \inf_{z\in B(y,2^{-k})} \mathcal{M}_N f(z) \lesssim 2^{\frac{kn}{p}} \|\mathcal{M}_N f\|_{L^p} = 2^{\frac{kn}{p}} \|f\|_{H^p} \tag{3.6}$$

thanks to Lemma **??**. As a consequence,

$$|\langle f_{k,r}, \varphi \rangle| \lesssim 2^{\frac{kn}{p} - \max(0,2kL)} \|f\|_{H^p} \int_{V_{k,r}\setminus V_{k,r+1}} \frac{\mathrm{d}y}{\langle y \rangle^{2n+1}}.$$

If we add this inequality over $r \in \mathbb{Z}$, then we obtain

$$\begin{aligned} \sum_{r\in\mathbb{Z}} |\langle f_{k,r}, \varphi \rangle| &\lesssim 2^{\frac{kn}{p} - \max(0,2kL)} \|f\|_{H^p} \int_{\mathbb{R}^n} \frac{\mathrm{d}y}{\langle y \rangle^{2n+1}} \\ &\sim 2^{\frac{kn}{p} - \max(0,2kL)} \|f\|_{H^p}. \end{aligned} \tag{3.7}$$

If $L > \frac{n}{2p}$, then this estimate is summable over $k \in \mathbb{Z}$.

Once we can prove that the series converges absolutely, we see that the series converges back to f thanks to (**??**). □

Remark that the power $2n+1$ in the above proof (see (**??**) for example) seems superfluous: This number will turn out important in Section **??**.

From Lemma **??**,

$$f = \sum_{r=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} f_{k,r} \right) \tag{3.8}$$

in $\mathcal{S}'(\mathbb{R}^n)$. We analyze the summand with r fixed.

Lemma 3.5. *Let $r \in \mathbb{Z}$. Then*

$$\left| \sum_{k=-\infty}^{\infty} f_{k,r}(x) \right| \lesssim 2^r$$

for all $x \in \mathbb{R}^n$.

Proof. Since each $f_{k,r}$ is supported on Ω_r, we may assume that $x \in \Omega_r$. We distinguish two cases:

- Let $x \in \Omega_{r+1}$. Choose $l_0, l_1 \in \mathbb{Z}$ so that $x \in (V_{l_0+1,r} \setminus V_{l_0,r}) \cap (V_{l_1+1,r+1} \setminus V_{l_1,r+1})$. Thanks to Lemma **??**(1), $l_0 \le l_1$. We further assume that $l_0 + 3 \le l_1$; otherwise we may simply use Lemma **??**

  Fix $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ so that $f_{k,r}(x) \neq 0$. Then $B(x, 2^{-k}) \cap (V_{k,r} \setminus V_{k,r+1}) \neq \emptyset$. Thus $l_0 \le k \le l_1 + 1$ according to Lemma **??**(2).

  Due to Lemma **??**(3), $f_{k,r}(x) = \Psi^k * \tilde{\Psi}^k * f(x) = \Phi^{k+1} * \Phi^{k+1} * f(x) - \Phi^k * \Phi^k * f(x)$ if $l_0 + 2 \le k \le l_1 - 1$. Hence thanks to Lemma **??**

  $$\sum_{k=l_0+2}^{l_1-1} f_{k,r}(x) = \Phi^{l_1} * \Phi^{l_1} * f(x) - \Phi^{l_0+2} * \Phi^{l_0+2} * f(x) = \mathrm{O}(2^r).$$

  We do not have to take into account the terms for $k \ge l_1 + 2$ or $k \le l_0 - 1$ since they vanish according to Lemma **??**(2). If we handle the terms for $l_0 \le k \le l_0 + 1$ and $l_1 \le k \le l_1 + 1$ using Lemma **??** again, then we obtain the desired result.
- Let $x \in \Omega_r \setminus \Omega_{r+1}$. Then let $l_1 = \infty$ and $x \in V_{l_0+1,r} \setminus V_{l_0,r}$ with $l_0 \in \mathbb{Z}$ in the above and go through the same argument.

□

We can generalize Lemma **??**, whose proof we omit.

Lemma 3.6. *Let $l_0, l_1, r \in \mathbb{Z}$ satisfy $l_0 < l_1$. Then*

$$\left| \sum_{k=l_0}^{l_1} f_{k,r}(x) \right| \lesssim 2^r$$

for all $x \in \mathbb{R}^n$, where the implicit constant does not depend on l_0 and l_1.

For an arbitrary set S, define an open set S_k by $S_k \equiv \{y \in \mathbb{R}^n : \mathrm{dist}(y, S) < 2^{1-k}\}$.

Lemma 3.7. *Let $l \in \mathbb{Z}$ and $x \in S_l \setminus S_{l+1}$.*

1. *Whenever $k < l$, $B(x, 2^{-k}) \subset S_k$.*
2. *Whenever $k \ge l + 2$, $B(x, 2^{-k}) \cap S_k = \emptyset$.*

Proof. Since $x \in S_l \setminus S_{l+1}$, $2^{-l} \le \mathrm{dist}(x, S) < 2^{1-l}$. Let $y \in B(x, 2^{-k})$.

(1) Using the triangle inequality, we obtain
$$\mathrm{dist}(y,S) \le |x-y| + \mathrm{dist}(x,S) \le 2^{-k} + 2^{1-l} \le 2^{1-k},$$
implying $y \in S_k$.

(2) Using the triangle inequality again, we obtain
$$\mathrm{dist}(y,S) \ge -|x-y| + \mathrm{dist}(x,S) > -2^{-k} + 2^{1-l} \ge 2^{1-k},$$
implying $y \notin S_k$.

□

Let S be a set. Set
$$F_S(x) \equiv \sum_{k=-\infty}^{\infty} \Psi^k * (\chi_{(V_{k,r} \setminus V_{k,r+1}) \cap S_k} \cdot \tilde{\Psi}^k * f)(x) \quad (x \in \mathbb{R}^n).$$
If S is bounded, then by the Fubini theorem, we see that F_S satisfies the same moment condition as Ψ^k.

Lemma 3.8. *For any set S and $r \in \mathbb{Z}$, $\|F_S\|_{L^\infty} \lesssim 2^r$.*

Proof. Let $x \in S$ and $k \in \mathbb{Z}$. Then $B(x, 2^{-k}) \subset S_k$ and hence
$$(V_{k,r} \setminus V_{k,r+1}) \cap S_k \cap B(x,2^{-k}) = (V_{k,r} \setminus V_{k,r+1}) \cap B(x,2^{-k}).$$
Thus
$$F_S(x) = \sum_{k=-\infty}^{\infty} \Psi^k * (\chi_{V_{k,r} \setminus V_{k,r+1}} \cdot \tilde{\Psi}^k * f)(x) = \mathrm{O}(2^r).$$
Suppose $x \in S_l \setminus S_{l+1}$ for some $l \in \mathbb{Z}$. Then thanks to Lemmas **??**, **??** and **??**,
$$\begin{aligned} F_S(x) &= \sum_{k=-\infty}^{l-1} \Psi^k * (\chi_{V_{k,r} \setminus V_{k,r+1}} \cdot \tilde{\Psi}^k * f)(x) \\ &\quad + \sum_{k=l}^{l+1} \Psi^k * (\chi_{(V_{k,r} \setminus V_{k,r+1}) \cap S_k} \cdot \tilde{\Psi}^k * f)(x) \\ &= \mathrm{O}(2^r). \end{aligned}$$

□

We slightly generalize Lemma **??**.

Let S be a set and $\kappa \in \mathbb{R}$. Set
$$F_{S,\kappa}(x) \equiv \sum_{k=-\infty}^{\infty} \chi_{(\kappa,\infty)}(k) \Psi^k * (\chi_{(V_{k,r} \setminus V_{k,r+1}) \cap S_k} \cdot \tilde{\Psi}^k * f)(x) \quad (x \in \mathbb{R}^n).$$

Lemma 3.9. *For any set S, $\kappa \in \mathbb{R}$ and $r \in \mathbb{Z}$, $\|F_{S,\kappa}\|_{L^\infty} \lesssim 2^r$.*

We do not prove Lemma **??** since it is similar to Lemma **??**.

Form the Whitney decomposition of $\Omega_r = \{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > 2^r\}$ for each $r \in \mathbb{Z}$. For $x \in \mathbb{R}^n$ and $r \in \mathbb{Z}$, we let $\rho_r(x) \equiv \mathrm{dist}(x, \partial\Omega_r)$. We let $\left\{B\left(\xi_{j,r}, \frac{\rho_{j,r}}{5}\right)\right\}_{j=1}^{\infty}$ be a maximal disjoint family, where $\rho_{j,r} \equiv \rho_r(\xi_{j,r})$ for $j \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then we have the following properties:

(1) $\Omega_r = \bigcup_{j=1}^{\infty} B\left(\xi_{j,r}, 2^{-1}\rho_{j,r}\right)$.

(2) Let $j \in \mathbb{N}$ and $r \in \mathbb{Z}$. Set

$$\mathcal{J}_{j,r} \equiv \left\{ \nu \in \mathbb{N} \cap (j,\infty) \,:\, B\left(\xi_{j,r}, \frac{3}{4}\rho_{j,r}\right) \cap B\left(\xi_{\nu,r}, \frac{3}{4}\rho_{\nu,r}\right) \neq \emptyset \right\}.$$

Then $\sharp \mathcal{J}_{j,r} \le 300^n$ and $7^{-1}\rho_{\nu,r} \le \rho_{j,r} \le 7\rho_{\nu,r}$ for each $\nu \in \mathcal{J}_{j,r}$.

Let $j \in \mathbb{N}$ and $k, r \in \mathbb{Z}$. We define $E_{j,k,r} \equiv B\left(\xi_{j,r}, 2^{-1}\rho_{j,r} + 2^{1-k}\right) \cap (V_{k,r} \setminus V_{k,r+1})$ if $B\left(\xi_{j,r}, 2^{-1}\rho_{j,r}\right) \cap (V_{k,r} \setminus V_{k,r+1}) \neq \emptyset$. If $B\left(\xi_{j,r}, 2^{-1}\rho_{j,r}\right) \cap (V_{k,r} \setminus V_{k,r+1}) = \emptyset$, then define $E_{j,k,r} \equiv \emptyset$. We have

$$\bigcup_{j=1}^{\infty} E_{j,k,r} = V_{k,r} \setminus V_{k,r+1} \quad (k, r \in \mathbb{Z}).$$

We set

$$R_{j,k,r} \equiv E_{j,k,r} \setminus \bigcup_{\nu=j+1}^{\infty} E_{\nu,k,r} \quad (j \in \mathbb{N}, k, r \in \mathbb{Z}).$$

We write

$$F_{j,k,r} \equiv \Psi^k * (\chi_{R_{j,k,r}} \cdot \tilde{\Psi}^k * f)$$

and

$$F_{j,r} \equiv \sum_{l=-\infty}^{\infty} F_{j,l,r}$$

for $j \in \mathbb{N}$ and $k, r \in \mathbb{Z}$. As before, we can check that the sum defining $F_{j,r}$ converges absolutely in $\mathcal{S}'(\mathbb{R}^n)$. The next lemma shows that the limit belongs to $L^\infty(\mathbb{R}^n)$. Also observe that

$$f = \sum_{(k,r)\in\mathbb{Z}^2} f_{k,r} = \sum_{(j,k,r)\in\mathbb{N}\times\mathbb{Z}^2} F_{j,k,r} = \sum_{(j,r)\in\mathbb{N}\times\mathbb{Z}} F_{j,r}.$$

Lemma 3.10. *For all $j \in \mathbb{N}$ and $r \in \mathbb{Z}$, $|F_{j,r}| \lesssim 2^r \chi_{B(\xi_{j,r}, 8\rho_{j,r})}$.*

Proof. The proof consists of two steps.

- Let us verify that $F_{j,r}$ vanishes outside $B(\xi_{j,r}, 5\rho_{j,r})$. Let $k \in \mathbb{Z}$ satisfy $R_{j,k,r} \neq \emptyset$. Then $B(\xi_{j,r}, 2^{-1}\rho_{j,r}) \cap (V_{k,r} \setminus V_{k,r+1}) \neq \emptyset$. Let $z \in B(\xi_{j,r}, 2^{-1}\rho_{j,r}) \cap (V_{k,r} \setminus V_{k,r+1})$. Then

$$\frac{3}{2}\rho_{j,r} \ge |\xi_{j,r} - z| + \mathrm{dist}(\xi_{j,r}, \partial\Omega_r) \ge \mathrm{dist}(z, \partial\Omega_r) \ge 2^{1-k},$$

so that $\rho_{j,r} \ge \frac{4}{3} \cdot 2^{-k}$. Thus, $B(\xi_{j,r}, 2^{-1}\rho_{j,r} + 2^{1-k}) \subset B\left(\xi_{j,r}, 2\rho_{j,r}\right)$. Since

$$\mathrm{supp}(F_{j,k,r}) \subset B\left(\xi_{j,r}, \frac{7}{2}\rho_{j,r} + 2^{1-k} + 2^{-k}\right) \subset B(\xi_{j,r}, 5\rho_{j,r}),$$

we obtain the desired result.

- Let us obtain the L^∞-bound of $F_{j,r}$. If $k \in \mathbb{Z}$ satisfies $2^{-k} \ge 2\rho_{j,r}$, then from the definition of $\rho_{j,r}$,

$$\sup_{z \in B(\xi_{j,r}, 2^{-1}\rho_{j,r})} \mathrm{dist}(z, \partial\Omega_r) = \frac{3}{2}\rho_{j,r} \le 2^{-k}$$

and hence $B(\xi_{j,r}, 2^{-1}\rho_{j,r}) \cap (V_{k,r} \setminus V_{k,r+1}) = \emptyset$. Namely, if $k \le -\log_2 \rho_{j,r} - 1$, then $B(\xi_{j,r}, 2^{-1}\rho_{j,r}) \cap (V_{k,r} \setminus V_{k,r+1}) = \emptyset$. From the definition of $\mathcal{J}_{j,r}$,

$$B(\xi_{\nu,r}, 2^{-1}\rho_{\nu,r} + 2^{1-k}) \subset B\left(\xi_{\nu,r}, \frac{3}{4}\rho_{\nu,r}\right)$$

for all $k \ge 10 - \log_2 \rho_{j,r}$ and $\nu \in \mathcal{J}_{j,r}$. Let

$$S \equiv \bigcup_{\nu \in \mathcal{J}_{j,r}} B(\xi_{\nu,r}, 2^{-1}\rho_{\nu,r}), \quad \tilde{S} \equiv S \cup B(\xi_{j,r}, 2^{-1}\rho_{j,r}).$$

Then we have

$$S_k \equiv \bigcup_{\nu \in \mathcal{J}_{j,r}} B(\xi_{\nu,r}, 2^{-1}\rho_{\nu,r} + 2^{1-k}), \quad (\tilde{S})_k \equiv S_k \cup B(\xi_{j,r}, 2^{-1}\rho_{j,r} + 2^{1-k})$$

and

$$R_{j,k,r} = \left\{ (\tilde{S})_k \cap (V_{k,r} \setminus V_{k,r+1}) \right\} \setminus \{ S_k \cap (V_{k,r} \setminus V_{k,r+1}) \}.$$

Thus

$$F_{j,r} = F_{\tilde{S}, 10 - \log_2 \rho_{j,r}} - F_{S, 10 - \log_2 \rho_{j,r}} + \sum_{-\log_2 \rho_{j,r} \le k \le -\log_2 \rho_{j,r} + 10} F_{j,k,r}.$$

It remains to use Lemma **??**.

□

We conclude the proof of Proposition **??**. Equality (**??**) is a consequence of Lemma **??**. Thanks to Lemma **??**, $f_{k,r}$ satisfies (**??**). It remains to prove (**??**). Using Lemma **??** again and the definition of Ω_r, we estimate

$$\begin{aligned}
\sum_{(j,r) \in \mathbb{N} \times \mathbb{Z}} (\|F_{j,r}\|_{L^\infty} \chi_{B(\xi_{j,r}, 2^{-1}\rho_{j,r})})^u &\lesssim \sum_{(j,r) \in \mathbb{N} \times \mathbb{Z}} 2^{ur} \chi_{B(\xi_{j,r}, 2^{-1}\rho_{j,r})} \\
&\lesssim \sum_{r=-\infty}^{\infty} 2^{ur} \chi_{\Omega_r} \\
&= \sum_{r=-\infty}^{\infty} 2^{ur} \chi_{(2^r, \infty]}(\mathcal{M}_N f) \\
&\lesssim (\mathcal{M}_N f)^u,
\end{aligned}$$

as required.

4. Applications to Hardy spaces based on other ball Banach spaces

Here we modify the proof especially (**??**) to obtain the decomposition results for distributions in Hardy spaces based on other ball Banach spaces. As we saw in Section **??**, it matters that the distribution vanishes weakly at infinity and that the distribution satisfies (**??**). Section **??** considers the weighted Hardy space $H^p(w)$ with $0 < p < \infty$ and $w \in A_1$. As an application of Section **??**, we consider Hardy spaces based on ball Banach function spaces. We can locate Sections **??**, **??** and **??** as further examples of Section **??**. Hardy spaces with weight in A_∞, variable Hardy spaces and Hardy–Morrey spaces are considered in Sections **??**, **??** and **??**, respectively. We will give a precise condition on L in Sections **??**, **??** and **??**. We need to define the above spaces by way of $\mathcal{M}_N$. It is known in [**?**] that the function spaces we are going to handle in this section do not depend on the choice of N as long as $N \gg 1$. This condition L is used to obtain the boundedness of operators. However, as we mentioned, the condition on L can be tightened since we are considering the decompositions of distributions. So, although we present some concrete conditions on L in Sections **??**, **??** and **??**, we still may assume that L is large enough.

We will make use of the Hardy–Littlewood maximal operator M. The space $L^0(\mathbb{R}^n)$ denotes the set of all complex/$[0,\infty]$-valued measurable functions considered modulo the difference on the set of measure zero. For $f \in L^0(\mathbb{R}^n)$, define a function Mf by

$$Mf(x) \equiv \sup_{B \in \mathcal{B}} \chi_B(x) m_B(|f|) \quad (x \in \mathbb{R}^n). \tag{4.1}$$

Here $m_B(f)$ stands for the average of a locally integrable or non-negative function f over B. The mapping $M : f \mapsto Mf$ is called the *Hardy–Littlewood maximal operator*. We also use the *powered Hardy–Littlewood maximal operator* $M^{(\eta)}$ defined by

$$M^{(\eta)}f(x) \equiv \sup_{B \in \mathcal{B}} \left(\chi_B(x) m_B(|f|^\eta)\right)^{\frac{1}{\eta}},$$

where $0 < \eta < \infty$ and $f \in L^0(\mathbb{R}^n)$. Together with the Hardy–Littlewood maximal operator, we need to recall the notion of weights as well as their fundamental properties, which will be done in Sections **??** and **??**. See [**?**] for more details on weights.

We remark that the same idea can be used for Hardy spaces based on other function spaces such as the ones considered in [**?, ?, ?, ?, ?**].

4.1. Weighted Hardy space $H^p(w)$ with $w \in A_1$. As the starting point, we seek to change $L^p(\mathbb{R}^n)$ by $L^p(w)$ for some good class of weights. Although we work in a rather special setting, this setting will be a core of our argument. By a weight we mean a function $w \in L^0(\mathbb{R}^n)$ which satisfies $0 < w(x) < \infty$ for almost all $x \in \mathbb{R}^n$. We write $w(A) \equiv \int_A w(x)\mathrm{d}x$ if A is a measurable set of $\mathbb{R}^n$. The space $L^p(w)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{L^p(w)} \equiv \|f w^{\frac{1}{p}}\|_{L^p} < \infty$ (cf. [**?**]).

To proceed further, we compare the weights w and 1. Here we introduce a general definition following the book [**?**, p. 402]. A weight w_1 is comparable to a weight w_2 if there exist $\alpha, \beta < 1$ such that $w_1(A) \le \beta w_1(B)$ for any measurable set A and any $B \in \mathcal{B}$ satisfying $A \subset B$ and $w_2(A) \le \alpha w_2(B)$. It is important that comparability is symmetric; w_1 is comparable to w_2 if and only if w_2 is comparable to w_1. In this case there exists $\delta > 0$ such that

$$\frac{w_1(A)}{w_1(B)} \lesssim \left(\frac{w_2(A)}{w_2(B)}\right)^\delta \tag{4.2}$$

and that

$$\frac{w_2(A)}{w_2(B)} \lesssim \left(\frac{w_1(A)}{w_1(B)}\right)^\delta \tag{4.3}$$

for any measurable set A and any $B \in \mathcal{B}$ satisfying $A \subset B$.

Let $0 < p < \infty$, w be a weight and $f \in \mathcal{S}'(\mathbb{R}^n)$. Define

$$\|f\|_{H^p(w)} \equiv \|\mathcal{M}_N f\|_{L^p(w)}.$$

The weighted Hardy space $H^p(w)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quantity $\|f\|_{H^p(w)}$ is finite. In the present paper, as long as $N \gg 1$, the definition of $H^p(w)$ does not depend on the choice of N.

As a preliminary and important step, we consider A_1-weights among other classes of weights. Recall that a locally integrable weight w is said be an A_1-weight, if there exists $C_0 > 0$ such that

$$Mw(x) \le C_0 w(x) \tag{4.4}$$

for a.e. $x \in \mathbb{R}^n$. The infimum of C_0 satisfying (**??**) is called the A_1-norm.

Let $\Gamma \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$. We estimate

$$|\Gamma^k * f(x)| \le A \inf_{y \in B(x,2^{-k})} |\Gamma^k * f(y)| \le \frac{A p_N(\Gamma)}{w(B(x,2^{-k}))^{\frac{1}{p}}} \|f\|_{H^p(w)}$$

using Lemma **??**. It follows from (**??**) and (**??**) that

$$\frac{w(B(x,1))}{w(B(x,2^{-k}))} \lesssim \left(\frac{|B(x,1)|}{|B(x,2^{-k})|} \right)^{\delta} = 2^{kn\delta}$$

for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z} \setminus \mathbb{N}$ and that

$$\frac{w(B(x,2^{-k}))}{w(B(x,1))} \gtrsim \left(\frac{|B(x,2^{-k})|}{|B(x,1)|} \right)^{\delta} = 2^{-kn\delta}$$

for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Also, it follows from (**??**) that

$$\langle x \rangle^{-n} w(B(1)) \lesssim w(B(x,1)) \lesssim \langle x \rangle^{n} w(B(1)).$$

Therefore,

$$|\Gamma^k * f(x)| \lesssim \frac{2^{\frac{kn\delta}{p}}}{w(B(x,1))^{\frac{1}{p}}} \|f\|_{H^p(w)} \lesssim 2^{\frac{kn\delta}{p}} \langle x \rangle^{\frac{n}{p}} \|f\|_{H^p(w)}. \tag{4.5}$$

Recall that $\Gamma \in \mathcal{S}(\mathbb{R}^n)$ is arbitrary. By letting $\Gamma = \tilde{\Psi}$, we learn that a counterpart to (**??**) still holds. Estimate (**??**) also shows that f vanishes weakly at infinity. As in [**?**], $A_1 \cap L^1(\mathbb{R}^n) = \emptyset$. Thus, Ω_r, the level set of $\mathcal{M}_N f$ at 2^r, can not coincide with $\mathbb{R}^n$, allowing us to use Lemma **??**. Therefore, the same conclusion with $L \gg 1$ as Theorem **??** holds.

THEOREM 4.1. *Let $0 < p < \infty$, $f \in H^p(w)$ with $w \in A_1$ and let $L \gg 1$. Then there exist a countable collection $\{f_j\}_{j=1}^{\infty}$ of $L^{\infty}_{\rm c}$-functions having moment of order L and a countable collection $\{B_j\}_{j=1}^{\infty} \subset \mathcal{B}$ satisfying* (**??**), (**??**) *and* (**??**).

4.2. Hardy spaces based on ball Banach function spaces. Based on Section **??**, we establish a general theory of the decomposition of distributions in Hardy spaces based on ball Banach function spaces.

DEFINITION 4.2 (Ball Banach function space). A mapping $\|\cdot\|_Y \to [0,\infty]$ is said to be a *ball Banach function norm* and the couple $(Y(\mathbb{R}^n), \|\cdot\|_Y)$ is said to be a *ball Banach function space* if $(Y(\mathbb{R}^n), \|\cdot\|_Y)$ satisfies (1)–(7) for all f, g, $f_j \subset L^0(\mathbb{R}^n)$, $j \in \mathbb{N}$, and $\lambda \in \mathbb{C}$.

(1) $(Y(\mathbb{R}^n), \|\cdot\|_Y)$ is a Banach space with the following property: $f \in Y(\mathbb{R}^n)$ if and only if $\|f\|_Y < \infty$.
(2) (Norm property):
  (A1) (Positivity): $\|f\|_Y \ge 0$.
  (A2) (Strict positivity) $\|f\|_Y = 0$ if and only if $f = 0$ a.e..
  (B) (Homogeneity): $\|\lambda f\|_Y = |\lambda| \cdot \|f\|_Y$.
  (C) (Triangle inequality): $\|f+g\|_Y \le \|f\|_Y + \|g\|_Y$.
(3) (Symmetry): $\|f\|_Y = \|\,|f|\,\|_Y$.
(4) (Lattice property): If $0 \le g \le f$ a.e., then $\|g\|_Y \le \|f\|_Y$.
(5) (Fatou property): If $0 \le f_1 \le f_2 \le \cdots$ and $\lim_{j\to\infty} f_j = f$, then $\lim_{j\to\infty} \|f_j\|_Y = \|f\|_Y$.
(6) For $B \in \mathcal{B}$, $\chi_B \in Y(\mathbb{R}^n)$.

(7) If $B \in \mathcal{B}$ and $f \in Y(\mathbb{R}^n)$, then $\chi_B f \in L^1(\mathbb{R}^n)$.

For a ball Banach function space $Y(\mathbb{R}^n)$, we let

$$Y'(\mathbb{R}^n) \equiv \left\{ f \in L^0(\mathbb{R}^n) \,:\, \|f\|_{Y'} \equiv \sup_{g \in Y,\, \|g\|_Y = 1} \|f \cdot g\|_{L^1} < \infty \right\}.$$

The space $Y'(\mathbb{R}^n)$ is called the Köthe dual of $Y(\mathbb{R}^n)$ and it is known that $Y'(\mathbb{R}^n)$ is a ball Banach space if $Y(\mathbb{R}^n)$ is a ball Banach space; see [**?**, Proposition 2.3]. Assume that $Y(\mathbb{R}^n)$ is a ball Banach function space such that M is bounded on $Y(\mathbb{R}^n)$ and $Y'(\mathbb{R}^n)$. Then there exists $\eta > 1$ such that $M^{(\eta)}$ is also bounded on $Y'(\mathbb{R}^n)$ according to [**?**, Corollary 6.1]. Thus, for all $f \in Y(\mathbb{R}^n)$,

$$\|f\|_{L^1(M^{(\eta)}\chi_{B(1)})} \le \|f\|_Y \|M^{(\eta)}\chi_{B(1)}\|_{Y'} \lesssim \|f\|_Y \|\chi_{B(1)}\|_{Y'} \sim \|f\|_Y. \tag{4.6}$$

We can develop the theory of the decomposition of Hardy spaces based on $Y(\mathbb{R}^n)$. But we can extend the class of linear spaces to some extent. Consider the power of $Y(\mathbb{R}^n)$: For $0 < p < \infty$, we define

$$\|f\|_{Y^{(p)}} \equiv (\||f|^p\|_Y)^{\frac{1}{p}}$$

for all $f \in L^0(\mathbb{R}^n)$. The *p-convexification* $Y^{(p)}(\mathbb{R}^n)$ of $Y(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{Y^{(p)}} < \infty$. For example, $(L^p)^{(u)}(\mathbb{R}^n) = L^{pu}(\mathbb{R}^n)$ for all $0 < u < \infty$ and $1 \le p \le \infty$.

Let $Y(\mathbb{R}^n)$ be as above and let $X(\mathbb{R}^n) \equiv Y^{(p)}(\mathbb{R}^n)$ for some $0 < p < \infty$. The X-based Hardy space $HX(\mathbb{R}^n)$ collects all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{HX} \equiv \|\mathcal{M}_N f\|_X$ is finite. The number N will do as long as $N \gg 1$. As is seen from (**??**), $HX(\mathbb{R}^n)$ is embedded into $H^p(w)$ for some $w \in A_1$. Therefore, the space $HX(\mathbb{R}^n)$ falls within the scope of Theorem **??**.

THEOREM 4.3. *Let $Y(\mathbb{R}^n)$ be a ball Banach function space such that M is bounded on $Y(\mathbb{R}^n)$ and $Y'(\mathbb{R}^n)$. Let $0 < p < \infty$ and define $X(\mathbb{R}^n) \equiv Y^{(p)}(\mathbb{R}^n)$. Then for any $f \in HX(\mathbb{R}^n)$ and $L \gg 1$, there exist a countable collection $\{f_j\}_{j=1}^\infty$ of $L^\infty_{\rm c}$-functions having moment of order L and a countable collection $\{B_j\}_{j=1}^\infty \subset \mathcal{B}$ satisfying* (**??**), (**??**) *and* (**??**).

4.3. A_∞-Weighted Hardy spaces. We expand Section **??** using Section **??**. A locally integrable weight w is said to be an *A_∞-weight*, if

$$[w]_{A_\infty} \equiv \sup_{B \in \mathcal{B}} m_B(w) \exp(-m_B(\log w)) < \infty.$$

The quantity $[w]_{A_\infty}$ is referred to as the *A_∞-constant*.

An important property of the class A_∞ is that any weight in A_∞ belongs to A_p for some $1 < p < \infty$. Let $1 < p < \infty$. A locally integrable weight w is an *A_p-weight*, if

$$[w]_{A_p} \equiv \sup_{B \in \mathcal{B}} m_B(w)(m_B(w^{-\frac{1}{p-1}}))^{p-1} < \infty.$$

It is remarkable that $w \in A_p$ if and only if M is bounded on $L^p(w)$. A direct consequence of the definition is that $w \in A_p$ if and only if $\sigma \in A_{p'}$, where $\sigma \equiv w^{-\frac{1}{p-1}}$. Remark also that $\{A_p\}_{p \in [1,\infty]}$ is nested: $A_1 \subset A_p \subset A_q \subset A_\infty$ if $1 \le p \le q \le \infty$.

Let $w \in A_\infty$ and $0 < p < \infty$. Based on Section **??**, we consider $H^p(w)$. Let $w \in A_\infty$, so that $w \in A_u$ for some $1 < u < \infty$. Then as we saw, M is bounded on $Y(\mathbb{R}^n) \equiv L^u(w)$ and on $Y'(\mathbb{R}^n) = L^{u'}(\sigma)$, where $\sigma \equiv w^{-\frac{1}{u-1}}$. Since $Y^{(p)}(\mathbb{R}^n) = L^{pu}(w)$ for all $0 < p < \infty$, the space $L^p(w)$ with $0 < p < \infty$ and $w \in A_\infty$ falls within the scope of Theorem **??**. In particular, Theorem **??** below can be used for another proof of the decomposition result in [**?**].

Theorem 4.4. *The same conclusion as Theorem* **??** *holds if we assume merely* $w \in A_\infty$ *in Theorem* **??**.

4.4. Variable Hardy spaces. For a measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) \equiv \bigcup_{\lambda>0} \{f \in L^0(\mathbb{R}^n) \,:\, \rho_p(\lambda^{-1} f) < \infty\},$$

where

$$\rho_p(f) \equiv \||f|^{p(\cdot)}\|_{L^1}$$

Moreover, for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ we define the *variable Lebesgue norm* $\|\cdot\|_{L^{p(\cdot)}}$ by

$$\|f\|_{L^{p(\cdot)}} \equiv \inf\left(\left\{\lambda > 0 \,:\, \rho_p(\lambda^{-1} f) \le 1\right\} \cup \{\infty\}\right).$$

Here we postulate the following conditions with some positive constants c_*, c^* and p_∞ independent of x and y:

- Local log-Hölder continuity condition:

$$|p(x) - p(y)| \le \frac{c_*}{\log(|x-y|^{-1})} \text{ for } x, y \in \mathbb{R}^n \text{ satisfying } |x-y| \le \frac{1}{2}, \tag{4.7}$$

- log-Hölder-type decay condition at infinity:

$$|p(x) - p_\infty| \le \frac{c^*}{\log(e + |x|)} \text{ for } x \in \mathbb{R}^n. \tag{4.8}$$

Assuming (**??**) and (**??**) as well as $0 < p_- \equiv \inf p(\cdot) \le p_+ \equiv \sup p(\cdot) < \infty$, we can define variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\mathcal{M}_N f \in L^{p(\cdot)}(\mathbb{R}^n)$. The number N will do as long as $N \gg 1$. Theorem **??** did not use the structure of the underlying space $L^p(\mathbb{R}^n)$ heavily except in (**??**) and in the proof of the fact that the distribution vanishes weakly at infinity. Modify slightly the proof of Theorem **??**, in particular (**??**), to have the following short proof of the key estimates of the decomposition theorems in [**?, ?**].

Theorem 4.5. *Assume that the exponent* $p(\cdot)$ *satisfies the above conditions. Let* $f \in H^{p(\cdot)}(\mathbb{R}^n)$ *and* $L \in \mathbb{Z} \cap [[\sigma_{p_-}], \infty)$. *Then there exist a countable collection* $\{f_j\}_{j=1}^\infty$ *of* $L^\infty_{\rm c}$*-functions having moment of order* L *and a countable collection* $\{B_j\}_{j=1}^\infty \subset \mathcal{B}$ *satisfying* (**??**), (**??**) *and* (**??**).

We may use Theorem **??** for another proof of Theorem **??**, since M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and on $L^{p'(\cdot)}(\mathbb{R}^n)$ as long as $p(\cdot)$ satisfies (**??**) and (**??**) as well as $1 < p_- \le p_+ < \infty$. Here $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ stands for the dual exponent.

4.5. Hardy–Morrey spaces. First of all, let us recall the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $0 < q \le p < \infty$. Define the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup\left\{|B|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(B)} \,:\, B \in \mathcal{B}\right\}$$

for $f \in L^0(\mathbb{R}^n)$. See [**?**] for example. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{\mathcal{M}^p_q}$ is finite. The *Hardy–Morrey space* $H\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{H\mathcal{M}^p_q} \equiv \|\mathcal{M}_N f\|_{\mathcal{M}^p_q}$ is finite. The number N will do as long as $N \gg 1$.

We recall the following facts:

(1) Thanks to [**?**], M is bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$ if $1 < q \le p < \infty$.
(2) In [**?**], the Köthe dual of $\mathcal{M}^p_q(\mathbb{R}^n)$ is specified if $1 < q \le p < \infty$.
(3) Thanks to [**?**], M is bounded on the Köthe dual of $\mathcal{M}^p_q(\mathbb{R}^n)$ if $1 < q \le p < \infty$.

Let $0 < q \le p < \infty$ again. Then from the above observation the space $\mathcal{M}^p_q(\mathbb{R}^n)$ falls under the scope of Theorem **??**.

Theorem 4.6. *Let $0 < q \le p < \infty$. Let $f \in H\mathcal{M}^p_q(\mathbb{R}^n)$ and $L \in \mathbb{Z} \cap [[\sigma_q], \infty)$. Then there exist a countable collection $\{f_j\}_{j=1}^\infty$ of $L^\infty_{\rm c}$-functions having moment of order L and a countable collection $\{B_j\}_{j=1}^\infty \subset \mathcal{B}$ satisfying* (**??**), (**??**) *and* (**??**).

Theorem **??** recovers the results in [**?, ?, ?**]. It is noteworthy that in the present paper we did not depend on the diagonal argument in [**?, ?**]. As we did for variable Hardy spaces, we may also reexamine the proof of Theorem **??** to prove Theorem **??**.

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Department of Mathematics, Chuo University, 1-13-27, Kasuga, 112-8551, Tokyo, Japan

Email address: yoshihiro-sawano@celery.ocn.ne.jp

Department of Mathematics, Chuo University, 1-13-27, Kasuga, 112-8551, Tokyo, Japan

Email address: a19.dad7@g.chuo-u.ac.jp

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