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Impact of Delays in Policy Implementation on Economic Stability under Sticky Prices

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Abstract

This study extends the New Keynesian model to examine the impact of delays in monetary policy implementation on economic stability. The extant literature indicates that if a central bank delays its response to the inflation rate, this policy lag may increase the number of positive roots in the model economy, implying that a time lag associated with inflation-targeting policy causes instability or eliminates indeterminacy. However, cases where a central bank considers multiple target variables but only one of them has a lag have not yet been examined. We analyze the case in which the inflation rate and output are the target variables of policy intervention, wherein a delay occurs in the central bank's response to output. We demonstrate that a policy lag may increase the number of negative roots in the dynamic system, implying that instability rather than indeterminacy may be eliminated by a policy lag.

Keywords New Keynesian model; Policy lag; Delay differential equation; Equilibrium determinacy

JEL Classification E32; E52; E63

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1 Introduction

A policy proposition derived from the standard New Keynesian (NK) model—an optimization model that considers price stickiness—explains that active inflation-targeting policy establishes local equilibrium determinacy. Here, active refers to the policy attitude wherein the central bank changes the nominal interest rate by more than a one-for-one in response to a one-unit change in the inflation rate. In contrast, passive refers to a policy attitude with which the interest rate's response is less than one-for-one. This proposition is called the "Taylor principle," named after the economist who performed pioneering research on the Federal Reserve Bank's policy from demonstrative and normative viewpoints.

In general, two-dimensional NK models include two non-predetermined variables (inflation rate and output). Accordingly, two roots with positive real parts are required to achieve equilibrium determinacy. If there are fewer than two roots with a positive real part, the equilibrium is indeterminate.

If the central bank implements a policy that considers not only changes in the inflation rate but also changes in output (inflation–output targeting), equilibrium determinacy can be achieved even if the policy is passive against inflation rate fluctuations (Bullard and Mitra 2002). Thus, the implementation of output targeting increases the possibility of determinacy.

Benhabib et al. (2003) examine the effect of a delay in monetary policy responses on equilibrium determinacy. Their model assumes that the nominal interest rate responds to the weighted average of previous inflation rates. In mathematical terms, this denotes *distributed lag.* Benhabib et al. (2003) prove the existence of a limit cycle, indicating that global indeterminacy of equilibrium can occur under such policies. In contrast, Tsuzuki (2014, 2015) assumes that the nominal interest rate responds to the inflation rate observed at a certain point in the past, denoting a *fixed lag.*[1] Models with a fixed lag are expressed as delay differential equation systems (differential-difference equation systems), whereas models with distributed lag are expressed as ordinary differential equation systems. In delay differential equation systems, the characteristic equations have an infinite number of roots in the complex plane. Therefore, equilibrium can become locally unstable; that is, if the number of

[1]Guerrini and Sodini (2013) propose a simple neoclassical growth model (Solow model) that considers a capital accumulation lag of this type.

roots with positive real parts is greater than two, the equilibrium is unstable.

Tsuzuki (2014) shows that if a fixed lag exists in the implementation of inflation targeting, the economy may become locally unstable, even if the interest rate has a sufficiently large response to the inflation rate. Additionally, Tsuzuki (2015) shows that policy lag could eliminate equilibrium indeterminacy by increasing the number of roots with positive real parts.

Thus, the studies on economic stability using NK models have shown that a delay in the central bank's response to the inflation rate has the effect of destabilizing the system or eliminating indeterminacy (essentially, it causes an increase in the number of roots with positive real parts) when the target variable of the policy intervention is only one and the number of time lag is also only one. However, cases where there are multiple policy target variables (inflation rate and output), and the number of fixed lags is only one, have not yet been examined. Tsuzuki (2016) and Shinagawa and Tsuzuki (2019) present NK models in which there is a time lag in the implementation of fiscal policy, but those studies also do not assume policy rules with multiple target variables.

This study extends the NK model to include both inflation rate and output as part of the monetary policy rule. We assume that a fixed lag is present in the implementation of output targeting and demonstrate that stabilization may occur because of an increase in the policy lag. As mentioned above, the previous literature shows that a time lag associated with inflation-targeting policy causes instability or eliminates indeterminacy (Tsuzuki 2014, 2015). However, our study shows that roots with negative real parts may increase due to an increase in the lag, implying that stability may be established by a policy lag, even if there is only one lag.

The remainder of this paper is organized as follows. In section 2, we construct the dynamic system of the model, while in section 3 we briefly analyze the case in which a policy lag is not present. Sections 4 and 5 present the stability analysis for the case with a policy lag. Finally, section 6 concludes the study.

2 The model

We consider a continuous-time NK model that follows Benhabib et al. (2003) and Tsuzuki (2014). The economy consists of household–firm units, final goods-producing firms, and the central bank. The household–firm units are continuously distributed in the interval $[0,1]$. Each household–firm unit produces a differentiated product and consumes the final goods. The final goods-producing firms produce the final goods by aggregating the differentiated products produced by household–firm units.

2.1 Final goods-producing firms

The final goods-producing firms aggregate differentiated products according to the Dixit–Stiglitz-type production function as follows:

$$y = \left[\int_0^1 y_j^{\alpha} dj \right]^{\frac{1}{\alpha}}, \tag{1}$$

where y is the amount of final good produced, y_j is the input of the differentiated product of type $j \in [0,1]$, and $0 < \alpha < 1$. The elasticity of substitution among the differentiated products is represented by $\phi \equiv 1/(1-\alpha) > 1$.

To identify the final goods sector as a single representative firm, we assume that the final goods market is perfectly competitive. Given the output y and price of product j, which we denote p_j, the representative firm minimizes the total cost—$\int_0^1 p_j y_j dj$—to yield the demand function for product j as follows:

$$y_j = \left(\frac{p_j}{p} \right)^{-\phi} y, \tag{2}$$

where p is the price index expressed as follows:

$$p = \left[\int_0^1 p_j^{1-\phi} dj \right]^{\frac{1}{1-\phi}}. \tag{3}$$

The inflation rate is given by $v \equiv \dot{p}/p$.

2.2 Household–firm units

Household–firm unit j obtains utility from consumption c_j and real money holding m_j. Similarly, it receives disutility from labor supply ℓ_j and price revisions v_j. We express the instantaneous utility function of household–firm unit j as follows:

$$u_j(c_j(t), m_j(t), \ell_j(t), v_j(t)) = \varepsilon \log c_j(t) + (1-\varepsilon) \log m_j(t) - \frac{\ell_j(t)^{1+\psi}}{1+\psi} - \frac{\eta}{2}(v_j(t) - v^*)^2,$$

where $t \in [0, \infty)$ denotes time, v^* represents the steady-state value of the inflation rate, and v_j denotes the price change rate of product j expressed as:

$$v_j \equiv \frac{\dot{p}_j}{p_j}. \tag{4}$$

In addition, $\psi > 0$ is the elasticity of the marginal disutility of labor supply, $\eta > 0$ is the scale parameter of the price revision cost, and $0 < \varepsilon < 1$. The existence of the price revision cost means that the price becomes sticky. Hence, η can also be interpreted as representing price stickiness.

Assuming that household–firm units are infinitely lived, we can express the lifetime utility of household–firm unit j as follows:

$$U_j(c_j, m_j, \ell_j, v_j, t) = \int_0^\infty e^{-\rho t} u_j(c_j(t), m_j(t), \ell_j(t), v_j(t))dt, \tag{5}$$

where $\rho > 0$ is the subjective discount rate.

Furthermore, the nominal assets of the household–firm unit j are represented as A_j. These assets A_j comprise money $M_j(\equiv pm_j)$ and bonds B_j; therefore, $A_j = M_j + B_j$. Denoting the nominal interest rate for bonds as R, the budget constraint equation for household–firm unit j can be represented as follows: $\dot{A}_j = p_j y_j + RB_j - pc_j - ps_j$, where s_j denotes lump-sum taxes (subsidies if negative). This equation can be rewritten in real terms as follows:

$$\dot{a}_j = \frac{p_j}{p} y_j + ra_j - c_j - s_j - Rm_j, \tag{6}$$

where a_j denotes the real assets held by the household–firm unit j, and $r \equiv R - v$ is the real interest rate.

Moreover, we assume that $1/\zeta > 0$ units of the labor force are required to produce one unit of all kinds of products. Accordingly, the production function of the household–firm unit j can be expressed as $y_j = \zeta \ell_j$. Under this technology, household–firm unit j faced with the demand function for product j in (2) determines the paths of c_j, m_j, and v_j that maximize lifetime utility in (5), subject to the restrictions in (4) and (6). Solving this optimization problem yields the following equations (see Appendix A.1):

$$\frac{\dot{c}_j}{c_j} = r - \rho, \tag{7}$$

$$m_j = \frac{1-\varepsilon}{\varepsilon}\frac{c_j}{R}, \tag{8}$$

$$\dot{v}_j = \rho(v_j - v^*) - \frac{\phi}{\eta\zeta^{1+\psi}} y_j^{1+\psi} + \frac{\varepsilon(\phi - 1)}{\eta}\frac{p_j y_j}{p c_j}. \tag{9}$$

Equations (7), (8), and (9) denote an Euler equation, the demand function for money, and New Keynesian Phillips curve, respectively. Economically significant solutions are also required to satisfy the transversality conditions, expressed as $\lim_{t\to\infty} e^{-\rho t}\mu_1(t)a_j(t) = 0$ and $\lim_{t\to\infty} e^{-\rho t}\mu_2(t)p_j(t) = 0$, where μ_1 and μ_2 are the costate variables of the state variables a_j and p_j, respectively.

Considering the symmetry between all types of household–firm units, the subscript j is dropped from all variables. Additionally, the assumption that $j \in [0, 1]$ leads to $m \equiv \int_0^1 m_j dj$ and $c \equiv \int_0^1 c_j dj$, where m and c are the aggregative values of real money balances and consumption, respectively. Finally, the market-clearing condition for the final goods market is:

$$y = c. \tag{10}$$

2.3 Central bank

We consider a situation in which the central bank manipulates the nominal interest rate according to deviations of the inflation rate v and output y from their steady-state values v^* and y^*. This assumption implies that the central bank's objective is to stabilize the economy.

We further assume that a policy lag exists in relation to the central bank's recognition of actual economic conditions. Recognition of the output level seems to take longer than the recognition of the inflation rate because, in most countries, the inflation rate is announced monthly, whereas GDP is announced quarterly. Thus, in this study, we assume that only output targeting experiences a delay. Accordingly, the monetary policy rule can be expressed as follows:

$$R(t) = R(v(t) - v^*, y(t-\tau) - y^*), \tag{11}$$

where $\tau \geq 0$ represents the policy lag. The function R is assumed to satisfy the following properties:

$$R(0,0) = \bar{R}, \quad \frac{\partial R}{\partial v(t)} > 0, \quad \frac{\partial R}{\partial y(t-\tau)} \geq 0,$$

where $\bar{R} > 0$ is the target nominal interest rate level that corresponds to a situation in which both inflation and output are consistent with their target levels.

If $\partial R/\partial y(t-\tau) = 0$, the policy rule in (11) becomes a simple interest rate rule that does not consider the output a target variable. In this case, a policy is referred to as active if $\partial R/\partial v(t) > 1$ and passive if $\partial R/\partial v(t) < 1$, consistent with previous studies.

2.4 Dynamic system

The differential equation system with two endogenous variables, y and v, can be derived from (7) and (9)–(11) as follows:[2]

$$\begin{aligned} \dot{y}(t) &= [R(v(t) - v^*, y(t-\tau) - y^*) - v(t) - \rho]y(t), \\ \dot{v}(t) &= \rho(v(t) - v^*) - \frac{\phi}{\eta\zeta^{1+\psi}} y(t)^{1+\psi} + \frac{\varepsilon(\phi - 1)}{\eta}. \end{aligned} \tag{12}$$

In the case of $\tau > 0$, the first equation becomes a *delay differential equation.*

Generally, delay differential equation systems require the variables' initial conditions not only at time $t = 0$ but also for $t - \tau \leq t < 0$. If these values are not set, the system will not operate. However, the initial values that household–firm units can determine at time $t = 0$ are only $y(0)$ and $v(0)$ because values for $t - \tau \leq t < 0$ are "past values." Therefore, the values for $t - \tau \leq t < 0$ should be considered as given, even if they are non-predetermined variables. Thus, as with the standard model, the equilibrium is locally determinate (locally stable) when the system has two roots with positive real parts. In contrast, the equilibrium is indeterminate if fewer than two roots have positive real parts and unstable (an equilibrium path does not exist) if more than two roots have positive real parts.

[2]The only role that (8) plays in our model is to determine the level of the lump-sum tax s_j through the money-market-clearing condition and the government's budget constraint. Therefore, (8) can be separated from the examined differential equation system.

3 A case without a policy lag

To clarify the effect of a policy lag on local equilibrium determinacy, we first briefly analyze the standard case that does not contain a policy lag ($\tau = 0$). The steady-state values of system (12) are given by $y^* = \zeta(\frac{\phi-1}{\phi})^{\frac{1}{1+\psi}}$, and $v^* = \bar{R} - \rho$. By linearizing system (12) around the steady-state (y^*, v^*), the equations become

$$\begin{bmatrix} \dot{\hat{y}}(t) \\ \dot{\hat{v}}(t) \end{bmatrix} = \boldsymbol{J}_0 \begin{bmatrix} \hat{y}(t) \\ \hat{v}(t) \end{bmatrix} \quad \text{with} \quad \boldsymbol{J}_0 = \begin{bmatrix} D_1 y^* & (D_2 - 1)y^* \\ -A_{21} & \rho \end{bmatrix}, \tag{13}$$

where $\hat{y}(t) \equiv y(t) - y^*$, $\hat{v}(t) \equiv v(t) - v^*$, $A_{21} \equiv (1+\psi)(\phi/(\eta\zeta^{1+\psi}))y^{*\psi} > 0$, $D_1 \equiv \partial R(0,0)/\partial y \geq 0$, and $D_2 \equiv \partial R(0,0)/\partial v > 0$.

The characteristic equation of this system can be expressed using matrix $\boldsymbol{J}_0$ as follows:

$$\Delta_0(\lambda) = \lambda^2 - \mathrm{tr}\,\boldsymbol{J}_0\lambda + \det \boldsymbol{J}_0 = 0,$$

where λ is the characteristic root. This equation has at least one positive root because $\mathrm{tr}\,\boldsymbol{J}_0 = (D_1 y^* + \rho) > 0$. Moreover, if $\det \boldsymbol{J}_0 = [D_1\rho + (D_2 - 1)A_{21}]y^* > 0$, two roots with positive real parts exist. Conversely, if $\det \boldsymbol{J}_0 < 0$, one positive and one negative real root exist. Accordingly, we obtain the following well-known results:

Proposition 1 *When $\tau = 0$, the equilibrium of system (12) is locally determinate if $D_2 > \bar{D} \equiv 1 - \frac{D_1\rho}{A_{21}}$ and locally indeterminate if $D_2 < \bar{D}$.*

This proposition is illustrated in Figure 1.[3] In the case of simple inflation targeting ($D_1 = 0$), Proposition 1 can be rewritten as follows:

Corollary 1 *When $\tau = 0$ and $D_1 = 0$, the equilibrium of system (12) is locally determinate if $D_2 > 1$ and locally indeterminate if $D_2 < 1$.*

This is the Taylor principle, which asserts that monetary policy must be active to achieve local equilibrium determinacy under inflation targeting. However, in the case of inflation–output targeting ($D_1 > 0$), the equilibrium can be locally determinate even if $D_2 < 1$ because $\bar{D} < 1$, implying that adding output as one of the central bank's target variables makes economic stability easier to achieve (Bullard and Mitra

[3]The same figure can also be observed in Chapter 4 in Galí (2015).

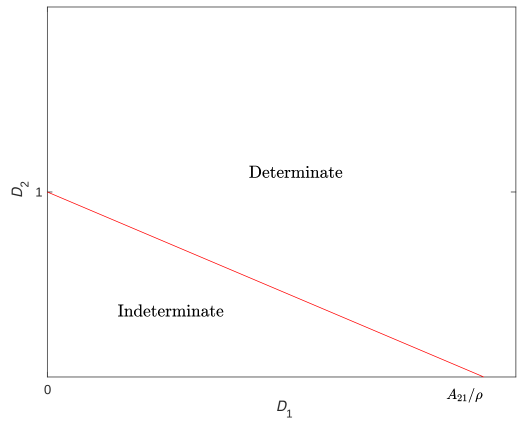


Figure 1 Equilibrium determinacy in the case of inflation–output targeting without a policy lag

2002). This result is attributed to the proportional relationship between v and y. Under inflation–output targeting, the nominal interest rate responds not only to the inflation rate v but also to output y. Therefore, the interest rate's response becomes greater than that in the case without output targeting. Consequently, the potential for achieving determinacy is higher.

4 A case with a policy lag

In this section, we assume $\tau > 0$ and consider the conditions for the local determinacy of the equilibrium.[4] Linearizing (12) around the steady-state (y^*, v^*), we obtain:

$$\begin{aligned}\dot{\hat{y}}(t) &= D_1 y^* \hat{y}(t-\tau) + (D_2 - 1) y^* \hat{v}(t),\\ \dot{\hat{v}}(t) &= -A_{21}\hat{y}(t) + \rho \hat{v}(t).\end{aligned}$$

Moreover, assuming trial solutions of the form $\hat{v}(t) = C_v e^{\lambda t}$ and $\hat{y}(t) = C_y e^{\lambda t}$ (where C_v and C_y are arbitrary constants and λ is an eigenvalue), we obtain

$$\begin{bmatrix}\dot{\hat{y}}(t)\\ \dot{\hat{v}}(t)\end{bmatrix} = \boldsymbol{J}_1 \begin{bmatrix}\hat{y}(t)\\ \hat{v}(t)\end{bmatrix} \quad \text{with} \quad \boldsymbol{J}_1 = \begin{bmatrix} D_1 y^* e^{-\lambda\tau} & (D_2-1)y^* \\ -A_{21} & \rho\end{bmatrix}.$$

[4] Matsumoto and Szidarovszky (2013) present details of the method used in this section.

The characteristic equation is now expressed as follows:

$$\begin{aligned}\Delta_1(\lambda) &= \lambda^2 - \operatorname{tr} \boldsymbol{J}_1\lambda + \det \boldsymbol{J}_1 \\ &= \lambda^2 - \rho\lambda + A_{21}(D_2 - 1)y^* + (\rho - \lambda)D_1 y^* e^{-\lambda\tau} = 0. \end{aligned} \tag{14}$$

This equation has an infinite number of roots owing to the existence of the exponential function $e^{-\lambda\tau}$.

4.1 Stability crossing

To examine how the signs of the roots of (14) change with an increase in τ, we first seek the points at which zero or pure imaginary roots appear. At these points, the number of roots with positive real parts can vary; in other words, the steady-state's dynamic properties can change. Let $z > 0$ denote the imaginary parts of the roots, and i indicate the imaginary unit. We seek the values of τ at which the real root $\lambda = 0$ or the imaginary roots $\lambda = \pm iz$ appear.

4.1.1 Real roots

When $\lambda = 0$ arises, the sign of a real root changes. However, if $D_2 \neq \bar{D}$, $\Delta_1(0) = [\rho D_1 + (D_2 - 1)A_{21}]y^* \neq 0$. Therefore, in this case, $\lambda = 0$ cannot be a root, and the number of positive real roots does not change. In contrast, if $D_2 = \bar{D}$, $\lambda = 0$ can be a root. However, because this condition does not depend on τ, the number of positive real roots also does not change in response to a change in τ.

4.1.2 Complex roots

When the roots $\lambda = \pm iz$ arise, the number of complex roots with positive real parts changes. Substituting $\lambda = \pm iz$ into (14) yields the following expression:

$$\Delta_1(\pm iz) = -z^2 \mp i\rho z + A_{21}(D_2 - 1)y^* + (\rho \mp iz)D_1 y^* e^{\mp i\tau z} = 0.$$

Furthermore, applying Euler's formula ($e^{\pm i\tau z} = \cos\tau z \pm i\sin\tau z$) to this equation yields

$$-z^2 \mp i\rho z + A_{21}(D_2 - 1)y^* + (\rho \mp iz)D_1 y^* \cos\tau z \mp i(\rho \mp iz)D_1 y^* \sin\tau z = 0.$$

For the equality to hold here, both the real and imaginary parts of the left-hand side expression must equal zero, that is,

$$-z^2 + A_{21}(D_2 - 1)y^* + \rho D_1 y^* \cos\tau z - zD_1 y^* \sin\tau z = 0, \tag{15}$$
$$-\rho z - zD_1 y^* \cos\tau z - \rho D_1 y^* \sin\tau z = 0. \tag{16}$$

Solving these equations for $\cos\tau z$ and $\sin\tau z$ yields

$$\cos\tau z = -\frac{\rho A_{21}(D_2 - 1)}{(\rho^2 + z^2)D_1}, \tag{17}$$
$$\sin\tau z = \frac{-z(\rho^2 + z^2) + zy^* A_{21}(D_2 - 1)}{D_1 y^*(\rho^2 + z^2)}, \tag{18}$$

where $2h\pi < \tau z < 2(1+h)\pi$, $\pi = 3.14159\ldots$, $h = 0, 1, 2, 3, \ldots$.

Moreover, the sum of squares of (17) and (18) yields the quartic equation of z as follows:[5]

$$z^4 + \chi z^2 + \frac{1}{4}\gamma = 0, \tag{19}$$

where

$$\chi \equiv \rho^2 - 2A_{21}(D_2 - 1)y^* - (D_1 y^*)^2, \tag{20}$$
$$\gamma \equiv 4y^{*2}\{A_{21}^2(D_2 - 1)^2 - (\rho D_1)^2\}. \tag{21}$$

Solving for z yields

$$z_1 = \sqrt{Z_+}, \quad z_2 = -\sqrt{Z_+}, \quad z_3 = \sqrt{Z_-}, \quad z_4 = -\sqrt{Z_-}, \tag{22}$$

where

$$Z_+ \equiv \frac{-\chi + \sqrt{\chi^2 - \gamma}}{2}, \quad Z_- \equiv \frac{-\chi - \sqrt{\chi^2 - \gamma}}{2}. \tag{23}$$

By definition, z should be a positive real number. Hence, both z_2 and z_4 can be ignored, while considering only z_1 and z_3.

[5] More precisely, the calculation yields the equation $(\rho^2 + z^2)(z^4 + \chi z^2 + \frac{1}{4}\gamma) = 0$. The solutions of the equation $\rho^2 + z^2 = 0$ are given by $z = \pm i\rho$; however, z should represent positive real numbers. Therefore, these solutions can be ignored.

4.2 Division of the (D_1, D_2)-plane

In this subsection, based on the analysis in the previous subsection, we divide the (D_1, D_2)-plane according to the characteristics of the roots.

From (22) and (23), the properties of z_1 and z_3 can be divided mainly into two types depending on whether $\chi > 0$ or $\chi < 0$. When $\chi > 0$, properties of z_1 and z_3 are classified as shown in Table 1, where "$-$" and "$+$" indicate negative and positive real numbers, respectively, and ξ denotes a complex number. When $\chi < 0$, the properties of z_1 and z_3 are classified as shown in Table 2.

	$\gamma \geq \chi^2$	$\chi^2 > \gamma > 0$	$\gamma = 0$	$\gamma < 0$
Z_+	ξ	$-$	0	$+$
Z_-	ξ	$-$	$-$	$-$
z_1	ξ	ξ	0	$+$
z_3	ξ	ξ	ξ	ξ

Table 1 Cases with $\chi > 0$

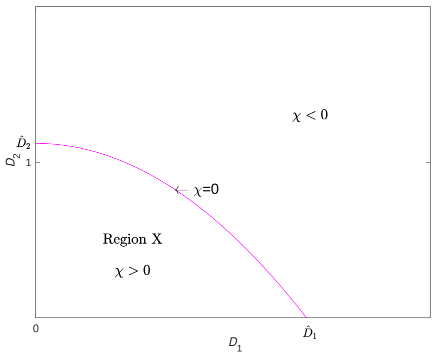
	$\gamma > \chi^2$	$\chi^2 \geq \gamma > 0$	$\gamma = 0$	$\gamma < 0$
Z_+	ξ	$+$	$+$	$+$
Z_-	ξ	$+$	0	$-$
z_1	ξ	$+$	$+$	$+$
z_3	ξ	$+$	0	ξ

Table 2 Cases with $\chi < 0$

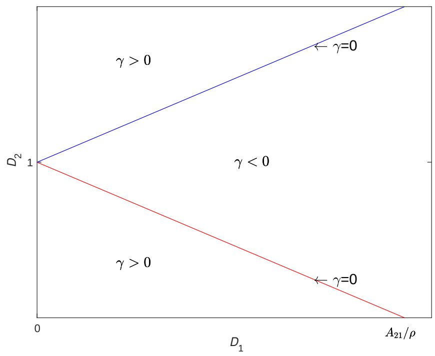
Based on Tables 1 and 2, the (D_1, D_2)-plane is divided into several regions. The sets (D_1, D_2) that establish $\chi = 0$ satisfy the following equation:

$$D_2 = -\frac{y^*}{2A_{21}} D_1^2 + \frac{\rho^2}{2A_{21} y^*} + 1.$$

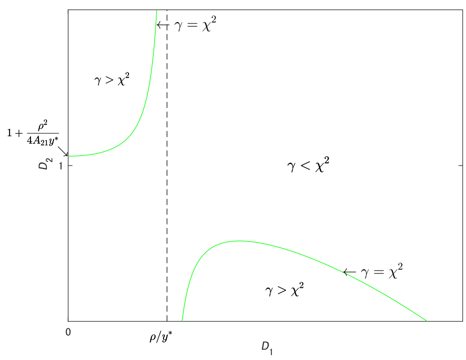
This relation appears as the downward curve in Figure 2(a), with D_1 on the horizontal axis and D_2 on the vertical axis. The intercepts $\hat{D}_1$ and $\hat{D}_2$ are defined as



(a) Curve $\chi = 0$



(b) Lines $\gamma = 0$



(c) Curves $\gamma = \chi^2$

Figure 2 Boundaries on the (D_1, D_2)-plane

follows:

$$\hat{D}_1 \equiv \frac{\sqrt{\rho^2 + 2A_{21}y^*}}{y^*} > 0,$$
$$\hat{D}_2 \equiv 1 + \frac{\rho^2}{2A_{21}y^*} > 1.$$

The region that establishes $\chi > 0$ is referred to as Region X.

Next, we depict curves that establish $\gamma = 0$ and $\gamma = \chi^2$ on the (D_1, D_2)-plane. The equation $\gamma = 0$ can be rewritten as

$$D_2 = \begin{cases} \frac{\rho}{A_{21}} D_1 + 1 & \text{if } D_2 > 1, \\ -\frac{\rho}{A_{21}} D_1 + 1 \equiv \bar{D} & \text{if } D_2 \leq 1. \end{cases}$$

Accordingly, we obtain Figure 2(b).

Moreover, equation $\gamma = \chi^2$ can be rewritten as follows:

$$D_2 = 1 + \frac{\rho^4 + (D_1 y^*)^4 + 2(\rho y^* D_1)^2}{4A_{21} y^* \{\rho^2 - (D_1 y^*)^2\}}.$$

The necessary and sufficient condition for $\gamma > \chi^2$ to hold is given by

$$4A_{21} y^* \{\rho^2 - (D_1 y^*)^2\}(D_2 - 1) > \rho^4 + (D_1 y^*)^4 + 2(\rho y^* D_1)^2.$$

Hence, when $D_1 y^* < (>) \rho$, the condition for $\gamma > \chi^2$ is

$$D_2 > (<) 1 + \frac{\rho^4 + (D_1 y^*)^4 + 2(\rho y^* D_1)^2}{4A_{21} y^* \{\rho^2 - (D_1 y^*)^2\}}. \tag{24}$$

Therefore, Figure 2(c) can be obtained. Note that when $D_1 y^* > \rho$, the right-hand side expression of (24) may become either positive or negative. When it is negative, the curve $\gamma = \chi^2$ exists in the negative region of D_2 for all $D_1 > \rho/y^*$.

4.3 Positional relationship between curves

This subsection considers the positional relationship between the curves shown in Figs. 2(a)–(c).

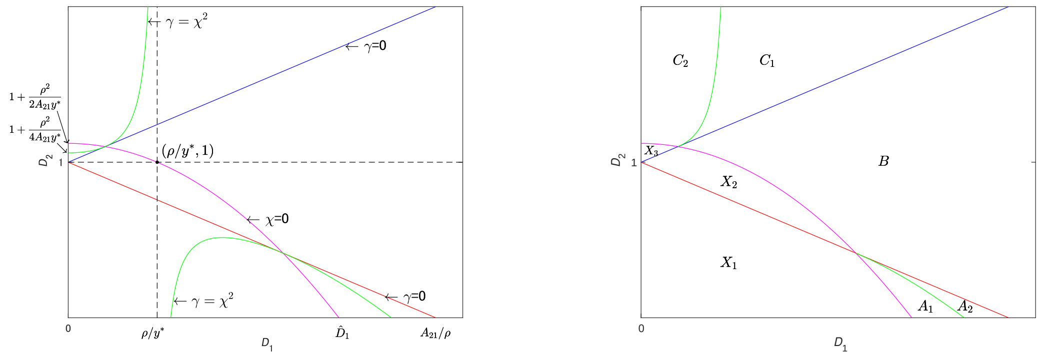
The intercept of the curve $\chi = 0$ in Figure 2(a) is $\hat{D}_2 = 1 + \frac{\rho^2}{2A_{21} y^*}$, and that of the curve $\gamma = \chi^2$ in Figure 2(c) is $D_2 = 1 + \frac{\rho^2}{4A_{21} y^*}$. Therefore, for $D_1 < \rho/y^*$, the curve $\chi = 0$ starts from a point located above the starting point of the curve $\gamma = \chi^2$, and the two curves intersect in the positive quadrant (Figure 3(a)).

At this intersection, $\gamma = 0$ holds because both $\chi = 0$ and $\gamma = \chi^2$ hold. Accordingly, the curves $\gamma = 0$ and $\gamma = \chi^2$ should have a common point. Moreover, because $\gamma < 0$ and $\gamma > \chi^2$ are incompatible, the curve $\gamma = \chi^2$ cannot pass below the line $\gamma = 0$. Therefore, these two curves have a point of tangency.

In summary, a point exists at which curves $\gamma = 0$, $\gamma = \chi^2$, and $\chi = 0$ pass through, and at that point, curves $\gamma = 0$ and $\gamma = \chi^2$ are in contact with each other.[6]

In addition, $\gamma = \chi^2 = 0$ is also established at the intersection of the downward segment of $\gamma = 0$ and the curve $\chi = 0$. A similar argument applies to the case in which $D_1 > \rho/y^*$.

[6] $\hat{D}_1 > \rho/y^*$ necessarily holds. Furthermore, the curve $\chi = 0$ passes through the point $(D_1, D_2) = (\rho/y^*, 1)$.



(a) Positional relationship between curves

(b) Eight regions

Figure 3 Partition of the (D_1, D_2)-plane

Some algebra shows that the intersections of curves $\chi = 0$ and $\gamma = 0$ are $\left(\frac{\rho(\sqrt{2}-1)}{y^*}, 1 + \frac{(\sqrt{2}-1)\rho^2}{y^* A_{21}}\right)$ and $\left(\frac{\rho(\sqrt{2}+1)}{y^*}, 1 - \frac{(\sqrt{2}+1)\rho^2}{y^* A_{21}}\right)$. The former coordinates always belong to the positive quadrant, corresponding to the upper intersection shown in Figure 3(a). Conversely, the condition that $A_{21}/\rho > (1 + \sqrt{2})\frac{\rho}{y^*}$ must be satisfied for the latter coordinates lie in the positive quadrant.[7] This condition is equivalent to the requirement that the intercept of the curve $\chi = 0$ on the horizontal axis is smaller than that of line $\gamma = 0$; that is, $\hat{D}_1 < A_{21}/\rho$.

In summary, eight regions arise as combinations of Figs. 2(a)–(c), as shown in Figure 3(b). In Region X, whether $\gamma > \chi^2$ or $\gamma < \chi^2$ does not affect the properties of z_1 and z_3, as shown in Table 1. When $A_{21}/\rho \leq (1+\sqrt{2})\frac{\rho}{y^*}$ holds and the intersection of the downward segment of $\gamma = 0$ and curve $\chi = 0$ does not exist in the positive quadrant, Regions A_1 and A_2 do not exist.

4.4 Stability change

Before moving on to the dynamic analysis of each region, we present a proposition that serves as the criterion for clarifying the direction of change in stability.

[7]This condition is equivalent to $(1 + \psi)(\phi - 1)/\eta > (1 + \sqrt{2})\rho^2$. Therefore, the condition is more likely to hold for larger values of ψ and ϕ and smaller values of ρ and η. Typically, this is satisfied because the subjective discount rate ρ takes a sufficiently small value.

4.4.1 Stability crossing points

If neither z_1 nor z_3 are positive real numbers, pure imaginary roots $\lambda = iz_1$ and iz_3 will not appear. Hence, in this case, a change in τ does not change the dynamic properties of the system. Conversely, if z_1 and/or z_3 are positive real numbers, the sign of the real parts of the complex roots changes at the point where $\lambda = iz_k$ $(k = 1, 3)$ holds.

For a given z, the value of τ that satisfies (17) and (18) exists in each interval $(2h\pi, 2(1+h)\pi)$, $h = 0, 1, 2, \ldots$. Let τ_h^k be the values of τ corresponding to z_k $(k = 1, 3)$ such that $2h\pi < \tau_h^k z_k < 2(1+h)\pi$. Different expressions for τ_h^k should be used depending on whether $2h\pi < \tau_h^k z_k \leq (1+2h)\pi$ or $(1+2h)\pi < \tau_h^k z_k < 2(1+h)\pi$. If $2h\pi < \tau_h^k z_k \leq (1+2h)\pi$, τ_h^k is defined as follows:

$$\tau_h^k = \frac{1}{z_k}\cos^{-1}\left[-\frac{\rho A_{21}(D_2-1)}{(\rho^2+z_k^2)D_1}\right] + \frac{2\pi h}{z_k}. \tag{25}$$

However , if $(1+2h)\pi < \tau_h^k z_k < 2(1+h)\pi$, $\cos\tau_h^k z_k$ belongs to the upward-sloping part of the cosine curve, which is not the range of the arccosine. Therefore, τ_h^k can be defined as follows.

$$\tau_h^k = \frac{1}{z_k}\left\{2\pi - \cos^{-1}\left[-\frac{\rho A_{21}(D_2-1)}{(\rho^2+z_k^2)D_1}\right]\right\} + \frac{2\pi h}{z_k}. \tag{26}$$

If z_1 is a positive real number, the following relation holds:

$$\begin{aligned} z_1^2 = Z_+ &= y^* A_{21}(D_2-1) - \rho^2 + \frac{\rho^2 + (D_1 y^*)^2 + \sqrt{\chi^2-\gamma}}{2} \\ &> y^* A_{21}(D_2-1) - \rho^2. \end{aligned}$$

Hence, from (18), $\sin\tau_h^1 z_1$ is negative, and $(1+2h)\pi < \tau_h^1 z_1 < 2(1+h)\pi$ holds for all $h = \{0, 1, 2, 3, \ldots\}$. Therefore, τ_h^1 is defined by (26).

If z_3 is a positive real number, the sign of $\sin\tau_h^3 z_3$ can be either positive or negative. From (22) and (23), we obtain the following equation:

$$z_3^2 = Z_- = y^* A_{21}(D_2-1) - \rho^2 + \varphi(D_1),$$

where $\varphi(D_1) \equiv \frac{\rho^2+(D_1y^*)^2-\sqrt{\chi^2-\gamma}}{2}$. From (18), $\sin\tau_h^3 z_3 < 0$ and $(1+2h)\pi < \tau_h^3 z_3 < 2(1+h)\pi$ hold if and only if $\varphi(D_1)$ is positive. Differentiating $\varphi(D_1)$ with respect to D_1 yields $\varphi'(D_1) = -\frac{2D_1y^{*2}(Z_-+\rho^2)}{\sqrt{\chi^2-\gamma}}$. When z_3 is a real number, Z_- must be positive;

therefore, $\varphi'(D_1) < 0$. Moreover, we can show that $\varphi(\rho/y^*) = 0$. Therefore, $\varphi(D_1)$ is positive only if $D_1 < \rho/y^*$.

The above result indicates that when $D_1 < \rho/y^*$, $\sin \tau_h^3 z_3$ is negative, and τ_h^3 is defined by (26). In contrast, when $D_1 \geq \rho/y^*$, $\sin \tau_h^3 z_3 > 0$ holds, and (25) should be used for τ_h^3. As shown in Tables 1 and 2, $z_3 > 0$ holds only when $\chi < 0$ and $\chi^2 \geq \gamma > 0$. This case corresponds to Regions A_2 and C_1 in Figure 3(b). In Region A_2, τ_h^3 is always defined by (26) because $D_1 > \rho/y^*$. However, in Region C_1, both cases can arise depending on the value of D_1.

In either case, when τ crosses τ_h^k, $h = 0, 1, 2, 3, \ldots$, the sign of the real parts of the complex roots changes. These points are referred to as *stability crossing points.*

4.4.2 Direction of crossing

The crossing directions in the sign of the complex roots can be examined as follows: When τ increases, if $\frac{d\,\mathrm{Re}\,\lambda}{d\tau}|_{\tau=\tau_h^k} > 0$, the sign changes from negative to positive. In contrast, if $\frac{d\,\mathrm{Re}\,\lambda}{d\tau}|_{\tau=\tau_h^k} < 0$, the sign changes from positive to negative. In other words, destabilization occurs if $\frac{d\,\mathrm{Re}\,\lambda}{d\tau}|_{\tau=\tau_h^k} > 0$, whereas stabilization occurs if $\frac{d\,\mathrm{Re}\,\lambda}{d\tau}|_{\tau=\tau_h^k} < 0$.

Using (14), we can prove the following Lemma.

Lemma 1 *For positive values of z_k, $k = 1, 3$, $\frac{d\,\mathrm{Re}\,\lambda}{d\tau}|_{\tau=\tau_h^1} > 0$ and $\frac{d\,\mathrm{Re}\,\lambda}{d\tau}|_{\tau=\tau_h^3} < 0$ hold for all $h = 0, 1, 2, 3, \ldots$.*

proof. See Appendix A.2.

Lemma 1 indicates that destabilization occurs when τ crosses τ_h^1. Conversely, stabilization occurs when τ crosses τ_h^3.

5 Dynamic analysis

5.1 Equilibrium determinacy in Regions X_1, X_3, A_1, and C_2

In Regions X_1, X_3, A_1, and C_2, both z_1 and z_3 are complex numbers. Hence, a stability crossing does not occur; that is, the local equilibrium determinacy is unaffected by variations in τ.

Proposition 1 (Figure 1) suggests that the equilibrium is locally indeterminate in Regions X_1 and A_1, irrespective of the value of τ. In Regions X_3 and C_2, the equilibrium is locally determinate, irrespective of the value of τ.

Accordingly, the following proposition can be stated.

Proposition 2 *1. When $D_1 \leq \frac{\rho(\sqrt{2}-1)}{y^*}$ and $D_2 > \frac{\rho}{A_{21}}D_1+1$ (Region X_3) or $D_1 \in [\frac{\rho(\sqrt{2}-1)}{y^*}, \frac{\rho}{y^*})$ and $D_2 > 1 + \frac{\rho^4+(D_1y^*)^4+2(\rho y^* D_1)^2}{4A_{21}y^*\{\rho^2-(D_1y^*)^2\}}$ (Region C_2), the equilibrium of system (12) is locally determinate for all $\tau \geq 0$.*

2. When $D_1 \leq \frac{\rho(\sqrt{2}+1)}{y^*}$ and $D_2 < \bar{D}$ (Region X_1) or $D_1 > \frac{\rho(\sqrt{2}+1)}{y^*}$ and $D_2 < 1 + \frac{\rho^4+(D_1y^*)^4+2(\rho y^* D_1)^2}{4A_{21}y^*\{\rho^2-(D_1y^*)^2\}}$ (Region A_1), the equilibrium of system (12) is locally indeterminate for all $\tau \geq 0$.

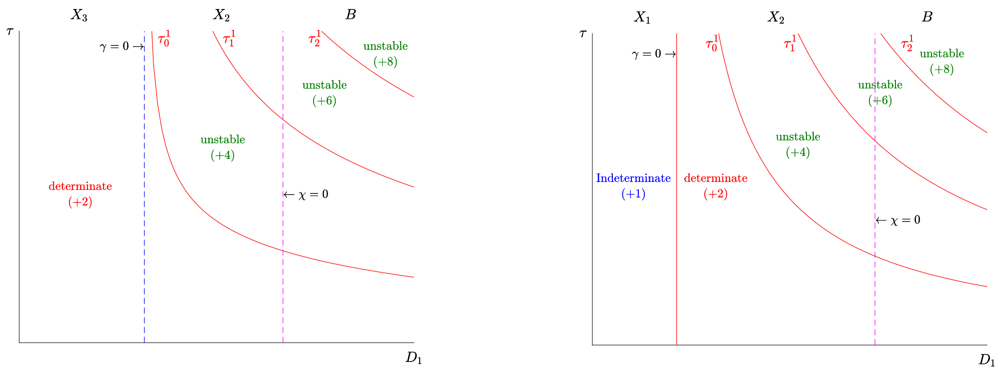
This proposition indicates that the interval of D_1 in which the lag has no effect is larger when D_2 has greater distances from 1. In particular, the policy lag associated with output targeting is highly likely to have no effect on the equilibrium determinacy when the monetary policy shows only a slight reaction to output.

5.2 Equilibrium determinacy in Regions X_2 and B

In Regions X_2 and B, z_3 is a complex number, and z_1 is a positive real number. Lemma 1 supports the conclusion that destabilization occurs at every stability crossing point τ_h^1, $h = 0, 1, 2, 3, \ldots$, when τ increases. Therefore, in these regions, the equilibrium is locally determinate for $0 \leq \tau < \tau_0^1$ and unstable for $\tau > \tau_0^1$. Thus, the following proposition can be derived.

Proposition 3 *When $D_1 > \max\left\{(D_2-1)\frac{A_{21}}{\rho}, -(D_2-1)\frac{A_{21}}{\rho}\right\}$ (Regions X_2 and B), a stability crossing point τ_0^1 exists such that the equilibrium is locally determinate for $0 \leq \tau < \tau_0^1$; in contrast, it is unstable for $\tau > \tau_0^1$.*

In this section, we present numerical examples. In Figure 4, we plot the graphs of τ_h^1, $h = 0, 1, 2$, in (26) with D_1 on the horizontal axis for a given D_2. The structural parameter values assumed here are based on those used by Tsuzuki (2014): $\phi = 21$, $\rho = 0.01$, $\eta = 200$, and $\psi = 1$. In addition, we set $\zeta = 0.01$. The graphs of τ_h^1 are downward curves on the (D_1, τ)-plane. On the boundary between Regions X_2 and X_3 or X_2 and X_1, τ_0^1 diverges to infinity because $z_1 = 0$ for $\gamma = 0$ and $\chi > 0$. In



(a) $1 < D_2 < 1 + \frac{(\sqrt{2}-1)\rho^2}{y^* A_{21}}$ **(b)** $D_2 < 1$

Figure 4 Equilibrium dynamics on the (D_1, τ)-plane

this figure, the numbers in parentheses indicate the number of roots with positive real parts. Figure 4(a) shows the case of $D_2 > 1$, in which the equilibrium is locally determinate under pure inflation targeting ($D_1 = 0$). Figure 4(b) shows the case of $D_2 < 1$, in which the equilibrium is locally indeterminate under pure inflation targeting. In both cases, when D_1 is sufficiently small, a stability crossing point does not exist; thus, a change in policy lag will never impact the equilibrium determinacy, as argued in Proposition 2.

However, when D_1 is large to some extent, the stability crossing point τ_0^1 emerges, and an increase in policy lag causes the equilibrium to change from determinate to unstable. Such a destabilizing effect arising from policy lag is consistent with Tsuzuki's (2014) findings. Furthermore, the larger the value of D_1, the smaller the threshold value of τ at which the stability changes from determinate to unstable. That is, when monetary policy is sensitive to output fluctuations, an increase in the policy lag associated with output targeting is more likely to cause economic instability.

This figure also clearly demonstrates that when a policy lag exists, the determinate equilibrium becomes unstable in response to an increase in D_1. In particular, Figure 4(a) highlights that although the equilibrium is determinate under simple inflation targeting, it may lose its stability for positive values of D_1 and τ. Therefore, introducing output targeting may trigger the instability of the equilibrium, thus creating volatility in the economy when a policy lag exists.

In the absence of a lag, an increase in D_1 necessarily contributes to stabilizing the economy, as shown in Proposition 1 and Figure 1. However, in cases where the lag is positive, this is not necessarily the case. Rather, the economy may become unstable because of an increase in D_1. The threshold value of D_1 is smaller for a larger value of τ; that is, when a larger policy lag exists, economic instability is more likely to result from output targeting.

5.3 Equilibrium determinacy in Regions A_2 and C_1

Finally, we examine the remaining cases. In Regions A_2 and C_1, both z_1 and z_3 are positive real numbers because $\chi > 0$ and $\chi^2 \geq \gamma > 0$ hold. In this case, two types of stability crossing points exist, and destabilization and stabilization may occur by increasing τ (see Lemma 1). The difference between Regions A_2 and C_1 is that in the absence of a policy lag, the equilibrium is locally indeterminate in Region A_2 (where one positive root exists), whereas in Region C_1, the equilibrium is locally determinate (where two positive roots exist). The following discussion explores the features of these regions in more detail.

5.3.1 Analysis of Region A_2

In Region A_2, both τ_h^1 and τ_h^3 are defined by (26). By differentiating (26) with respect to z_k,[8] we obtain

$$\frac{d\tau_h^k}{dz_k} = -\frac{\tau_h^k}{z_k} + \frac{2\rho A_{21}(D_2 - 1)}{(\rho^2 + z_k^2)^2 D_1 \sqrt{1 - \left(\frac{\rho A_{21}(D_2-1)}{(\rho^2+z_k^2)D_1}\right)^2}}. \tag{27}$$

In Region A_2, $D_2 < 1$ holds; hence, $d\tau_h^k/dz_k < 0$. Since $z_1 > z_3$, we obtain $\tau_h^1 < \tau_h^3$, $h = 0, 1, 2, 3, \ldots$. Accordingly, when τ increases from zero, the first-appearing stability crossing point is τ_0^1. At τ_0^1, the number of complex roots with positive real parts increases by two, and the equilibrium becomes locally unstable.

However, the next stability crossing point can be either τ_0^3 or τ_1^1. Because $z_1 > z_3$, the increment of τ_h^1 is smaller than that of τ_h^3 for an increase in h (see (26)). Therefore, τ_h^1 appears more frequently than τ_h^3. In other words, destabilization occurs more frequently. Hence, the equilibrium becomes locally unstable for a sufficiently large τ.

[8] The derivative of the function $y = \cos^{-1}(x)$ is given by $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$.

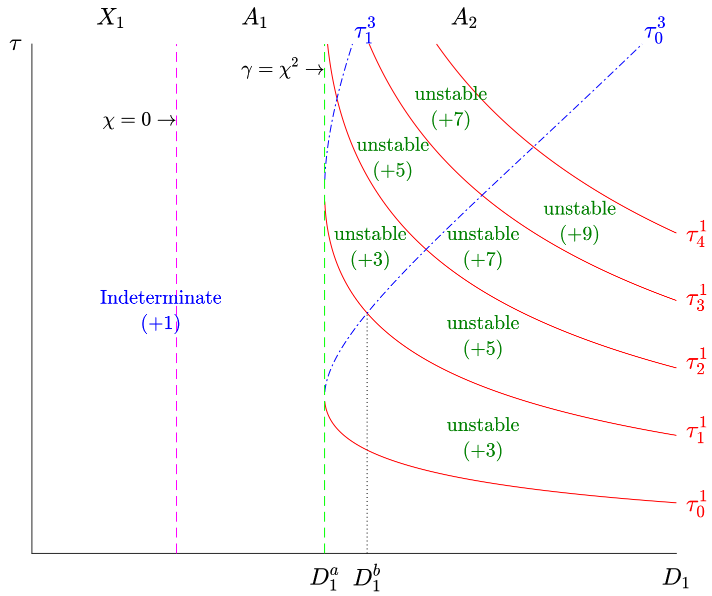


Figure 5 Stability changes in Region A_2

Figure 5 plots the graphs of τ_h^1 and τ_h^3 for $D_2 < 1 - \frac{(\sqrt{2}+1)\rho^2}{y^* A_{21}}$ with D_1 on the horizontal axis. In Region A_2, each τ_h^1 is a downward curve, and τ_0^3 is an upward curve. This region's remarkable phenomenon is observed for $D_1 \in (D_1^a, D_1^b)$. In this case, $\tau_0^1 < \tau_0^3 < \tau_1^1$ holds. The equilibrium determinacy changes from indeterminate to unstable at the first stability crossing point τ_0^1 when τ is increased from zero. However, it changes from unstable to indeterminate at the second stability crossing point τ_0^3. That is, stabilization occurs. The equilibrium changes are as follows: indeterminate–unstable–indeterminate.

Regarding $D_1 > D_1^b$, τ_0^3 is larger than τ_1^1. The sign of the real parts of a pair of complex roots changes from positive to negative when τ passes τ_0^3. However, such a change does not affect the determinacy of equilibrium because five or more complex roots with positive real parts already exist. The equilibrium is locally unstable for all $\tau > \tau_0^1$ and never becomes indeterminate again as a result of increasing τ.

In Region A_2, policy lag has complex effects on the equilibrium determinacy property, as shown above. However, the determinate equilibrium will never be achieved because the number of roots with positive real parts is always odd. Next, we examine Region C_1.

5.3.2 Analysis of Region C_1

In Region C_1, τ_h^3 is defined by (25) for $D_1 \geq \rho/y^*$ and by (26) for $D_1 < \rho/y^*$. In this region, it is uncertain whether τ_0^1 or τ_0^3 is larger. The first-appearing stability crossing point can be either τ_0^1 or τ_0^3. The increment of τ_h^1 is smaller than that of τ_h^3 with an increase in h; that is, τ_h^1 appears more frequently than τ_h^3. Therefore, for sufficiently large values of τ, the equilibrium is unstable.

When $\gamma = 0$, $z_3 = 0$; therefore, τ_h^3 diverges to infinity. Then, $\tau_h^3 > \tau_h^1$ holds in the neighborhood of the boundary between Regions C_1 and B. Accordingly, for a sufficiently large value of D_1, the first-appearing stability crossing point is τ_0^1 when τ increases from zero. The equilibrium becomes locally unstable at the first stability crossing point. The next stability crossing point can be either τ_0^3 or τ_1^1. If it is τ_0^3, the equilibrium can recover its stability by further increasing τ.

Figure 6 plots the graphs of τ_h^1 and τ_h^3 for $D_2 > 1 + \frac{(\sqrt{2}+1)\rho^2}{y^* A_{21}}$ with D_1 on the horizontal axis.[9] In this figure, both cases of $\tau_0^1 < \tau_0^3$ and $\tau_0^3 < \tau_0^1$ can be observed. When D_1 is sufficiently small and located in the interval (D_1^c, D_1^d), $\tau_0^3 < \tau_0^1$ holds and the first-appearing stability crossing point is τ_0^3. At this point, the equilibrium determinacy changes from determinate to indeterminate. The second-appearing stability crossing point is τ_0^1, at which the determinacy changes from indeterminate to determinate. That is, the equilibrium changes are as follows: determinate–indeterminate–determinate.

When D_1 is larger than D_1^d, $\tau_0^1 < \tau_0^3$ holds. In this case, the first-appearing stability crossing point becomes τ_0^1, and destabilization occurs first. For $D_1 \in (D_1^d, D_1^e)$, $\tau_0^1 < \tau_0^3 < \tau_1^1$ holds. In this case, increasing τ from zero causes the equilibrium determinacy to change from determinate to unstable at the first stability crossing point τ_0^1, and changes from unstable to determinate at the second stability crossing point τ_0^3. The equilibrium changes are as follows: determinate–unstable–determinate. This indicates that the stability lost by increasing τ is recovered by further increasing τ. We must emphasize that an increase in policy lag may not only cause destabilization, as pointed out in the previous literature (e.g., Tsuzuki 2014) but may also lead to stabilization if the monetary policy has multiple target variables.

[9] More precisely, we depict the case of $D_2 > 1 + \frac{\rho^2}{2A_{21}} y^*$ in Figure 6; therefore, Region X_3 does not appear.

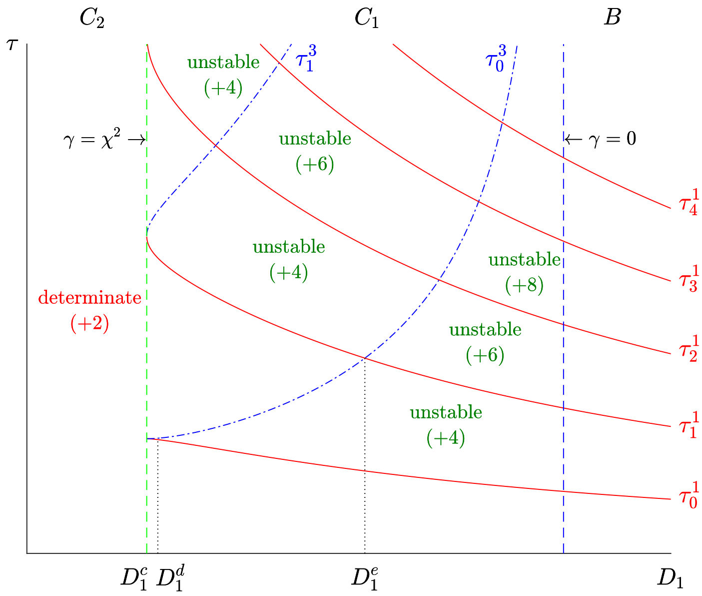


Figure 6 Stability changes in Region C_1

Finally, for $D_1 > D_1^e$, τ_0^3 is larger than τ_1^1. In this case, an increase in τ also has a stabilizing effect. The sign of the real parts of a pair of complex roots changes from positive to negative when τ is increased and passes τ_0^3. However, such a change does not influence the determinacy of equilibrium because at least six roots already have positive real parts when τ passes τ_0^3. Once lost, the determinacy of equilibrium cannot be recovered by increasing τ.

Figure 6 also highlights the destabilizing effect of the output targeting on the equilibrium determinacy. That is, when a policy lag exists, an equilibrium that is determinate without output targeting may lose its stability by increasing D_1. The threshold value of D_1 that triggers instability is not monotone with respect to τ owing to the coexistence of the stabilizing and destabilizing effects resulting from the policy lag.

The results obtained from the above analysis can be summarized as follows.

Proposition 4 *1. When $D_1 > \frac{\rho(\sqrt{2}+1)}{y^*}$ and* $\max\left\{0, 1 + \frac{\rho^4+(D_1y^*)^4+2(\rho y^* D_1)^2}{4A_{21}y^*\{\rho^2-(D_1y^*)^2\}}\right\} < D_2 < \bar{D} \equiv 1 - \frac{\rho D_1}{A_{21}}$ *(Region A_2), the equilibrium is locally indeterminate for $0 \le \tau < \tau_0^1$. For $\tau > \tau_0^1$, two typical cases may arise: (i) In the case that $\tau_0^1 < \tau_0^3 < \tau_1^1$, the equilibrium changes as follows: unstable–indeterminate–*

unstable–$\cdots$. (ii) In the case that $\tau_0^1 < \tau_1^1 < \tau_0^3$, the equilibrium changes as follows: unstable–unstable–unstable–$\cdots$.

2. *When $D_1 > \frac{\rho(\sqrt{2}-1)}{y^*}$ and $1 + \frac{\rho^4+(D_1 y^*)^4+2(\rho y^* D_1)^2}{4A_{21}y^*\{\rho^2-(D_1 y^*)^2\}} \geq D_2 \geq 1 + \frac{\rho D_1}{A_{21}}$ (Region C_1), if $\tau_0^3 > \tau_0^1$, the equilibrium is locally determinate for $0 \leq \tau < \tau_0^1$. For $\tau > \tau_0^1$, two typical cases may arise: (i) In the case that $\tau_0^1 < \tau_0^3 < \tau_1^1$, the equilibrium changes as follows: unstable–determinate–unstable–$\cdots$. (ii) In the case that $\tau_0^1 < \tau_1^1 < \tau_0^3$, the equilibrium changes as follows: unstable–unstable–unstable–$\cdots$.*

   Conversely, if $\tau_0^3 < \tau_0^1$, the equilibrium determinacy changes to indeterminate by increasing τ at first, and recovers its determinacy by further increasing τ.

Figure 7 summarizes the dynamic properties that typically arise in our model. For Regions X_1, X_3, A_1, and C_2, a policy lag does not affect their dynamic properties; however, Regions X_2, A_2, B, and C_1 can change dynamic properties in response to an increase in policy lag. In Regions X_2 and B, a policy lag necessarily causes instability. However, in Region C_1, a policy lag may contribute to achieving equilibrium determinacy.

Previous literature (e.g., Tsuzuki 2015) indicates that a central bank can eliminate indeterminacy by introducing a policy lag. In contrast, our results demonstrate that, in some cases, a policy lag can achieve determinacy by eliminating instability.

6 Conclusion

This study examined the effect of inflation–output targeting policies using an NK framework that considers a recognition or implementation lag in output targeting.

Previous studies have demonstrated that an increase in the time lag associated with inflation targeting could eliminate equilibrium indeterminacy or cause instability. By contrast, this study finds that an increase in a policy lag associated with output targeting can effectively eliminate instability.

In the conventional models that do not consider the existence of a policy lag, output targeting is regarded as a complement to inflation targeting; that is, equilibrium determinacy is established if the interest rate responds to the output in order to compensate for the interest rate's low response to the inflation rate. However,

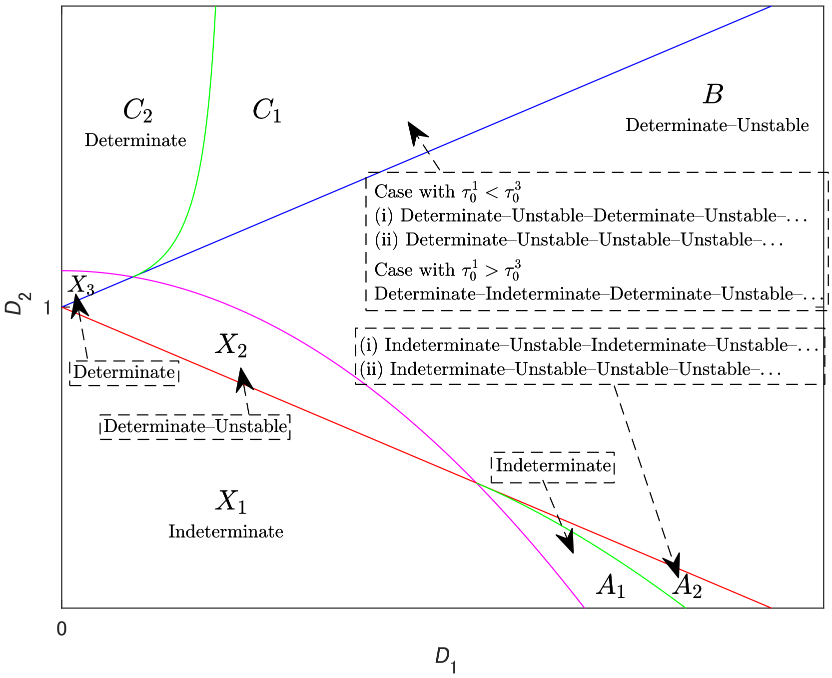


Figure 7 Summary of changes in dynamics

we showed that when there is a delay in the response to fluctuations in the output, a large response of the interest rate to the output does not necessarily stabilize the economy. This suggests that policymakers should carefully select target variables, considering the presence and length of a lag. Specifically, if D_2 is at a level that is included in Region C_1 in Figure 7, determinacy may be achieved by setting the level of D_1 appropriately even if a delay occurs in the central bank's response to output.

When time lags are present in both inflation targeting and output targeting, the economic model becomes a system of differential equations with two fixed delays. The properties of this type of dynamical system can be examined using the mathematical methods developed by Gu et al. (2005), Lin and Wang (2012), and Matsumoto and Szidarovszky (2012). The analysis of such cases could be a future research topic.

Another direction of research involves the analysis of global dynamics. This study discusses local dynamics. When the steady-state is locally unstable, no path exists that converges to the steady-state. The paths that continue to move away from the steady-state do not satisfy the transversality conditions of the household–firm's optimization problem. However, if a stable attractor such as a limit cycle exists around the steady-state, the equilibrium will be globally indeterminate because all points on the cycle naturally satisfy the transversality conditions. The analysis of global dynamics remains a topic for future research.

A Appendix

A.1 Dynamic optimization

The current-value Hamiltonian of the dynamic optimization problem in Section 2.2 is expressed as follows:

$$
\begin{aligned}
\mathcal{H}_j(c_j, m_j, v_j, a_j, p_j, \mu_1, \mu_2) = \varepsilon \log c_j + (1-\varepsilon)\log m_j &- \frac{1}{1+\psi}\left[\frac{1}{\zeta}\left(\frac{p_j}{p}\right)^{-\phi} y\right]^{1+\psi} \\
&- \frac{\eta}{2}(v_j - v^*)^2 + \mu_1\left[\frac{p_j}{p}\left(\frac{p_j}{p}\right)^{-\phi} y + ra_j - c_j - s_j - Rm_j\right] \\
&+ \mu_2 v_j p_j,
\end{aligned}
$$

where μ_1 and μ_2 are the costate variables of the state variables a_j and p_j, respectively. The first-order conditions for optimality are as follows:

$$\frac{\partial \mathcal{H}_j}{\partial c_j} = \varepsilon\frac{1}{c_j} - \mu_1 = 0, \tag{A.1}$$

$$\frac{\partial \mathcal{H}_j}{\partial m_j} = (1-\varepsilon)\frac{1}{m_j} - \mu_1 R = 0, \tag{A.2}$$

$$\frac{\partial \mathcal{H}_j}{\partial v_j} = -\eta(v_j - v^*) + \mu_2 p_j = 0, \tag{A.3}$$

$$\dot{\mu}_1 = \rho\mu_1 - \frac{\partial \mathcal{H}_j}{\partial a_j} = (\rho - r)\mu_1, \tag{A.4}$$

$$\dot{\mu}_2 = \rho\mu_2 - \frac{\partial \mathcal{H}_j}{\partial p_j} = \rho\mu_2 - y_j^{\psi}\frac{\phi}{\zeta^{1+\psi}}\frac{y_j}{p_j} - \mu_1(1-\phi)\frac{y_j}{p} - \mu_2 v_j. \tag{A.5}$$

Combining (A.1) and (A.4) yields (7). Furthermore, (A.1) and (A.2) yield Eq. (8). Moreover, (A.3) yields the following two expressions:

$$\mu_2 = \frac{\eta(v_j - v^*)}{p_j}, \quad \dot{\mu}_2 = \frac{\eta\dot{v}_j}{p_j} - \frac{\eta(v_j - v^*)}{p_j}v_j.$$

Substituting these equations and (A.1) into (A.5) yields (9).

A.2 Proof of Lemma 1

For calculational convenience, we examine the sign of $\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}|_{\tau=\tau_h^k}$ instead of that of $\frac{d\,\mathrm{Re}\,\lambda}{d\tau}|_{\tau=\tau_h^k}$.

The differentiation of both sides of (14) with respect to τ yields

$$\{2\lambda - \rho - D_1 y^* e^{-\lambda\tau} - \tau(\rho-\lambda)D_1 y^* e^{-\lambda\tau}\}\frac{d\lambda}{d\tau} = (\rho-\lambda)D_1 y^* \lambda e^{-\lambda\tau},$$

or equivalently

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda-\rho)e^{\lambda\tau}}{(\rho-\lambda)D_1 y^*\lambda} - \frac{1}{(\rho-\lambda)\lambda} - \frac{\tau}{\lambda}. \tag{A.6}$$

By substituting (14) into (A.6) to eliminate $e^{\lambda\tau}$, the equation becomes

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{-(2\lambda-\rho)}{[\lambda^2-\rho\lambda+A_{21}(D_2-1)y^*]\lambda} - \frac{1}{(\rho-\lambda)\lambda} - \frac{\tau}{\lambda}.$$

When $\lambda = iz_k$, the third term of the right-hand side expression becomes a pure imaginary number and hence can be ignored in the discussion here.

Let the real part of the first term of the right-hand side expression be θ_1, and the imaginary part be ν_1. Likewise, let the real part of the second term of the right-hand side expression be θ_2, and the imaginary part be ν_2. Then, the following equalities hold:

$$\frac{-2iz_k+\rho}{[-z_k^2 - i\rho z_k + D_1 + A_{21}(D_2-1)y^*]iz_k} = \theta_1 + i\nu_1,$$

$$-\frac{1}{i\rho z_k + z_k^2} = \theta_2 + i\nu_2.$$

Expanding these equations produces two simultaneous equation systems in which the unknown variables are θ_1 and ν_1, and θ_2 and ν_2, respectively; that is,

$$\begin{cases} \rho - \theta_1\rho z_k^2 - \nu_1 z_k^3 + \nu_1 z_k A_{21}(D_2-1)y^* = 0 \\ -2z_k + \theta_1 z_k^3 - \theta_1 z_k A_{21}(D_2-1)y^* - \nu_1\rho z_k^2 = 0 \end{cases}$$

$$\begin{cases} \rho z_k\theta_2 + \nu_2 z_k^2 = 0 \\ z_k^2\theta_2 - \nu_2\rho z_k + 1 = 0 \end{cases}$$

Solving these systems yields

$$\theta_1 = \frac{2\{z_k^2 - A_{21}(D_2-1)y^*\} + \rho^2}{\{z_k^2 - A_{21}(D_2-1)y^*\}^2 + \rho^2 z_k^2},$$

$$\theta_2 = -\frac{1}{z_k^2+\rho^2}.$$

Therefore, the real part of (A.6) can be expressed as follows:

$$\begin{aligned}\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_h^k} &= \theta_1+\theta_2\\ &= \frac{\Xi}{[\{z_k^2-A_{21}(D_2-1)y^*\}^2+\rho^2 z_k^2](z_k^2+\rho^2)},\end{aligned} \tag{A.7}$$

where

$$\begin{aligned}\Xi &\equiv 2\{z_k^2-A_{21}(D_2-1)y^*\}(z_k^2+\rho^2)+\rho^2(z_k^2+\rho^2)-\{z_k^2-A_{21}(D_2-1)y^*\}^2-\rho^2 z_k^2\\ &= z_k^4+2z_k^2\rho^2-2A_{21}(D_2-1)y^*\rho^2+\rho^4-A_{21}^2(D_2-1)^2y^{*2}.\end{aligned} \tag{A.8}$$

As the denominator of (A.7) is necessarily positive, we only need to observe the sign of the numerator Ξ.

Using (20) and (21), (A.8) can be rewritten as follows: $\Xi = z_k^4+2z_k^2\rho^2+\chi\rho^2-\frac{1}{4}\gamma$. Further, by substituting (19) into this equation and using equation $2z_k^2+\chi=(2-k)\sqrt{\chi^2-\gamma}$ $(k=1,3)$ obtained from (22) and (23), we get $\Xi=(2-k)(\rho^2+z_k^2)\sqrt{\chi^2-\gamma}$. For real and positive values of z_k, $k=1,3$, $\sqrt{\chi^2-\gamma}>0$. Therefore, Lemma 1 holds. ■

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