Revisiting the Aggregation Problem in Input-Output Analysis

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1. Introduction: Aggregation Problems

Aggregation plays an essential role in connecting functional relationships in micro and macro models in economic analyses. First, we consider the optimizing behavior of rational consumers in microeconomic theory. They are, subject to budget constraints, assumed to maximize utility in deciding their consumption. Therefore, the following consumption function can be derived for individual consumers:

$$c_i = f_i(y_i, m_i), \ (i = 1, 2, \cdots, n),$$

where c_i denotes consumption, y_i denotes income, m_i denotes individual consumer i's assets, and n is the number of consumers in a community, such as a nation. On the contrary, macroeconomic variables such as national consumption C, national income Y and national assets M are estimated by aggregating consumers' microvariables (usually by summing them); a consumption function, $C = F(Y, M)$, is obtained in a macroeconomic model. There are two types of aggregation problems in this procedure. One is how to statistically aggregate microvariables (c_i, y_i, *and* m_i) into macrovariables (C, Y, *and* M), and the other is how to aggregate microrelationships, $c_i = f_i(y_i, m_i)$, ($i = 1, 2, \cdots, n$), into the $C = F(Y, M)$ macrorelationship. The former is concerned with the measurement of macroeconomic statistics, such as the System of National Accounts (SNA) developed by the United Nations in empirical economic analysis; the latter relates to aggregation procedures dealt with by the aggregation theory.

It is still challenging to derive "consistent"[1)] macroeconomic relationships from

1) Consistency is a key notion in aggregation theory. It will be discussed in Section 3.

microeconomic relationships by aggregation methods because functional forms of microeconomic relationships are under very restrictive conditions: additive separability. In addition, it is difficult to consider the effects of income or assets distribution in macroeconomic relationships. Structural equations in macromodels are often derived from microeconomic theories by using analogies. Therefore, the behavior of a representative consumer or firm is often considered. The argument that microeconomic foundations should be emphasized in macroeconomic theories shows that aggregation problems remain unresolved in economic analysis.

Second, we consider the optimizing behavior of competitive individual firms. They face two constraints: technological and market constrains. The former is production functions:

$$x_j = g_j(l_j, k_j), \ (\mathrm{j}=1, 2, \cdots, m)$$

where x_j is the level of firm j's output, l_j is the level of its labor input, and k_j is the level of its capital input. The latter is the market prices of their outputs and inputs.

Each firm determines its input and output levels to maximize profits. On the contrary, a macromodel has usually a production function $X = G(L, K)$, where X is national output, L national employment, and K national capital, corresponding to microvariables x_j, l_j and k_j respectively, in macroeconomic analysis. In this case the same aggregation problems as mentioned above also arise.

On the production side, however, we have Leontief's input-output model, which explicitly takes into account the interindustry flow of commodities in the production system. The input-output model shows in detail how much of each commodity's output is used in the production of other commodities as an intermediate input. The aggregation over commodities is mainly used in this input-output analysis. A aggregation theory is considered to have begun in this area of economic analysis. This is caused by the linearity characteristic of the input-output model.

Many macroeconomic models used in empirical macroeconomic policies involve input-output models in many cases because great importance is attached to interindustry analysis. Input-output models are mainly used for assessing the effects of final demands on sectoral outputs, although they have been also used in order to assess the impacts of fluctuations in crude oil import prices or unit wage costs on the configuration of prices. They also play an important essential role in the determination of sectoral prices as well as sectoral outputs in multisectoral econometric models which have been recently constructed in many countries. However, Input-output tables used in these analyses are usually highly aggregated.

The aggregation problem in input-output analysis thus far attacked is mainly confined to the quantity model relating output levels to final demand levels. Therefore, the purpose of this study paper is to reexamine the aggregation problem in the input-output price model relating sectoral prices to direct factor costs in comparison with the quantity model connecting output levels with final demand levels.[2)]

The rest of the study is organized as follows. The notations and basic equations are presented in Section 2. Section 3 summarizes the main results obtained in the aggregation problems in the input-output analysis. Sections 4 and 5 present our analyses of the price model. Section 6 concludes the study.

2. Notations and Analysis Framework

We assume that an input-output model consisting of n industries is consolidated into an input-output model with m (< n) sectors. We call the former the original model and the latter the aggregated model. The r-th sector of the aggregated model includes the n_r industries of the original model, and

$$\sum_{r=1}^{m} n_r = n.$$

The values of each industrial output are assumed to be evaluated at constant producers' prices of the base period (t = 0). The notations are defined as follows: the superfix t attached on each notation indicates the period unless otherwise mentioned.

Notations of the original model:

$$x^t = \begin{bmatrix} x_1^t \\ x_2^t \\ \cdot \\ \cdot \\ \cdot \\ x_m^t \end{bmatrix}, \; x_r^t = \begin{bmatrix} x_{r1}^t \\ x_{r2}^t \\ \cdot \\ \cdot \\ \cdot \\ x_{rn_r}^t \end{bmatrix}, \; (r = 1, \cdots, m) \text{ denotes the output vector;}$$

$$f^t = \begin{bmatrix} f_1^t \\ f_2^t \\ \cdot \\ \cdot \\ \cdot \\ f_m^t \end{bmatrix}, \; f_r^t = \begin{bmatrix} f_{r1}^t \\ f_{r2}^t \\ \cdot \\ \cdot \\ \cdot \\ f_{rn_r}^t \end{bmatrix}, \; (r = 1, \cdots, m) \text{ denotes the final demand vector;}$$

2) Morishima, M. and F. Secton (1961) deals with factor costs in interindustrial aggregation.

$$v^t = \begin{bmatrix} v_1^t \\ v_2^t \\ \cdot \\ \cdot \\ \cdot \\ v_m^t \end{bmatrix}, \; v_r^t = \begin{bmatrix} v_{r1}^t \\ v_{r2}^t \\ \cdot \\ \cdot \\ \cdot \\ v_{rn_r}^t \end{bmatrix},$$ $(r = 1, \cdots, m)$ is the value added vector evaluated at current prices in period t;

$$p^t = \begin{bmatrix} p_1^t \\ p_2^t \\ \cdot \\ \cdot \\ \cdot \\ p_m^t \end{bmatrix}, \; p_r^t \begin{bmatrix} p_{r1}^t \\ p_{r2}^t \\ \cdot \\ \cdot \\ \cdot \\ p_{rn_r}^t \end{bmatrix},$$ $(r\ 1, \cdots, m)$ is the vector of output price indices; and

$$a^t = (a_{ri,sj}^t) \begin{bmatrix} r = 1, \cdots, m, \; s = 1, \cdots, m \\ i = 1, \cdots, n_r, \; j = 1, \cdots, n_s \end{bmatrix}$$ is the technical coefficient matrix,

where $a_{ri,sj}^t = x_{ri,sj}^t / x_{sj}^t$, and $x_{ri,sj}^t$ denotes the value of the output of industry ri, which is used as an intermediate input by industry sj to produce x_{sj}^t. For economic predictions, the technical coefficients are assumed to remain fixed throughout the entire period, as discussed in Section 3.

Notations of the aggregated model are denoted by upper case letters corresponding to the lower case letters in the original model:

$X^t = (X_r^t)$ denotes the output vector,

$F^t = (F_r^t)$ denotes the final demand vector,

$V^t = (V_r^t)$ denotes the value-added vector at current prices,

$P^t = (P_r^t)$ denotes the vector of the output price indices,

$A^t = (A_{r,s}^t)$ denotes the technical coefficient matrix.

The basic equations of both models are as follows:

$$x^t = a^t x^t + f^t \tag{1}$$

$$X^t = A^t X^t + F^t \tag{2}$$

$$p^t = a^{t\prime} p^t + \hat{x}^{t^{-1}} v^t \tag{3}$$

$$P^t = A^{t\prime} P^t + \hat{X}^{t^{-1}} V^t \tag{4}$$

Equations (1) and (2) express the relationship between final demand and output levels in both models, while Equations (3) and (4) the relationship between the ratios of value added to outputs and output price indices. A hat sign over a vector indicates a diagonal matrix whose diagonal elements consist of the corresponding elements of the vector. A prime over a vector or matrix indicates the transposition of the vector or matrix.

Hereafter, we call the former equations quantity equations and the latter equations price equations.

The relations among the variables in both models depend on the aggregation methods adopted in the aggregated model. Here we consider the aggregation methods usually used in empirical analyses. Therefore, we define the aggregation matrices or aggregators as follows:

$$\mathrm{G} = \begin{bmatrix} e(n_1), & o(n_2), & \cdots, & o(n_m) \\ o(n_1), & e(n_2), & \cdots, & o(n_m) \\ \cdot & \cdot & & \cdot \\ \cdot & & & \\ \cdot & & & \\ o(n_1), & o(n_2), & \cdots, & e(n_m) \end{bmatrix} \tag{5}$$

where $e(n_i)$ is a unit row vector of n_i dimensions and $o(n_i)$ is a zero row vector of n_i dimensions.

$$H^t = \begin{bmatrix} w_1^t, & o'(n_1), & \cdots, & o'(n_1) \\ o'(n_2), & w_2^t, & \cdots, & o'(n_2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ o'(n_m), & o'(n_m), & \cdots, & w_m^t \end{bmatrix} \tag{6}$$

where a column vector $w_i^{t\prime} = (w_{i1}^t, w_{i2}^t, \cdots, w_{in_i}^t)$, $(i = 1, 2, \cdots, \mathrm{m})$, and $w_{ri}^t = x_{ri}^t / \sum_{i=1}^{n_r} x_{ri}^t = (x_{ri}^t / X_r^t)$, $(\mathrm{r} = 1, 2, \cdots, \mathrm{m})$, are the composition ratios of the outputs of elementary industries in the r-th sector.

$$K^t = \begin{bmatrix} \bar{w}_1^t, & o'(n_1), & \cdots, & o'(n_1) \\ o'(n_2), & \bar{w}_2^t, & \cdots, & o'(n_2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ o'(n_m), & o'(n_m), & \cdots, & \bar{w}_m^t \end{bmatrix} \tag{7}$$

where $0 \leq \bar{w}_{ri}^t \leq 1$ and $\sum_{i=1}^{n_r} \bar{w}_{ri}^t = 1$, $(\mathrm{r} = 1, 2, \cdots, m)$.

Variables in the original model are related to those in the aggregated model by the following aggregators.

$$X^t = \mathrm{G}x^t \text{ or } X_r^t = e(n_r)x_r^t \tag{8a}$$

$$F^t = Gf^t \text{ or } F_r^t = e(n_r)f_r^t \tag{8b}$$

$$A^t = Ga^tH^t \text{ or } A_{rs}^t = e(n_r)a_{rs}^t w_s^t \tag{8c}$$

where a_{rs}^t is a sub-matrix of a^t,

$$a^t_{rs} = \begin{bmatrix} a^t_{r1,s1}, & a^t_{r1,s2}, & \cdots, & a^t_{r1,sn_s} \\ a^t_{r2,s1}, & a^t_{r2,s2}, & \cdots, & a^t_{r2,sn_s} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a^t_{rn_r,s1}, & a^t_{rn_r,s2}, & \cdots, & a^t_{rn_r,sn_s} \end{bmatrix} \quad (8c')$$

$$V^t = Gv^t \text{ or } V^t_r = e(n_r)v^t_r \quad (8d)$$

$$P^t = K^{t'}p^t \text{ or } P^t_r = \bar{w}^{t'}_r p^t_r \quad (8e)$$

The following relations between G, H^t, X^t and x^t can be easily obtained from the definitions of G and H^t.

$$GH^t = I \quad (9a)$$

$$x^t = H^t X^t \quad (9b)$$

$$H^{t'} = \widehat{X}^{t^{-1}} G\widehat{x}^t \quad (9c)$$

Equations (8a), (8b), and (8c) indicate that the levels of output, final demand, and value added for each sector in the aggregated model are respectively the sum of the levels of output, final demand, and value added of corresponding sector industries in the original model. The input coefficient A^t_{rs} is the ratio of the sum of the outputs of all industries in the r-th sector, which are absorbed in all industries in the s-th sector, to the sum of the outputs of all industries in the s-th sector. A^t_{rs} can be interpreted in two ways:

$$A^t_{rs} = \sum_{j=1}^{n_s}\left[\sum_{i=1}^{n_r} a^t_{ri,sj}\right] w^t_{sj} = \sum_{j=1}^{n_s} a^t_{r\cdot,sj} w^t_{sj} \quad (8c'')$$

where $a^t_{r\cdot,sj} = \sum_{i=1}^{n_r} a^t_{ri,sj} = \left(\sum_{i=1}^{n_r} x^t_{ri,sj}\right) / x^t_{sj}$.

$$A^t_{rs} = \sum_{i=1}^{n_r}\left[\sum_{j=1}^{n_s} a^t_{ri,sj} w^t_{sj}\right] = \left[\sum_{i=1}^{n_r} a^t_{ri,s\cdot}\left(x^t_{ri} / X^t_r\right)\right]\left(X^t_r / X^t_s\right), \quad (8c''')$$

where $a^t_{ri,s\cdot} = \left(\sum_{j=1}^{n_s} x^t_{ri,sj}\right) / x^t_{ri}$.

The construct (8c″) indicates that A^t_{rs} is a weighted average of the total input coefficients of all industries in the r-th sector sold to each industry in the s-th sector with the output share of each industry in the s-th sector as the weights. $a^t_{r\cdot,sj}$, which is the sum of the input coefficients of all industries in the r-th sector to the j-th industry in the s-th sector, is called the “derived technical coefficient” by Theil (1957) and “a semi-aggregate coefficient” by Fisher (1958). On the contrary, (8c‴) indicates that A^t_{rs} is a weighted average of the sum of the ratios of output sold to all industries in the s-th sector by each industry in the r-th sector with the output shares of each industry in the r-th sector as the weights. We assume that $a^t_{ri,s\cdot}$*.is* ”the derived output distribution coefficient”, which shows the sum of the ratios of the output of the i-th industry in the r-th sector sold to each industry in the s-th sector to the total output of the i-th industry in the r-th sector.

It is possible to use various types of weight matrices, such as K^t, but we consider only two cases. In the first case, we assume $K^t = H^0$: the Laspeyres price indices. In the other case we assume $K^t = H^t$: the Paasche price indices.

3. Main Findings

We can predict the outputs of industries x^t by solving the original quantity equation (Equation 1), assuming that the technical coefficient matrix a^t remains fixed at the base year value ($a^t = a^0 = a$), when the values of the final demand, f^t, are given. Then, the sectoral output X^t can be obtained using Equation (8a). Similarly, we can predict the sectoral output, X^t, of the aggregated model according to Equation (2), assuming that the aggregated technical coefficient matrix A^t remains fixed at the base year value ($A^t = A^0 =$ A), when the values of the final demand, F^t, are given. If the sectoral outputs based on the aggregated model are equal to those based on the original model, then the aggregation is said to be consistent. An aggregator H^0 may usually be used to obtain the aggregated technical coefficient matrix A^0, because it is reasonable to assume that H^t is not known in advance. The predicted output vector, $\tilde{X}^t$, based on the original model, can be written in the form

$$\tilde{X}^t = Gx^t = G[I-a]^{-1}f^t. \text{[3]} \tag{10}$$

The predicted output vector, X^t, based on the aggregated model, can be written in the form,

$$\begin{aligned} X^t &= [I-A^0]^{-1}F^t = [I-A^0]^{-1}Gf^t \\ &= [I-GaH^0]^{-1}Gf^t \\ &= G[I-aU]^{-1}f^t, \text{[4]} \end{aligned} \tag{11}$$

where $U = H^0G$.

Therefore, the aggregation bias in the prediction based on the aggregated quantity equation (Equation 2) may be defined as follows:

$$\begin{aligned} e^t &= X^t - \tilde{X}^t \\ &= G[(I-aU)^{-1} - (I-a)^{-1}]f^t \\ &= Bf^t \end{aligned} \tag{12}$$

where $B = G[(I-aU)^{-1} - (I-a)^{-1}]$. (13)

It can be easily shown that matrix B, the aggregation bias matrix, is subject to the following restriction,

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3) Theil (1957) considers a stochastic model. But the stochastic part does not affect the non-stochastic bias, so that we drop the disturbance in this paper.

4) Ara (1959) proves that the aggregated model is stable if the original model is. Therefore, provided that the original model is stable, expanding and rearranging $[I\text{-}A]^{-1}$ gives (11).

$$B(1-a)H^0 = 0. \qquad (14)$$

Equations (12), (13), and (14) are called "the fundamental aggregation equations of input-output analysis" by Theil (1957).

If the aggregation is consistent, the results from Equation (13) is given by:

$$(I-aU)^{-1} = (I-a)^{-1}$$

$$aU = a.$$

Premultiplying by G gives

$$GaU = Ga$$

$$AG = Ga. \qquad (15)$$

Ara (1959) proved that Equation (15) is a necessary and sufficient condition for the consistency of the aggregation.

According to Equation (15), the following conditions are satisfied,

$$\sum_{i=1}^{n_r} a_{ri,s1} = \sum_{i=1}^{n_r} a_{ri,s2} = \cdots = \sum_{i=1}^{n_r} a_{ri,sn_s}, \begin{bmatrix} r = 1,\cdots,m \\ s = 1,\cdots,m \end{bmatrix} \qquad (16)$$

These conditions imply that the sum of input coefficients of each industry in the r-th sector is the same as industries in the s-th. That is to say, all derived technical coefficients of industries in the same sector are equal. However, it should be noted that these conditions do not necessarily mean that each industry in the same sector has a homogeneous input structure. The condition that the total sum of all input coefficients of each industry is equal in the same sector is necessary, but not sufficient. This means that the equality of ratios of value added to output in the same sector is necessary, but not sufficient.

The conditions mentioned above are all concerned with technical input coefficients. As far as final demand f^t is concerned, no aggregation bias exists in the following special cases:

First, if f^t changes in proportion to f^0, There is no aggregation bias.

Under this condition f^t can be written in the following form,

$$f^t = kf^0, \text{ (from (1))}$$

$$= k(I-a)x^0, \text{ (from (9b))}$$

$$= k(I-a)H^0X^0, \qquad (17)$$

where k is a scalar representing a proportional factor. Substituting (17) into (12) and using (14) yields

$$e^t = kB(I-a)H^0X^0$$

$$= 0. \qquad (18)$$

When final demand grows proportionally, we can use consolidated input-output tables without aggregation bias to analyze the output levels relative to final demand levels. The industrial composition of the final demand component, such as increases in inventories and exports, usually changes rapidly over time; therefore, aggregation bias may become large unless a proper grouping of industries is adopted.

Second, Equation (12) shows that industries with no final demand do not affect the aggregation bias e^t. It may be said that industries whose outputs are mainly used as intermediate inputs cause less aggregation bias than those whose outputs are mainly supplied to final users.[5)]

4. Fundamental Price Aggregation Equations of Input-Output Analysis

This section considers the aggregation problem of price equations (3) and (4). We define vectors of ratios of value added to outputs v^t_* and V^t_* as follows:

$$v^t_* = \hat{x}^{t^{-1}} v^t \tag{19a}$$

$$V^t_* = \hat{X}^{t^{-1}} V^t. \tag{19b}$$

Usually, v^t_{ri}, an element of v^t, comprises consumption of fixed capital, indirect taxes less subsidies, compensation of employees and operating surplus. If we consider crude oil complementary, it will be included in v^t_{ri}. If crude oil is included, its corresponding part is the product of the crude oil import price and the amount of crude oil used per unit output.

The price equations' aggregation problem can be expressed as follows. We can obtain sectoral prices in period t in two different ways, as in the case of the quantity equations.

First, we can obtain p^t by the price equation, (3), in the original model, when the vector of value-added ratios v^t_* is given. Then, sectoral prices can be obtained by aggregating p^t with aggregator K^t. Second, we can predict sectoral prices P^t by using Equation (4) in the aggregated model if the value-added ratios vector, V^t_*, is given. Our purpose is to derive equations indicating bias corresponding to Theil's fundamental aggregation equations of input-output analysis and to find the necessary and sufficient conditions for consistent price aggregation.

The predicted values of sectoral prices based on the original model can be written in the form,

$$\tilde{P}^t = K^{t'} (I - a^{t'})^{-1} v^t_*. \tag{20}$$

The predicted values by the aggregated model can also be written in the form,

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5) This supports the aggregation that firms which supply parts are usually consolidated into the industry which uses the parts as materials.

$$P^t = (I - A^{t\prime})^{-1} V_*^t. \tag{21}$$

Substituting (19b), (8d), (19a) and (9c) into Equation (21),

$$\begin{aligned} P^t &= (I - A^{t\prime})^{-1} \hat{X}^{t-1} G v^t \\ &= (I - A^{t\prime})^{-1} \hat{X}^{t-1} G \hat{x}^t v_*^t \\ &= (I - A^{t\prime})^{-1} H^{t\prime} v_*^t. \end{aligned} \tag{22}$$

Therefore, from (20) and (22), the vector of the aggregation bias can be written in the form:

$$\begin{aligned} e^t &= P^t - \tilde{P}^t \\ &= \{(I - A^{t\prime})^{-1} H^{t\prime} - K^{t\prime} (I - a^{t\prime})^{-1}\} v_*^t \\ &= D v_*^t, \end{aligned} \tag{23}$$

where $D = (I - A^{t\prime})^{-1} H^{t\prime} - K^{t\prime} (I - a^{t\prime})^{-1}$. (24)

We call this matrix D "aggregation bias matrix." By postmultiplying D by $(I - a')G'$, we can easily obtain the following restriction concerning D:

$$D(I - a^{t\prime}) G' = 0. \tag{25}$$

Equations (23), (24), and (25) correspond to Theil's fundamental aggregation equations, (12), (13), and (14), respectively. Hence, we call these three equations "the fundamental price aggregation equations of input-output analysis."

Although a^t and A^t are used in Equations (23), (24), and (25), a^0 and A^0 are typically used in empirical studies. In fact, we do not know x^t in advance, so we cannot use A^t in the prediction of P^t. If $a^0 (= a)$ and A^0 are used in place of a^t and A^t respectively, Equations (23) and (24) can be rewritten in the following form:

$$e^t = D^0 v_*^t \tag{23'}$$

$$D^0 = (I - A^{0\prime})^{-1} H^{t\prime} - K^{t\prime} (I - a')^{-1}. \tag{24'}$$

Equation (25), however, does not necessarily hold, because

$$D^0 (I - a') G' = (I - A^{0\prime})^{-1} (I - A^{t\prime}) - I. \tag{26}$$

If $A^0 = A^t$, that is $H^0 = H^t$, then $D^0 (I - a') G' = 0$.

Consider the case of the Paasche price indices, $K^t = H^t$. In this case, the following theorem can be proven:

Theorem 1: The aggregation of price equations is consistent if and only if

$$H^t A^0 = a H^t \tag{27}$$

Proof: If the aggregation is consistent, e^t in (23′) must be zero for any value of v_*^t. Therefore, we must have $D^0 = 0$. If $H^t = K^t$ is substituted into Equation (24′), we obtain

$$(I-A^{0\prime})^{-1}H^{t\prime}=H^{t\prime}(I-\alpha^{\prime})^{-1}.$$

Premultiplying by $(I-A^{0^{\prime}})$ and postmultiplying by $(I-\alpha^{\prime})$ gives

$$H^{t\prime}(I-\alpha^{\prime})=(I-A^{0\prime})H^{t\prime}$$
$$A^{0\prime}H^{t\prime}=H^{t\prime}\alpha^{\prime}$$
$$H^{t}A^{0}=\alpha H^{t}.$$

Next, we prove the sufficiency.
If $H^{t}A^{0}=\alpha H^{t}$, then

$$A^{0\prime}H^{t\prime}=H^{t\prime}\alpha^{\prime}.$$

Adding $-H^{t^{\prime}}$ to both sides and rearranging gives

$$-H^{t\prime}+A^{0\prime}H^{t\prime}=H^{t\prime}\alpha^{\prime}-H^{t\prime}$$
$$(I-A^{0\prime})H^{t\prime}=H^{t\prime}(I-\alpha^{\prime})$$

Premultiplying by $(I-A^{0^{\prime}})^{-1}$ and postmultipling by $(I-\alpha^{\prime})^{-1}$ gives

$$(I-A^{0\prime})^{-1}H^{t\prime}=H^{t\prime}(I-\alpha^{\prime})^{-1}.$$

This means $D^{0}=0$.
(Q.E.D)

Condition (27) corresponds to Condition (15) of the quantity equations. However, the critical difference between the two conditions is that Condition (15) depends only on the base period information, while Condition (27) depends on period t information. This increases the difficulty of price aggregating analysis. Let us investigate what this condition means economically. If we explicitly show each element on both sides of Equation (27), it can be written as follows:

$$\left(\begin{array}{l}
w^{t}_{11}A^{0}_{11},w^{t}_{11}A^{0}_{12},\cdots,w^{t}_{11}A^{0}_{1s},\cdots,w^{t}_{11}A^{0}_{1m}\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
w^{t}_{1n_1}A^{0}_{11},w^{t}_{1n_1}A^{0}_{12},\cdots,w^{0}_{1n_1}A^{0}_{1s},\cdots,w^{t}_{1n_1}A^{0}_{1m}\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
w^{t}_{r1}A^{0}_{r1},w^{t}_{r1}A^{0}_{r2},\cdots,w^{t}_{r1}A^{0}_{rs},\cdots,w^{t}_{r1}A^{0}_{rm}\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
w^{t}_{rn_r}A^{0}_{r1},w^{0}_{rn_r}A^{0}_{r2}\cdots,w^{t}_{rn_r}A^{0}_{rs},\cdots,w^{t}_{rn_r}A^{0}_{rm}\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
w^{t}_{m1}A^{0}_{m1},w^{t}_{m1}A^{0}_{m2},\cdots,w^{t}_{m1}A^{0}_{ms},\cdots,w^{t}_{mn_m}A^{0}_{mm}\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
w^{t}_{mn_m}A^{0}_{m1},w^{t}_{mn_m}A^{0}_{m2},\cdots,w^{t}_{mn_m}A^{0}_{ms},\cdots,w^{t}_{mn_m}A^{0}_{mm}
\end{array}\right)=$$

$$\begin{bmatrix} \sum_{j=1}^{n_1} a_{11,1j} w_{1j}^t, \sum_{j=1}^{n_2} a_{11,2j} w_{2j}^t, \cdots, \sum_{j=1}^{n_s} a_{11,sj} w_{sj}^t, \cdots \sum_{j=1}^{n_m} a_{11,mj} w_{mj}^t \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=1}^{n_1} a_{1n_1,1j} w_{1j}^t, \sum_{j=1}^{n_2} a_{1n_1,2j} w_{2j}^t, \cdots, \sum_{j=1}^{n_s} a_{1n_1,sj} w_{sj}^t, \cdots, \sum_{j=1}^{n_m} a_{11,mj} w_{mj}^t \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=1}^{n_1} a_{r1,1j} w_{1j}^t, \sum_{j=1}^{n_2} a_{r1,2j} w_{2j}^t, \cdots, \sum_{j=1}^{n_s} a_{r1,sj} w_{sj}^t, \cdots, \sum_{j=1}^{n_m} a_{ri,mj} w_{mj}^t \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=1}^{n_1} a_{rn_r,1j} w_{1j}^t, \sum_{j=1}^{n_2} a_{rn_r,2j} w_{2j}^t, \cdots, \sum_{j=1}^{n_s} a_{rn_r,sj} w_{sj}^t, \cdots, \sum_{j=1}^{n_m} a_{rn_r,mj} w_{mj}^t \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=1}^{n_1} a_{m1,1j} w_{1j}^t, \sum_{j=1}^{n_2} a_{m1,2j} w_{1j}^t, \cdots, \sum_{j=1}^{n_s} a_{m1,sj} w_{sj}^t, \cdots, \sum_{j=1}^{n_m} a_{m1,mj} w_{mj}^t \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=1}^{n_1} a_{mn_m,1j} w_{1j}^t, \sum_{j=1}^{n_2} a_{mn_m,2j} w_{2j}^t, \cdots, \sum_{j=1}^{n_s} a_{mn_m,sj} w_{sj}^t, \cdots, \sum_{j=1}^{n_m} a_{mn_m,mj} w_{mj}^t \end{bmatrix} \quad (27')$$

From Equation (27′), following relations can be obtained,

$$\frac{\sum_{j=1}^{n_s} a_{r1,sj} w_{sj}^t}{w_{r1}^t} = \frac{\sum_{j=1}^{n_s} a_{r2,sj} w_{sj}^t}{w_{r2}^t} = \cdots = \frac{\sum_{j=1}^{n_s} a_{rn_r,sj} w_{sj}^t}{w_{rn_r}^t} \ (= A_{rs}^0), \quad (28)$$

(r, s = 1,2,⋯,m).

When matrix notations of (6) and (8c′) are used, these relations can be rewritten in the following form,

$$a_{rs} w_s^t = A_{rs}^0 w_r^t, \ (\text{r, s} = 1, 2, \cdots, \text{m}). \quad (28')$$ [6)]

Substituting $w_{ri}^t = x_{ri}^t / \sum_{i=1}^{n_r} x_{ri}^t (= x_{ri}^t / X_r^t)$ into (28) gives

$$\frac{\sum_{j=1}^{n_s} a_{r1,sj} x_{sj}^t}{x_{r1}^t} = \frac{\sum_{j=1}^{n_s} a_{r2,sj} x_{sj}^t}{x_{r2}^t} = \cdots = \frac{\sum_{j=1}^{n_s} a_{rn_r,sj} x_{sj}^t}{x_{rn_r}^t}, \ (\text{r, s} = 1, \cdots, \text{m}) \quad (29)$$

the $\sum_{j=1}^{n_s} a_{ri,sj} x_{sj}^t$ in the numerator is the sum of the value of the outputs of the i-th industry in the r-th sector sold to industries in the s-th sector. Therefore, the relations in (29) imply that the ratios of the total values of outputs of each industry in the r-th sector sold

6) If the original model is aggregated into one sector model, that is m = 1, then $w_s^t = w_r^t$ and (28′) will become $aw^t = A_{11}^0 w^t$. A_{11}^0 is a scalar and an eigenvalue of the technical coefficient matrix a, and w^t is the corresponding eigenvector. The condition of consistent aggregation into a one-sector model that McManns points out is $a'w = \lambda w$ in the case of quantity equations. Since a is a square matrix, we have the same eigenvectors and eigenvalues for w and its transpose a'. However, it should be noted that a weight vector w^t is no longer related to the outputs of industries.

to industries in the s-th sector to its total outputs are equal. We call the composition ratios of the values of output sold to other industries relative to the total output value of an industry "output structures."[7] Thus, Equation (29) implies that industries in the same sector have a homogeneous output structure in the aggregated model.

Next, we assume that the relations in (29) hold. Let the common ratio in (29) be denoted as $\bar{A}_{rs}(X^t_s / X^t_r)$: (29) can be written in a matrix form as follows

$$a_{rs}x^t_s = \bar{A}_{rs}(X^t_s / X^t_r)x^t_r, \text{ (r, s = 1, 2, ⋯, m).} \tag{30}$$

Hence,

$$a_{rs}w^t_s = \bar{A}_{rs}w^t_r, \text{ (r, s = 1, 2, ⋯, m).} \tag{31}$$

Let a matrix whose (r, s) element is $\bar{A}_{rs}$ be denoted by $\bar{A}$. Then, (31) can be written in the following form,

$$H^{t\prime}\bar{A} = aH^{t\prime}. \tag{32}$$

However, the following result can be obtained from (8c").

$$\begin{aligned} A^t_{rs} &= \sum_{i=1}^{n_r}\left[\sum_{j=1}^{n_s} a_{ri,sj}w^t_{sj}\right] \\ &= \sum_{i=1}^{n_r}\left(\bar{A}_{rs}w^t_{ri}\right) \\ &= \bar{A}_{rs}. \text{ (r, s = 1, 2, ⋯, m)} \end{aligned} \tag{33}$$

That is, $\bar{A} = A^t$. This shows that the same output structure in Equation (29) is a necessary but not sufficient condition for the consistency of the price aggregation. This finding is attributed to the fact that the output structures in (29) are in relation to activities in period t. It should be noted that the output structures of industries in the same sectors in the original model are not necessarily homogeneous.

Let us consider the case of Laspeyres price indices, $K^t = H^0$. In this case the bias matrix D^0 becomes

$$D^0 = (I - A^{0\prime})^{-1}H^{t\prime} - H^{0\prime}(I - a^\prime)^{-1}. \tag{34}$$

As in the case of the Paasche price indices, restriction (25) does not hold either. If the aggregation is consistent, we obtain the following from (34):

$$H^0A^0 = aH^t. \tag{35}$$

This condition is necessary but not sufficient, and it is very difficult to interpret

7) Morishima, M. and F. Seton (1961) define them as producer's "output quotas". They conceptually correspond to the SNA supply tables.

$H^0A^0 = aH^t$ economically.

5. First Order Approximation and Special Cases

We focused on the conditions of the technical coefficient in the previous section. We shift our attention to the growth patterns of industries and the ratios of value added to outputs.

We expand the inverted matrices in (24′),

$$D^0 = \{H^{t\prime} + A^{0\prime}H^{t\prime} + (A^{0\prime})^2 H^{t\prime} + \cdots\} - \{K^{t\prime} + K^{t\prime}a' + K^{t\prime}(a')^2 + \cdots\},$$

and we take the first-order terms

$$D_1^0 = (H^{t\prime} - K^{t\prime}) + (A^{0\prime}H^{t\prime} - K^{t\prime}a'). \tag{36}$$

This is the first-order aggregation bias matrix.

First, we consider the Paasche price indices, $H^t = K^t$. Then, D_1^0 can be written in the following form,

$$D_1^0 = A^{0\prime}H^{t\prime} - H^{t\prime}a'. \tag{37}$$

If the aggregation is consistent, then $D_1^0 = 0$, because (27) holds. Therefore, the first-order aggregation bias $e_1^t = D_1^0 v_*^t$ vanishes. Therefore, we consider the case in which aggregation is not consistent.

As a special case, we assume that the ratios of value added to the output of industries in the same sectors are equal in period t. Then we have

$$v_*^t = G'V_*^t. \tag{38}$$

Using the equations (9c) and (19a) gives

$$H^{t\prime}v_*^t = V_*^t. \tag{39}$$

If we substitute these into the right-hand side of e_1^t, we ge the following results:

$$\begin{aligned} e_1^t &= (A^{0\prime}H^{t\prime} - H^{t\prime}a')v_*^t \\ &= A^{0\prime}H^{t\prime}v_*^t - H^{t\prime}a'v_*^t \\ &= A^{0\prime}V_*^t - H^{t\prime}a'G'V_*^t \\ &= (H^{0\prime} - H^{t\prime})a'G'V_*^t \\ &= (H^{0\prime} - H^{t\prime})a'v_*^t. \end{aligned} \tag{40}$$

Therefore, if $H^0 = H^t$, the first-order approximation bias vanishes; that is, when industries outputs increase proportionately, we have no first-order approximation bias. This result is economically understandable because the relative prices of industrial outputs in the same sectors will not significantly change when industries grow at the same rate, and the value-added ratios will be equal within the sectors.

Second, we consider Laspeyres price indices, and D_1^0 can be written in the following form,

$$D_1^0 = (H^{t\prime} - H^{0\prime}) + (A^{0\prime} H^{t\prime} - H^{0\prime} a'). \tag{41}$$

Even if the aggregation is consistent, D_1^0 does not become a zero matrix. The first-order approximation bias can then be expressed as follows:

$$e_*^t = (H^{t\prime} - H^{0\prime}) v_*^t + H^{0\prime} a' (G' V_*^t - v_*^t). \tag{42}$$

Therefore, if the value-added ratios of industries in the same sectors are equal in period t, that is, $v_*^t = G' V_*^t$, then we have

$$e_1^t = (H^{t\prime} - H^{0\prime}) v_*^t. \tag{43}$$

As in the case of the Paasche price indices, the first-order aggregation bias vanishes if the outputs of industries increase proportionately.

Third, we consider the case in which industries grow proportionately. In this case, we have,

$$H^0 = H^t \tag{44}$$

and the same weights for the Paasche and Laspeyres price indices. The fundamental price aggregations can then be written in the following form,

$$e^t = D^0 v_*^t,$$
$$D^0 = (I - A^{0\prime})^{-1} H^{0\prime} (I - a')^{-1}, \tag{45}$$
$$D^0 (I - a') G' = 0. \tag{46}$$

From Theorem 1, it is evident that aggregation is consistent only and only if

$$H^0 A^0 = a H^0. \tag{47}$$

Condition (47) implies that industries in the same sectors have homogeneous output structures in period 0. In this case, the condition of homogeneous output structures is sufficient and necessary. If the value-added ratios of industries in the same sectors are equal, the first-order approximation bias becomes zero, as previously mentioned.

Finally, we consider the case in which the value-added ratios change in proportion to those in period 0:

$$v_*^t = \lambda v_*^0, \tag{48}$$

where λ is a scalar.

Then,

$$
\begin{aligned}
e^t &= D\lambda v_*^0 \\
&= \lambda D(I-a')p^0 \\
&= \lambda D(I-a')G'P^0, \text{ (from (46))} \\
&= 0.
\end{aligned}
$$

We have no aggregation bias in this case.

6. Conclusion

In empirical studies, highly aggregated input-output tables are usually used for both quantity and price analyses. Homogeneity of input structures in the base period or a proportional increase in the final demand is required for consistent aggregation in quantity analysis. However, in price analysis, the homogeneity of output structures in period t or a proportionate increase in output is required for consistent aggregation. Therefore, when we use an aggregated input-output table for both quantity and price analyses simultaneously in a multisectoral econometric model or the assessment of an increase in crude oil import price or wages, homogeneity of input structure in the base period and output structures in period t should be carefully examined and taken into account in the consolidation of industries into sectors. In particular, when abrupt or large changes in the relative prices or commodity composition of final demand are expected in period t, the aggregation bias mentioned above will not be negligible. In particular, difficulty arises in price analysis because the homogeneity of output structures not in the base period but in prediction period t is required for consistency. Furthermore, the type of index number is closely related to the aggregation bias, as shown in the previous sections.

Input-output tables are now compiled and used for economic policies and projections in many countries. For example, the Japanese government has compiled and published input-output tables every five years since 1955. The latest input-output table for 2015 is recorded in a 509×391 matrix form based on the most detailed classification (basic sectors). They are also aggregated into three types of tables: minor aggregated classifications (187 sectors), medium aggregated classifications (107 sectors), and major aggregated classifications (37 sectors).

Aggregation problems in input-output analyses have been explored relatively well in theory, but, in empirical analyses, few experiments on aggregation bias have been conducted and reported in the literature. Therefore, more empirical studies on aggregation bias should be conducted in the future.

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