Theoretical Studies in Unawareness and Discovery Process

by

Yoshihiko Tada

A Dissertation

Submitted to Graduate School of Economics

Chuo University

In Fulfillment of the Requirements

For Doctor of Economics

March 2023

Preface

In 2012, when I entered Chuo University, I was far from being an honors student. However, I earned a Bachelor of Arts in Economics in 2016, a Master of Arts in Economics in 2018, and will successfully complete my PhD in Economics in 2023. My research life has taken off not only through my endeavors, but with the support of several others. I would like to thank the faculty and staff of the Faculty of Economics, Graduate School of Economics, and Graduate School of Strategic Management at Chuo University, for their generous support. I would also like to express my gratitude to my peers, Wataru Akutsu, Kohei Takahashi, Kiriya Tachibana, Yusuke Sugimoto, and Yosuke Fukuoka, along with members of the Hakumon Seiyukai (alumni association of Chuo University), for their extended support. Finally, I would like to thank my family for their patience and cooperation.

Abstract

This thesis explores two themes: a process of discovery in simultaneous-move games with unawareness and a reconsideration of non-trivial unawareness in standard information structures.[1]

The first theme discusses the discovery process in simultaneous-move games with unawareness. Games with unawareness assume that at least one player is unaware of her or his own actions or the opponents' actions. In this scenario, suppose that a player unaware of the opponents' actions observes such actions. Then, she or he may be surprised by the opponents' moves and adjust her or his subjective game accordingly.

Schipper (2021) proposed discovery processes as such updating models. He demonstrated that any rationalizable discovery process converges to some extensive-form game with unawareness possessing a rationalizable self-confirming equilibrium. However, he does not show that the plays of all agents converge to some self-confirming equilibrium. That is, he fails to demonstrate a convergence of plays to a specific solution concept.

Chapter 3 examines whether players can play to converge to a particular solution concept via a discovery process in simultaneous-move games with unawareness. Before conducting analysis, this study generalizes the concept of *closedness under rational behavior* (CURB) to simultaneous-move games with unawareness and models the *myopic discovery processes* allowing all players to respond optimally to their opponents' previous moves. A key result demonstrates that any myopic discovery process converges to specific revised subjective games with a *common realizable CURB set* of which all agents are aware. Then, in the games, supports of players' myopic best responses constitute a subset of the common realizable CURB set.

Chapter 4 explores discoveries of actions in *coordination games with unawareness*. In Schipper (2021) and Chapter 3, through a process of discovery, each player adds opponents' unnoticed actions to her or his revised subjective game. However, in situations where successful coordination is crucial, such as those in coordination games with unawareness, not only must players in their subjective games know their opponents' unnoticed actions, but they must also be able to imitate such actions. This thesis models an *imitative discovered game*

[1] In this thesis, standard information structures include not only partitional information structures where an information function is partitional but also non-partitional information structures where an information function is not partitional.

in which each player adds the opponents' actions to their own set of actions. We also demonstrate the existence of a *successful-coordination equilibrium* in which coordination thrives in the subsequent stage game.

The second theme reexamines unawareness in the standard model of a single agent's information structure where an information function may be not partitional. As pointed out by Dekel, Lipman, and Rustichini (1998), in standard information structures assuming Plausibility, KU Introspection, and AU Introspection, non-trivial unawareness (meaning that there is an event an agent is unaware of) cannot be represented even if an information function is not partitional.[2] Hence, models of unawareness structures proposed by Heifetz, Meier, and Schipper (2006) have become the mainstream in unawareness studies. However, changing the definitions and assumptions of knowledge and unawareness operators allows as to examine non-trivial unawareness in standard information structures. Chapter 5 models unawareness in standard information structures similar to Ewerhart (2001) and characterizes the knowledge and unawareness operators. Chapter 6 explores the relationship between Symmetry and AU Introspection.[3] According to Modica and Rustichini (1994, 1999), Dekel, Lipman, and Rustichini (1998), and Chen, Ely, and Luo (2012), Symmetry and AU Introspection must be equivalent to Negative Introspection in standard information structures under several assumptions. However, if Necessitation does not hold, the equivalence may not hold. This chapter relaxes Necessitation and demonstrates the conditions under which Symmetry and AU Introspection are equivalent in the presence of non-trivial unawareness. Chapter 7 reexamines the definition of the knowledge operator and the information function. Although the standard information function assumes that all information sets are not empty, Necessitation holds true even if some information sets are empty. That is, agents know all events even if they obtain a nonempty information set. This property is counterintuitive. This thesis redefines the knowledge operator such that the agent knows no event if the given information set is empty. In this case, Necessitation fails under non-trivial unawareness, whereas Monotonicity always holds. The knowledge operator in Chapter 7 is related to its Chapter 6 counterpart. Other aspects of unawareness have also been identified.

Finally, Chapter 8 concludes the dissertation with suggestions for further research.

[2]Plausibility means that if an agent is unaware of some event, then she or he does not know it, and she or he does not know that she or he does not know it. KU Introspection means that there is no event that an agent knows that she or he is unaware of. AU Introspection means that if an agent is unaware of some event, then the agent is unaware that she or he is unaware of it.

[3]Symmetry means that an agent is unaware of some event if and only if the agent is unaware of the complement event.

Acknowledgement

This dissertation is an edited version of the following discussion papers.

- Chapter 3 is based on the author's working papers titled "Unawareness of Actions Closedness under Rational Behavior in Static Games with Unawareness" (IERCU Discussion Paper No.336, Chuo University), "Discovery Process in Normal-Form Games with Unawareness: Cognitive Stability and Closedness under Rational Behavior" (IERCU Discussion Paper No. 343, Chuo University), and "Unawareness of Actions and Myopic Discovery Process in Simultaneous-Move Games with Unawareness" (IERCU Discussion Paper, No. 365; IERCU Discussion Paper, No. 370, Chuo University). This chapter ’ s preliminary version is presented at the 2018 Japanese Economic Association Autumn Meeting (at Gakushuin University; on September 8-9, 2018; in Tokyo), the 2021 Japanese Economic Association Spring Meeting (at Kwansei Gakuin University; on May 15-16, 2021; virtually), and the 2022 Asian Meeting of the Econometric Society in East and South-East Asia (at Keio University and University of Tokyo; on August 8-10, 2022; in Tokyo, through a hybrid format).

- Chapter 4 is based on the author's working paper titled "Coordination and Imitation under Unawareness" (IERCU Discussion Paper, No. 366; IERCU Discussion Paper, No. 371, Chuo University). A preliminary version of this chapter is presented at the 2022 Japanese Economic Association Autumn Meeting (at Keio University; on October 15-16, 2022; in Tokyo through a hybrid format).

- Chapter 5 is based on the author's working papers titled "Aumann Structure with Complete Lattice and Unawareness: Constructive Approach" (IERCU Discussion Paper No. 347, Chuo University), "Unawareness and Reverse Symmetry: Aumann Structure with Complete Lattice" (IERCU Discussion Paper No.351; IERCU Discussion Paper No.352; IERCU Discussion Paper No. 353, Chuo University), and "Non-Trivial Unawareness in (Non-)Partitional Standard Information Structures" (IERCU Discussion Paper No. 359, Chuo University).

- Chapter 6 is based on the author's working paper titled "Note: AU Introspection and Symmetry under Non-Trivial Unawareness" (IERCU Discussion Paper No. 357, Chuo University). A preliminary version of this chapter is presented at the Japan Association for Evolutionary Economics Annual Meeting (at Doshisha University; on March 26-27, 2022; virtually).

- Chapter 7 is based on the author's working papers titled "Mathematically Characterization of the Knowledge Structure of the Information Illiterate" (IERCU Discussion Paper No. 356, Chuo University), and "Is "Unawareness Leads to Ignorance" Trivial?" (IERCU Discussion Paper No. 358, Chuo University). A preliminary version of this chapter is presented at the Japan Association for Evolutionary Economics Annual Meeting (at Doshisha University; on March 26-27, 2022; virtually).

Each working paper and chapter received helpful comments from several academicians and anonymous reviewers regarding content, style, and paper structure. I express my gratitude towards Yasuo Sasaki, Norimasa Kobayashi, Masakazu Fukuzumi, Taisuke Matsubae, Bart Lipman, Spyros Galanis, and Satoshi Fukuda. I would also like to thank the examiners, Hirokazu Takizawa, Toichiro Asada, Yoji Taniguchi, and Ryuichiro Ishikawa, who reviewed this dissertation and provided a variety of comments.

Contents

Preface i

Abstract ii

Acknowledgement iv

I Introduction Part 1

1 Introduction 2
- 1.1 The Object of This Study 2
- 1.2 Related Literature 4
- 1.3 Thesis Structure 5

2 Theoretical Review and Motivation of the Thesis 7
- 2.1 Introduction 7
- 2.2 The Inception of the Study of Unawareness 8
  - 2.2.1 Information Structures and Common Knowledge 8
  - 2.2.2 Agreement Theorem 10
  - 2.2.3 No-Trade Theorem 11
  - 2.2.4 Triviality of Unawareness 12
  - 2.2.5 Unawareness Structures 13
- 2.3 Games with Unawareness 16
  - 2.3.1 Mathematical Formulations of Games 17
  - 2.3.2 Solution Concepts 20
  - 2.3.3 Discovery Process 20
  - 2.3.4 Motivation of PART II 21
- 2.4 Unawareness in Standard Information Structures 22
  - 2.4.1 Motivation of PART III 22

II Discovery of Actions in Simultaneous-Move Games

with Unawareness **23**

3 Unawareness of Actions and the Myopic Discovery Process in Simultaneous-Move Games with Unawareness **24**
3.1 Introduction 24
3.2 Preliminaries 26
3.2.1 Simultaneous-Move Games with Unawareness 26
3.2.2 Generalization of the CURB Concept 27
3.3 Myopic Discovery Process 31
3.4 Discovery Process and Other Solution Concepts 39
3.4.1 Relationships among Other Solution Concepts 39
3.4.2 Discovery and Equilibrium Notions 44
3.5 Discussion 48
3.5.1 A CURB Block Game and Economy of Cognitive Costs . 48
3.5.2 Adaptive Play 49
3.5.3 Limitations 50
3.5.4 Related Literature 50

4 Coordination and Imitation under Unawareness **52**
4.1 Introduction 52
4.2 Preliminaries 54
4.3 Successful-Coordination Equilibrium 55
4.4 Discovery and Imitation of Actions 55
4.5 Discussion 62
4.5.1 Relationship with the Generalized Nash Equilibrium . . . 62
4.5.2 Unawareness of Actions versus Lack of Conception 65
4.5.3 Related Literature 67

III Reexamining Unawareness in Standard Information Structures **68**

5 Non-Trivial Unawareness in Standard Information Structures **69**
5.1 Introduction 69
5.2 Preliminaries 70
5.3 Generalized Knowledge Operator 71
5.4 A Generalization of the Unawareness Operator 74
5.5 Generalized Triviality Theorems 78
5.6 Conclusion 79

6 Relationships between AU Introspection and Symmetry **81**
6.1 Introduction 81
6.2 Preliminaries 83
6.3 Triviality Theorems 84
6.4 Main Theorem 85
6.5 Concluding Remarks 88

7 Unaware Non-Decision Makers **90**
7.1 Introduction 90
7.2 Preliminaries 92
7.2.1 Information Structure 92
7.2.2 Standard Knowledge Operator 92
7.3 Knowledge Operator of Non-Decision Makers 93
7.4 Relationship with the Unawareness Operator 96
7.5 Concluding Remarks 100

IV Conclusion Part **101**

8 Concluding Remarks **102**
8.1 Conclusion 102
8.2 Further Research 103

Part I

Introduction Part

Chapter 1

Introduction

> "I am wiser than this human being. For probably neither of us knows anything noble and good, but he supposes he knows something when he does not know, while I, just as I do not know, do not even suppose that I do. I *am* likely to be a little bit wiser than he in this very thing: that whatever I do not know, I do not even suppose I know."
>
> *Apology of Socrates* 21d
> West and West (1998: 70)

1.1 The Object of This Study

This thesis examines two themes: (i) implementing discovery processes in simultaneous-move games with unawareness and (ii) characterizing unawareness in standard information structures. First, games with unawareness assume that players are unaware of others' actions. In such models, some players' beliefs about their opponents' plays might not include the opponents' actions of which they are unaware. When an opponent chooses an action of which players are unaware, they may be surprised by the opponent's choice and revise their subjective views and beliefs about opponents' plays accordingly. In games with unawareness, Schipper (2021) models *discovery processes* that are update processes of players' subjective games. These models assume that if actions of which some player is unaware are played, then all players add such actions into their subjective games in the next-stage game. Schipper (2021) shows that in any extensive-form game with unawareness, if all players select rationalizable strategies, then every rationalizable discovery process converges into some game with unawareness, possessing a rationalizable self-confirming equilibrium. However, he does not show that through any (rationalizable) discovery process, players' choices converge to some equilibrium or any other solution concepts. Moreover, in his model focusing on coordination games, although each player adds the discovered opponents' actions into their action sets based on those players' subjective views, the discovered actions are not added into each player's action set; in other words, his model does not assume that opponents' actions are imitated. PART II of this thesis considers such issues and models another discovery process that converges to some solution concept. Chapter 3 generalizes the concept of *closedness under rational behavior* (CURB)[1] to simultaneous-move games with unawareness, models the *myopic discovery processes* allowing all players to best respond to their opponents' previous moves, and shows that any myopic discovery process converges to specific revised subjective games with a *common realizable CURB set* of which all agents are aware. Then, in the games, supports of players' myopic best responses constitute a subset of the common realizable CURB set. Furthermore, it models the imitation of opponents' actions in coordination games with unawareness. Chapter 4 discusses coordination games with unawareness, defines a *successful-coordination equilibrium*, and models an *imitative discovered game* in which each player adds the opponents' actions to their own set of actions. We also demonstrate the existence of a successful-coordination equilibrium in which coordination thrives in the subsequent stage game.

The second theme examined by this study is characterizing unawareness in standard information structures where an information function might be not partitional. In such structures, given several assumptions about knowledge and unawareness, then unawareness is trivial in that an agent is aware of all event, as shown by Modica and Rustichini (1994, 1999) and Dekel, Lipman, and Rustichini (1998). However, unawareness can be discussed in information structure models if we relax some assumptions. Ewerhart (2001) presents a subjective state space that is a proper subset of the objective state space, assumes that an unaware agent does not know all the states in the complementary set, and characterizes the knowledge operator and unawareness operator. Meanwhile, Fukuda (2021) proposes excluding AU Introspection.[2] Both studies show that unawareness is not trivial in their frameworks. Hence, PART III of this thesis reexamines non-trivial unawareness in standard information structures. Chapter 5 models unawareness in (non-partitional) standard information structures similar to Ewerhart (2001) and characterizes the knowledge and unawareness operators. In the model, the knowledge operator might not satisfy Necessitation or Monotonicity. Moreover, we show that Symmetry does not hold when unawareness is not trivial.[3] Chapter 6 explores the relationship between Symmetry and AU Introspection, and shows the equivalence of them. Chapter 7 redefines

[1] We call a generalization of a CURB set a *realizable CURB set*.

[2] AU Introspection means that if an agent is unaware of some event, then the agent is unaware that she or he is unaware of it.

[3] Symmetry means that an agent is unaware of some event if and only if the agent is unaware of the complement event.

the knowledge operator differently from Chapter 5 such that the agent knows no event if the given information set is empty. In this case, Necessitation fails under non-trivial unawareness, whereas Monotonicity always holds.

1.2 Related Literature

Since Aumann (1976), standard game theory has supposed *common knowledge* of a game's structures.[4] Although this assumption is unrealistic, if theoretical conclusions are consistent with real-world decision-making outcomes, relaxing this assumption may no longer be needed. However, the assumption leads to a counterintuitive conclusion. Aumann (1976) shows that under the *common prior assumption* and the common knowledge of all players' posterior probabilities, all players' posterior must be the same. Milgrom and Stokey (1982) prove that if all players are risk-averse and it is common knowledge in speculative trade that an endowment is Pareto-optimal ex ante, then no players trade. Moreover, Rubinstein (1989) shows that common knowledge is assumed by the modeler and not formed by the communication exchanged among the agents. Therefore, subsequent studies have attempted to relax the common knowledge assumption.

The first model they considered was a *non-partitional information structure*, as shown in the studies of Geanakoplos (2021), Samet (1990), and Shin (1993). In this model, some agents may receive an incorrect information set. However, when introducing *unawareness* that means second-order ignorance, Modica and Rustichini (1994, 1999) and Dekel, Lipman, and Rustichini (1998) show that unawareness is trivial under several assumptions.[5]

To avoid this issue, Heifetz, Meier, and Schipper (2006) propose a model of *unawareness structures*. The interpretation of unawareness in their model differs from that in a model of standard information structures. Unawareness in standard information structures including non-partitional information structures is interpreted as a *lack of knowledge*, which suggests second-order ignorance. If agents are unaware of some event, then they do not know the event and are unaware that they do not know the event. By contrast, in unawareness structures, unawareness is interpreted as a *lack of conception*. In this model, not only do agents fail to recognize the objective state space, they also recognize the subjective state space, which is harder to describe than the objective state space. In other words, agents cannot recognize a specific conception in the objective state space. For example, von Neumann is both a physicist and a game theorist. However, someone unfamiliar with game theory may only know that he is a physicist and be unaware that he was a game theorist in the first place. Then, such an individual is aware that von Neumann is a physicist but not aware that he is a game theorist. Unawareness structures can represent non-trivial

[4] Lewis (1969) was the first to informally define common knowledge. Aumann (1976) defined common knowledge as follows: Some event is common knowledge if all players know the event, they know that they know it, they know that they know that they know it, and so on.

[5] Unawareness is trivial in that an agent is aware of every event. In other word, there is no event that the agent is unaware of. In this thesis, a series of results that unawareness is trivial is henceforth referred to as the *Triviality Theorems*.

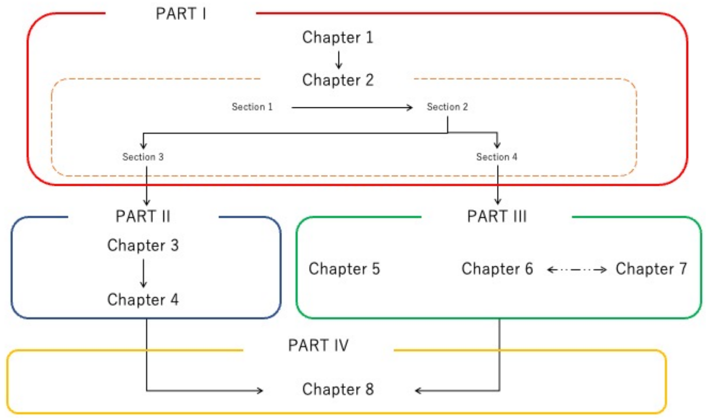


Figure 1.1: Structure of this thesis

unawareness. Therefore, subsequent mainstream studies of unawareness have used unawareness structures in their discussions, including Heifetz, Meier, and Schipper (2008, 2013a), Schipper (2013), Galanis (2013, 2018), and Galanis and Kotronis (2021).[6] Moreover, this idea of the lack of conception is introduced into game theory, with the resulting game models named *games with unawareness*, including the studies of Feinberg (2021), Heifetz, Meier, and Schipper (2013b), Halpern and Rêgo (2014), Rêgo and Halpern (2012), Grant and Quiggin (2013), Čopič and Galeotti (2006), and Meier and Schipper (2014).[7]

1.3 Thesis Structure

This dissertation organizes four parts and eight chapters. Figure 1.1 depicts the structure of this thesis. The remainder of this thesis is organized as follows. Chapter 2 describes the theoretical background, including a theoretical review of common knowledge and unawareness. It also models simultaneous-move games with unawareness and discusses the motivation of the thesis.

PART II discusses the discoveries of actions in simultaneous-move games with unawareness. Chapter 3 models a *myopic discovery process* in which each player best responds to opponents' actual actions in the immediately previous-stage game. First, this chapter generalizes the CURB (closedness under rational behavior) concept that is one of the set-valued solution concepts in simultaneous-move games with unawareness. Next, this study shows that through any myopic

[6] Another strand of the literature is similar in spirit to those arguments. For instance, Li (2009) presents a model of unawareness using product structures. Heinsalu (2012) shows that Li's (2009) work is equivalent to Fagin and Halpern (1988). Meanwhile, Heinsalu (2014) discusses universal type structures under unawareness.

[7] Among studies of unawareness, Karni and Vierø (2013, 2017), Vierø (2021), and Piermont (2017) discuss decision theory under unawareness. Similarly, Filiz-Ozbay (2012) and Auster (2013) discuss contract theory under unawareness.

discovery process, the support of all players' choices converges to an action subset of some CURB set. Chapter 4 assumes that opponents' actions in *coordination games with unawareness* can be imitated. In the model of imitation, when some player observes an opponent's action of which the player is unaware, the player adds such an action into both the opponent's action set and her or his own action set in the subjective game in the next-stage game with unawareness. Before presenting the analysis, this study provides a novel solution concept named the *successful-coordination equilibrium* that is a refinement of the *generalized Nash equilibrium*. This chapter shows that every next-stage game must have a successful-coordination equilibrium.

PART III reexamines non-trivial unawareness in standard information structures where an information function might be non-partitional. Chapter 5 revises the standard definitions of knowledge operator and unawareness operator along the lines of Ewerhart (2001). This chapter shows that our unawareness operator does not satisfy Symmetry[8] under non-trivial unawareness. Chapter 6 characterizes Symmetry and studies the relationships between Symmetry and AU Introspection under non-trivial unawareness. This chapter shows that Symmetry and AU Introspection are equivalent under non-trivial unawareness. Chapter 7 considers unaware non-decision makers. This chapter supposes that for some state, an information set corresponding to the state might be empty. The author interprets that such an empty information set means that the agent cannot recognize anything. Based on such an interpretation, a novel knowledge operator is defined and the properties of the knowledge operator and unawareness operator are shown. A knowledge operator in Chapter 7 is related its Chapter 6 counterpart. Note that definitions of knowledge operators between Chapters 5 and 7 are different. Hence, in each chapter, a knowledge operator leads to different properties. Under non-trivial unawareness, the knowledge operator introduced in Chapter 5 does not satisfy Necessitation or Monotonicity, whereas the knowledge operator introduced in Chapters 6 and 7 does not satisfy only Necessitation. Also, properties of unawareness operators between their chapters are different.

[8]Symmetry is one of the properties of the unawareness operator. It means that an agent is unaware of some event if and only if the agent is unaware of the complement event.

Chapter 2

Theoretical Review and Motivation of the Thesis

2.1 Introduction

Standard game theory usually assumes *common knowledge* of the structures of games.[1] Informally, the common knowledge of some events means that all agents know the event, know that they know the event, know that they know that they know the event, and so on ad infinitum. Even if common knowledge is an unrealistic assumption, if it adequately explains real economic phenomena, there may be no need to relax that assumption. However, the assumption of common knowledge leaves theoretically counterintuitive results, as shown by Milgrom and Stokey (1982). Hence, since their work, previous studies have attempted to relax the common knowledge assumption. The concept of unawareness was proposed during their research. Before discussing the two themes of games with unawareness (PART II) and the characterization of unawareness in standard information structures (PART III), this chapter provides a theoretical review of research on common knowledge and unawareness.[2]

[1] As aforementioned, Lewis (1969) was the first to introduce an informal definition of common knowledge and Aumann (1976) theoretically defines it. The concept of common knowledge is mainly used in studies of knowledge and equilibrium (e.g., Aumann (1987), Aumann and Brandenburger (1995), and Brandenburger (1992)).

[2] Several scholars have reviewed common knowledge and unawareness, including Geanakoplos (1992, 1994). Schipper (2014) presents a historical review of unawareness and Schipper (2015) reviews studies concerning unawareness using modal logic.

2.2 The Inception of the Study of Unawareness

2.2.1 Information Structures and Common Knowledge

First, information structures and common knowledge are mathematically formulated.[3] Let Ω be the finite state space and denote a state by $\omega \in \Omega$. Let I be the set of agents. Given $i \in I$, let $\langle \Omega, P_i \rangle$ be i's information structure. $P_i : \Omega \to 2^{\Omega} \setminus \{\emptyset\}$ is i's information function.[4] At any $\omega \in \Omega$, i receives an information set $P_i(\omega)$. The standard information function is assumed to have the following properties.

P1 For any $\omega \in \Omega$, $\omega \in P_i(\omega)$.

P2 For any $\omega, \omega' \in \Omega$, if $\omega' \in P_i(\omega)$, then $P_i(\omega') \subseteq P_i(\omega)$.

P3 For any $\omega, \omega' \in \Omega$, if $\omega' \in P_i(\omega)$, then $P_i(\omega') \supseteq P_i(\omega)$.

P1 means that given any state, i receives some information set possessing the state. P2 means that for any state in some given information set, the information set corresponding to the state is a subset of the given information set. P3 means that for any state in some given information set, the information set corresponding to the state is a super set of the given information set.

Next, let $\mathscr{P}_i = \{P_\lambda\}_{\lambda \in \Lambda}$ be i's partition on Ω, where Λ is an index set. That is,

1. $\bigcup_{P \in \mathscr{P}_i} P = \Omega$; and
2. For any $P, P' \in \mathscr{P}_i$, if $P \cap P' \neq \emptyset$, then $P = P'$.

Then, i's information function is partitional if and only if given any $\omega \in \Omega$, there exists a member of partition $P_\lambda \in \mathscr{P}_i$ such that $\omega \in P_\lambda$ and $P_\lambda = P_i(\omega)$. Then, the following remark holds.

Remark 1. *i's information function P_i satisfies P1, P2, and P3 if and only if P_i is partitional.*

Proof. First, suppose that P_i satisfies P1, P2, and P3. Then, from P1, $\bigcup_{\omega \in \Omega} P_i(\omega) = \Omega$. Pick $\omega, \omega' \in \Omega$. If $P_i(\omega) \cap P_i(\omega')$ is not empty, then there exists $\omega'' \in P_i(\omega) \cap P_i(\omega')$. Then, from P2 and P3, $P_i(\omega) = P_i(\omega') = P_i(\omega'')$. That is, P_i is partitional.

Next, suppose that P_i is partitional. Then, it is obvious that P1, P2, and P3 hold. $\square$

By Remark 1, in partitional information structures, the information function must satisfy P1-3.

---

[3]Related work in this section includes Aumann (1976), Milgrom (1981), Cave (1983), and Bacharach (1985).

[4]Standard discourse presupposes that any given information set is not empty. In Chapter 7, the assumption is relaxed, that is, there may exist some ω with $P_i(\omega) = \emptyset$.

Next, let us define i's knowledge operator $K_i : 2^\Omega \to 2^\Omega$. Let $E \subseteq \Omega$ be an event and $\neg E = \Omega \setminus E$. Here, given any $E \subseteq \Omega$, $K_i(E)$ is formulated as follows:

$$\begin{cases} \omega \in K_i(E) \text{ if } P_i(\omega) \subseteq E; \text{ and} \\ \omega \notin K_i(E) \text{ otherwise.} \end{cases}$$

Given any ω, if some event E is a superset of agent i's information set at ω, then $\omega \in K_i(E)$ holds. Hence, this is interpreted as "at ω, i knows the event E." If the event E is not a superset of i's information set at ω, $\omega \notin K_i(E)$ holds and this is interpreted as "at ω, agent i does not know the event E." Let $\neg K_i(E) = \Omega \setminus K_i(E)$. Moreover, let $K_iK_i(E) = K_i(K_i(E))$ meaning that i knows that i knows an event E, and let $K_i\neg K_i(E) = K_i(\neg K_i(E))$ meaning that i knows that i does not know an event E. Then, as is well known, the standard knowledge operator has the following properties.

Remark 2. Given the information structure $\langle \Omega, P_i \rangle$, K_i satisfies the following.

K1 Necessitation:

$K_i(\Omega) = \Omega$.

K2 Monotonicity:

$E \subseteq F \Longrightarrow K_i(E) \subseteq K_i(F)$.

K3 Conjunction:

$K_i(E \cap F) = K_i(E) \cap K_i(F)$.

K4 Truth:

If P1 holds, then $K_i(E) \subseteq E$.

K5 Positive Introspection:

If P2 holds, then $K_i(E) \subseteq K_iK_i(E)$.

K6 Negative Introspection:

If P3 holds, then $\neg K_i(E) \subseteq K_i\neg K_i(E)$.

Proof.

K1 By definition, $K_i(\Omega) \subseteq \Omega$. Pick any $\omega \in \Omega$, then $P_i(\omega) \subseteq \Omega$ obviously holds. That is, $\omega \in K_i(\Omega)$. Hence, $\Omega \subseteq K_i(\Omega)$.

K3 First, given any $\omega \in K_i(E \cap F)$, then $P_i(\omega) \subseteq E \cap F$. That is, $P_i(\omega) \subseteq E$ and $P_i(\omega) \subseteq F$. Then, because $\omega \in K_i(E)$ and $\omega \in K_i(F)$, $\omega \in K_i(E) \cap K_i(F)$. Hence, $K_i(E \cap F) \subseteq K_i(E) \cap K_i(F)$.

Next, given any $\omega \in K_i(E) \cap K_i(F)$, then $P_i(\omega) \subseteq E$ and $P_i(\omega) \subseteq F$; that is, $P_i(\omega) \subseteq E \cap F$ holds. Then, since $\omega \in K_i(E \cap F)$, $K_i(E) \cap K_i(F) \subseteq K_i(E \cap F)$.

K2 Suppose $E \subseteq F$. From K3, $K_i(E) = K_i(E \cap F) = K_i(E) \cap K_i(F) \subseteq K_i(F)$.

K4 Suppose P1. Given any $\omega \in K_i(E)$, then $P_i(\omega) \subseteq E$. From P1, since $\omega \in P_i(\omega)$, $\omega \in E$. Hence, $K_i(E) \subseteq E$.

K5 Suppose P2. Given any $\omega \in K_i(E)$, then $P_i(\omega) \subseteq E$. Here, from P2, for any $\omega' \in P_i(\omega)$, since $P_i(\omega') \subseteq P_i(\omega)$, $P_i(\omega') \subseteq E$; that is, $\omega' \in K_i(E)$. Hence, $P_i(\omega) \subseteq K_i(E)$; that is, $\omega \in K_i K_i(E)$. Therefore, $K_i(E) \subseteq K_i K_i(E)$.

K6 Suppose P3. Given any $\omega \in \neg K_i(E)$, then since $\omega \notin K_i(E)$, $P_i(\omega) \nsubseteq E$. Given any $\omega' \in P_i(\omega)$, from P3, since $P_i(\omega') \supseteq P_i(\omega)$, $P_i(\omega') \nsubseteq E$. Therefore, $\omega' \in \neg K_i(E)$; that is, $P_i(\omega) \subseteq \neg K_i(E)$. Then, since $\omega \in K_i \neg K_i(E)$, $\neg K_i(E) \subseteq K_i \neg K_i(E)$.

□

Necessitation means that i knows the whole state space at any state. Monotonicity means that if i knows some event, then she or he knows an event that is a super set of the event. Conjunction means that i knows an intersection of two events if and only if she or he knows both events. Truth means that if i knows some event then the event is true. Positive Introspection means that if i knows some event, then i knows that i knows it. Negative Introspection means that if i does not know some event, then i knows that i does not know it.

Now, suppose that each agent i's information function P_i is partitional. Given some $E \subseteq \Omega$, E is (first-order) mutual knowledge at $\omega \in \Omega$ if and only if for any $i \in I$, $\omega \in K_i(E)$; that is, $\omega \in \bigcap_{i \in I} K_i(E)$. Let $K^1(E) = \bigcap_{i \in I} K_i(E)$. E is nth-order mutual knowledge at ω if and only if $\omega \in K^n(E) = \bigcap_{k=1}^{n} (K^1)^k(E)$. Then, common knowledge is defined as follows.

Definition 2.2.1. An event $E \subseteq \Omega$ is *common knowledge* at $\omega \in \Omega$ if and only if $\omega \in \bigcap_{k=1}^{\infty} (K^1)^k(E)$. Let $CK(E) = \bigcap_{k=1}^{\infty} (K^1)^k(E)$.

As is well known, another definition of common knowledge exists. Given some $F \subseteq \Omega$, F is *self-evident* if and only if for any $\omega \in F$ and any $i \in I$, $P_i(\omega) \subseteq F$. Yet another definition is shown next.

Definition 2.2.2. An event $E \subseteq \Omega$ is common knowledge at $\omega \in \Omega$ if and only if there exists a *self-evident event* F such that $\omega \in F \subseteq E$.

Although the proof is omitted, the two definitions are equivalent.

Remark 3. Definition 2.2.1 and Definition 2.2.2 are equivalent.

2.2.2 Agreement Theorem

This section discusses the Agreement Theorem provided by Aumann (1976).[5] Consider again a finite set of states Ω and the set of agents I. For any $i \in I$, let

[5] Related work in this section includes Geanakoplos and Polemarchakis (1982) and Sebenius and Geanakoplos (1983).

$\mu_i : 2^\Omega \to [0, 1]$ be i's probability measure. Given any $E, F \subseteq \Omega$ with $\mu_i(F) \neq 0$, a conditional probability is defined as $\mu_i(E|F) = \frac{\mu_i(E \cap F)}{\mu_i(F)}$. Suppose a *common prior*: for any $\omega \in \Omega$ and $i, j \in I$, $\mu_i(\{\omega\}) = \mu_j(\{\omega\})$. Let us assume that each agent's information function is common knowledge. Here, at $\omega \in \Omega$, denote i's posterior of E by $\mu_i(E|P_i(\omega))$. Let $E(q_i) = \{\omega \in \Omega | \mu_i(E|P_i(\omega)) = q_i\}$ be the event that i assigns probability q_i to E. Then, although Aumann's (1976) formulation is a two-person model, we can generalize his Agreement Theorem as follows.

Theorem 2.2.1 (Aumann 1976)**.** Suppose that there exists a common prior on Ω. If $CK(\cap_{i \in I} E(q_i)) \neq \emptyset$, then for any $i, j \in I$, $q_i = q_j$.

The Agreement Theorem means that if the posterior probabilities of all agents with the same prior are commonly known, then each agent's posterior is the same as that of the others. In other words, "people with the same priors cannot agree to disagree" (Aumann, 1976: 1236).

2.2.3 No-Trade Theorem

This subsection shows the No-Trade Theorem provided by Milgrom and Stokey (1982). Let us consider l commodities. Denote the set of traders by I and the set of commodity bundles by $X \subseteq \mathbb{R}^l_+$. Each trader i has the partitional information function P_i on the finite state space Ω. Here, for any $i \in I$,

- $e_i : \Omega \to X$ is i's endowment function and $e_i(\omega) \in X$ is i's endowment at ω;
- $u_i : \Omega \times X \to \mathbb{R}$ is i's utility function; and
- $\mu_i : 2^\Omega \to [0, 1]$ is i's probability measure.

Given any $x \in X$, i's expected utility is denoted by

$$Eu_i(\cdot, x) = \sum_{\omega \in \Omega} \mu_i(\{\omega\}) u_i(\omega, x)$$

and i' s expected utility at ω is denoted by

$$E[u_i(\cdot, x)|\omega] = \sum_{\omega' \in P_i(\omega)} \frac{\mu_i(\{\omega'\})}{\mu_i(P_i(\omega))} u_i(\omega', x).$$

Given any $\omega \in \Omega$, suppose that $u_i(\omega, x)$ is strictly concave in x for each ω. This means that i is risk-averse. Let $t_i : \Omega \to X$ be i's trade function. At ω, a trade is feasible if and only if

- For any $i \in I$, $e_i(\omega) + t_i(\omega) \geq 0$; and
- $\sum_{i \in I} t_i(\omega) \leq 0$.

Finally, suppose a common prior and that all agents are rational and that this is common knowledge. Then, Milgrom and Stokey (1982) show their No-Trade Theorem, which they prove using the Agreement Theorem.

Theorem 2.2.2 (Milgrom and Stokey 1982)**.** In the l-commodities market, assume that every trader is risk-averse. Here, suppose that the followings are common knowledge at ω.

- $e = (e_i)_{i \in I}$ is Pareto-optimal ex ante.
- Trade is feasible.
- Every trader prefers trade to zero trade.

Then, no traders trade.

This result means that no traders trade even if they obtain private information.

2.2.4 Triviality of Unawareness

The unrealistic result shown in the No-Trade Theorem is derived from the assumption of common knowledge. Therefore, previous studies have tried to relax the assumption of common knowledge and to formulate agents' ignorance. Geanakoplos (2021) provides a framework of *non-partitional information structures*. In his model, the information function might not be partitional; that is, P1, P2, or P3 might not hold. Samet (1990) and Shin (1993) also discuss agents' ignorance in the non-partitional model.

However, even if we consider non-partitional information structures, it may be impossible to discuss non-trivial unawareness. The following shows the Triviality Theorem proven by Dekel, Lipman, and Rustichini (1998). First, given i's unawareness operator $U_i : 2^\Omega \to 2^\Omega$, suppose the following properties.

- Plausibility:
  $U_i(E) \subseteq \neg K_i(E) \cap \neg K_i \neg K_i(E)$.
- KU Introspection:
  $K_i U_i(E) = \emptyset$.
- AU Introspection:
  $U_i(E) \subseteq U_i U_i(E)$.

Plausibility means that if i is unaware of some event, then i does not know it and i does not know that i does not know it. KU Introspection means that there is no event that i knows that she or he is unaware of. AU Introspection means that if i is unaware of some event, then i is unaware that i is unaware of it. Then, Dekel, Lipman, and Rustichini (1998) prove that the following theorem holds.[6]

[6]Other studies proving trivial unawareness include Modica and Rustichini (1994, 1999) and Chen, Ely, and Luo (2012).

Theorem 2.2.3 (Dekel, Lipman, and Rustichini 1998)**.** Suppose U_i satisfies Plausibility, KU Introspection, and AU Introspection. Then:

- Triviality:

  If K_i satisfies Necessitation, then $U_i(E) = \emptyset$ for any $E \subseteq \Omega$.

- Unawareness Leads to Ignorance:

  If K_i satisfies Monotonicity, then for any $E, F \subseteq \Omega$, $U_i(E) \subseteq \neg K_i(F)$.

Proof. Pick any $E \subseteq \Omega$. From Plausibility, KU Introspection, and AU Introspection, $U_i(E) \subseteq U_i U_i(E) \subseteq \neg K_i U_i(E) \cap \neg K_i \neg K_i U_i(E) = \neg\emptyset \cap \neg K_i \neg\emptyset = \neg K_i(\Omega)$.

Here, suppose Necessitation; then, since $K_i(\Omega) = \Omega$, $U_i(E) \subseteq \neg K_i(\Omega) = \emptyset$.

Suppose Monotonicity. For any $F \subseteq \Omega$, since $K_i(F) \subseteq K_i(\Omega)$, $\neg K_i(\Omega) \subseteq \neg K_i(F)$. That is, $U_i(E) \subseteq \neg K_i(\Omega) \subseteq \neg K_i(F)$. $\square$

Triviality means that no event makes an agent unaware. Unawareness Leads to Ignorance means that if an agent is unaware of some event, then she or he knows no event.

Therefore, Necessitation and Monotonicity must be relaxed to discuss non-trivial unawareness. However, in the standard information structures, Necessitation and Monotonicity must hold according to the definition of the knowledge operator even if an information function is not partitional. See the proof of Remark 2. Hence, the standard information structures cannot discuss non-trivial unawareness when assuming Plausibility, KU Introspection, and AU Introspection.

2.2.5 Unawareness Structures

As shown in the previous subsection, under the standard assumptions, non-trivial unawareness cannot be discussed. To avoid this issue, Heifetz, Meier, and Schipper (2006) propose a novel framework of unawareness named *unawareness structures*.[7]

The interpretation of unawareness in unawareness structures differs from that in the standard information structures in which unawareness shows higher-order ignorance. That is, if an agent is unaware of some event, then they do not know the event and do not know that they do not know the event. By contrast, in unawareness structures, unawareness is interpreted as a *lack of conception*. For example, the cholera bacterium existed before Koch discovered it, but people did not know about its existence until that discovery. Then, people were unaware of the cholera bacterium. Unawareness structures assume that each agent recognizes a subjective state space with different expressive power. This paper formulates unawareness structures and characterizes the

[7]Other studies that interpret unawareness as a lack of conception include Heifetz, Meier, and Schipper (2008, 2013a), Li (2009), Heinsalu (2012), Schipper (2013, 2014, 2015), Galanis (2013, 2018), Galanis and Kotronis (2021), and Fukuda (2021).

knowledge operator and unawareness operator thereof based on Heifetz, Meier, and Schipper (2006).

Let $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ be a family of non-empty state spaces, where Λ is an index set. $\mathcal{S}$ is a complete lattice of disjoint spaces and there exists a partial order $\succeq$ on $\mathcal{S}$. Let $\Sigma = \bigcup_{S \in \mathcal{S}} S$ be the generalized state space. For any two spaces $S, S' \in \mathcal{S}$, $S \succeq S'$ means that "S has the same expressive power as S' or S is more expressive than S'." For any $S, S' \in \mathcal{S}$, let $r^S_{S'} : S \to S'$ be a surjective projection. For any $\omega \in S$, $r^S_{S'}(\omega) \in S'$ is a restriction of ω. Here, let $r^S_{S'}(\omega) = \omega_{S'}$. Then, the description of $\omega_{S'}$ is coarser than that of ω. For example, let $\mathcal{S}$ be the family of cholera bacteria. S is the state space after Koch discovered cholera bacteria, whereas S' is that beforehand. Then, given $\omega \in S$ and $\omega' \in S'$, the following interpretation is provided. ω means that Alice contracted a disease caused by cholera bacteria, whereas ω' means that Alice fell ill. Given the three spaces $S, S', S'' \in \mathcal{S}$, if $S \succeq S' \succeq S''$, then $r^S_{S''} = r^{S'}_{S''} \circ r^S_{S'}$. For any $B \subseteq S$, $B_{S'} = \{\omega_{S'} | \text{For any } \omega \in B,\ \omega_{S'} = r^S_{S'}(\omega)\}$.

Next, we consider events on the generalized state space. Given any space $S \in \mathcal{S}$ and an event $B \subseteq S$, let $B^\uparrow = \bigcup_{S' \in \mathcal{S}: S' \succeq S} \{\omega' \in S' | \exists \omega \in B,\ r^{S'}_S(\omega') = \omega\}$. $E \subseteq \Sigma$ is an event satisfing $E \subseteq B^\uparrow$, where $S \in \mathcal{S}$ and $B \subseteq S$. Then, we call B the basis of E and S the base space of E. Let $S(E)$ be the base space of E. Given S and B, if $B^\uparrow$ is an event, $\neg B^\uparrow = \Sigma \setminus B^\uparrow$ is defined by $(S \setminus B)^\uparrow$. Define $\neg S^\uparrow = \emptyset^S$ and $\neg \emptyset^S = S^\uparrow$.

Now, let us consider agent i's possibility correspondence $\Pi_i : \Sigma \to 2^\Sigma \setminus \{\emptyset\}$. Heifetz, Meier, and Schipper (2006) assume the following properties.[8]

(0) Confinedness:

Given $\omega \in S$, there exists S' such that $S \succeq S'$ and $\Pi_i(\omega) \subseteq S'$.

(1) Generalized Reflexivity:

For any $\omega \in \Sigma$, $\omega \in (\Pi_i(\omega))^\uparrow$.

(2) Stationarity:

If $\omega' \in \Pi_i(\omega)$, then $\Pi_i(\omega') = \Pi_i(\omega)$.

(3) Projections Preserve Awareness:

If $\omega \in S$, $\omega \in \Pi_i(\omega)$ and $S \succeq S'$, then $\omega_{S'} \in \Pi_i(\omega_{S'})$.

(4) Projections Preserve Ignorance: If $\omega \in S$ and $S \succeq S'$, then $(\Pi_i(\omega))^\uparrow \subseteq (\Pi_i(\omega_{S'}))^\uparrow$.

(5) Projections Preserve Knowledge: If $S \succeq S' \succeq S''$, $\omega \in S$, and $\Pi_i(\omega) \subseteq S'$, then $(\Pi_i(\omega))_{S'} = \Pi_i(\omega_{S'})$.

Let us now formulate the knowledge operator on the generalized state space. Given $E \subseteq \Sigma$, i's knowledge operator is defined as follows.

[8] Galanis (2013) and Galanis and Kotronis (2021) relax Projections Preserve Knowledge.

$$K_i^*(E) = \begin{cases} \{\omega \in \Sigma | \Pi_i(\omega) \subseteq E\} & \text{if it is nonempty; and} \\ \emptyset^{S(E)} & \text{otherwise.} \end{cases}$$

Then, the knowledge operator has the following properties.

Proposition 2.2.1 (Heifetz, Meier, and Schipper 2006)**.** The knowledge operator K_i^* has the following properties.

K1* Necessitation:

$K_i^*(\Sigma) = \Sigma$.

K2* Monotonicity:

$E \subseteq F$ implies $K_i^*(E) \subseteq K_i^*(F)$.

K3* Conjunction:

$K_i^*(\bigcap_\lambda E_\lambda) = \bigcap_\lambda K_i^*(E_\lambda)$.

K4* Truth:

$K_i^*(E) \subseteq E$.

K5* Positive Introspection:

$K_i^*(E) \subseteq K_i^* K_i^*(E)$.

The properties of the knowledge operator in unawareness structures are similar to those in standard information structures. However, there are two differences. One is about Necessitation. Strictly speaking, in unawareness structures, the knowledge operator does not satisfy Necessitation, but only Necessitation for "least expressive tautology," that is the union of all states across all spaces in the lattice.[9] The other is about Negative Introspection. Negative Introspection may not hold in unawareness structures.

We are now in a position to discuss the unawareness operator. Let $U_i^*(E) = \neg K_i^*(E) \cap \neg K_i^* \neg K_i^*(E)$, and $A_i^*(E) = \neg U_i^*(E)$. Then, the following properties hold.

Proposition 2.2.2 (Heifetz, Meier, and Schipper 2006)**.**

U1 KU Introspection:
$K_i^* U_i^*(E) = \emptyset^{S(E)}$.

U2 AU Introspection:
$U_i^*(E) = U_i^* U_i^*(E)$.

U3 Weak Necessitation:
$A_i^*(E) = K_i^*(S(E)^\uparrow)$.

U4 Strong Plausibility:
$U_i^*(E) = \bigcap_{n=1}^\infty (\neg K_i^*)^n(E)$.

[9] The author would like to thank an anonymous referee for pointing this out.

U5 Weak Negative Introspection:
$\neg K_i^*(E) \cap A_i^* \neg K_i^*(E) = K_i^* \neg K_i^*(E)$.

U6 Symmetry:
$U_i^*(E) = U_i^*(\neg E)$.

U7 A-Conjunction:
$\bigcap_\lambda A_i^*(E_\lambda) = A_i^*(\bigcap_\lambda E_\lambda)$.

U8 AK-Self-Reflection:
$A_i^* K_i^*(E) = A_i^*(E)$.

U9 AA-Self-Reflection:
$A_i^* A_i^*(E) = A_i^*(E)$.

U10 A-Introspection:
$K_i^* A_i^*(E) = A_i^*(E)$.

U1–4 are proposed by Dekel, Lipman, and Rustichini (1998); U6–9 by Modica and Rustichini (1994, 1999); and U5–9 by Halpern (2001). Heifetz, Meier, and Schipper (2006) show an additional property, U10. In unawareness structures, non-trivial unawareness can be discussed.

Since Heifetz, Meier, and Schipper (2006), most studies of unawareness have used unawareness structures. Heifetz, Meier, and Schipper (2013a) and Galanis (2013, 2018) use unawareness structures and discuss and generalize Aumann's Agreement Theorem (Aumann, 1976) and the No-Trade Theorem (Milgrom and Stokey, 1982). Heifetz, Meier, and Schipper (2008) propose canonical models of unawareness. Galanis (2011) considers unawareness of theorems using a logical approach, while Galanis (2013) discusses unawareness of theorems using a set-theoretical approach, provides a property named Awareness Leads to Knowledge, and shows that a knowledge operator in a more expressive state space can better describe an agent's knowledge than a knowledge operator in a less expressive state space. This result means that Galanis' model allows agents to disagree on whether opponents know about some event. Li (2009) proposes a product of the state-space model, called an information structure with unawareness. Heinsalu (2012) discusses the relationship between the work of Fagin and Halpern (1988) and Li (2009).

2.3 Games with Unawareness

Since Heifetz, Meier, and Schipper (2006), unawareness has been interpreted as a lack of conception. This notion has been introduced to game theory. Game theory with unawareness assumes a lack of conception of situations faced by unaware agents. An unaware player can be unaware of agents, actions, payoffs, and types. Pioneering work on games with unawareness includes Feinberg (2021), Heifetz, Meier, and Schipper (2013b), Halpern and Rêgo(2014), Ozbay (2007), Grant and Quiggin (2013), and Meier and Schipper (2014).

Their models differ from standard Bayesian games, which usually assume only payoff uncertainty. While all players share the common prior of types, no

Alice / Bob	L	R
U	3, 3	0, 5
B	5, 0	1, 1

Table 2.1: A prisoner's dilemma game

Alice / Bob	L	R
U	4, 4	2, 5
B	5, 2	0, 0

Table 2.2: A chicken game

player knows the true value of the other players' payoff. However, all players know all actions and all types of players. Let us consider two games, namely, a prisoner's dilemma game (depicted in Table 2.1) and a chicken game (depicted in Table 2.2).
Alice and Bob face a strategic situation. In Bayesian games, although they do not know which game is true between Table 2.1 and Table 2.2, they know that the true game is either Table 2.1 or Table 2.2.

By contrast, games with unawareness assume that some player is unaware of a subset of the action set. For example, both players face a prisoner's dilemma game, but Bob does not know that he has an action R; that is, he believes that he faces the game in Table 2.3.

Expressions exist in Bayesian games that mean that players ignore actions. Harsanyi (1967) presents the ignorance of actions by assigning extremely low payoffs to such actions. For example, if Bob does not know R, then $-\infty$ is assigned to his payoffs in (U, R) and (B, R), as shown in Table 2.4. However, as pointed out by Meier and Schipper (2014), if Bob is irrational, he might choose R. Hence, Harsanyi's representation may be unsuitable for discussing the ignorance of actions. Models of games with unawareness must be used to express genuine unawareness.

2.3.1 Mathematical Formulations of Games

Let us focus on simultaneous-move situations and formulate standard Bayesian games before modeling games with unawareness.

Definition 2.3.1. A standard Bayesian game $\Gamma^B = (I, (A_i)_{i\in I}, (T_i)_{i\in I}, (p_i)_{i\in I}, (u_i)_{i\in I})$ is defined as follows.

- I is the set of players. $i \in I$ is one of the players.
- A_i is the set of i's actions. Let $A = \times_{i\in I} A_i$.
- T_i is the set of i's types. Let $T = \times_{i\in I} T_i$.
- $p_i : T \to [0, 1]$ is i's probability measure of opponents' types.[10]

[10]A common prior assumption is that for any $t \in T$ and $i, j \in I$, $p_i(t) = p_j(t)$.

Alice / Bob	L
U	3, 3
B	5, 0

Table 2.3: Bob is unaware of R.

Alice / Bob	L	R
U	3, 3	0, $-\infty$
B	5, 0	1, $-\infty$

Table 2.4: Bob does not know R in the Harsanyi-style model.

- $u_i : A \times T_i \to \mathbb{R}$ is i's utility function.

The above definition indicates that the action set is fixed. That is, no agent can be unaware of some actions.

Next, we define games with unawareness using type-based approaches. In simultaneous-move games with unawareness, type-based approaches have been adopted by Meier and Schipper (2014) and Perea (2022). Meier and Schipper (2014) model Bayesian games with unawareness based on Heifetz, Meier, and Schipper's (2013a) probabilistic unawareness structures. By contrast, Perea's (2022) formulation of type spaces are similar to Harsanyi's (1967) type spaces. Crucial differences between Meier and Schipper (2014) and Perea (2022) exist. The former assume that players' types are directly associated with their views of games, whereas the latter believes that although players' types are associated with their beliefs about the views of games, those types cannot be associated with the views themselves. For simplicity, we focus on the Perea-style model of games with unawareness.[11]

Before formulating the Perea-style model, note the following. Consider a standard finite simultaneous-move game $G = (I, A, u)$. I is a finite set of players and $I_{-i} = I \backslash \{i\}$. $A = \times_{i \in I} A_i$, where A_i is the non-empty finite set of i's actions and each element of the set is $a_i \in A_i$. Let $A_{-i} = \times_{j \in I_{-i}} A_j$. $u = (u_i)_{i \in I}$, where $u_i : A \to \mathbb{R}$ is i's utility function. Denote i's mixed action on A_i by $m_i \in M(A_i)$, where $M(A_i)$ is the set of i's mixed actions, and a mixed action profile on A by $m = (m_i)_{i \in I} \in \times_{i \in I} M(A_i)$. We denote i's expected utility for $m \in \times_{i \in I} M(A_i)$ by $Eu_i(m)$.

For any standard simultaneous-move game G, let $V = \times_{i \in I}(2^{A_i} \setminus \{\emptyset\})$ be the set of possible *views* of G (i.e., the set of a Cartesian product of a non-empty action subset). Like most previous works, this study assumes that the set of players is commonly known and that each player's utility for each action profile is the same among all possible views. Let $v \in V$ be one (possible) view and A_i^v be the set of i's actions in v. Let $A_{-i}^v = \times_{j \in I_{-i}} A_j^v$. When a player i is given v, i is aware of $a \in v$ and unaware of $a \in A \setminus v$. For any $v, v' \in V$, v is contained in v', denoted as $v \subseteq v'$, if A_i^v is a subset of $A_i^{v'}$ for any $i \in I$; that is, $A_i^v \subseteq A_i^{v'}$.

[11] Sadzik (2021) offers another discussion of Bayesian games with unawareness. He assumes the existence of a common prior.

Definition 2.3.2 (Perea 2022)**.** A simultaneous-move games with unawareness in the Perea style, $\Gamma^P = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$, is defined as follows.

- T_i is a finite and non-empty set of i's type.
- $v_i : T_i \to V$ is i's view function. For any i and $v \in V$, let $T_i^v \subseteq T_i$ satisfy $v_i(t_i) \subseteq v$ for any $t_i \in T_i^v$. Let $T_{-i}^v = \times_{j\in I_{-i}} T_j^v$.
- $b_i : T_i \to \bigcup_{v\in V} \Delta(T_{-i}^v \times A_{-i}^v)$ with $b_i(t_i) \in \Delta(T_{-i}^{v_i(t_i)} \times A_{-i}^{v_i(t_i)})$ is i's belief function. Here, $\Delta(T_{-i}^v \times A_{-i}^v)$ is the set of probability measures over $T_{-i}^v \times A_{-i}^v$. Given $t_i \in T_i$ and $j \in I_{-i}$, at $b_i(t_i)$, if j's type is t_j, then $v_j(t_j) \subseteq v_i(t_i)$.
- For any $(i, t_i) \in I \times T_i$, i's belief $b_i(t_i)$ only assigns positive probability to opponent j's type-action pairs $(t_j, a_j) \in T_j^{v_i(t_i)} \times A_j^{v_j(t_j)}$.

Let us call G the objective game (in Γ^P). The objective game can be interpreted as the "true game" in Γ^P.[12] Each player i's type t_i describes their view about the game and belief about the opponents' types and their possible choices. At t_i, $v_i(t_i) = v$ means that i is aware of v and unaware of $A \setminus v$, while $b_i(t_i)$ assigns zero or positive probability to each $((t_j)_{j\in I_{-i}}, (a_j)_{j\in I_{-i}}) \in T_{-i}^{v_i(t_i)} \times A_{-i}^{v_i(t_i)}$. Then, $b_i(t_i)$ means that for any opponent $j \in I_{-i}$ and a pair (t_j, a_j) with positive probability assigning to it, i believes that j's type $(t_j)_{j\in I_{-i}}$, that each j's view is $v_j(t_j)$, and that j chooses a_j with positive probability.

In Perea's (2022) framework, each player's beliefs describe not only the opponents' beliefs but also the opponents' plays. By contrast, simplicity, this dissertation supposes that players' beliefs describe only the opponents' beliefs but not the opponents' choices. Although Chapters 3 and 4 provide non-probabilistic models, this chapter proposes probabilistic model not Perea version as follows.[13]

Definition 2.3.3. Given any standard game G, let $\Gamma^U = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ be a probabilistic version of a simultaneous-move game with unawareness as follows: for each $i \in I$,

- T_i is a finite and non-empty set of i's type.
- $v_i : T_i \to V$ is i's view function. For any i and $v \in V$, $T_i^v \subseteq T_i$ satisfies $v_i(t_i) \subseteq v$ for any $t_i \in T_i^v$. Let $T_{-i}^v = \times_{j\in I_{-i}} T_j$.
- $b_i : T_i \to \bigcup_{v\in V} \Delta(T_{-i}^v)$ with $b_i(t_i) \in \Delta(T_{-i}^{v_i(t_i)})$ is i's belief function. Here, $\Delta(T_{-i}^v)$ is the set of probability measures over T_{-i}^v, that is a probability measure over some subset of T_{-i}. For each $t_i \in T_i$ and $t_{-i} = (t_j)_{j\in I_{-i}} \in T_{-i}$, $b_i(t_{-i}|t_i) \geq 0$ implies that $v_j(t_j) \subseteq v_i(t_i)$ for all $j \in I_{-i}$, where $b_i(t_{-i}|t_i)$ is the probability that $b_i(t_i)$ is assigned to t_{-i}. Given any $t_i \in T_i$

[12] Although Perea (2022) calls G a base game, this paper uses the term the objective game in line with the previous literature.

[13] This formulation is similar to Kobayashi and Sasaki (2021). This non-probabilistic version is first defined by Sasaki (2022), although he focuses on multicriteria games with unawareness.

and $t_j \in T_j$, denote by $b_i(t_j|t_i)$ the probability that $b_i(t_i)$ is assigned to t_j.

This thesis uses a non-probabilistic version of simultaneous-move games with unawareness in Definition 2.3.3.

2.3.2 Solution Concepts

The main solution concepts in games with unawareness have two approaches: equilibrium notions (Feinberg 2021; Čopič and Galeotti 2006; Ozbay 2007; Halpern and Rêgo 2014; Rêgo and Halpern 2012; Grant and Quiggin 2013; Meier and Schipper 2014; Sasaki 2017; Schipper 2021; Kobayashi and Sasaki 2021) and rationalizability notions (Heifetz, Meier, and Schipper 2013b, 2021; Perea 2022; Guarino 2020).[14]

Halpern and Rêgo (2014) generalize a Nash equilibrium in (extensive-form) games with unawareness and name it a generalized Nash equilibrium. A generalized Nash equilibrium is interpreted as an "equilibrium in beliefs." However, it has some problems. Games with unawareness assume unawareness of actions. Hence, each player's belief about opponents' plays should exclude actions that she or he is unaware of. However, as pointed out by Schipper (2014), if an opponent implements an action that some players are unaware of, other players are surprised at the play and they might revise their beliefs about the action set that opponents possess. Then, an equilibrium before playing might not be an equilibrium after playing. That is, it is not an "equilibrium in beliefs" in the next-stage game.

To avoid this issue, previous work has proposed two approaches. The first is steady-state equilibrium notions, as proposed by Sasaki (2017) and Schipper (2021). This approach selects only one equilibrium; then, if that equilibrium is played, it is also played in the next stage. Sasaki (2017) refines the generalized Nash equilibrium and proposes a generalized Nash equilibrium with stable belief hierarchies. Schipper (2021) generalizes the self-confirming equilibrium provided by Fudenberg and Levine (1993).[15] The second set of the two approaches proposed by previous work is classified as rationalizability approaches, as exemplified by Heifetz, Meier, and Schipper (2013b, 2021), Perea (2022), and Guarino (2020). This approach is suitable for games in which beliefs have not yet been formed. Additionally, since the rationalizable action set is supports of all mixed strategy equilibria, the difficulty of disproving the probability distribution in mixed action equilibria can be overcome.

2.3.3 Discovery Process

Suppose that two players, Alice and Bob, face the objective outcome, as shown in Table 2.1. Assume that Bob is aware of all actions, whereas Alice is unaware

[14] Chapter 3 discusses solution concepts, a generalized Nash equilibrium, a generalized Nash equilibrium with stable belief hierarchies, self-confirming equilibrium, rationalizability, and closedness under rational behavior.

[15] Both notions can be interpreted as the *equilibrium in correct beliefs*.

of Bob's action R; that is, she believes that a true game is that shown in Table 2.3. If Bob implements R, then Alice observes R. She is surprised at the play and might revise her view of the game. Then, how does she revise her subjective view? Schipper (2021) provides one answer. He models a discovery process in which agents add observed actions into their subjective views. For example, if Bob plays R, then Alice adds R into her view and replaces her view in Table 2.3 with that in Table 2.1. Schipper (2021) assumes that all players implement rationalizable strategies and shows that the rationalizable discovery process in any extensive-form game with unawareness converges to some extensive-form structure possessing a rationalizable self-confirming equilibrium.

2.3.4 Motivation of PART II

While Schipper (2021) yields insightful results, he only shows convergence to some structures of games, but not convergence to a self-confirming equilibrium. In other words, in the rationalizable discovery process, players might not play a rationalizable self-confirming equilibrium in the converged game. Is there a model of discovery in which all participants' plays converge to some solution concept that is equilibrium notions or set-valued notions? Chapter 3 attempts to answer this question by modeling *myopic discovery processes* in which all players best respond to opponents' plays in a previous-stage game. The chapter generalizes the CURB concept,[16] which is a set-valued solution concept, before analyzing the myopic discovery processes. The result shows that the supports of plays throughout any myopic discovery process converge to some generalized CURB set called a *realizable CURB set*.

Schipper (2021) and Chapter 3 of this thesis suppose that all players update their subjective views of a game by adding into *only* their opponents' action sets. This update is suitable when each player's action set is not supposed to be equivalent to opponents' action sets. However, in coordination games, it is natural to suppose that players add observed actions of which they are unaware into not only opponents' action sets but also their own action sets. In other words, we must assume that opponents' actions can be imitated. Chapter 4 examines *coordination games with unawareness* and supposes that opponents' actions can be *imitated*. Before presenting the analysis, this thesis proposes a concept of *successful-coordination equilibrium*. The main result in Chapter 4 shows that after only one play, the next-stage game must possess a successful-coordination equilibrium.

[16]The concept of CURB is proposed by Basu and Weibull (1991). The notion is a refinement of rationalizability.

2.4 Unawareness in Standard Information Structures

2.4.1 Motivation of PART III

As described in Section 2.2, when assuming Plausibility, KU Introspection, and AU Introspection, standard information structures cannot discuss non-trivial unawareness even if the information structures include non-partitional models. This is why information structures are no longer used as frequently as in earlier studies. However, non-trivial unawareness can be discussed by relaxing the assumptions or revising the definitions of knowledge operator in standard information structures. For example, Ewerhart (2001) assumes that an agent can perceive a proper subset of the state space but cannot perceive the complementary set. The author then formulates the knowledge operator and unawareness operator in his framework. Fukuda (2021) suggests the possibility of discussing non-trivial unawareness by relaxing AU Introspection. PART III of this thesis considers non-trivial unawareness in standard information structures by reformulating the knowledge operator and unawareness operator.

Chapter 5 refers to the idea of Ewerhart (2001) and defines the knowledge operator that Necessitation and Monotonicity might not hold. In that chapter, unawareness is not trivial. Moreover, Chapter 5 shows that Symmetry and Negative Introspection must be equivalent under non-trivial unawareness as well as under trivial unawareness. In other words, under non-trivial unawareness, Symmetry does not hold. Chapter 6 focuses on Symmetry. Modica and Rustichini (1994, 1999) show that Symmetry and Negative Introspection are equivalent supposing Necessitation, Monotonicity, Truth, and Positive Introspection. Chen, Ely, and Luo (2012) show that Negative Introspection is equivalent to AU Introspection and KU Introspection. Moreover, they show that if Necessitation, Monotonicity, Truth, and Positive Introspection hold, then Negative Introspection, AU Introspection, and Symmetry are equivalent. However, those studies do not examine the relationships and equivalence between Symmetry and AU Introspection without Negative Introspection. Chapter 6 shows the equivalence of the two properties. Excluding only Necessitation, this study shows that Symmetry is equivalent to AU Introspection under non-trivial unawareness. Chapter 7 supposes that the information set for some state may be empty. Then, this empty information set is interpreted as the agent obtaining no information. However, for the standard knowledge operator, if a given information set is empty, then the agent knows all events. To represent that if an empty information set is given, then the agent is unaware of every event, this thesis revises the definition of the knowledge operator. The knowledge operator in this chapter excludes only Necessitation.

Part II

Discovery of Actions in Simultaneous-Move Games with Unawareness

Chapter 3

Unawareness of Actions and the Myopic Discovery Process in Simultaneous-Move Games with Unawareness

3.1 Introduction

This chapter (i) generalizes the concept of CURB (closedness under rational behavior) to simultaneous-move games with unawareness, (ii) models the myopic discovery process, and (iii) proves the main theorem that the plays of all agents converge to some CURB set in a discovered game in which every player does not need to revise their subjective games further.

In spite of the wide variety in the amount of knowledge among people, a human community seems to be relatively stable, showing some regularity in behavior. However, people do not always choose particular regular actions from the outset. They often start by recognizing their decision-making or interactive situation and find a specific solution through trial and error. The main purpose of this chapter is to demonstrate how agents discover and play a specific solution under a lack of knowledge or understanding using discovery processes in simultaneous-move games with unawareness.

Studies of unawareness analyze decision-making and interactive situations assuming a lack of knowledge or understanding. Models of unawareness assume that agents are unaware of events, choices, opponents' plays, and so on. In games with unawareness, the outcomes of players' implementations might differ from their predictions of the opponents' play because they previously did not know the actions of some opponents. Therefore, as players observe opponents' plays, they may be surprised by the plays and may realize the error of their subjective views about the game's situation. Thus, prior beliefs about opponents' plays may differ from those reformulated thereafter. Consequently, players' next-stage game might change. Schipper (2021) proposes models of discovery processes to analyze belief revision and replay through the revision process.[1] In a discovery process, players add opponents' actions of which they are unaware in each stage game. Schipper (2021) focuses on extensive-form games with unawareness and shows that a rationalizable discovery process in which every player implements rationalizable actions converges to some revised game possessing a rationalizable self-confirming equilibrium.

However, Schipper (2021) does not show that a rationalizable discovery process converges to a self-confirming equilibrium. In other words, there is no model of discovery converging to some solution concept. This study adopts the idea of discovery processes and discusses the convergence of plays. To do so, we introduce and generalize the concept of CURB, which is a set-valued solution concept. The concept of CURB combines the characteristics of an equilibrium notion and rationalizability notion as follows.

- When players implement actions in some CURB set, they do not have an incentive to deviate from that CURB set as in an equilibrium. Hence, we can find a characteristic of stable conventions in the concept of CURB.
- In a CURB set, players can only disconfirm their beliefs upon observing a pure strategy out of the set, as in the set of rationalizable strategy profiles.

The concept of CURB possessing the above features might be a convergence of implementations by players who best respond to opponents' immediately preceding plays. Indeed, Hurkens (1995) and Young (1998) show that in a standard game, adaptive plays converge to a minimal CURB set, considering the possibility of error. Although this paper does not assume the possibility of error, it shows that myopic plays in which all players best respond to opponents' preceding plays converge to some CURB set in simultaneous-move games with unawareness.

However, in some simultaneous-move game with unawareness, some CURB set of the objective game might not be CURB in some player's subjective game because the player is unaware of a subset of the CURB set. Hence, we have to try to generalize the CURB notion to simultaneous-move games with unawareness. The point is that each player has several types with different subjective games,

[1] A discovery process is different from a learning process. Learning is the process of updating probability distributions. However, as shown by Schipper (2013), unawareness of an event is different from assigning probability 0 to that event. Any event of which agents are unaware is not included in the subjective state space. Hence, the agent cannot assign a probability to the event. If such an event occurs, the subjective state space must be expanded. In games with unawareness, the set of actions must also be increased when actions of which players were unaware are played. Therefore, learning cannot be applied to games with unawareness. Discovery processes are alternative models to learning processes that have been proposed to avoid this issue.

and only actions chosen by the "actual" type can be realized. We call the actions that the actual type of a player can choose "realizable actions." This study focuses only on CURB sets on the realizable action set, namely, *realizable CURB sets*. The main theorem is that players' *myopic play* converges to some realizable CURB set.[2] When myopic players implement actions on some realizable CURB set and do not deviate from that CURB set, then all players' actions in the CURB set are stable.[3]

First, simultaneous-move games with unawareness are formulated in Section 3.2 based on a type-based approach. Additionally, this section generalizes the concept of CURB. Section 3.3 models the *myopic discovery processes* and analyzes the dynamics of those processes and shows the main result. The discussion on the relationship between the CURB notion and other solution concepts and the features of discovery processes is provided in Section 3.4. Section 3.5 considers the *block game* notion in (simultaneous-move) games with unawareness, adaptive plays, and the limitations of this work as well as review the related literature.

3.2 Preliminaries

3.2.1 Simultaneous-Move Games with Unawareness

This section defines simultaneous-move games with unawareness, which are type-based models, and generalizes the concept of CURB to those games. Let $G = (I, A, u)$ be a standard finite simultaneous-move game. I is a finite set of players and $I_{-i} = I \setminus \{i\}$. $A = \times_{i \in I} A_i$, where A_i is the non-empty finite set of i's actions, and each element of the set is $a_i \in A_i$. Let $A_{-i} = \times_{j \in I_{-i}} A_j$. $u = (u_i)_{i \in I}$, where $u_i : A \to \mathbb{R}$ is i's utility function. Denote i's mixed action on A_i by $m_i \in M(A_i)$, where $M(A_i)$ is the set of i's mixed actions, and a mixed action profile on A by $m = (m_i)_{i \in I} \in \times_{i \in I} M(A_i)$. We denote i's expected utility for m by $Eu_i(m)$.

First, simultaneous-move games with unawareness is defined.[4] For any standard simultaneous-move game G, let $V = \times_{i \in I}(2^{A_i} \setminus \{\emptyset\})$ be the set of possible *views* of G (i.e., the set of Cartesian products of non-empty action subsets). Like most previous works, this paper assumes that the set of players is commonly known and that each player's utility for each action profile is the same among all possible views. Let $v \in V$ be a (possible) view or a block[5] and A_i^v be the set of i's actions in v. Let $A_{-i}^v = \times_{j \in I_{-i}} A_j^v$. Here, when player i is

[2] Some realizable CURB set might not be CURB in some player's subjective game. Hence, we must distinguish whether the set is CURB in all subjective games of all players.

[3] However, this study does not show that plays converge to specific actions in the CURB set. Players might choose to alternate actions over the CURB set.

[4] The definitions are similar to those of Perea (2022), with three major differences. First, this thesis assumes that the "actual type" of players is given, whereas Perea's (2022) model does not. Second, the player types in Perea (2022) include not only the opponents' subjective views but also their choices, whereas the types in this study do not. Third, Perea (2022) considers probabilistic beliefs, whereas this study's players always have point beliefs.

[5] A block is a Cartesian product of non-empty subsets of actions.

given v, i is aware of $a \in v$, and unaware of $a \in A \setminus v$. For any $v, v' \in V$, v is contained in v', denoted as $v \subseteq v'$, if A_i^v is a subset of $A_i^{v'}$ for any $i \in I$; that is, $A_i^v \subseteq A_i^{v'}$. Let $M(A_i^v) = \{m_i \in M(A_i) | \Sigma_{a_i \in A_i^v} m_i(a_i) = 1\}$. Given any $\delta, \delta' \in \bigcup_{v \in V} \bigcup_{X \in 2^I \setminus \{\emptyset\}} \times_{i \in X} M(A_i^v)$, $\delta \equiv \delta'$ means that δ and δ' have the same supports and probabilities. Therefore, δ and δ' are equivalent.

Let $\Gamma = (G, (T_i)_{i \in I}, (v_i)_{i \in I}, (b_i)_{i \in I})$ be a simultaneous-move game with unawareness that is a non-probabilistic version of Definition 2.3.3 as follows:[6] for each $i \in I$,

- T_i is a finite and non-empty set of i's types, one of which is their actual type t_i^*.
- $v_i : T_i \to V$ is i's view function.
- $b_i : T_i \to T_{-i}$ is i's belief function, where $T_{-i} = \times_{j \in I_{-i}} T_j$. If $b_i(t_i) = (t_j)_{j \in I_{-i}}$, then, for each $j \in I_{-i}$, $v_j(t_j)$ must be contained in $v_i(t_i)$. Moreover, given any $(i, t_i) \in I \times T_i$, when $b_i(t_i) = (t_j)_{j \in I_{-i}}$, then let $b_i(t_i)(j) = t_j$ for each $j \in I_{-i}$.

Let us call G an objective game (in Γ). An objective game can be interpreted as the "true game" in Γ.[7] Given the formulation above, i's type t_i describes i's view of the game and belief about opponents' types. At t_i, $v_i(t_i) = v$ means that i is aware of v and unaware of $A \setminus v$, while $b_i(t_i) = (t_j)_{j \in I_{-i}}$ means that at t_i, i believes that the others' types are $(t_j)_{j \in I_{-i}}$ and that each j's view is $v_j(t_j)$. Given $(i, t_i) \in I \times T_i$, we denote a sequence of specific types of players by $t_{i_1}, t_{i_2}, \ldots, t_{i_h}, \ldots$, where t_{i_h} is a specific type of i_h, $t_{i_1} = t_i$, and for any $h \geq 2$, $t_{i_h} = b_{i_{h-1}}(t_{i_{h-1}})(i_h)$. Let us call such a sequence a *sequence of players and types induced by belief functions.* We say that t_i leads to t_j if and only if there exists a subsequence $t_{i_1}, \ldots, t_{i_h}$ such that $t_{i_1} = t_i$ and $t_{i_h} = t_j$. This thesis supposes $\bigcup_{i \in I} T_i = \bigcup_{i \in I} \{t_{i_h}^*\}_{h \geq 1; t_{i_1}^* = t_i^*}$.

The set of i's *realizable actions* $A_i^{v_i(t_i^*)}$ may be a proper subset of i's *full* action set A_i. In such a scenario, they cannot play $a_i \in A_i \setminus A_i^{v_i(t_i^*)}$. Let $\times_{i \in I} A_i^{v_i(t_i^*)}$ be the *realizable action set*, which some players may not perceive.

3.2.2 Generalization of the CURB Concept

We generalize the CURB concept, which is a set-valued concept, to simultaneous-move games with unawareness. the concept of CURB is proposed by Basu and Weibull (1991). It has the features of both an equilibrium and a rationalizability notion. A rationalizable action is a support of all mixed equilibria, whereas a CURB set is a subset of the supports of such equilibria. In a mixed equilibrium,

[6] This non-probabilistic formulation is first defined by Sasaki (2022). In his model, G is a multicriteria game.

[7] The term "objective game" is used by Halpern and Rêgo (2014). Feinberg (2021) refers to such a game as the "modeler's normal-form game" and Perea (2022) calls it the "base game." In this context, we follow Halpern and Rêgo (2014).

players might be unable to disconfirm a specific distribution of mixed actions, because they only observe a pure action. On the contrary the concept of CURB makes us free from such considerations.

First, we define *generalized strategies*. For any $i \in I$, let $s_i : T_i \to \bigcup_{t_i \in T_i} M(A_i^{v_i(t_i)})$ with $s_i(t_i) \in M(A_i^{v_i(t_i)})$ for all $t_i \in T_i$. Then, given t_i, $s_i(t_i) \in M(A_i^{v_i(t_i)})$ is i's *local action* at t_i. We denote i's generalized strategy by $s_i = (s_i(t_i))_{t_i \in T_i}$ and the *generalized strategy profile* by $s = (s_i)_{i \in I}$. In a generalized strategy profile s, each player i's *actual play* is $s_i(t_i^*)$. Let us call $m_i \in M(A_i)$ with $m_i \equiv s_i(t_i^*)$ i's *objective outcome* induced from s and the profile $m = (m_i)_{i \in I}$ the objective outcome induced from s.

Second, we generalize the CURB concept proposed by Basu and Weibull (1991) to simultaneous-move games with unawareness. Basu and Weibull (1991) define the concept of CURB on a standard game G, whereas this section defines it on each view. Given any standard simultaneous-move game G, any possible view $\hat{v} \in V$, and mixed action profile $m \in \times_{i \in I} M(A_i^{\hat{v}})$, let

$$\beta_i^{\hat{v}}(m_{-i}) = \{a_i \in A_i | a_i \in \mathrm{supp}(m_i) \text{ such that } m_i \in \arg \max_{x \in M(A_i^{\hat{v}})} Eu_i(x, m_{-i})\}$$

be the set of i's pure-action best responses to $m_{-i} \in \times_{j \in I_{-i}} M(A_j^{\hat{v}})$. For any $v \subseteq \hat{v}$, let

$$\beta_i^{\hat{v}}(A_{-i}^v) = \bigcup_{m_{-i} \in \times_{j \in I_{-i}} M(A_j^{\hat{v}}) : \exists m'_{-i} \in \times_{j \in I_{-i}} M(A_j^v),\ m_{-i} \equiv m'_{-i}} \beta_i^{\hat{v}}(m_{-i})$$

be the set of i's optimal actions under beliefs in $\times_{j \in I_{-i}} M(A_j^v)$ and $\beta^{\hat{v}}(v) = \times_{i \in I} \beta_i^{\hat{v}}(A_{-i}^v)$.

Then, CURB is defined as follows:

Definition 3.2.1. Given a standard simultaneous-move game G and $\hat{v} \in V$, $C \subseteq \hat{v}$ is a *CURB set* on $\hat{v}$ if $\beta^{\hat{v}}(C) \subseteq C$. C is a *minimal CURB set* on $\hat{v}$ if C is CURB on $\hat{v}$ and every proper subset of C is not CURB on $\hat{v}$. Moreover, $C \subseteq \hat{v}$ is *fixed under rational behavior* (FURB) in $\hat{v}$ if $\beta^{\hat{v}}(C) = C$. C is a *minimal FURB set* on $\hat{v}$ if C is FURB on $\hat{v}$, and every proper subset of C is not FURB on $\hat{v}$.

As pointed out by Basu and Weibull (1991), the rationalizable action set on $\hat{v}$ is a maximum FURB set on $\hat{v}$.

Basu and Weibull (1991) show that every standard game has a minimal CURB set. In the present context, the next proposition holds.

Remark 4 (Basu and Weibull 1991)**.** Given any standard game, every possible view has a minimal CURB set.

In standard games, we only need to consider a CURB set on the full action set. However, since a given possible view for each player may be different from the full action set in games with unawareness, "realizable" CURB sets might differ for standard games and games with unawareness. Hence, we must distinguish the CURB concept between the two models. In the CURB concept under unawareness, we define a CURB set on the realizable action set, called a *realizable CURB set*, as follows.

Definition 3.2.2. Given a simultaneous-move game with unawareness Γ, let $v^* = \times_{i \in I} A_i^{v_i(t_i^*)}$ be the realizable action set. $C \in V$ is a *realizable CURB set* if $C \subseteq v^*$ and $\beta^{v^*}(C) \subseteq C$. C is a *minimal realizable CURB set* if it is CURB on v^* and every proper subset of C is not CURB on v^*.

Realizable CURB notions have the following property.

Lemma 3.2.1. Every simultaneous-move game with unawareness Γ has a minimal realizable CURB set; it is non-empty.

Proof. Let us construct a game $G' = (N, A', u')$ such that the following assumptions hold:

- N is common in Γ.
- $A' = \times_{i \in I} A_i^{v_i(t_i^*)}$.
- For any $i \in I$, $u_i' : A' \to \mathbb{R}$ such that $u_i(a) = u_i'(a)$ for any $a \in A'$.

Following Basu and Weibull (1991), there must be a (minimal) CURB set $C \subseteq A'$ in G'. In other words, there exists a set C such that $\beta'(C) \subseteq C$ in G', where $\beta'(\cdot)$ is defined in G'. Since $\beta'(C)$ is defined on $A' = \times_{i \in I} A_i^{v_i(t_i^*)}$, C is a minimal realizable CURB set. $\square$

Given Γ, some realizable CURB set $C \in V$ may be $C \subseteq v_i(t_i)$ for any $(i, t_i) \in I \times T_i$. However, the set might not be CURB in $v_i(t_i)$ at some (i, t_i). Given a realizable CURB set, we distinguish between a case in which the realizable CURB set is CURB in every $v_i(t_i)$ for any $(i, t_i) \in I \times T_i$ and the case in which it is not.

Definition 3.2.3. In a simultaneous-move game with unawareness Γ, $C \in V$ is a *common (minimal) realizable CURB set* if C is a (minimal) realizable CURB set and for any $(i, t_i) \in I \times T_i$, $C \subseteq v_i(t_i)$. C is a *common (minimal) CURB set* if for any $(i, t_i) \in I \times (T_i \setminus \{t_i^*\})$, C is (minimal) CURB in $v_i(t_i)$ and for any i, $\beta_i^{v_i(t_i^*)}(A_{-i}^C) \subseteq A_i^C$.

A common realizable CURB set is a subset in each subjective view of each player, but the set might not be CURB in some view. By contrast, a common CURB set is CURB in each subjective view other than the player's actual view[8] and each player's action thereof best responds to some action of opponents in the CURB set in the player's actual view.

[8] An agent's actual view means the agent's view at her or his actual type.

Example 1. Let us consider that two players, Alice (A) and Bob (B), face the following objective game:[9]

$v^O =$

A / B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, 0
a_2	5, 0	1, 1	0, 0
a_3	0, 0	0, 0	2, 2

.

Here, if Alice in her actual type is unaware of her own action a_2, then her view is as follows:

$v^A =$

A / B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, 0
a_3	0, 0	0, 0	2, 2

,

If Bob in his actual type is unaware of his own action b_2, then his view is as follows:

$v^B =$

A / B	b_1	b_3
a_1	3, 3	0, 0
a_2	5, 0	0, 0
a_3	0, 0	2, 2

.

Let us suppose that Alice believes that Bob's view is the same as hers; that is, they both believe that they hold the same view v^A. On the contrary, Bob believes that Alice's view is the same as his; that is, again, they both believe that they hold the same view v^B.

Here, we formulate this game (with unawareness) $\Gamma = (v^O, (T_A, T_B), (v_A, v_B), (b_A, b_B))$ as follows:

$T_A = \{t_A^*, t_A\}$, and $T_B = \{t_B^*, t_B\}$;
For t_A^*, $v_A(t_A^*) = v^A$, and $b_A(t_A^*) = t_B$;
For t_A, $v_A(t_A) = v^B$, and $b_A(t_A) = t_B^*$;
For t_B^*, $v_B(t_B^*) = v^B$, and $b_B(t_B^*) = t_A$; and
For t_B, $v_B(t_B) = v^A$, and $b_B(t_B) = t_A^*$.

This formulation is depicted in Figure 3.1.

Since Alice's realizable actions are a_1 and a_3, while Bob's realizable actions are b_1 and b_3, the realizable action set is shown in the following table.

$v^R =$

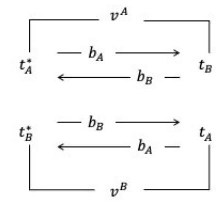
A / B	b_1	b_3
a_1	3, 3	0, 0
a_3	0, 0	2, 2

.

Then, there exists three CURB sets on the realizable action set (i.e., three realizable CURB sets):

[9] This example is similar to Example 3 in Schipper (2018). The idea of the similar example is borrowed from his example.

A | B	b_1	b_2	b_3
a_1	3, 3	0, 5	0, 0
a_3	0, 0	0, 0	2, 2



A | B	b_1	b_3
a_1	3, 3	0, 0
a_2	5, 0	0, 0
a_3	0, 0	2, 2

Figure 3.1: Example 1

$C^1 = \{a_1\} \times \{b_1\}$;
$C^2 = \{a_3\} \times \{b_3\}$; and
$C^3 = \{a_1, a_3\} \times \{b_1, b_3\}$.

Here, C^3 is the maximum FURB set. Since, $C^1, C^2, C^3 \subseteq v^A$ and $C^1, C^2, C^3 \subseteq v^B$, every realizable CURB set is a common realizable CURB set. Moreover, C^2 is the only unique common CURB set because it is CURB on v^A and v^B. $\square$

3.3 Myopic Discovery Process

Standard game models study the convergence to a minimal CURB set using a learning model or an adaptation model; see, for example, Hurkens (1995) and Young (1998). Previous studies that adopt standard models show that when all players best respond to opponents' preceding plays under assuming the possibility of error, their plays converge to some minimal CURB set. We can also predict that in dynamics of simultaneous-move games with unawareness, all agents' implementations converge to some generalized CURB set when they best respond to opponents' preceding plays. To prove this prediction, this study models a *myopic discovery process* in this section.

A *discovery process* represents an update process through which each player revises their own belief about the game's structure and opponents' plays. This model was first introduced into games with unawareness by Schipper (2021). He analyzes a *rationalizable discovery process* in extensive-form models based on Heifetz, Meier, and Schipper (2013b). This study models a discovery process in simultaneous-move games with unawareness based on our framework. Although the definition, at first glance, may seem different from that of Schipper (2021), both are essentially the same.

Definition 3.3.1. Consider a simultaneous-move game with unawareness $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ and a generalized strategy $s = (s_i)_{i\in I}$ thereof. Then, $\Gamma' = (G, (T_i')_{i\in I}, (v_i')_{i\in I}, (b_i')_{i\in I})$ is a *discovered game* associated with (Γ, s) if: for any $(i, t_i) \in I \times T_i$, and any sequence of players $i_1, i_2, \ldots, i_h, \ldots,$ with a sequence of types induced by belief functions in Γ, $t_{i_1}, t_{i_2}, \ldots, t_{i_h}, \ldots,$ where $t_{i_1} = t_i$, in Γ, there exists $t_i' \in T_i'$ and a sequence of types induced by belief functions in Γ', $t'_{i_1}, t'_{i_2}, \ldots, t'_{i_h}, \ldots,$ where $t'_{i_1} = t'_i$, such that for any $h \geq 1$,

$$v'_{i_h}(t'_{i_h}) = \times_{j\in I}[A_j^{v_{i_h}(t_{i_h})} \cup \mathrm{supp}(s_j(t_j^*))],$$

where t_j^* is j's actual type in Γ, and

$$b'_{i_h}(t'_{i_h})(i_{h+1}) = t'_{i_{h+1}}.$$

Note that Γ' is a novel simultaneous-move game with unawareness. Moreover, for some Γ and Γ', it may be that $T \not\subseteq T'$ and $T' \not\subseteq T$, or $T \cap T' = \emptyset$.

In a discovered game, a player's subjective view is the union of her or his previous view and her or his actual play. Additionally, the immediately preceding game's types and belief functions are replaced in the discovered game.

Example 2. Consider the following objective game played by Colin (C) and David (D):

$v^0 =$

C / D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_2	5, 0	1, 1	1, 0
c_3	-1, 0	0, 1	2, 2

;

and two possible views as follows:

$v^1 =$

C / D	d_2	d_3
c_1	0, 5	0, -1
c_3	0, 1	2, 2

and $v^2 =$

C / D	d_1	d_3
c_1	3, 3	0, -1
c_2	5, 0	1, 0

.

Let us formulate the game with unawareness $\Gamma = (v^0, (T_C, T_D), (v_C, v_D), (b_C, b_D))$ as follows:

$T_C = \{t_C^*, t_C\}$, and $T_D = \{t_D^*, t_D\}$;
For t_C^*, $v_C(t_C^*) = v^1$, and $b_C(t_C^*) = t_D$;
For t_C, $v_C(t_C) = v^2$, and $b_C(t_C) = t_D^*$;
For t_D^*, $v_D(t_D^*) = v^2$, and $b_D(t_D^*) = t_C$; and
For t_D, $v_D(t_D) = v^1$, and $b_D(t_D) = t_C^*$.

This formulation is depicted in Figure 3.2.

Here, suppose that Colin and David play a generalized strategy profile:

$$s = ([s_C(t_C^*) = c_1, s_C(t_C) = c_2)], [s_D(t_D^*) = d_1, s_D(t_D) = d_3]).$$

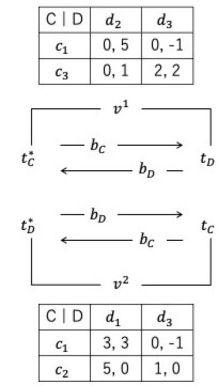


Figure 3.2: First stage game in Example 2

The objective outcome is (c_1, d_1) induced by s.

Let $\hat{\Gamma}$ be the discovered game associated with (Γ, s). Then, each player's type set in $\hat{\Gamma}$ is $\hat{T}_C = \{\hat{t}_C^*, \hat{t}_C\}$ and $\hat{T}_D = \{\hat{t}_D^*, \hat{t}_D\}$, where

$$\hat{b}_C(\hat{t}_C^*) = \hat{t}_D;$$

$$\hat{b}_C(\hat{t}_C) = \hat{t}_D^*;$$

$$\hat{b}_D(\hat{t}_D^*) = \hat{t}_C;$$

$$\hat{b}_D(\hat{t}_D) = \hat{t}_C^*;$$

$$\hat{v}^1 = \hat{v}_C(\hat{t}_C^*) = \hat{v}_D(\hat{t}_D) = \begin{array}{|c|c|c|c|}\hline \text{C / D} & d_1 & d_2 & d_3 \\ \hline c_1 & 3,\ 3 & 0,\ 5 & 0,\ \text{-1} \\ \hline c_3 & \text{-1},\ 0 & 0,\ 1 & 2,\ 2 \\ \hline \end{array} \ ; \text{ and}$$

$$v^2 = \hat{v}_C(\hat{t}_C) = \hat{v}_D(\hat{t}_D^*).$$

Then, the discovered game is as depicted in Figure 3.3. □

Next, let us define a discovery process.

Definition 3.3.2. A *discovery process* $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \ldots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \ldots)$ is defined as follows:

- For any λ, $\Gamma^\lambda = (G, (T_i^\lambda)_{i \in I}, (v_i^\lambda)_{i \in I}, (b_i^\lambda)_{i \in I})$,

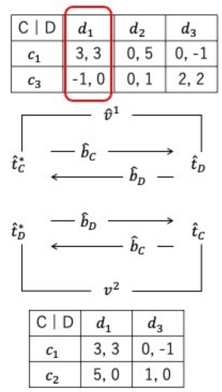


Figure 3.3: The discovered game in Example 2

- $s^0 = \phi$ (i.e., unspecified), while for any $\lambda \geq 1$, s^λ is a played generalized strategy profile in Γ^λ, and
- For any $\lambda \geq 2$, Γ^λ is a discovered game associated with $(\Gamma^{\lambda-1}, s^{\lambda-1})$.

Let us call Γ^1 the *initial game* with unawareness (in P).

Each stage game is a simultaneous-move game with unawareness, and each discovered game is connected to a pair of the immediately preceding game and the generalized strategy profile played in it. From Definition 3.3.1, Definition 3.3.2 implicitly assumes perfect recall. If we ignore this assumption, some players may forget some action at λ even if they are aware of the action at $\lambda - 1$.

This study assumes that every player implements a pure action. Meanwhile, standard game models might assume that every player implements and observes a mixed action. By contrast, in games with unawareness, it does not seem appropriate that every player implements and observes a mixed action because under unawareness, players cannot observe the frequency of their opponents' actions in each stage of the game during any discovery process.[10]

In a discovery process, cautious players might carefully revise their beliefs about the game, opponents' plays and rationalities, and payoff uncertainty (e.g., they might play rationalizable strategies). However, in the real world, agents must pay a higher cost for revising such beliefs and implementing rationalizable strategies. If players are myopic, they do not pay a high cost for revising their beliefs. This section explains the myopic discovery process in which every player best responds to opponents' preceding plays.

First, a strategy of myopic play in a discovered game is defined as follows.

[10]I thank an anonymous referee for pointing this out.

Definition 3.3.3. Suppose that a generalized strategy profile s is played in Γ. Let Γ' be a discovered game associated with (Γ, s). Then, s' is a *myopic best response* in Γ' if for any $(i, t'_i) \in I \times T'_i$,

$$s'_i(t'_i) \in \arg \max_{x \in M(A_i^{v'_i(t'_i)})} Eu_i(x, (s_j(t^*_j))_{j \in I_{-i}}),$$

where for any $j \in I_{-i}$, $t^*_j \in T_j$ is j's actual type in Γ, and $s_j(t^*_j)$ is j's actual play in s.

Our definition of discovered games, Definition 3.3.1, implicitly assumes that each agent is aware of the opponents' payoff after discovery. For Example 2, when Colin and David observe (c_1, d_1) in the first-stage game. Then, Colin is aware of a utility for (c_3, d_1) after her updating even if the profile is not observed by her. This is a weird assumption. However, the process of myopic best responses mitigates this weirdness because players only use their own preferences after new strategy profiles are added in the discovered game. This is in sharp contrast with the rationalizable notion.

In a myopic best response, all players best respond to the opponents' preceding plays in all subjective views. In Example 2, suppose Colin and David plays $s = ([s_C(t^*_C) = c_1, s_C(t_C) = c_2)], [s_D(t^*_D) = d_1, s_D(t_D) = d_3])$ in the initial game. Then, the objective outcome is (c_1, d_1), and the discovered game associated with (Γ, s) is $\hat{\Gamma}$. Since myopic best responses are the best responses to the opponents' immediately preceding plays, in $\hat{\Gamma}$, the myopic (pure) best response is $s' = ([s_C(\hat{t}^*_C) = c_1, s_C(\hat{t}_C) = c_2)], [s_D(\hat{t}^*_D) = d_1, s_D(\hat{t}_D) = d_2])$.

Second, we provide a myopic discovery process.

Definition 3.3.4. Any discovery process $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \dots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \dots)$ is called a *myopic discovery process* if for any $\lambda \geq 2$, s^λ is a myopic best response at λ. A discovery process P is said to converge to Γ if there exists h such that for any $\lambda \geq h$, $\Gamma^\lambda = \Gamma$.

In a myopic discovery process, all agents implement myopic best responses in every discovered game. A convergence of discovery process means that all players' update stops in some discovered game.

From the above formulations, we can show the convergence of a CURB set as follows.

Theorem 3.3.1. Given any simultaneous-move game with unawareness Γ, every myopic discovery process, P, converges to a discovered game, possessing a common realizable CURB set. Thus, a subset of the supports of all agents' myopic best responses converges to a common realizable CURB set.

Before proving this theorem, we show the following lemma.

Lemma 3.3.1. Given any standard game $G = (I, A, u)$, any view $v \in V$, and any mixed action $m \in \times_{i \in I} M(A_i)$. Let us define β', B^k, and $\mathscr{B}^k$ as follows:

- $\beta'(\cdot) = \beta^v(\cdot)$ is an operator giving the set of best responses on v.

- A sequence of B^k is defined as follows: $B^1 = \beta'(\text{supp}(m))$, $B^2 = \beta'(B^1)$, $B^3 = \beta'(B^2)$, …, $B^k = \beta'(B^{k-1})$, …. For any k, $B^k \in V$.
- For any k, $\mathscr{B}^k = \times_{i \in I}(\bigcup_k A_i^{B^k})$.

Then, the following properties hold.

1. There exists a natural number h such that for any $k \geq h$, $\mathscr{B}^k = \mathscr{B}^h$.
2. For any k, if $\mathscr{B}^{k+1} \subseteq \mathscr{B}^k$, then $\beta'(\mathscr{B}^k) \subseteq \mathscr{B}^k$.

Proof. First property obviously holds, because the action set A is finite.

Suppose that for some k, $\mathscr{B}^{k+1} \subseteq \mathscr{B}^k$, and $\beta'(\mathscr{B}^k) \not\subseteq \mathscr{B}^k$. Then, $B^{k+1} = \beta'(B^k) \not\subseteq \mathscr{B}^k$. However, since $\mathscr{B}^{k+1} \subseteq \mathscr{B}^k$, $B^{k+1} \subseteq B^k$ should hold. It is a contradiction. Hence, $\beta'(\mathscr{B}^k) \subseteq \mathscr{B}^k$. □

Proof of Theorem 3.3.1. Since we are considering a myopic discovery process, we only have to focus on the realizable action set. Let us denote the realizable action set by $v^* = \times_{i \in I} A_i^{v_i(t_i^*)}$. Consider any objective outcome in the initial game $m \in \times_{i \in I} M(A_i^{v^*})$. Let $\beta'(\cdot) = \beta^{v^*}(\cdot)$ denote an operator giving the set of best responses on the realizable action set and we can define a sequence of β^λ as follows: $\beta^1(m) = \text{supp}(m)$, $\beta^2(m) = \beta' \circ \beta^1(m)$, $\beta^3(m) = \beta' \circ \beta^2(m)$, …, $\beta^\lambda(m) = \beta' \circ \beta^{\lambda-1}(m)$, …. For any λ, let $\mathscr{B}^\lambda = \times_{i \in I}(\bigcup_\lambda A_i^{\beta^\lambda(m)})$ be the set of observed actions by all players until λ. Then, since the set of all players' actions is finite, by property 1 of Lemma 3.3.1, there exists a natural number n such that for any $\lambda \geq n$, $\mathscr{B}^\lambda = \mathscr{B}^n$. That is, updates by myopic plays converge. Moreover, by property 2 of Lemma 3.3.1, since $\beta'(\mathscr{B}^\lambda) \subseteq \mathscr{B}^\lambda$ for any $\lambda \geq n$, the set of observed actions is CURB and the CURB set is a common realizable CURB set from Definition 3.3.1. □

Many intuitively appealing adaptive processes eventually settle to a minimal CURB set (e.g., Hurkens, 1995; Young, 1998). Theorem 3.3.1 thus adds to this strand of the literature by highlighting the importance of the CURB set. However, through this process, it converges to a generalized CURB set and, not necessarily, a "minimal" one, as in Hurkens (1995) and Young (1998). [11]

Example 2 (Continued)**.** Let Γ be an initial game (i.e., a game at $\lambda = 1$). Then, the realizable action set is as follows:

$v^R =$

C / D	d_1	d_3
c_1	3, 3	0, -1
c_3	-1, 0	2, 2

.

v^R has three CURB sets, $C^1 = \{c_1\} \times \{d_1\}$, $C^2 = \{c_3\} \times \{d_3\}$, and $C^3 = \{c_1, c_3\} \times \{d_1, d_3\}$.[12]

[11] Hurkens (1995) and Young (1998) show the convergence of a minimal CURB set using the adaptive plays proposed by Young (1993). Section 3.5.2 discusses a discovery process with adaptive plays.

[12] C^2 is a mutual CURB set which we will define in Definition 3.4.3.

Let us focus on two generalized strategy profiles:

$s_1 = ([s_C(t_C^*) = c_1, s_C(t_C) = c_2], [s_D(t_D^*) = d_1, s_D(t_D) = d_2])$; and
$s_2 = ([s_C(t_C^*) = c_3, s_C(t_C) = c_2], [s_D(t_D^*) = d_3, s_D(t_D) = d_3])$.[13]

First, this study focuses on the former strategy profile, s_1. The objective outcome is (c_1, d_1). Since Colin is unaware of d_1, he is surprised and revises his view as follows:

$v^{1'} =$

C / D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_3	-1, 0	0, 1	2, 2

.

Then, at $\lambda = 2$, the discovered game is $\Gamma' = (G, (T'_C, T'_D), (v'_C, v'_D), (b'_C, b'_D))$, where

$T'_C = \{t_C^{2*}, t_C^2\}$, and $T'_D = \{t_D^{2*}, t_D^2\}$;

$v'_C(t_C^{2*}) = v^{1'}$, and $b'_C(t_C^{2*}) = t_D^2$;

$v'_C(t_C^2) = v^2$, and $b'_C(t_C^2) = t_D^{2*}$;

$v'_D(t_D^{2*}) = v^2$, and $b'_D(t_D^{2*}) = t_C^2$; and

$v'_D(t_D^2) = v^{1'}$, and $b'_D(t_D^2) = t_C^{2*}$.

This formulation is depicted in Figure 3.4. At $\lambda = 2$, when they play the myopic best response, the generalized strategy profile is

$$s_1^2 = ([s_C^2(t_C^{2*}) = c_1, s_C^2(t_C^2) = c_1], [s_D^2(t_D^{2*}) = d_1, s_D^2(t_D^2) = d_1]).$$

Neither player is surprised at their opponent's actual play. Hence, the next-stage game, at $\lambda = 3$, is the same as that in Γ'. In Γ', the objective outcome induced from the play s_1^2 is (c_1, d_1). The support of the objective outcome, $\{c_1\} \times \{d_1\}$, is a subset of a realizable CURB set, C^1.

Second, let us focus on the latter generalized strategy profile, s_2. The objective outcome is (c_3, d_3). Since David is unaware of c_3, he is surprised and revises his view as follows:

$v^{2'} =$

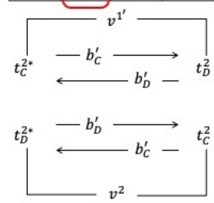
C / D	d_1	d_3
c_1	3, 3	0, -1
c_2	5, 0	1, 0
c_3	-1, 0	2, 2

.

Then, at $\lambda = 2'$, $\Gamma'' = (G, (T''_C, T''_D), (v''_C, v''_D), (b''_C, b''_D))$, where

[13] As will be described in section 3.4.1, the two generalized strategy profiles s_1 and s_2 are generalized Nash equilibria.

C | D	d_1	d_2	d_3
c_1	3, 3	0, 5	0, -1
c_3	-1, 0	0, 1	2, 2



C | D	d_1	d_3
c_1	3, 3	0, -1
c_2	5, 0	1, 0

Figure 3.4: The discovered game Γ' associated with (Γ, s_1)

$T''_C = \{t_C^{2'*}, t_C^{2'}\}$, and $T''_D = \{t_D^{2'*}, t_D^{2'}\}$;

$v''_C(t_C^{2'*}) = v^1$, and $b''_C(t_C^{2'*}) = t_D^{2'}$;

$v''_C(t_C^{2'}) = v^{2'}$, and $b''_C(t_C^{2'}) = t_D^{2'*}$;

$v''_D(t_D^{2'*}) = v^{2'}$, and $b''_D(t_D^{2'*}) = t_C^{2'}$; and

$v''_D(t_D^{2'}) = v^1$, and $b''_D(t_D^{2'}) = t_C^{2'*}$.

This formulation is depicted in Figure 3.5. At $\lambda = 2'$, when they play the myopic best response, the generalized strategy profile is

$$s_2^{2'} = ([s_C^{2'}(t_C^{2'*}) = c_3, s_C^{2'}(t_C^{2'}) = c_3], [s_D^{2'}(t_D^{2'*}) = d_3, s_D^{2'}(t_D^{2'}) = d_3]).$$

Neither player discovers their opponent's actual play. Hence, the next-stage game, at $\lambda = 3'$, is the same as that in Γ''.[14] In Γ'', a support of the objective outcome, $\{c_3\} \times \{d_3\}$, is a subset of a common CURB set, C^2 in Γ''. $\square$

[14]Moreover, the generalized strategy profile $s_2^{2'}$ is a cognitively stable generalized Nash equilibrium. See section 3.4.1.

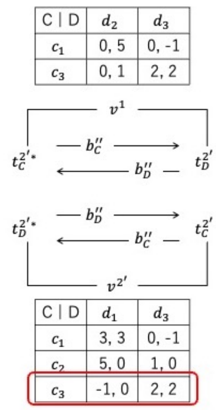


Figure 3.5: The discovered game Γ'' associated with (Γ, s_2)

3.4 Discovery Process and Other Solution Concepts

3.4.1 Relationships among Other Solution Concepts

This section discusses the relationship between the CURB concept and other solution concepts such as the generalized Nash equilibrium, the self-confirming equilibrium, and rationalizability.

(a) Equilibrium Notions

Among the equilibrium concepts so far studies and discussed, this study focuses on the generalized Nash (Halpern and Rêgo, 2021; Sasaki, 2017) and self-confirming equilibria (Schipper, 2021; Kobayashi and Sasaki, 2021) in this study.

Given any generalized strategy profile s, let $\mu = (\mu_i)_{i \in I}$ be the belief system in s, where for any $(i, t_i) \in I \times T_i$, $\mu_i(t_i) \in M(A_{-i}^{v_i(t_i)})$ and $\mu_i = (\mu_i(t_i))_{t_i \in T_i}$. This study allows correlated beliefs as well as Schipper (2021) and Kobayashi and Sasaki (2021). Therefore, the generalized Nash equilibrium is defined as follows. In a simultaneous-move game with unawareness Γ, s^* is a generalized Nash equilibrium if there exists the belief system μ such that for any $(i, t_i) \in I \times T_i$,

1. $s_i^*(t_i) \in \arg\max_{x \in M(A_i^{v_i(t_i)})} Eu_i(x, \mu_i(t_i))$, and
2. $\mu_i(t_i) \equiv (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}}$.

A generalized Nash equilibrium is best interpreted as "an equilibrium in beliefs" (Halpern and Rêgo, 2014: 50). However, as pointed out by Schipper

(2014), a generalized Nash equilibrium can consist of wrong beliefs. In that case, each player would revise their own beliefs about a game's structure and opponents' play, and may not play the same generalized Nash equilibrium.

To avoid this scenario, studies have discussed refinements of the generalized Nash equilibria in terms of steady-state notions: the stable belief hierarchies notion, cognitive stability notion, and self-confirming notion.

First, we consider the stable belief hierarchies notion and cognitive stability notion, as proposed by Sasaki (2017).

- The generalized Nash equilibrium s^* has stable belief hierarchies if the belief system μ satisfies that for any $(i, t_i) \in I \times T_i$, $\mu_i(t_i) \equiv (s_j^*(t_j^*))_{j \in I_{-i}}$.
- The generalized Nash equilibrium s^* is cognitively stable if for any $(i, t_i) \in I \times T_i$, $s_i^*(t_i) \equiv s_i^*(t_i^*)$.

The former notion means that every player's belief about opponents' plays is correct, while the latter means that every player's decision on their arbitrary type is equivalent to their actual play.

Sasaki (2017) expresses this as follows.

Remark 5 (Sasaki 2017)**.** The generalized strategy profile s^* is a generalized Nash equilibrium with stable belief hierarchies if and only if s^* is a cognitively stable generalized Nash equilibrium.

In this section, such a generalized Nash equilibrium is called a cognitively stable generalized Nash equilibrium, which has the following property.[15]

Remark 6. For any simultaneous-move game with unawareness Γ, let s^* be a cognitively stable generalized Nash equilibrium. Then, the objective outcome $m^* \equiv (s_i^*(t_i^*))_{i \in I}$ is a Nash equilibrium on the realizable action set.

Proof. Suppose that s^* is a cognitively stable generalized Nash equilibrium, and let m^* be the objective outcome induced from s^*; that is, for any $(i, t_i) \in I \times T_i$, $s_i^*(t_i) \equiv m_i^*$. Then, $m^* \equiv m \in \times_{i \in I} M(A_i^{v_j(t_j)})$ for any $(j, t_j) \in I \times T_j$. Assume that $m' \in \times_{i \in I} M(A_i^{v_i(t_i^*)})$ with $m' \equiv m^*$ is not a Nash equilibrium on the realizable action set. In other words, there exists $i \in I$ such that

$$m_i^* \equiv m_i' \notin \arg \max_{x \in A_i^{v_i(t_i^*)}} Eu_i(x, m_{-i}').$$

Then, for any $(i, t_i) \in I \times T_i$,

$$m_i^* \equiv m_i' \equiv m_i'' \notin \arg \max_{x \in A_i^{v_i(t_i)}} Eu_i(x, \mu_i(t_i)),$$

where $\mu_i \equiv m_{-i}' \equiv m_{-i}^*$. However, since s^* is a cognitively stable generalized Nash equilibrium, this is a contradiction. Therefore, m' is a Nash equilibrium on the realizable action set. $\square$

[15] A cognitively stable generalized Nash equilibrium can be interpreted as an *equilibrium in correct beliefs*.

The relationship between the concept of CURB and cognitive stability notion has the following property.

Proposition 3.4.1. Any simultaneous-move game with unawareness possessing a common CURB set has a cognitively stable generalized Nash equilibrium.

Proof. Assume that any simultaneous-move game with unawareness has a common CURB set $C \in V$. That is, for any $(i, t_i) \in I \times T_i$, C is CURB on $v_i(t_i)$. Then, following Basu and Weibull (1991), a Nash equilibrium on $v_i(t_i)$, $m^* \in \times_{j \in I} M(A_j^{v_i(t_i)})$ exists, satisfying $m^* \equiv m' \in \times_{j \in I} M(A_j^C)$. Suppose that m' is not a Nash equilibrium on v. Put simply, $(i, a_i) \in I \times A_i^{v_i(t_i)}$ exists, such that $Eu_i(a_i, m'_{-i}) > Eu_i(m')$. However, since C is a common CURB set, this is a contradiction. Therefore, m' is a Nash equilibrium on C. Thus, since $(i, t_i) \in I \times T_i$ is arbitrary, $m^* \equiv m'$ is a Nash equilibrium on $v_i(t_i)$. Hence, s^* with $s_i^*(t_i) \equiv m_i^*$ for any $i \in I$ and $t_i \in T_i$ is a cognitively stable generalized Nash equilibrium. □

Proposition 3.4.1 suggests one of the conditions for the existence of a cognitively stable generalized Nash equilibrium in any game with unawareness. The contraposition is that if no cognitively stable generalized Nash equilibrium exists, then no common CURB set exists. This means that if some players cannot perceive any CURB set in the realizable action set, then they are surprised about an actual play because each player's belief about opponents' plays is wrong. The proposition suggests a condition for all players' stable plays; that is, rational players do not deviate from a specific play.[16]

The following corollary is obvious from the above proof of Proposition 3.4.1.

Corollary 3.4.1. Given any simultaneous-move game with unawareness, a common CURB set includes the support of some cognitively stable generalized Nash equilibrium.

Second, let us consider the self-confirming equilibrium proposed by Fudenberg and Levine (1993). Schipper (2021) generalizes a rationalizable self-confirming equilibrium to include extensive-form games with unawareness. Kobayashi and Sasaki (2021) focus on simultaneous-move games with unawareness and discuss a rationalizable self-confirming equilibrium using epistemic models. The present study discusses the k-self-confirming equilibrium, which means that all players in the k-th order mutually believe that all their beliefs are correct. The following definition of self-confirming equilibria is based on Kobayashi and Sasaki (2021). s^* is 0-self-confirming equilibrium if for any $i \in I$, $s_i^*(t_i^*) \in \arg\max_{x \in M(A_i^{v_i(t_i^*)})} Eu_i(x, (s_j^*(t_j^*))_{j \in I_{-i}})$. s^* is a k-self-confirming equilibrium ($k \geq 1$) if there exists the belief system μ such that for any $h = 1, \ldots, k$ and $i_h \in I$, where $t_{i_1} = t_i^*$,

1. $s_{i_h}^*(t_{i_h}) \in \arg\max_{x \in M(A_{i_h}^{v_{i_h}(t_{i_h})})} Eu_{i_h}(x, \mu_{i_h}(t_{i_h}))$, and
2. $\mu_{i_h}(t_{i_h}) \equiv (s_j^*(t_j^*))_{j \in I_{-i_h}}$.

[16] I thank Masakazu Fukuzumi for this suggestion.

(b) Rationalizable Notions

Rationalizability is proposed by Bernheim (1984) and Pearce (1984). Heifetz, Meier, and Schipper (2013b) generalize Pearce's extensive-form rationalizability to games with unawareness. This notion is typically used to overcome the issue of equilibrium notions.

As pointed out by Basu and Weibull (1991), because the rationalizable action set is the maximum FURB set, we can use β to define rationalizability under unawareness. In any simultaneous-move game with unawareness Γ, let $t^* = (t_i^*)_{i\in I}$ be the actual type profile. Given any type profile $t = (t_i)_{i\in I} \in \times_{i\in I} T_i$, let $R(t) = \times_{i\in I} R_i(t_i)$, where $R_i(t_i) \subseteq A_i^{v_i(t_i)}$, and let $R_{-i}(t_{-i}) = \times_{j\in I_{-i}} R_j(t_j)$. Then, the rationalizability under unawareness is defined as follows.

Definition 3.4.1. In a simultaneous-move game with unawareness Γ, $R = (R(t^*), (R(t_i, b_i(t_i)))_{(i,t_i)\in I\times T_i})$ is the rationalizable strategy if and only if for any $(i, t_i) \in I \times T_i$,

$$R_i(t_i) = \beta_i^{v_i(t_i)}(R_{-i}(t_{-i})).$$

For each $(i, t_i) \in I \times T_i$, $R_i(t_i)$ means that the set of best responses to the opponents' rationalizable actions in the i's belief about the opponents' views. Let us call $R_i(t_i)$, i's rationalizable action set at t_i, and $R(t)$ is the rationalizable action set at t.

Then, the following remark represents a relationship between rationalizability and a FURB notion.

Remark 7. In a simultaneous-move game with unawareness Γ, for any $(i, t_i) \in I \times T_i$, if $b_i(t_i)(j)$ leads to t_i for any $j \in I_{-i}$, then $\beta^{v_i(t_i)}(R(t_i, b_i(t_i))) = R(t_i, b_i(t_i))$.

Proof. In Γ, given $(i, t_i) \in I \times T_i$, and suppose that $b_i(t_i)(j)$ leads to t_i for any $j \in I_{-i}$, then for any $j \in I_{-i}$, $v_i(t_i) = v_j(b_i(t_i)(j))$. By the definition of β, $\beta^{v_i(t_i)}(R(t_i, b_i(t_i))) = \beta_i^{v_i(t_i)}(R_{-i}(b_i(t_i))) \times_{j\in I_{-i}} \beta_j^{v_i(t_i)}(R_{-j}(b_j(b_i(t_i)(j))))$. Then, by Definition 3.4.1, $\beta_i^{v_i(t_i)}(R_{-i}(b_i(t_i))) = R_i(t_i)$, and for any $j \in I_{-i}$, $\beta_j^{v_i(t_i)}(R_{-j}(b_j(b_i(t_i)(j)))) = R_j(b_i(t_i)(j))$. Since $R_i(t_i) \times_{j\in I_{-i}} R_j(b_i(t_i)(j))$, $\beta^{v_i(t_i)}(R(t_i, b_i(t_i))) = R(t_i, b_i(t_i))$. □

Kobayashi and Sasaki (2021) propose a k-rationalizable self-confirming equilibrium as follows. In a simultaneous-move game with unawareness Γ, s^* is a 0-rationalizable self-confirming equilibrium if it is a 0-self-confirming equilibrium. s^* is a 1-rationalizable self-confirming equilibrium if it is a 1-self-confirming equilibrium, and $\mathrm{supp}[(s_i^*(t_i^*))_{i\in I}] \subseteq R(t^*)$. s^* is a k-rationalizable self-confirming equilibrium ($k \geq 2$) if it is a k-self-confirming equilibrium, $\mathrm{supp}[(s_i^*(t_i^*))_{i\in I}] \subseteq R(t^*)$, and for any $i \in I$, and $h = 1, \ldots, k$, $\mathrm{supp}[(s_i^*(t_i^*))_{i\in I}] \subseteq R(t_{i_h}, b_{i_h}(t_{i_h}))$, where $t_{i_1} = t_i^*$. Kobayashi and Sasaki (2021) provide the following remark.

Remark 8 (Kobayashi and Sasaki 2021)**.** s^* is an ∞-rationalizable self-confirming equilibrium if and only if s^* is a cognitively stable generalized Nash equilibrium.

(c) Example

This section compares the CURB concept with other solution concepts.

Example 1 (Continued)**.** Let us reconsider Example 1. First, this thesis considers the four generalized (pure) Nash equilibria in the game:

$$\begin{aligned}
s_1 &= ([s_A(t_A^*) = a_1, s_A(t_A) = a_2], [s_B(t_B^*) = b_1, s_B(t_B) = b_2]);\\
s_2 &= ([s_A(t_A^*) = a_1, s_A(t_A) = a_3], [s_B(t_B^*) = b_3, s_B(t_B) = b_2]);\\
s_3 &= ([s_A(t_A^*) = a_3, s_A(t_A) = a_2], [s_B(t_B^*) = b_1, s_B(t_B) = b_3]); \text{ and}\\
s_4 &= ([s_A(t_A^*) = a_3, s_A(t_A) = a_3], [s_B(t_B^*) = b_3, s_B(t_B) = b_3]).
\end{aligned}$$

Then, the cognitively stable generalized Nash equilibrium is only s_4.

Second, we consider the self-confirming (pure) equilibria in the game. A 0-self-confirming equilibrium and ∞-self-confirming equilibrium exist. The 0-self-confirming equilibrium is only

$$s_5 = ([s_A(t_A^*) = a_1, s_A(t_A) = a_1], [s_B(t_B^*) = b_1, s_B(t_B) = b_1]).$$

The 0-self-confirming equilibrium is not the k-self-confirming equilibrium ($k \geq 1$) because at k, Alice's a_1 does not best respond to Bob's b_1, while Bob's b_1 does not respond best to Alice's a_1.

By contrast, s_4 is the only ∞-self-confirming equilibrium. When comparing a cognitively stable generalized Nash equilibrium with an ∞-self-confirming equilibrium, both these equilibria are the same.

Third, we consider rationalizability. Given three tuples, $t^1 = (t_A^*, t_B^*)$, $t_2 = (t_A^*, t_B)$, and $t_3 = (t_A, t_B^*)$, the rationalizable strategy is $R = (R(t_1), R(t_2), R(t_3))$, and the pure rationalizable actions at t_2 and t_3 are as follows:

$$\begin{aligned}
&R(t^2) = \{a_1, a_3\} \times \{b_2, b_3\}; \text{ and}\\
&R(t^3) = \{a_2, a_3\} \times \{b_1, b_3\}.
\end{aligned}$$

Then, at t_1,

$$R(t^1) = \beta_A^{v_i(t_A^*)}(\{b_2, b_3\}) \times \beta_B^{v_i(t_B^*)}(\{a_2, a_3\}) = \{a_1, a_3\} \times \{b_1, b_3\}.$$

Here, it is obvious that the 0-self-confirming equilibrium s_5 is a 0-rationalizable self-confirming equilibrium and that the ∞-self-confirming equilibrium s_4 is an ∞-rationalizable self-confirming equilibrium.

Let us compare the CURB concept with the other notions in this example. First, we compare the CURB notion with the equilibrium notion. From Proposition 3.4.1, any common CURB set includes a support for the objective outcome, induced from the cognitively stable generalized Nash equilibrium. Since the ∞-rationalizable self-confirming equilibrium and cognitively stable generalized Nash equilibrium are the same, C^2 includes a support for the objective outcome (a_3, b_3), induced from the cognitively stable generalized Nash equilibrium and ∞-rationalizable self-confirming equilibrium, s^4. By contrast, a common realizable CURB set C^1, which is not a common CURB set, is a support for the objective outcome (a_1, b_1), induced from the 0-rationalizable self-confirming equilibrium s^5. At C^1 or s^5, each player is rational but their first-order belief is irrational play.[17] Moreover, they might be certain about the opponent's irrationality.

Next, we consider the relationship with rationalizability. Given a type t^1, $R(t^1) = C^3$. As shown by Basu and Weibull (1991), a rationalizable action set is equivalent to a maximum FURB set. By contrast, two minimal CURB sets, C^1 and C^2, are subsets of the rationalizable action set; that is, any minimal CURB set is a refined notion of rationalizability. □

As shown in the above example, a realizable CURB concept is related to other solution concepts; that is, the CURB concept has similar characterizations to the other concepts.

3.4.2 Discovery and Equilibrium Notions

This section discusses the relationships between discovery processes and equilibrium notions.

(a) Rationalizable Discovery Process and Self-Confirming Equilibrium

Schipper (2021) models rationalizable discovery processes in which all players implement rationalizable actions in each stage of the game. This thesis formulates rationalizable discovery processes based on Perea (2022) as follows.

Definition 3.4.2. A discovery process $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \dots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \dots)$ is a *rationalizable discovery process* if for any λ, R^λ is the set of rationalizable strategies.

Schipper (2021) models a rationalizable discovery process based on Heifetz, Meier, and Schipper (2013b) and shows that every rationalizable discovery process (in any extensive-form game with unawareness) converges to a self-confirming game in which every rational player does not need further revision, and possesses some (0-)rationalizable self-confirming equilibrium.

However, this study's framework provides a result different from Schipper's (2021).

Example 3. Let us consider two agents, Elena and Filip. We assume they face a zero-sum game:

$v =$

E / F	f_1	f_2
e_1	1, -1	-1, 1
e_2	-1, 1	1, -1

.

Here, suppose that in a zero-sum game with unawareness, there exists a view as shown below:

[17]Rational play means that players maximize their utility.

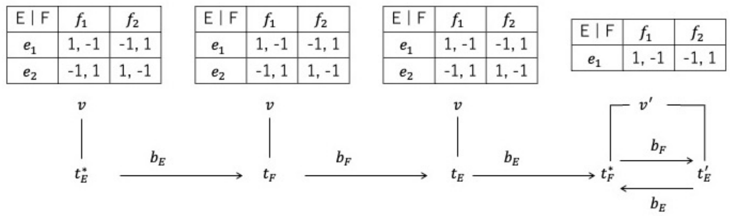


Figure 3.6: Example 3

$v' =$

E / F	f_1	f_2
e_1	1, -1	-1, 1

;

$T_E = \{t_E^*, t_E, t'_E\}$ and $T_F = \{t_F^*, t_F\}$;
For t_E^*, $v_E(t_E^*) = v$ and $b_E(t_E^*) = t_F$;
For t_E, $v_E(t_E) = v'$ and $b_E(t_E) = t_F^*$;
For t'_E, $v_E(t'_E) = v'$ and $b_E(t'_E) = t_F^*$;
For t_F^*, $v_F(t_F^*) = v'$ and $b_F(t_F^*) = t'_E$; and
For t_F, $v_F(t_F) = v$ and $b_F(t_F) = t_E$.

This formulation is depicted in Figure 3.6.

Then, a strategy profile that is uniquely rationalizable is

$$s = ([s_E(t_E^*) = e_1, s_E(t_E) = e_2, s_E(t'_E) = e_1], [s_F(t_F^*) = f_2, s_F(t_F) = f_1]);$$

that is, the actual play is (e_1, f_2). In the play, Filip's belief is correct, whereas Elena's belief is wrong because she predicts that Filip plays f_1. However, both players are aware of the actual play; that is, no discoveries occur. Hence, in the next-stage game, they play the same s. Then, there is no n-rationalizable self-confirming equilibrium for any natural number n (i.e., no 0-rationalizable self-confirming equilibrium exists) because in Elena's actual subjective view v, a unique self-confirming equilibrium is that both players assign probability $\frac{1}{2}$ to each of their actions, while in Filip's actual subjective view v', a unique self-confirming equilibrium is (e_1, f_2).

By contrast, myopic discovery processes avoid the above crucial issue in Harsanyi-Perea style. If an objective outcome (e_1, f_2) is observed in the first stage game, in next-stage game, Elena best responds to Filip's previous play f_2. Then, she choose e_2. Hence, in the second-stage game, Filip observes the novel action of Elena e_2 and revises his subjective view. Then, both players' subjective views are replaced with v in the third-stage game. □

(b) Myopic Discovery Process and Cognitive Stability

This section considers cognitively stable generalized Nash equilibria in myopic discovery processes. First, this study provides the *mutual CURB concept* that each player's actual view has the same CURB set.

Definition 3.4.3. In any simultaneous-move game with unawareness Γ, $C \in V$ is a *mutual CURB set* if for any $i \in I$, C is a non-empty CURB set in $v_i(t_i^*)$.

A mutual CURB concept has the following property.

Lemma 3.4.1. Every mutual CURB set is a realizable CURB set.

Proof. Given any mutual CURB set, $C \in V$, $C \subseteq v_i(t_i^*)$ for any $i \in I$. Suppose that C is not a realizable CURB set; that is, there exists some i such that $\beta_i^*(A_{-i}^C) \not\subseteq A_i^C$ in the realizable action set. Since the realizable action set is defined by $\times_{i \in I} A_i^{v_i(t_i^*)}$, $\beta_i^{v_i(t_i^*)}(A_{-i}^C) \not\subseteq A_i^C$. This contradicts that C is a mutual CURB set. Hence, C is a realizable CURB set. ☐

Lemma 3.4.2. In a simultaneous-move game with unawareness, if a mutual CURB set is present in every view, then a common CURB set exists.

Proof. Suppose that the mutual CURB set C is present in every view in a simultaneous-move game with unawareness. Suppose that for some $(i, t_i) \in I \times T_i$, C is not CURB in $v_i(t_i)$. Since for some $j \in I$ $v_i(t_i) \subseteq v_j(t_j^*)$, where t_j^* is j's actual type and t_j^* leads to t_i, C is not CURB in $v_i(t_i^*)$. This is a contradiction. Therefore, the mutual CURB set is CURB in every view in the game. Then, from Lemma 3.4.1, since the mutual CURB set is a realizable CURB set, the set is a common CURB set. ☐

When relating the mutual CURB concept to steady-state equilibrium notions, this paper shows the condition for converging to a discovered game possessing some steady-state equilibrium. Moreover, we can show the condition for converging a game such that every equilibrium is a steady-state equilibrium. This is proven in the following theorems.

Proposition 3.4.2. In any simultaneous-move game with unawareness, if there exists a mutual CURB set such that the CURB set is CURB in every view in the game with unawareness, then there exists a cognitively stable generalized Nash equilibrium.

Proof. Suppose that some mutual CURB set is present in every view in a simultaneous-move game with unawareness. From Lemma 3.4.2, the mutual CURB set is a common CURB set. Then, from Proposition 3.4.1, a cognitively stable generalized Nash equilibrium exists. ☐

Theorem 3.4.1. Suppose a simultaneous-move game with unawareness Γ has a mutual CURB set. Then, a myopic discovery process converging to a discovered game possessing a cognitively stable generalized Nash equilibrium exists.

Proof. Suppose that a mutual CURB set C in Γ exists. From Lemma 3.4.1, C is a realizable CURB set. From Theorem 3.3.1, a myopic discovery process P converging to C exists. Since C is a common realizable CURB set from Lemma 3.4.2, from Proposition 3.4.2, a cognitively stable generalized Nash equilibrium exists. ☐

Corollary 3.4.2. Suppose that every realizable CURB set is a mutual CURB set in Γ. Then, every myopic discovery process converges to a discovered game possessing a cognitively stable generalized Nash equilibrium.

The process considered herein starts from an arbitrary generalized strategy profile. The convergence result holds even if the starting point is not necessarily a generalized Nash equilibrium.[18]

Next, let us consider the relationship with a Nash equilibrium in an objective game. Sasaki (2017) discusses the relationships between a cognitively stable generalized Nash equilibrium and Nash equilibrium in an objective game in simultaneous-move games with unawareness. Sasaki (2017) shows the following proposition.

Proposition 3.4.3 (Sasaki 2017)**.** Given any simultaneous-move game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type, then every cognitively stable generalized Nash equilibrium induces an objective outcome that is a Nash equilibrium in an objective game G.

Proof. Given any simultaneous-move game with unawareness Γ, suppose that for any $i \in I$, $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type. Suppose that the generalized strategy profile s^* is a cognitively stable generalized Nash equilibrium. From Remark 6, the objective outcome induced from a cognitively stable generalized Nash equilibrium is a Nash equilibrium of the realizable action set. Since every player is aware of their own actions in the objective game, the realizable action set is equivalent to the action set of the objective game. Hence, since every Nash equilibrium of the realizable action set is a Nash equilibrium in the objective game, the support of the objective outcome, induced by a cognitively stable generalized Nash equilibrium, is a Nash equilibrium in the objective game. □

Combining Theorem 3.3.1 and Proposition 3.4.3, we can show the following theorem.

Theorem 3.4.2. In any simultaneous-move game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, where t_i^* is i's actual type, then any myopic discovery process converges to a discovered game such that any cognitively stable generalized Nash equilibrium induces an objective outcome that is a Nash equilibrium in an objective game G.

[18] Tada (2018) discusses a revision process in which players play a generalized Nash equilibrium in each round and conjectures that the process converges to a cognitively stable generalized Nash equilibrium, if there is any. However, the conjecture wrongly assumes that players play a generalized Nash equilibrium in each round. This study yields a result in the spirit of that paper, but under the condition in which players play their myopic best responses.

3.5 Discussion

3.5.1 A CURB Block Game and Economy of Cognitive Costs

In our model, some myopic discovery processes do not converge to a discovered game possessing a common CURB set. Some players may be certain of their opponents' irrationality. However, using the *block game* notion (e.g., Myerson and Weibull, 2015) of a smaller game than each player's subjective game, players can reconstruct a block game possessing a common CURB set from a discovered game to which a myopic discovery process converges, which then allows them to ascertain each other's rationality.

Let us consider the case in which a discovered game possesses a realizable CURB set. When all players implement a generalized strategy profile so that the objective outcome is in the realizable CURB set, if they are rational, they do not perform actions outside the realizable CURB set. Thus, all the actions in the complementary set of the realizable CURB set are redundant for them. Therefore, each player excludes the actions in the complementary set to economize the cognitive costs of the true structure of the game. If they economize those cognitive costs, their subjective games are the smallest games in which the action set is a common realizable CURB set. The following definition represents the "economy of knowledge" about a game's structure.

Definition 3.5.1. Given any game with unawareness, $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$, and any common realizable CURB set, $C \in V$ in Γ, $\Gamma' = (G, (T'_i)_{i\in I}, (v'_i)_{i\in I}, (b'_i)_{i\in I})$ is an *economized game* by C in Γ if for any $(i, t_i) \in I \times T_i$, there exists $t'_i \in T'_i$ so that

- $v'_i(t'_i) = C$; and
- For any $(j, t_j) \in I_{-i} \times T_j$ with $b_j(t_j)(j) = t_j$, there exists $t'_j \in T'_j$ so that $b'_j(t'_j)(j) = t'_j$, and $v'_j(t'_j) = C$.

Then, $G^C = (I, C, u^C)$ is called a *realizable CURB block game* with C, where $u^C = (u_i)^C_{I\in I}$, and $u^C_i : C \to \mathbb{R}$ so that for any $a \in C$, $u^C_i(a) = u_i(a)$.

In Example 2, when Colin and David play s_1 in the initial game and s^2_1 in the next stage of the game, since the objective outcome induced by s^2_1 is (c_1, d_1), the realizable CURB block game with C^1 is $G^{C^1} = (I, C^1, (u^{C^1}_C, u^{C^1}_D))$. Thus, in the economized game, Γ^{C^1}, all subjective games are G^{C^1}.

The following remark is obvious.

Remark 9. Given any Γ, and let C is a common realizable CURB set in Γ. Then, an economized game Γ^C made by C has a cognitively stable generalized Nash equilibrium.

In Γ^{C^1} in Example 2, there exists a unique generalized Nash equilibrium such that Colin and David play c_1 and d_1 in each subjective game, respectively.

Thus, from the definition of cognitive stability, the generalized Nash equilibrium is cognitively stable.

When Γ is a discovered game to which a myopic discovery process converges, every subjective game is a realizable CURB block game with a CURB set such that supports for players' actual actions converge in the process. Hence, a rationalizable discovery process is a search process for larger subjective games, whereas this paper's myopic discovery process is a search process for common, smaller subjective games (i.e., realizable CURB block games).

3.5.2 Adaptive Play

This study considers myopic agents and myopic play. In the model, each player best responds to opponents' actual plays in the previous stage of the game. However, a bounded rational agent may be unable to provide their best response to opponents' strategies. Young (1993) analyzes *adaptive play models* that do not allow participants to exactly play their best responses to previous plays. This subsection discusses a generalization of adaptive plays to simultaneous-move games with unawareness.

First, we define adaptive plays in a discovered game.

Definition 3.5.2. Let Γ' be a discovered game from Γ and $\varepsilon > 0$ be an error rate such that ε is sufficiently small. The generalized strategy profile s' is an *adaptive play* in Γ' if for any $(i, t'_i) \in I \times T_i$, with probability $1 - \varepsilon$, a player i chooses the best response to i's beliefs $\mu'_i(t'_i) \equiv (s^*_j(t^*_j))_{j \in I_{-i}}$ such that s^* is a generalized strategy profile played in Γ, and t^*_j is j's actual type in Γ; further, with probability ε, i chooses an action in $A_i^{v'_i(t'_i)}$ at random.

The following discovery process with an adaptive play based on Definition 3.5.2 is proposed.

Definition 3.5.3. Any discovery process $P = (\langle \Gamma^1, s^0 \rangle, \langle \Gamma^2, s^1 \rangle, \dots, \langle \Gamma^\lambda, s^{\lambda-1} \rangle, \dots)$ is an *adaptive discovery process* if for any $\lambda \geq 2$, s^λ is an adaptive play profile at λ.

In a game without unawareness, Hurkens (1995) and Young (1998) use an adaptive play notion and show the convergence to a minimal CURB set. The proof of Theorem 3.3.1 focuses on the realizable action set. Additionally, we can conjecture the following.

Conjecture 1. Given any simultaneous-move game with unawareness, in any adaptive play, the supports of the objective outcome, induced by adaptive plays, converge to a common minimal realizable CURB set.

Informal proof. Given any simultaneous-move game with unawareness and any adaptive discovery process, we must focus on the realizable action set as per Theorem 3.3.1. Based on Hurkens (1995) and Young (1998), adaptive plays converge to a minimal CURB set of the realizable action set. Then, the set is a common minimal realizable CURB set.[19] □

[19] The exact proofs are beyond the capabilities of the author and are omitted.

3.5.3 Limitations

This research has the following limitations:

1. In a game with unawareness, in a generalized Nash equilibrium or under a rationalizable strategy, each player may be convinced that they are playing a higher-order subjective game or that their opponents are unaware of certain actions. However, in certain plays, each player may discover actions of which they were unaware, which may confirm that their subjective game was wrong. Here, the question of why the player was convinced that their higher-order subjective game was correct in the initial game with unawareness arises. In Example 2, two cognitively unstable generalized Nash equilibria, s_1 and s_2, exist in the initial game. This study does not provide a suitable answer for which equilibrium Colin and David play when they both implement a generalized Nash equilibrium play.

   This study's discovery process and those of previous work explain how to build each player's subjective game under unawareness; however, they do not explain how to do so in an initial game with unawareness. This issue provides a direction for future research on games with unawareness.

2. While each player pays attention to opponents' subjective games in the initial game with unawareness, they do not pay attention to them in a discovered game. This thesis does not provide a suitable answer to the question why each player ceases to pay attention.

3. Models of discovery processes suppose that each player recognizes opponents' plays and actions of which they were previously unaware. However, the assumptions may be too strict. For example, most preschool-aged children would be unable to understand conversations among adults, or, at least, would not be able to have the same conversations. Further research could aim to relax this assumption and reconstruct the models of discovery processes.

4. Section 3.4 shows that Schipper's (2021) result might not hold in this study's framework. Specifically, in this model, some rationalizable discovery process might not converge to any (simultaneous-move) game with unawareness possessing a self-confirming equilibrium. This chapter thus proposes the following open question: what are the conditions for satisfying the result of Schipper (2021) in the framework of this study?

3.5.4 Related Literature

Growing Awareness

Studying discovery processes entails analyzing growing awareness or updating awareness. Karni and Vierø (2013, 2017) and Vierø (2021) discuss decision making under unawareness and propose a reverse Bayesian model. As pointed out by Schipper (2013), an agent unaware of an event differs from an agent who assigns probability 0 to that event. This means that an unaware agent cannot assign a probability to an event of which they are unaware. Given such an event, the models presented by Karni and Vierø (2013, 2017) and Vierø (2021) discuss the way to revise such agents' beliefs.

Galanis and Kotronis (2021) analyzes a model where all agents announce prices to each other and generalize the results of Geanakoplos and Polemarchakis (1982) and Ostrovsky (2012). They suppose that updating awareness is minimal and that a true state is never excluded. Traders eventually agree on the price of the security. Moreover, if the security is separable, traders agree on the correct price and aggregate their information.

CURB Notions

Basu and Weibull (1991) were the first to introduce CURB notions into standard game models. CURB notions in dynamic models are discussed by Hurkens (1995), Young (1998), and Grandjean, Mauleon, and Vannetelbosch (2017). Voorneveld, Kets, and Norde (2005) discuss the axiom and properties of minimal CURB sets. Pruzhansky (2003) shows that in extensive games with perfect information and a finite horizon, only one minimal CURB set exists. Benisch, Davis, and Sandholm (2010) provide algorithms for computing CURB sets. Asheim, Voorneveld, and Weibull (2016) discuss the epistemic robustness of CURB in epistemic models.

Chapter 4

Coordination and Imitation under Unawareness

4.1 Introduction

This chapter (i) focuses on *coordination games with unawareness* (specifically, symmetrical games with unawareness), (ii) introduces a *successful-coordination equilibrium* to coordination games with unawareness, and (iii) models a *discovered game with imitation* and relates it to a successful-coordination equilibrium.

A game with unawareness is a model in which players are unaware of the structures of the game, especially their own actions. Since players might be unaware of their own actions or the opponents' actions, they might make incorrect beliefs and they might not best respond to the opponents' actual plays. If unaware players observe opponents' plays that they are unaware of, then they might notice their unawareness and attempt to revise their subjective games by adding their opponents' actions to their action sets. Schipper (2021) and Chapter 3 discuss such situations, and propose a model of discovery and update named a *discovery process.* A model of discovery process assumes that if unnoticed actions that players are unaware of is played, then in the next stage game, players revise their subjective games by adding the unnoticed actions to their action sets, renew beliefs about the opponents' play, and best respond to them. Schipper (2021) models discovery processes in extensive-form games with unawareness based on Heifetz, Meier, and Schipper (2013b), and shows that if all players choose rationalizable actions, then their update converges to some subjective games possessing a rationalizable self-confirming equilibrium that all players are aware of. Chapter 3 models discovery processes in simultaneous-move games with unawareness in a simplified version of Perea (2022), and shows that if all players best respond to their choices in a previous-stage game, then their update converges to some subjective games possessing a common realizable CURB set that is CURB in realizable action set, and supports of their plays is a subset of the CURB set in the subjective games.

The above studies deal with the discovery processes of general games. However, one issue arises when considering coordination games under unawareness. Let us consider the following example. In a coordination game, there are two players, Alice and Bob, and two actions, X and Y. Here, suppose that Alice can play only X and Bob can play only Y. Moreover, suppose that both players commonly believe this, that is, their beliefs are (X, Y). For the belief, Alice's best response is X, and Bob's best response is Y. Hence, they play (X, Y). That is, (X, Y) is an equilibrium because it is their *correct beliefs* and their best responses to the beliefs. Since their beliefs are correct, they are convinced of their coordination failure in the equilibrium. In this case, there is no discovery, because Alice is aware that Bob has Y and Bob is aware that Alice has X. Thus, their coordination failures cannot be resolved using models of discovery processes based on Schipper (2021) or Chapter 3. Suppose now the labeling is such that, for the coordination to be successful, both have to play actions of the same labeling. In Alice and Bob's example, for the coordination to be successful, Alice must be able to choose Y or Bob must be able to choose X. That is, each other must *imitate* the opponent's action.

This study investigates the case in which players observe their opponents' action that they could not play and supposes that they can imitate the action of their opponents. Games are constructed based on the above assumptions and let *imitative discovered games* be the reconstructed games. By assuming imitation, we can consider successful coordination under unawareness.

However, to consider successful coordination under unawareness, we must reconsider equilibrium concepts. As shown in the above example, in some equilibrium coordination might not be successful. Hence, this thesis introduces a novel equilibrium concept, named the *successful-coordination equilibrium*. The successful-coordination equilibrium, a specific solution concept in coordination games with unawareness, deals only with successful coordination. The solution concept is characterized to show that a successful-coordination equilibrium must exist in any imitative discovered game.

Section 4.2 models a (pure) coordination game with unawareness. Next, Section 4.3 introduces a successful-coordination equilibrium to coordination games with unawareness and characterizes it. Section 4.4 models imitative discovered games and shows that any imitative discovered game has a successful-coordination equilibrium. Additionally, this study introduces a *block game* notion to coordination games with unawareness. In any imitative discovered game, players can remove the redundant actions that were not played to form a *coordination block game*. Finally, Section 4.5 discusses the relationship between successful coordination and cognitively stable generalized Nash equilibria. A cognitively stable generalized Nash equilibrium, which is a generalization of a Nash equilibrium, is interpreted as the equilibrium with the correct beliefs. However, in some coordination games with unawareness, the cognitively stable generalized Nash equilibrium can induce coordination failure. This shows that the correctness of beliefs differs from the accuracy of subjective games. Section 4.5 also discusses the assumptions of discoveries and imitations, which are specific to unawareness, and examines these two assumptions in more details.

4.2 Preliminaries

This section formulates a *coordination game with unawareness*, which is a more strictly symmetrical game with unawareness. Let $G = (I, A, u)$ be a standard n-person pure coordination game.[1] $I = \{1, \dots, n\}$ is a finite set of players and $I_{-i} = I \setminus \{i\}$. $A = \times_{i\in I} A_i$, where A_i is a non-empty finite set of actions of i and $A_1 = \cdots = A_n$. Let $a_i \in A_i$ be i's action. $u = (u_i)_{i\in I}$, where $u_i : A \to \mathbb{R}$ is the utility function of i. For any $a = (a_1, \dots, a_n) \in A$, $u_1(a) = \cdots = u_n(a) > 0$ if $a_1 = \cdots = a_n$ (in terms of the labeling), while $u_1(a) = \cdots = u_n(a) = 0$ otherwise.

Coordination games with unawareness is defined based on Chapter 3 of this thesis. For any standard pure coordination game G, let $V = \times_{i\in I}(2^{A_i} \setminus \{\emptyset\})$ be the set of possible *views* or *blocks* of G. Similar to most studies, this thesis assumes that the set of players is commonly known and that each player's utility for each action profile does not depend on awareness. Given $v \in V$, let A_i^v be the set of actions of i in $v = \times_{j\in I} A_j^v$. Here, when a player i is given v, i is aware of $a \in v$ and unaware of $a \in A \setminus v$. For any $v, v' \in V$, v is contained in v', denoted as $v \subseteq v'$, if A_i^v is a subset of $A_i^{v'}$ for any $i \in I$; that is, $A_i^v \subseteq A_i^{v'}$.

Let $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ be a coordination game with unawareness, which is described as follows: for each $i \in I$,

- T_i is a finite and non-empty set of i's types, one of which is the actual type t_i^*.
- $v_i : T_i \to V$ is i's view function.
- $b_i : T_i \to T_{-i}$ is the belief function of i, where $T_{-i} = \times_{j\in I_{-i}} T_j$. If $b_i(t_i) = (t_j)_{j\in I_{-i}}$, then for each $j \in I_{-i}$, $v_j(t_j) \subseteq v_i(t_i)$.

Let us call G an *objective game*, which can be interpreted as a "true game" in Γ. i's type, t_i, describes her or his view of the game and belief about opponents' types. Given t_i, $v_i(t_i) = v$ implies that i is aware of v and unaware of $A \setminus v$, and $b_i(t_i) = (t_j)_{j\in I_{-i}}$ means that at t_i, i believes that the other players' types are $(t_j)_{j\in I_{-i}}$. Simultaneously, i believes that each view of j is $v_j(t_j)$. Let $b_i(t_i)(j)$ be j's type in $b_i(t_i)$. Each player may be unaware of some types of players, including their own. Given $(i, t_i) \in I \times T_i$, we denote a sequence of players' types induced by the belief functions as $t_{i_1}, t_{i_2}, \dots, t_{i_h}, \dots$, where $t_{i_1} = t_i$, and for any $h \geq 2$, $t_{i_h} = b_{i_{h-1}}(t_{i_{h-1}})(i_h)$. We say that t_i leads to t_j if and only if there exists a subsequence $t_{i_1}, \dots, t_{i_h}$ such that $t_{i_1} = t_i$ and $t_{i_h} = t_j$. This thesis supposes $\bigcup_{i\in I} T_i = \bigcup_{i\in I}\{t_{i_h}^*\}_{h\geq 1; t_{i_1}^* = t_i^*}$.

For simplicity, this study only focuses on pure actions. For any $i \in I$, let i's *generalized strategy* be $s_i : T_i \to A_i$. Then, given t_i, $s_i(t_i) \in A_i^{v_i(t_i)}$ is i's *local action* at t_i. Let $s = (s_i)_{i\in I}$ be a *generalized strategy profile.* For any s, $s_i(t_i^*)$ is i's *actual play.* The set of players' realizable play $A_i^{v_i(t_i^*)}$ may be a proper subset of i's *full* action set A_i. Then, the player i cannot implement $a_i \in A_i \setminus A_i^{v_i(t_i^*)}$.

[1] This study only focuses on pure coordination games.

4.3 Successful-Coordination Equilibrium

This section proposes a novel equilibrium concept in (coordination) games with unawareness, namely, the *successful-coordination equilibrium*.

Definition 4.3.1. In a coordination game with unawareness Γ, s^* is a *successful-coordination equilibrium* if

1. For any $i \in I$ and $t_i \in T_i$,

$$s_i^*(t_i) \in \arg \max_{x \in A_i^{v_i(t_i)}} u_i(x, (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}});$$

2. For any $i \in I$ and $t_i \in T_i$, $s_i^*(t_i) = s_i^*(t_i^*)$; and

3. $s_1^*(t_1^*) = \cdots = s_n^*(t_n^*)$.

The first condition requires that players best respond to their beliefs about opponents' plays and the second condition requires that all players' beliefs are correct.[2] The combination of the first condition and second condition can be interpreted as the equilibrium in correct beliefs. However, as described below, some equilibrium in correct beliefs might be a coordination failure. Hence, we need the third condition that requires that the coordination be successful.

We can easily deduce the following remark and proposition.[3]

Remark 10. In any coordination game *without* unawareness, any successful-coordination equilibrium is a Nash equilibrium and vice versa.

Proposition 4.3.1. Suppose that $\bigcap_{i \in I} \bigcap_{t_i \in T_i} v_i(t_i) \neq \emptyset$ in a coordination game with unawareness Γ. If some $a \in \bigcap_{i \in I} \bigcap_{t_i \in T_i} v_i(t_i)$ is a Nash equilibrium in G, then a successful-coordination equilibrium exists.

Proof. Suppose that $\bigcap_{i \in I} \bigcap_{t_i \in T_i} v_i(t_i) \neq \emptyset$ in Γ and that some $a = (a_1, \dots, a_n) \in \bigcap_{i \in I} \bigcap_{t_i \in T_i} v_i(t_i)$ is a Nash equilibrium in G. As G is a coordination game, the Nash equilibrium in G satisfies $a_1 = \cdots = a_n$. For any $(i, t_i) \in I \times T_i$, let $s_i(t_i) = a_i$. Then, the generalized strategy profile s satisfies the conditions of Definition 4.3.1. That is, s is a successful-coordination equilibrium. $\square$

4.4 Discovery and Imitation of Actions

The previous section defined a successful-coordination equilibrium. However, such an equilibrium may not exist in some coordination games with unawareness.

Remark 11. The following example shows that a successful-coordination equilibrium may not exist.

[2] As explained later, the first condition is a definition of generalized Nash equilibria and the second condition is a definition of cognitive stability.

[3] Proposition 4.3.1 is a special case of Sasaki (2017, Proposition 2).

Example 4. Consider two people, Alice and Bob. They face the following coordination game, which is an objective game:

$v^0 =$

Alice / Bob	X	Y
X	1, 1	0, 0
Y	0, 0	1, 1

In v^0, there exist two pure-strategy Nash equilibria, (X, X) and (Y, Y).

Here, let us assume the followings about Alice's belief about this game:

- Alice can implement her action X, but she cannot implement the other action Y because she does not know how to play Y.
- Alice knows that Bob has two actions X and Y.
- Alice knows that Bob can choose only Y and that Bob does not know how to play X; hence, she knows that Bob cannot choose X.
- Alice supposes that Bob believes that it is common knowledge that she can choose only X, Bob can choose only Y, and the others' actions cannot be played.

Additionally, let us assume the following about Bob's belief about this game:

- Bob can implement action Y, but he cannot implement the other action X because he does not know how to play X.
- Bob knows that Alice has two actions X and Y.
- Bob knows that Alice can choose only X and that she does not know how to play Y; hence, he knows that Alice cannot choose Y.
- Bob supposes that Alice believes that it is common knowledge that she can choose only X, Bob can choose only Y, and the others' actions cannot be played.

Then, Alice's first-order view of this game is as follows:

$v^1 =$

Alice / Bob	X	Y
X	1, 1	0, 0

;

Bob's first-order view of this game is as follows:

$v^2 =$

Alice / Bob	Y
X	0, 0
Y	1, 1

; and

Both players' second- or higher-order views of this game are as follows:

$v^3 =$

Alice / Bob	Y
X	0, 0

.

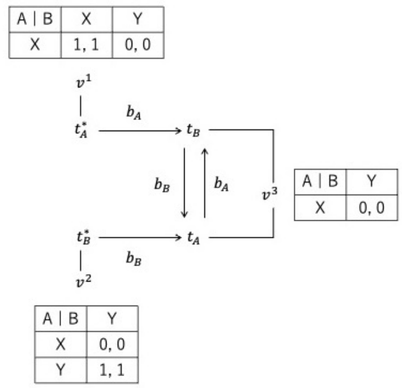


Figure 4.1: The first stage game in Example 4

The mathematical formulation of this example is as follows. Denote Alice by A and Bob by B. Suppose that $T_A = \{t_A^*, t_A\}$ and $T_B = \{t_B^*, t_B\}$ such that

$v_A(t_A^*) = v^1$ and $b_A(t_A^*) = t_B$;
$v_A(t_A) = v^3$ and $b_A(t_A) = t_B$;
$v_B(t_B^*) = v^2$ and $b_B(t_B^*) = t_A$; and
$v_B(t_B) = v^3$ and $b_B(t_B) = t_A$.

This formulation is depicted in Figure 4.1.

Suppose that each player is rational. No player needs to believe that their opponent is rational. Alice then plays X in v^1 as the best response to (X, Y) in v^3. Additionally, Bob plays Y in v^2 as the best response to (X, Y) in v^3. Their beliefs and decisions consist of the following generalized strategy profile: $s^* = ([s_A(t_A^*) = X, s_A(t_A) = X], [s_B(t_B^*) = Y, s_B(t_B) = Y])$. In the strategy profile, both players' beliefs are correct and they best respond to the beliefs.

However, each player knows that the equilibrium play is not a Nash equilibrium in each first-order subjective view. Alice is aware of the Nash equilibrium (X, X) in v^1 and Bob is aware of the Nash equilibrium (Y, Y) in v^2. In the equilibrium, coordination is not successful. Hence, they are aware of a coordination failure. $\square$

Noteworthy about this example is that we cannot use models of discovery processes. A discovery process is a process of updating models under unawareness proposed by Schipper (2021) and Chapter 3. In their frameworks, if a player observes some opponent action that she or he is unaware of, then she or he adds the action to the opponent action set in her or his subjective game. However, in Example 4, Alice is aware of Y and Bob is aware of X, meaning there is no discovery. Hence, they cannot update their subjective games and coordination cannot be successful through their discovery process.

To resolve the above issue, this study presents a model of discoveries and *imitations* under unawareness based on the model of *endogenously* discovered games (Schipper, 2021: Chapter 3) as follows:[4] [5]

Definition 4.4.1. Consider a coordination game with unawareness $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ and a generalized strategy profile $s = (s_i)_{i\in I}$ thereof. Then, $\Gamma' = (G, (T_i')_{i\in I}, (v_i')_{i\in I}, (b_i')_{i\in I})$ is an *imitative discovered game* associated with (Γ, s) if: for any $(i, t_i) \in I \times T_i$, and any sequence of players $i_1, i_2, \ldots, i_h, \ldots$, with a sequence of types introduced by belief functions $(b_i)_{i\in I}$, $t_{i_1}, t_{i_2}, \ldots, t_{i_h}, \ldots$, where $t_{i_1} = t_i$, in Γ, there exists $t_i' \in T_i'$ and a sequence of types in Γ', $t'_{i_1}, t'_{i_2}, \ldots, t_{i_h}, \ldots$, where $t'_{i_1} = t'_i$, such that for any $h \geq 1$,

1. $v'_{i_h}(t'_{i_h}) = \times_{j\in I}[A_j^{v_{i_h}(t_{i_h})} \bigcup_{k\in I} \mathrm{supp}(s_k(t_k^*))]$, where t_k^* is k's actual type in Γ; and
2. $b'_{i_h}(t'_{i_h})(i_{h+1}) = t'_{i_{h+1}}$.

Note that Γ' is a novel coordination game with unawareness. Moreover, it may be that $T \not\subseteq T'$ and $T' \not\subseteq T$, or $T \cap T' = \emptyset$.

When both players observe each other's play, the first condition suggests that each player not only gains knowledge of their opponent's feasible actions, but also discovers (or "learns") a way of playing such actions. The second condition suggests that as supposed by each player, every player commonly believes that players not only gain knowledge of the other's feasible actions but also discover a way of playing such actions.

Example 4 (Continued)**.** Suppose Alice and Bob play $s^* = ([s_A(t_A^*) = X, s_A(t_A) = X], [s_B(t_B^*) = Y, s_B(t_B) = Y])$. Then, according to the first condition, Alice adds Bob's action Y to not only as Bob's choice but also as Alice's choice in her subjective view v^1 and Bob adds Alice's action X not only as Alice's choice but also as Bob's choice in his subjective view v^2. Moreover, as both players suppose that each of them commonly believes that they gain knowledge of each other's feasible actions and discover a way of playing such actions according to the second condition, both players add actions X and Y to their respective choice in each other's second or any higher-order view v^3. Then, each agent's first and any higher-order views are replaced with v^0.

This imitative discovered game $\Gamma' = (G, (T_A', T_B'), (v_A', v_B'), (b_A', b_B'))$ is formulated as follows:

$T_A = \{t_A'\}$ and $T_B = \{t_B'\}$;
$v_A'(t_A') = v^0$ and $b_A'(t_A') = t_B'$; and

[4] Karni and Vierø (2013, 2017) discuss the cases in which agents discover their own new feasible actions. However, in their model, such actions are not *endogenously* discovered but rather *exogenously* discovered. In other words, such actions are given to agents by modelers.

[5] Unlike Schipper (2021) and Chapter 3 of this thesis, this part does not deal with *discovery processes*. As indicated by one of the main results, only one imitation update is required in this model.

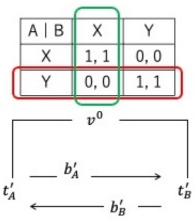


Figure 4.2: The imitative discovered game in Example 4

$v'_B(t'_B) = v^0$ and $b'_B(t'_B) = t'_A$.

This formulation is depicted in Figure 4.2. In Γ', Alice and Bob can choose two actions, X and Y. □

Interestingly, any imitative discovered game has the following property.

Proposition 4.4.1. Given any n-person coordination game with unawareness Γ and any generalized strategy profile s, an imitative discovered game Γ' associated with (Γ, s) has a successful-coordination equilibrium.

Proof. Suppose that s^* is played by all agents in Γ. For any $i \in I$, $a_i = s^*_i(t^*_i)$ is observed and imitated by them. Then, in the imitative discovered game Γ' associated with (Γ, s), for any $(j, t'_j) \in I \times t'_j$, $a_i \in A_j^{v'_j(t'_j)}$. Hence, $s' = ((s'_j(t'_j))_{t'_j \in T'_j})_{j \in I}$ with $s'_j(t'_j) = a_i$ for any $j \in I$ is a successful-coordination equilibrium. □

The above proposition means that for any actual play, next-stage game must have a successful-coordination equilibrium. In Example 4, two successful-coordination equilibria exist in Γ': $s'_1 = (s'_A(t'_A) = X, s_B(t'_B) = X)$ and $s'_2 = (s'_A(t'_A) = Y, s_B(t'_B) = Y)$.

After players have discovered revised subjective games in the imitative discovered game, to which set of actions do they pay attention? It seems to be redundant that a player rationalizes their actions based on their subjective view. Let us consider the following example.

Example 5. Consider the following objective game played by Colin (C) and David (D):

$v_O =$

Colin / David	α	β	γ	δ
α	1, 1	0, 0	0, 0	0, 0
β	0, 0	1, 1	0, 0	0, 0
γ	0, 0	0, 0	1, 1	0, 0
δ	0, 0	0, 0	0, 0	1, 1

.

Now, suppose that Colin believes the following view is a common belief:

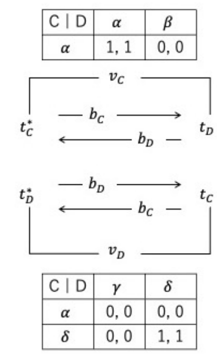


Figure 4.3: The first stage game in Example 5

$$v_C = \begin{array}{|c|c|c|} \hline \text{Colin / David} & \alpha & \beta \\ \hline \alpha & 1,\ 1 & 0,\ 0 \\ \hline \end{array} \ .$$

By contrast, David believes the following view as a common belief:

$$v_D = \begin{array}{|c|c|c|} \hline \text{Colin / David} & \gamma & \delta \\ \hline \beta & 0,\ 0 & 0,\ 0 \\ \hline \delta & 0,\ 0 & 1,\ 1 \\ \hline \end{array} \ .$$

Let us formulate this game $\Gamma = (G, (T_C, T_D), (v_C, v_D), (b_C, b_D))$ as follows:

$T_C = \{t_C^*, t_C\}$ and $T_D = \{t_D^*, t_D\}$;
For t_C^*, $v_C(t_C^*) = v_C$ and $b_C(t_C^*) = t_D$;
For t_C, $v_C(t_C) = v_D$ and $b_C(t_C) = t_D^*$;
For t_D^*, $v_D(t_D^*) = v_D$ and $b_D(t_D^*) = t_C$; and
For t_D, $v_D(t_D) = v_C$ and $b_D(t_D) = t_C^*$.

This formulation is depicted in Figure 4.3.

Suppose that both players implement a generalized strategy profile $s^* = ([s_C(t_C^*) = \alpha, s_C(t_C) = \beta], [s_D(t_D^*) = \gamma, s_D(t_D) = \alpha])$. In the strategy profile, the actual play is (α, γ). Then, the imitative discovered game $\Gamma' = (G, (T'_C, T'_D), (v'_C, v'_D), (b'_C, b'_D))$ is formulated as follows:

$T'_C = \{t'_C, t''_C\}$ and $T'_D = \{t'_D, t''_D\}$;
For t'_C, $v'_C(t'_C) = v'_C$ and $b'_C(t'_C) = t''_D$;
For t''_C, $v'_C(t''_C) = v'_D$ and $b'_C(t''_C) = t'_D$;
For t'_D, $v'_D(t'_D) = v'_D$ and $b'_D(t'_D) = t''_C$; and
For t''_D, $v'_D(t''_D) = v'_C$ and $b'_D(t''_D) = t'_C$, where

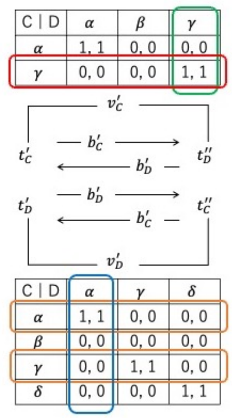


Figure 4.4: The imitative discovered game in Example 5

$v'_C =$

Colin / David	α	β	γ
α	1, 1	0, 0	0, 0
γ	0, 0	0, 0	1, 1

, and

$v'_D =$

Colin / David	α	γ	δ
α	1, 1	0, 0	0, 0
β	0, 0	0, 0	0, 0
γ	0, 0	1, 1	0, 0
δ	0, 0	0, 0	1, 1

.

This formulation is depicted in Figure 4.4. Then, Colin knows that two Nash equilibria exist in v'_C, namely, (α, α) and (γ, γ), whereas David knows that three Nash equilibria exist in v'_D, namely, (α, α), (γ, γ), and (δ, δ). Colin is unaware that David's view is v'_D and David is unaware that Colin's view is v'_C.

Each player must select one of those equilibria in each other's view. Here, let us focus on David. Although he knows there are three equilibria, it seems odd that he includes all the equilibria in his choices because δ is played by neither Colin nor David. □

After players imitate opponents' plays and revise their views, they might exclude redundant actions that nobody plays. Then, they might reconstruct their subjective views to exclude such actions. To provide such a representation, we use the *block game* notion proposed by Myerson and Weibull (2015). A block is a Cartesian product of non-empty subsets of players' actions. We first focus on the actions that each player observes and imitates. Then, we define a *coordination block game* as follows.[6]

[6]Chapter 3 proposed a similar notion as a realizable CURB block game in which the block is CURB.

Definition 4.4.2. Given any n-person coordination game with unawareness $\Gamma = (G, (T_i)_{i\in I}, (v_i)_{i\in I}, (b_i)_{i\in I})$ and any block $B = \times_{i\in I} B_i \in V$, $G^B = (I, B, u^B)$ is a *coordination block game* if

1. $B_1 = \cdots = B_n$; and
2. $u^B = (u_i^B)_{i\in I}$, where $u_i^B(a) = u_i(a)$ for any $i \in I$ and $a \in B$.

Here, B is called a *coordination block*.

Thus, the following proposition holds.

Proposition 4.4.2. Given any n-person coordination game with unawareness Γ and any generalized strategy profile s, let $B \in V$ be a block such that for any $i \in I$, $A_i^B = \bigcup_{j\in I} s_j(t_j^*)$. Then, the block game $G^B = (I, B, u^B)$ is a coordination block game.

Proof. The proof is straightforward. □

In Example 5, as Colin and David play (α, γ), they focus on α and γ. That is, David excludes δ from his choices. Then, the coordination block is $\{\alpha, \gamma\} \times \{\alpha, \gamma\}$ and the coordination block game is

$v_B =$

Colin / David	α	γ
α	1, 1	0, 0
γ	0, 0	1, 1

.

Hence, David can restrict his choices. Then, in their equilibrium selection, both players will focus on α and γ.

In some coordination games (with unawareness), not only are some players unaware of some choices but also might the set of choices be too large. Hence, in the first play, players might be unable to select a specific (successful coordination) equilibrium or restrict action sets to a coordination block. However, by discovering and imitating only those actions taken in the first play, players can restrict their actions to specific coordination blocks.

4.5 Discussion

4.5.1 Relationship with the Generalized Nash Equilibrium

A successful-coordination equilibrium is related to a generalized Nash equilibrium. This section considers the relationships among a successful-coordination equilibrium, a generalized Nash equilibrium, and cognitive stability. We first define the generalized pure Nash equilibrium proposed by Halpern and Rêgo (2014) as follows.

Definition 4.5.1 (Halpern and Rêgo 2014)**.** s^* is a generalized (pure) Nash equilibrium if for any $i \in I$ and $t_i \in T_i$,

$$s_i^*(t_i) \in \arg\max_{x\in A_i^{v_i(t_i)}} u_i(x, (s_j^*(b_i(t_i)(j)))_{j\in I_{-i}}).$$

A generalized Nash equilibrium is interpreted as an equilibrium in beliefs. However, as shown by Schipper (2014), because games with unawareness assume unawareness of players' actions, a generalized Nash equilibrium might consist of wrong beliefs. Then, players who have such wrong beliefs might revise their subjective games and might choose different actions from the immediately preceding stage in the game. To avoid such issues, Sasaki (2017) proposes the notion of cognitive stability or stable belief hierarchies. This notion expresses the requirement that in an equilibrium satisfying cognitive stability or stable belief hierarchies, all participants' beliefs about opponents' plays are correct. Although Sasaki (2017) distinguishes stable belief hierarchies from cognitive stability, he shows that the two notions are equivalent. Let us define cognitive stability as follows.

Definition 4.5.2 (Sasaki 2017)**.** A generalized Nash equilibrium s^* is cognitively stable if for any $i \in I$ and $t_i \in T_i$,

$$s_i^*(t_i) = s_i^*(t_i^*).$$

Cognitive stability means that all players' beliefs about opponents' plays are correct. In a cognitively stable generalized Nash equilibrium, all players' local actions are the same. This means that each player's belief is correct. To compare it with a successful-coordination equilibrium, the definition of a cognitively stable generalized Nash equilibrium is rewritten as follows.

Remark 12. s^* is a cognitively stable generalized Nash equilibrium if for any $i \in I$ and $t_i \in T_i$,

1. $s_i^*(t_i) \in \arg\max_{x \in A_i^{v_i(t_i)}} u_i(x, (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}})$; and
2. $s_i^*(t_i) = s_i^*(t_i^*)$.

A cognitively stable generalized Nash equilibrium can be interpreted as the equilibrium in correct beliefs. These two conditions of the definition of a cognitively stable generalized Nash equilibrium are the same in the first and second conditions of Definition 4.3.1. Hence, the following remark is true.

Remark 13. Under unawareness, every successful-coordination equilibrium is a cognitively stable generalized Nash equilibrium.

This is clear from Definition4.3.1 and Remark 12. In an imitative discovered game, a cognitively stable generalized Nash equilibrium must be a successful-coordination equilibrium. However, in some coordination game with unawareness, the opposite does not hold true; that is, in some cognitively stable generalized Nash equilibrium, coordination might fail. As seen in Example 4, a generalized strategy profile $s^* = ([s_A(t_A^*) = X, s_A(t_A) = X], [s_B(t_B^*) = Y, s_B(t_B) = Y])$ satisfies the definition of a cognitively stable generalized Nash equilibrium, but does not satisfy the definition of a successful-coordination equilibrium.

Cognitive stability is the concept of checking the correctness of the beliefs in a played equilibrium. In any cognitively stable generalized Nash equilibrium, all players confirm that their beliefs are correct. Then, they need not revise their beliefs as well as their subjective views. This means that stability of beliefs and stability of subjective views are entangled with each other in the concept of cognitively stable generalized Nash equilibrium.

However, we must distinguish between the stability of beliefs and stability of subjective views in a coordination game. In coordination games with unawareness, players who play a cognitively stable equilibrium might fail to coordinate and feet that they need to revise their subjective views as shown in Example 4.

Next, this study examines the mathematical relationships between the successful-coordination equilibrium and cognitive stability. This relationship has the following properties.

Proposition 4.5.1. In a coordination game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, i.e., the set of actions in the objective game, then every cognitively stable generalized Nash equilibrium is a successful-coordination equilibrium and vice versa.

Before proving this proposition, we refer to Sasaki's (2017) proposition. Although his proposition includes a generalized mixed strategy profile, his result is here restricted to pure strategies.

Proposition 4.5.2 (Sasaki 2017)**.** In a simultaneous-move game with unawareness Γ, for any $i \in I$, if $A_i^{v_i(t_i^*)} = A_i$, then in any cognitively stable generalized Nash equilibrium, the actual plays of all players are Nash equilibria in G.

Proof. Suppose that for any $i \in I$, $A_i^{v_i(t_i^*)} = A_i$. Pick any cognitively stable generalized Nash equilibrium s^*. Then, for any $(i, t_i) \in I \times T_i$, $u_i(s_i^*(t_i), (s_j^*(b_i(t_i)(j)))_{j \in I_{-i}}) = u_i(s_i^*(t_i), (s_j^*(t_j^*))_{j \in I_{-i}}) = u_i((s_j^*(t_j^*))_{j \in I})$. In other words, every participant's actual play best responds to others' actual plays. Suppose $((s_j^*(t_j^*))_{j \in I})$ is not a Nash equilibrium in G; that is, there exist $(i, a_i) \in I \times A_i$ such that $a_i \neq s_i^*(t_i^*)$ and $u_i(a_i, (s_j^*(t_j^*))_{j \in I_{-i}}) > u_i((s_j^*(t_j^*))_{j \in I})$. However, since s^* is a cognitively stable generalized Nash equilibrium, this is a contradiction. Hence, $((s_j^*(t_j^*))_{j \in I})$ is a Nash equilibrium in G. Because t_i^* denotes i's actual type, $((s_j^*(t_j^*))_{j \in I})$ refers to all players' actual plays. $\square$

Proof of Proposition 4.5.1. Suppose that for any $i \in I$, $A_i^{v_i(t_i^*)} = A_i$. A coordination game with unawareness is a special case of a simultaneous-move game with unawareness. Therefore, from Proposition 4.5.2, in every cognitively stable generalized Nash equilibrium s^*, the actual plays of all players $(s_i^*(t_i^*))_{i \in I}$ form a Nash equilibrium in the standard coordination game G. In any standard coordination game, a Nash equilibrium $a^* = (a_1^*, \dots a_n^*)$ satisfies $a_1^* = \cdots = a_n^*$. From the definition of an actual play, for any $i \in I$, since $s_i(t_i^*) = a_i^*$, $s_1^*(t_1^*) = \cdots = s_n^*(t_n^*)$. Therefore, s^* satisfies every condition of Definition 4.3.1.

The opposite clearly holds true by Remark 13. $\square$

4.5.2 Unawareness of Actions versus Lack of Conception

Studies of unawareness distinguish between two approaches: one is a lack of conception (e.g., Heifetz, Meier, and Schipper, 2006) and the other is a lack of knowledge (e.g., Geanakoplos, 2021).

1. *Lack of conception*: The cholera bacterium was discovered by Koch in 1884. It had existed before 1884, but people were unaware of its existence. Therefore, before 1884, people infected with cholera bacteria battled the disease without realizing they were infected.

2. *Lack of knowledge*: As recently shown, some COVID-19-infected patients are asymptomatic and remain unaware that they are asymptomatically infected unless they undergo PCR testing, at which point they learn that asymptomatic patients with COVID-19 exist.

The original motivation for analyzing games with unawareness was to analyze what happens in situations where players lack conception of their game structures. However, the author wonders whether the models of games with unawareness are really dealing with "lack of conception." In most models, players are only assumed to be unaware of their actions and to simply add the discovered actions to their action set upon discovery. This formulation seems to be closer to a model of "lack of knowledge" rather than "lack of conception."

However, even when focusing on unawareness of actions, we may seriously consider the difference between "lack of conception" and "lack of knowledge." For example, if a meeting place is a well-known location such as Big Ben, people can travel there without needing directions. However, if it is not a widely known place, people may not know how to journey there, even if they are told how to. Many people may then ask their opponents to change the meeting place to a more recognizable location. Knowing those choices influences decision making; however, whether the choices are understandable and feasible also influences decision making. From this perspective, it is important how accurately players can observe opponents' actions.

This study assumes that if agents observe opponents' actions of which they were unaware, then they can understand and imitate those actions. However, as shown above, players do not necessarily understand opponents' actions. When opponents choose unnoticed actions of which players were unaware, their play can be classified and discussed as follows.

Unawareness that opponents have already made a decision

When opponents implement actions of which the agent is unaware, the latter cannot recognize such actions. Then, the following two cases may exist:

- Games are not completed: Shiso Kanakuri, who was a marathon runner at the 1912 Summer Olympics, fell sick with sunstroke during the competition. He did not wake up until the day after the race, meaning that he involuntarily abstained. However, his decision to abstain was not communicated to the Olympic Committee. Shiso's competition time did not stop until March 21, 1967 when he officially crossed the finish line. In other words, his decisions were not tied to the outcome of the game.

- Not realizing being in a game's situation: Companies advertise their products and consumers decide whether to buy such products based on the advertisements. However, companies may use subliminal effects in their advertising. Consumers ignorant of advertising strategies using subliminal effects may not realize that they are being put in a game situation with the company.

Awareness of opponents' decision making, but unawareness of what opponents have decided

Players are aware that their opponents are making a decision when their opponents take actions of which they are unaware. However, it is not always possible to know exactly how those opponents have played.

- Misrecognition: Let us assume there are three entrances: east, west, and south. Of the three, you know the east and west entrances exist, but do not know the south entrance exists. If I go to the south entrance and you go to the east entrance, we cannot meet. Then, you may be misled into thinking that I went to the west entrance because you do not know the south entrance exists.

- Not recognized as symbols: Famous sites such as the White House and Big Ben sound familiar—even to first-time visitors to the area. However, if you have never heard of a company name or a niche restaurant, you may not know where to find it—even by looking at a map. For example, Lake Kawaguchi, located at the foot of Mt. Fuji, is one of the most famous lakes in Japan, but you might not know of it, even if you know where Mt. Fuji is. Then, you may not understand whether I am at Lake Kawaguchi.

Awareness of what opponents have played, but unawareness of their way of playing

Even if we know opponents' choices exactly, we may be unable to imitate them. In other words, we may never reach there—even in the case of famous places such as the White House, Big Ben, and Mt. Fuji. When watching a game of soccer or baseball, only a limited number of people can imitate players' moves. For us to imitate opponents' behavior, we also need to recognize how they did it.

However, it is not necessary to recognize it exactly. If it is a tall building such as Big Ben, it will stand out and we can travel there. In other words, the ability to imitate the behavior of others depends on the ease of imitation.

Of the three types described above, the first and second are caused by a lack of conception, whereas the third is due to a lack of knowledge. The study of discovery processes is clearly the third.[7] As shown by the three types above, there is a strong assumption under unawareness that players can know exactly what opponents are doing and imitate their behavior.

As pointed out by Schipper (2014), unawareness means a lack of conception rather than a lack of knowledge. By contrast, in most games with unawareness, unawareness of actions means a lack of knowledge rather than a lack of conception.[8]

4.5.3 Related Literature

The model in this chapter is based on the literature on growing awareness, updating awareness, and discoveries, including Karni and Vierø (2013, 2017), Schipper (2021), Galanis and Kotronis (2021), and Chapter 3 in this thesis. Previous studies indicate that agents additionally know such information as states, events, consequences, and actions of which they *were* previously unaware. The model in this chapter refers to the growing awareness of how opponents play their games rather than the growing awareness of opponents' plays.

[7] Most games with unawareness also seem to assume the third type of unawareness.

[8] The spirit of the lack of the conception of actions can be seen in such studies as Heifetz, Meier, and Schipper (2013b) and Halpern and Rêgo (2014), who discuss one's awareness of the unawareness of actions. In their frameworks, players know that opponents undertake some action, but do not know what those actions are.

Part III

Reexamining Unawareness in Standard Information Structures

Chapter 5

Non-Trivial Unawareness in Standard Information Structures

5.1 Introduction

In standard discussions of partitional information structures, common knowledge entails counterintuitive conclusions such as the Agreement Theorem of Aumann (1976) and the No-Trade Theorem of Milgrom and Stokey (1982). To solve this issue, Geanakoplos (2021) proposes *non-partitional information structures*. However, under several assumptions, unawareness is trivial, meaning that there is no unawareness of events in standard information structures including Geanakoplos' (2021) information structure. Modica and Rustichini (1994, 1999) show that if the knowledge operator satisfies Necessitation, Monotonicity, Truth, and Positive Introspection and the unawareness operator is defined by second-order ignorance, then Symmetry and Negative Introspection are equivalent. Dekel, Lipman, and Rustichini (1998) suppose that the unawareness operator satisfies Plausibility, KU Introspection, and AU Introspection and show that Necessitation leads to Triviality, which is Negative Introspection. Moreover, if Monotonicity holds, then Unawareness Leads to Ignorance holds. This property means that if an agent is unaware of some event, then the agent knows no event. Based on the above conclusions, the unawareness structure model has become the dominant model in previous studies, leading the information structure model to be used less frequently.

This does not mean, however, that non-trivial unawareness cannot be discussed in the standard information structure model. Ewerhart (2001) succeeds in expressing non-trivial unawareness of events by assuming that an agent's subjective state space is a proper subset of the objective state space and that the agent cannot recognize the complementary set. Fukuda (2021) focuses on the equivalence between AU Introspection and Negative Introspection, as proven by Chen, Ely, and Luo (2012), and suggests that non-trivial unawareness can be discussed by not assuming AU Introspection. This thesis conjectures that non-trivial unawareness can be discussed in other ways in (non-partitional) standard information structures.

Assuming that agents cannot recognize the complement of their subjective state space including a non-partitional model, Ewerhart (2001) characterizes the knowledge operator and the unawareness operator and generalizes the Agreement Theorem. Specifically, he discusses Plausibility, KU Introspection, and AU Introspection of the unawareness operator. We use his model with a restriction that the complement of agents' subjective state spaces is not state-dependent. In this setting, the study examines other properties of the unawareness operator than Ewerhart (2001) examines. First, this chapter defines the knowledge operator that does not satisfy Necessitation, Monotonicity, and Conjunction in the setting where the agent's subjective state space is a proper subset of the objective state space. Then, unawareness is not trivial; that is, there is some event of which the agent is unaware. Next, this thesis shows that Symmetry and Negative Introspection are equivalent. As mentioned above, Modica and Rustichini (1994, 1999) prove that if Necessitation, Monotonicity, Truth, and Positive Introspection hold and unawareness is defined by a second-order unknown, then the equivalence of Symmetry and Negative Introspection holds. By contrast, when the subjective state space is a proper subset of the objective state space (i.e., Necessitation and Monotonicity do not hold), Symmetry does not hold. Then, Symmetry is equivalent to not only Negative Introspection but also Necessitation and Monotonicity. Finally, this chapter generalizes the Triviality Theorems that unawareness is trivial under several assumptions, as shown by Modica and Rustichini (1994, 1999), Dekel, Lipman, and Rustichini (1998), and Chen, Ely, and Luo (2012).

The remainder of this chapter is organized as follows. Section 5.2 defines the standard information structure and Section 5.3 redefines the generalized knowledge operator in contrast to the standard knowledge operator. Section 5.4 characterizes the unawareness operator based on the generalized knowledge operator. Section 5.5 generalizes the Triviality Theorems. Finally, Section 5.6 concludes.

5.2 Preliminaries

This section defines the standard information structure $\langle \Omega, P \rangle$. Let Ω be the objective state space and $\omega \in \Omega$ be a state. $P : \Omega \to 2^\Omega$ is the information function.

Let us introduce the agent's subjective state space $Z \subseteq \Omega$ as follows:

$$Z = \bigcup_{\omega \in \Omega} P(\omega).$$

At this point, the following assumptions can be made about the information function.

P0 For any $\omega \in \Omega$, $P(\omega) \neq \emptyset$.

P1 For any $\omega \in Z$, $\omega \in P(\omega)$.

P2 For any $\omega, \omega' \in \Omega$, if $\omega' \in P(\omega)$, then $P(\omega') \subseteq P(\omega)$.

P3 For any $\omega, \omega' \in \Omega$, if $\omega' \in P(\omega)$, then $P(\omega') \supseteq P(\omega)$.

P0 states that every information set is non-empty; that is, the agent must obtain some information set. P1 suggests that for any state in the subjective state space, the information set contains that state. P2 means that every state in some information set makes the subset of the information set. P3 means that every state in some information set makes the superset of the information set. In what follow, we assume P0. Then, the following remark holds.

Remark 14. Suppose that the information function P satisfies P1. Then, $Z = \Omega$ if and only if

P1* For any $\omega \in \Omega$, $\omega \in P(\omega)$.

Proof. When $Z = \Omega$, this is obvious. Suppose that $Z \neq \Omega$; that is, Z is a proper subset of Ω. Then, for any $\omega \in \Omega \setminus Z$, $\omega \notin P(\omega)$ is obvious. Hence, P1* does not hold. □

Standard discussions about information structures assume the partitional information function with P1*, P2, and P3. This study examines the information function with relaxed assumptions, which we call *partially partitional* information function.

Definition 5.2.1. Given the information structure $\langle \Omega, P \rangle$, P is partitional if and only if P satisfies P1*, P2, and P3. Moreover, P is partially partitional if and only if P satisfies P1, P2, and P3.

5.3 Generalized Knowledge Operator

This section generalizes the knowledge operator. As shown by previous work (e.g., Modica and Rustichini, 1994, 1999; Dekel, Lipman, and Rustichini, 1998; Chen, Ely, and Luo, 2012), a standard knowledge operator has the following issue: under several assumptions, unawareness is trivial in the standard information structure even if an information function is not partitional, which means that the agent must be aware of all events in the structure. Thus the knowledge operator must be redefined to discuss non-trivial unawareness.

To do so, we refer to Ewerhart (2001). His model assumes that the agent is aware of all events in the subjective state space, but aware of no state outside the subjective state space. The idea of a generalized knowledge operator in this study is basically built on Ewerhart's model. However, in his model, the subjective state space may depend on each state, whereas the subjective state space in this study is fixed.

Let us denote the knowledge operator as $K : 2^{\Omega} \to 2^{\Omega}$, and let $E \subseteq \Omega$ be an event. Also let Z be the subjective state space. Then, for any $\omega \in \Omega$ and $E \subseteq \Omega$, the knowledge operator K is generalized as follows:

$$\begin{cases} \omega \in K(E) \text{ if } P(\omega) \subseteq E \text{ and } E \subseteq Z; \text{ and} \\ \omega \notin K(E) \text{ otherwise.} \end{cases}$$

This means that if $E \nsubseteq Z$, then $K(E) = \emptyset$.

This generalized knowledge operator works in the same way as the standard knowledge operator if any event is a subset of the subjective state space. However, if an event is not included in the subjective state space, then the agent knows nothing about it.

The generalized knowledge operator has the following properties.

Proposition 5.3.1. Given the information structure $\langle \Omega, P \rangle$, K satisfies the followings.

K1 Necessitation:

$Z = \Omega$ if and only if $K(\Omega) = \Omega$.

K2 Monotonicity:

$Z = \Omega$ if and only if for any $E, F \subseteq \Omega$, if $E \subseteq F$, then $K(E) \subseteq K(F)$.

K3 Conjunction:

$Z = \Omega$ if and only if for any $E, F \subseteq \Omega$, $K(E \cap F) = K(E) \cap K(F)$.

K4 Truth: (Ewerhart 2001)

If P1 holds, then $K(E) \subseteq E$.

K5 Positive Introspection: (Ewerhart 2001)

If P2 holds, then $K(E) \subseteq KK(E)$.

K6 Negative Introspection:

If P3 holds, $Z = \Omega$ if and only if $\neg K(E) \subseteq K \neg K(E)$.

Proof. K1 : $K(\Omega) \subseteq \Omega$ is obvious according to the definition of the generalized knowledge operator. First, suppose $Z = \Omega$. For any $\omega \in \Omega$, because $P(\omega) \subseteq \Omega$, $\omega \in K(\Omega)$. Hence, since $\Omega \subseteq K(\Omega)$, $K(\Omega) = \Omega$.

Next, suppose that $Z \neq \Omega$; that is, Z is a proper subset of Ω. Then, since $\Omega \nsubseteq Z$, from the definition of the generalized knowledge operator, $K(\Omega) = \emptyset$. Therefore, $K(\Omega) \neq \Omega$.

K3 : First, suppose $Z = \Omega$. Given any $\omega \in K(E \cap F)$, then $E, F \subseteq Z$, and $P(\omega) \subseteq E$ and $P(\omega) \subseteq F$ hold. Then, from the definition of the generalized knowledge operator, since $\omega \in K(E)$ and $\omega \in K(F)$, $\omega \in K(E) \cap K(F)$; that is, $K(E \cap F) \subseteq K(E) \cap K(F)$. Next, given any $\omega \in K(E) \cap K(F)$, then $E, F \subseteq Z$, and $P(\omega) \subseteq E$ and $P(\omega) \subseteq F$. That is, $P(\omega) \subseteq E \cap F$. Then, since $\omega \in K(E \cap F)$, $K(E) \cap K(F) \subseteq K(E \cap F)$. Hence, $K(E \cap F) \subseteq K(E) \cap K(F)$.

Next, suppose $Z \neq \Omega$; that is, Z is a proper subset of Ω. Let us suppose $E = Z$ and $F = \Omega$. Then, $K(E \cap \Omega) = K(E) \neq \emptyset$, whereas $K(E) \cap K(\Omega) = \emptyset$ because $K(\Omega) = \emptyset$ from the definition of the generalized knowledge operator.

K2 : First, suppose that $Z = \Omega$ given any $E, F \subseteq \Omega$ with $E \subseteq F$. Then, the following remark holds.

Remark 15. K3 implies K2.

This proof is as follows. From K3, $K(E) = K(E \cap F) = K(E) \cap K(F) \subseteq K(F)$.

Next, suppose $Z \neq \Omega$, $E = Z$ and $F = \Omega$. Then, $K(E) \neq \emptyset$ and $K(\Omega) = \emptyset$ from the definition of the generalized knowledge operator. Therefore, $K(E) \nsubseteq K(F)$.

K4 : Suppose that P1 holds. If $K(E)$ is empty, the assertion obviously holds. Pick any $\omega \in K(E)$. Then, $P(\omega) \subseteq E$ and $E \subseteq Z$ hold. From P1, $\omega \in P(\omega)$, $K(E) \subseteq E$.

K5 : Suppose that P2 holds. If $K(E)$ is empty, the assertion obviously holds. Pick any $\omega \in K(E)$. Then, $P(\omega) \subseteq E$ and $E \subseteq Z$ are satisfied. Pick any $\omega' \in P(\omega)$. Then, from P2, since $P(\omega') \subseteq P(\omega)$, $P(\omega') \subseteq E$. Therefore, $\omega' \in K(E)$; that is, $P(\omega) \subseteq K(E)$. Hence, $K(E) \subseteq KK(E)$.

K6 : First, suppose that $Z = \Omega$ and P3 hold. Suppose $\omega \in \neg K(E)$. Then, because $\omega \notin K(E)$, $P(\omega) \nsubseteq K(E)$. For any $\omega' \in P(\omega)$, from P3, since $P(\omega') \supseteq P(\omega)$, $P(\omega') \nsubseteq E$; that is, $\omega' \notin K(E)$. That is, $\omega' \in \neg K(E)$. Hence, $\neg K(E) \subseteq K\neg K(E)$.

Next, suppose that $Z \neq \Omega$ and $E = \Omega$. Then, from the definition of the generalized knowledge operator, $K(\Omega) = \emptyset$; that is, $\neg K(\Omega) = \Omega$. Therefore, $\neg K(\Omega) = \Omega \nsubseteq \emptyset = K\neg K(\Omega)$.

□

Ewerhart (2001) shows Truth, Positive Introspection, and $K(E) \cap K(F) \subseteq K(E \cap F)$.

These properties have received the following standard interpretations in the previous studies. Necessitation means that in all states, the agent knows that some state in Ω occurs. Monotonicity means that given any event and its superset, if the agent knows the event, then they must know the superset. Conjunction means that given several events, the agent knows each event if and only if they know the conjunction of such events. Truth means that if the agent knows some event, then the knowledge is true. Positive Introspection means that if the agent knows some event, then they know that they know the event. Finally, Negative Introspection means that if the agent does not know an event, then they know that they do not know the event.

The above properties are well known in previous works. However, the standard knowledge operator satisfies Necessitation, Monotonicity, and Conjunction, whereas this study's knowledge operator does not satisfy them if the agent's subjective state space is a proper subset of the objective state space. Moreover, Negative Introspection needs P3 in standard discussions, while the generalized knowledge operator requires not only P3 but also $Z = \Omega$ to satisfy Negative Introspection.

In the subjective state space, those properties hold as follows.

Remark 16.

K1' $K(Z) = Z$.

Moreover, for any $E, F \subseteq Z$,

K2' If $E \subseteq F$, then $K(E) \subseteq K(F)$; and

K3' $K(E \cap F) = K(E) \cap K(F)$.

Proof. K1' $K(Z) \subseteq Z$ is obvious. Given any $\omega \in Z$, then since $P(\omega) \subseteq Z$, $\omega \in K(Z)$; that is, $Z \subseteq K(Z)$. Hence, $K(Z) = Z$.

Given any $E, F \subseteq Z$.

K3' : First, if $\omega \in K(E \cap F)$, $P(\omega) \subseteq E \cap F \subseteq E \subseteq Z$ and $P(\omega) \subseteq E \cap F \subseteq F \subseteq Z$ hold. That is, $\omega \in K(E)$ and $\omega \in K(F)$. Therefore, $K(E \cap F) \subseteq K(E) \cap K(F)$.

Next, if $\omega \in K(E) \cap K(F)$, then, since $P(\omega) \subseteq E \subseteq Z$ and $P(\omega) \subseteq F \subseteq Z$, $P(\omega) \subseteq E \cap F \subseteq Z$. That is, $\omega \in K(E \cap F)$. Hence, $K(E \cap F) \supseteq K(E) \cap K(F)$. Therefore, $K(E \cap F) = K(E) \cap K(F)$.

K2' : This is straightforward from K3'.

□

5.4 A Generalization of the Unawareness Operator

This section provides the generalized unawareness operator $U : 2^\Omega \to 2^\Omega$ based on the generalized knowledge operator. The generalized unawareness operator is defined as

$$U(E) = \neg K(E) \cap \neg K \neg K(E),$$

as in Modica and Rustichini (1994). Moreover, the generalized awareness operator $A : 2^\Omega \to 2^\Omega$ is defined as $A(E) = \neg U(E)$. Then, a condition of trivial or non-trivial unawareness is shown below.

Lemma 5.4.1. Given the information structure $\langle \Omega, P \rangle$, suppose that P is partially partitional. Then, U satisfies the following: for any $E \subseteq \Omega$,

- Triviality: $Z = \Omega$ if and only if $A(E) = \Omega$; and
- Non-Triviality: $Z \neq \Omega$ if and only if $A(E) = K(E)$. Moreover, $E \not\subseteq Z$ if and only if $A(E) = \emptyset$.

Proof. Let us suppose that the information function is partially partitional. First, we prove Triviality. Suppose $Z = \Omega$. From P3, Negative Introspection holds. That is, $U(E) = \emptyset$ for any $E \subseteq \Omega$. Hence, $A(E) = \Omega$. In the case that $Z \neq \Omega$, since Negative Introspection does not hold, there exists some event $E \subseteq \Omega$ such that $A(E) \neq \Omega$.

We next turn to Non-Triviality. Here, first, given $E = \emptyset$, $A(\emptyset) = K(\emptyset) = \emptyset$. Second, given any non-empty set $E \subseteq Z$ such that there exists $\omega \in \Omega$ with $P(\omega) \subseteq E$, from Truth, $K(E) \subseteq E \subseteq Z$ and $K(E) \neq \emptyset$. Then, since $\neg K(E) \not\subseteq Z$, $K\neg K(E) = \emptyset$. Hence, $A(E) = K(E) \cup K\neg K(E) = K(E) \cup \emptyset = K(E)$. Finally, given $E \not\subseteq Z$, from the definition of the generalized knowledge operator, $K(E) = \emptyset$; that is, $\neg K(E) = \Omega$. Then, since $K\neg K(E) = \emptyset$, $A(E) = \emptyset$. $\square$

Triviality is obviously equivalent to Negative Introspection.

Given the generalized unawareness operator, Ewerhart (2001) weakens Monotonicity as follows.

Remark 17 (Ewerhart 2001)**.** If $E \subseteq F$, then $K(E) \cap A(F) \subseteq K(F)$.

Proof. For any $E, F \subseteq \Omega$, suppose $E \subseteq F$ given $\omega \in K(E) \cap A(F)$. Then, $\omega \in K(E)$ and $\omega \in A(F)$. First, suppose $F \subseteq \Omega$. Then, from Non-Triviality, $A(F) = K(F)$. From K2', because $K(E) \subseteq K(F)$, $K(E) \cap A(F) = K(E) \cap K(F) \subseteq K(F)$. Next, suppose $F \not\subseteq Z$. Then, from Non-Triviality, $A(F) = \emptyset$. Hence, $K(E) \cap A(F) = \emptyset \subseteq K(F)$. $\square$

The generalized awareness/unawareness operator has the following properties.

Proposition 5.4.1. Given the information structure $\langle \Omega, P \rangle$, suppose that P is partially partitional. Then, the generalized unawareness operator satisfies the following properties.

U1 KU Introspection:

$KU(E) = \emptyset$.

U2 AU Introspection:

$U(E) \subseteq UU(E)$.

U3 Weak Necessitation:

$Z = \Omega$ if and only if for any $E \subseteq \Omega$, $A(E) = K(Z)$.

U4 Strong Plausibility:

$U(E) = \bigcap_{n=1}^{\infty} (\neg K)^n(E)$.

U5 Weak Negative Introspection:

$\neg K(E) \cap A\neg K(E) = K\neg K(E)$.

U6 Symmetry:

$Z = \Omega$ if and only if for any $E \subseteq \Omega$, $U(E) = U(\neg E)$.

U7 A-Conjunction:

$Z = \Omega$ if and only if $A(\cap_\lambda E_\lambda) = \cap_\lambda A(E_\lambda)$.

U8 AK-Self-Reflection:

$AK(E) = A(E)$.

U9 AA-Self-Reflection:

$AA(E) = A(E)$.

U10 A-Introspection:

$KA(E) = A(E)$.

Proof. When $Z = \Omega$, it is obvious that all the properties hold. Suppose $Z \neq \Omega$; then, from Lemma 5.4.1, $A(E) = K(E)$; that is, $U(E) = \neg K(E)$.

U1 $KU(E) = K\neg K(E)$. When $E \subseteq Z$, from Truth, since $K(E) \subseteq Z$, $\neg K(E) \not\subseteq Z$; that is, $K\neg K(E) = \emptyset$. When $E \not\subseteq Z$, since $A(E) = K(E) = \emptyset$, $U(E) = \Omega$; that is, $K(\Omega) = \emptyset$. Hence, $KU(E) = \emptyset$ for any E.

U2 $UU(E) = \neg KU(E) \cap \neg K\neg KU(E)$. From U1, $\neg KU(E) \cap \neg K\neg KU(E) = \Omega \cap \neg K(\Omega) = \Omega$; that is, $UU(E) = \Omega$. Hence, $U(E) \subseteq UU(E)$.

U3 When $Z \neq \Omega$, from Non-Triviality, K1', K2', and Truth, given any $E \subsetneqq Z$, $A(E) = K(E) \subseteq E \subsetneqq Z = K(Z)$.

U4 First, for any $E \subseteq \Omega$, $\neg K\neg K(E) = \Omega$. Next, $\neg K\neg K\neg K(E) = \Omega$. By repeating this process, $\bigcap_{n=2}^{\infty} (\neg K)^n(E) = \Omega$. Hence, $U(E) = \neg K(E) = \neg K(E) \bigcap_{n=2}^{\infty} (\neg K)^n(E) = \bigcap_{n=1}^{\infty} (\neg K)^n(E)$.

U5 From U1, $K\neg K(E) = KU(E) = \emptyset$ and $A\neg K(E) = K\neg K(E) = \emptyset$. Hence, $\neg K(E) \cap A\neg K(E) = K\neg K(E)$.

U6 Given any non-empty subset $E \subseteq Z$, since $Z \neq \Omega$, from Non-Triviality, $A(E) = K(E) \neq \emptyset$, whereas $A(\neg E) = \emptyset$; that is, $U(E) \neq U(\neg E)$.

U7 From $Z \neq \Omega$ and Non-Triviality, $A(E) = K(E)$. Let us assume $E = E_1 \cup E_2$. Then, from K3, there might exist $\emptyset \neq E_1 \subseteq Z$ and $E_2 \not\subseteq Z$. Then, $A(E) = K(E) = K(E_1 \cap K_2) \neq \emptyset = K(E_1) \cap K(E_2) = A(E_1) \cap A(E_2)$; that is, there exist indexes such that $A(\cap_\lambda E_\lambda) \neq \cap_\lambda A(E_\lambda)$.

U8–10 Since P is partially partitional, K satisfies Truth and Positive Introspection. Hence, $A(E) = K(E) = KK(E) = KA(E) = AA(E) = AK(E)$.

$\square$

Proposition 5.4.2. Given the information structure $\langle \Omega, P \rangle$, suppose that P is partially partitional. Then, the generalized unawareness operator satisfies the following property.

U2' Reverse AU Introspection:

If $Z = \Omega$ or $E \not\subseteq Z$, then $UU(E) \subseteq U(E)$.

Moreover, if $Z \neq \Omega$ holds, then the following properties hold.

U6' Reverse Symmetry:

1. If $E \subseteq Z$, then $U(E) \subseteq U(\neg E)$.
2. If $E \not\subseteq Z$, then $U(E) \supseteq U(\neg E)$.
3. If $E \not\subseteq Z$ and $\neg E \not\subseteq Z$, then $U(E) = U(\neg E)$.

U7' Partial A-Conjunction:

For any λ, $E_\lambda \subseteq Z$, $A(\cap_\lambda E_\lambda) = \cap_\lambda A(E_\lambda)$.

Proof. Given any non-empty subset $E \subseteq \Omega$.

U2' First, suppose $Z = \Omega$; then, from Triviality, $UU(E) = \emptyset \subseteq U(E)$. Next, suppose $E \not\subseteq Z$. Then, from Non-Triviality, since $U(E) = \Omega$, $UU(E) = U(\Omega) = \Omega = U(\Omega)$; that is, $UU(E) \subseteq U(E)$.

U6' Suppose $Z \neq \Omega$.

1. If $E \subseteq Z$, then $\neg E \not\subseteq Z$. Hence, $U(\neg E) = \Omega$ from Non-Triviality. Then, $U(E) = K(E) \subseteq \Omega = U(\neg E)$.
2. If $E \not\subseteq Z$, from Non-Triviality, $U(E) = \Omega$. Hence, $U(\neg E) \subseteq \Omega = U(E)$. Here, suppose $\neg E \not\subseteq Z$; then, $U(\neg E) = \Omega = U(E)$.

U7' From Non-Triviality, since $A(E) = K(E)$, $A(\cap_\lambda E_\lambda) = K(\cap_\lambda E_\lambda)$ and $\cap_\lambda A(E_\lambda) = \cap_\lambda A(E_\lambda)$. Since for any λ, $E_\lambda \subseteq Z$, from K3', $A(\cap_\lambda E_\lambda) = K(\cap_\lambda E_\lambda) = \cap_\lambda K(E_\lambda) = \cap_\lambda A(E_\lambda)$. $\square$

KU Introspection, AU Introspection, Weak Necessitation, and Strong Plausibility are proposed by Dekel, Lipman, and Rustichini (1998); Symmetry, A-Conjunction, AK-Self-Reflection, and AA-Self-Reflection by Modica and Rustichini (1994, 1999); Weak Negative Introspection, Symmetry, A-Conjunction, AK-Self-Reflection, and AA-Self-Reflection by Halpern (2001); A-Introspection by Heifetz, Meier, and Schipper (2006); and Reverse AU Introspection by Fukuda (2021).

When the agent's subjective state space is equivalent to the objective state space, all the properties hold. However, if they are not equivalent, then KU Introspection, Strong Plausibility, Weak Negative Introspection, AK-Self-Reflection, AA-Self-Reflection, and A-Introspection hold, while Weak Necessitation and A-Conjunction do not hold. In particular, Symmetry does not hold. Let us call this property that Symmetry does not hold *Reverse Symmetry*. This property is the result in contrast to Modica and Rustichini (1994). They show that if the (standard) knowledge operator satisfies Necessitation, Monotonicity, Truth, and Positive Introspection, then Symmetry is equivalent to Negative Introspection. By contrast, in our model, Necessitation, Monotonicity, Symmetry, and Negative introspection are equivalent.

This chapter assumes that when the subjective state space is a proper subset of the objective state space, the agent is aware of the subjective state space but not its complement. This can be formulated as follows. If $Z \neq \Omega$, then $A(Z) \neq A(\Omega \backslash Z)$. That is, Reverse Symmetry is merely a formulation of the assumption.[1] Previous work proves Symmetry or assumes it, including Modica and Rustichini (1994, 1999), Halpern (2001), Heifetz, Meier, and Schipper (2006, 2008, 2013a), Li (2009), and Sadzik (2021). However, except for Modica and Rustichini (1994, 1999) and Chen, Ely, and Luo (2012), no other studies discuss Symmetry. Hence, characterizing Symmetry in the standard information structure needs to be reconsidered.[2]

5.5 Generalized Triviality Theorems

Previous work proves trivial unawareness in standard information structures, including Modica and Rustichini (1994), Dekel, Lipman, and Rustichini (1998), and Chen, Ely, and Luo (2012). This section shows the generalizations of their proposed Triviality Theorems.

Theorem 5.5.1 (Modica and Rustichini 1994)**.** Given $\langle \Omega, P \rangle$, suppose P is partially partitional. Then, Negative Introspection is equivalent to Symmetry.

Proof. This is straightforward from Propositions 5.3.1 and 5.4.1. $\square$

This theorem holds regardless of whether a subjective state space is equal to the objective state space. If a subjective state space is equal to the objective state space, then both Negative Introspection and Symmetry hold; whereas if a subjective state space is a proper subset of the objective state space, then neither Negative Introspection nor Symmetry does not hold.

Theorem 5.5.2 (Dekel, Lipman, and Rustichini 1998)**.** Given $\langle \Omega, P \rangle$, suppose P is partially partitional. Then, the following properties are equivalent.

1. $Z = \Omega$.

[1] See Theorem 5.5.1.

[2] Chapter 6 reconsiders and characterizes Symmetry.

2. Triviality:

   $U(E) = \emptyset$.

3. Unawareness Leads to Ignorance:

   For any $E, F \subseteq Z$, $U(E) \subseteq \neg K(F)$.

Proof. Assume that P is partially partitional. First, suppose property 1. Then, from Lemma 5.4.1, Triviality holds.

Next, suppose property 2. It is obvious that property 3 holds.

Finally, suppose property 3. Here, let us assume $Z \neq \Omega$. Then, from Non-Triviality, $U(E) = \neg K(E)$. Let $E = \Omega$ and $\emptyset \neq F \subseteq Z$. Then, $U(\Omega) = \Omega$, whereas from a partially partitional information function, a definition of the knowledge operator, and Truth, since $\emptyset \neq K(F) \subseteq F \subseteq Z$, $\neg K(F) \nsubseteq Z$ and $\neg K(F) \neq \Omega$. That is, there exists F such that $U(\Omega) \nsubseteq K(F)$. This is a contradiction. Hence, $Z = \Omega$; that is, property 1 holds. □

Theorem 5.5.3 (Chen, Ely, and Luo 2012)**.** Given $\langle \Omega, P \rangle$, suppose P is partially partitional. Then, $X = \Omega$ if and only if Symmetry, AU Introspection, and KU Introspection are equivalent.

Proof. First, suppose $X = \Omega$; then, from Proposition 5.4.1, Symmetry, AU Introspection, and KU Introspection hold.

Next, suppose $X \neq \Omega$; then, Proposition 5.4.1, AU Introspection, and KU Introspection hold, but Symmetry does not hold. □

In contrast to Modica and Rustichini (1994), Triviality Theorems of Dekel, Lipman, and Rustichini (1998) and Chen, Ely, and Luo (2012) depend on the subjective state space. If the subjective state space is equivalent to the objective state space, their Triviality Theorems hold and they do not hold otherwise.

5.6 Conclusion

This chapter redefines the knowledge operator and unawareness operator in standard information structures and characterizes those two operators. The knowledge operator in this study is defined in relation to an agent's subjective state space. Given any event not included in the subjective state space, we suppose that the agent cannot know that event. Then, we can show that the agent is unaware of the event, that is, non-trivial unawareness.

This chapter first characterizes the unawareness operator. Several properties are the same as in previous work. However, Weak Necessitation, Symmetry, and A-Conjunction do not hold under non-trivial unawareness. In particular, it is interesting that Symmetry does not hold. Modica and Rustichini (1994) show that if Necessitation, Monotonicity, Truth, and Positive Introspection hold, then Symmetry and Negative Introspection are equivalent. By contrast, even if the model can exclude Necessitation and Monotonicity under non-trivial unawareness, Symmetry is equivalent to Negative Introspection.

Few studies have characterized Symmetry, with its relationship with Negative Introspection primarily studied by such research as Modica and Rustichini (1994, 1999) and Chen, Ely, and Luo (2012). Thus, a series of Symmetry studies are needed. We reconsider the characterization of Symmetry in the next chapter.

Chapter 6

Relationships between AU Introspection and Symmetry

6.1 Introduction

Focusing on standard state-space models using a set-theoretical approach, this chapter discusses the relationship between AU Introspection and Symmetry for non-trivial unawareness (i.e., there is an event an agent is unaware of). As pointed out by previous studies, in standard state-space models, several assumptions lead to Triviality (i.e., an agent is aware of everything). Modica and Rustichini (1994) show the equivalence between Negative Introspection and Symmetry (Theorem 6.3.1). Dekel, Lipman, and Rustichini (1998) show that, if state-space models satisfy Necessitation, Plausibility, KU Introspection, and AU Introspection, then there is no event that some agent is unaware of. Chen, Ely, and Luo (2012) investigate the relationship between Negative Introspection and AU Introspection. They show that Negative Introspection is equivalent to AU Introspection when assuming Necessitation (Theorem 6.3.3). From their results, it is evident that AU Introspection is equivalent to Symmetry. In fact, Chen, Ely, and Luo (2012) show a generalization of Dekel, Lipman, and Rustichini (1998) and the aforementioned equivalence (Theorem 6.3.4) (below, such results are called Triviality Theorems.)

To avoid such an issue, Heifetz, Meier, and Schipper (2006) propose unawareness structure models. Their models assume that different agents perceive different disjoint subjective state spaces by defining the (generalized) state space that is a union set of disjoint state spaces as a lattice structure. Then, if some agent's subjective state space is "less expressive" than other state space, then we can say that the agent (denoted as "she" for convenience) is unaware of such state space. In other words, we can represent non-trivial unawareness.

Since Heifetz, Meier, and Schipper (2006), mainstream research has focused on unawareness structure models.

However, the abovementioned research results do not mean that we cannot discuss non-trivial unawareness in standard state-space models. Ewerhart (2001) proposes models of non-trivial unawareness by assuming that an agent is aware of her subjective state spaces to be a proper subset of the objective state space, and that the unaware agent does not know all states in the complementary set. Fukuda (2021) suggests that we can discuss non-trivial unawareness with Necessitation by excluding AU Introspection. We believe non-trivial unawareness should be reconsidered in the standard state-space models. Common to all Triviality Theorems is the assumption of Necessitation. In other words, Necessitation may lead to trivial unawareness. In fact, as pointed out by Dekel, Lipman, and Rustichini (1998), if we do not assume Necessitation, then we can discuss non-trivial unawareness.

This chapter attempts to exclude the assumption of Necessitation. Necessitation holds if and only if the agent knows the whole state space, no matter what state is given. In other words, Necessitation does not hold if and only if given some state, the agent does not know the whole state space even if she might know it given other states. This may seem an irrational perception. However, even if one knows the whole state space in advance, one may forget that knowledge when the time comes. For example, we know that a person may be infected with COVID-19 and not show any symptoms. However, in everyday life, the agent might not be aware of whether or not she is infected with COVID-19, or if infected, whether or not she has symptoms. Therefore, an asymptomatic infected person, even if infected, will go about her daily life in the same way without being aware of COVID-19, as long as she has no fever. In other words, the infected person forgets the knowledge of COVID-19 in a state in which she is asymptomatically infected. Our assumption excluding Necessitation is the basis for addressing the abovementioned cases.

Let us use Modica and Rustichini's definition of unawareness. Then, unlike Modica and Rustichini (1994), Symmetry might not be equivalent to Negative Introspection, and unlike Chen, Ely, and Luo (2012), AU Introspection might not be equivalent to Negative Introspection. However, since there are no studies on equivalence between AU Introspection and Symmetry yet, it remains to be seen whether equivalence holds even if the assumption of Necessitation is removed. This chapter aims to analyze whether the equivalence of AU Introspection and Symmetry holds when the assumption of Necessitation is relaxed. Our main result shows that if the knowledge operator satisfies Monotonicity, Truth, and Positive Introspection, then Modica and Rustichini's definition of unawareness leads to the equivalence of AU Introspection and Symmetry (Theorem 6.4.1).

We cannot directly prove the equivalence between AU Introspection and Symmetry. To do so, several properties are required, for example, KU Introspection and AA-Self Reflection of the unawareness operator. Therefore, these properties (Lemmas 6.4.1, 6.4.2, 6.4.3, and 6.4.4) must be proved before proving Theorem 6.4.1. In the proofs of these lemmas, we find that Necessitation is not required, that is, the equivalence of AU Introspection and Symmetry holds without Necessitation. Therefore, when excluding Necessitation, Negative Introspection is equivalent to neither AU Introspection nor Symmetry; however, AU Introspection and Symmetry are equivalent (Corollary 6.4.1). Our result implies that the non-triviality of unawareness consists of both AU Introspection and Symmetry, because Triviality is equivalent to Negative Introspection. However, note that Corollary 6.4.1 is a generalization of Theorem 6.3.4, but not Theorems 6.3.2 and 6.3.3. We use Modica and Rustichini's definition of unawareness and Positive Introspection, whereas Theorems 6.3.2 and 6.3.3 are based on plausible unawareness relaxing Modica and Rustichini's definition, and do not suppose Positive Introspection. Hence, our main corollary does not generalize all Triviality Theorems.

The rest of this chapter is organized as follows. The next subsection highlights related works in the literature. Section 6.2 introduces standard state-space models following the studies of Dekel, Lipman, and Rustichini (1998) and Chen, Ely, and Luo (2012) and properties of the knowledge/unawareness operator. Section 6.3 overviews the Triviality Theorems of Modica and Rustichini (1994), Dekel, Lipman, and Rustichini (1998), and Chen, Ely, and Luo (2012). Section 6.4 provides and proves our main theorem that AU Introspection is equivalent to Symmetry and generalizes a proof of Theorem 6.3.4. The last section concludes.

6.2 Preliminaries

Let us consider a standard state-space model, such as that of Dekel, Lipman, and Rustichini (1998) or Chen, Ely, and Luo (2012), $\langle \Omega, K, U \rangle$, where

- Ω is a state space. Any $E \subseteq \Omega$ is an event, and $\neg E = \Omega \setminus E$.
- $K : 2^{\Omega} \to 2^{\Omega}$ is the knowledge operator. Given any event $E \subseteq \Omega$, a set $K(E)$ is interpreted as the agent possessing K knows that event E occurs.
- $U : 2^{\Omega} \to 2^{\Omega}$ is the unawareness operator. Given any event E, a set $U(E)$ is interpreted as the agent possessing U is unaware whether event E occurs.

In a partitional state-space model, it is well known that the knowledge operator K satisfies the following properties:

K1 Necessitation: $K(\Omega) = \Omega$;

K2 Monotonicity: if $E \subseteq F$, then $K(E) \subseteq K(F)$;

K3 Truth: $K(E) \subseteq E$;

K4 Positive Introspection: $K(E) \subseteq KK(E)$; and

K5 Negative Introspection: $\neg K(E) \subseteq K\neg K(E)$.

Here, by K5, $\neg K \neg K(E) = \emptyset$ in a partitional state-space model. This means that it is impossible for an agent not to know an event, or that the agent does not know that she does not know the event. In other words, any higher-order lack of knowledge does not hold.

Previous studies on unawareness attempt to relax Negative Introspection and provide the following axioms of the unawareness operator:

U0 Modica and Rustichini's definition: $U(E) = \neg K(E) \cap \neg K \neg K(E)$;

U1 Plausibility: $U(E) \subseteq \neg K(E) \cap \neg K \neg K(E)$;

U2 KU Introspection: $KU(E) = \emptyset$;

U3 AU Introspection: $U(E) \subseteq UU(E)$; and

U4 Symmetry: $U(E) = U(\neg E)$.

U0 and U4 are proposed by Modica and Rustichini (1994) and U1–3 are provided by Dekel, Lipman, and Rustichini (1998).

Following Chen, Ely, and Luo (2012), we name and define trivial and non-trivial unawareness as follows:

U5 Triviality: $\forall E \subseteq \Omega$, $U(E) = \emptyset$; and

U6 Non-Triviality: $\exists E \subseteq \Omega$ subject to $U(E) \neq \emptyset$.

Remark 18. Under U0, K5 if and only if U5.

Finally, we define the awareness operator as $A(E) = \neg U(E)$.

6.3 Triviality Theorems

Modica and Rustichini (1994), Dekel, Lipman, and Rustichini (1998), and Chen, Ely, and Luo (2012) present the following theorems about trivial unawareness:

Theorem 6.3.1 (Modica and Rustichini 1994)**.** If $\langle \Omega, K, U \rangle$ satisfies K1–4 and U0, then K5 and U4 are equivalent.

Theorem 6.3.2 (Dekel, Lipman, and Rustichini 1998)**.** If $\langle \Omega, K, U \rangle$ satisfies K1 and U1–3, then U5 is satisfied.

Theorem 6.3.3 (Chen, Ely, and Luo 2012)**.** If $\langle \Omega, K, U \rangle$ satisfies K1—3 and U1, K5 if and only if U3

Theorem 6.3.4 (Chen, Ely, and Luo 2012)**.** If $\langle \Omega, K, U \rangle$ satisfies K1–4 and U0, K5 if and only if U3 if and only if U4.

Note that Theorems 6.3.1 and 6.3.4 use Modica and Rustichini's definition, U0, whereas, Theorems 6.3.2 and 6.3.3 use plausible unawareness, U1. Moreover, Theorems 6.3.2 and 6.3.3 do not need Positive Introspection, K4.

The following sketch provides an outline of a proof of Theorem 6.3.4 provided by Chen, Ely, and Luo (2012):

The outline of the proof of Theorem 6.3.4.

1. By Theorem 6.3.1, K5 and U4 are equivalent.
2. By Theorem 6.3.3, K5 and U3 are equivalent.
3. By 1 and 2, U3 and U4 are equivalent.

Hence, K5, U3, and U4 are equivalent. □

Theorem 6.3.4 is a generalization of Theorems 6.3.1 and 6.3.2, and we use Theorems 6.3.1 and 6.3.3 to prove Theorem 6.3.4. Theorem 6.3.1 suggests equivalence between Negative Introspection and Symmetry; Theorem 6.3.3 suggests equivalence between Negative Introspection and AU Introspection; and Theorem 6.3.4 suggests equivalence between AU Introspection and Symmetry. In proof of Theorem 6.3.4, AU Introspection and Symmetry are not directly equivalent. This proof is related to Necessitation. However, is Negative Introspection necessary to prove the equivalence between AU Introspection and Symmetry? Can we directly prove this equivalence without Negative Introspection? We explore this issue in the next section.

6.4 Main Theorem

In this section, we explore the proof of equivalence between AU Introspection and Symmetry without Negative Introspection. We show the following theorem.

Theorem 6.4.1. If $\langle \Omega, K, U \rangle$ satisfies K2–4 and U0, then U3 is equivalent to U4.

This theorem does not use Necessitation. In other words, Necessitation is not necessary for this theorem. Theorem 6.4.1 implies that AU Introspection is equivalent to Symmetry. Put differently, Negative Introspection is not necessary for this equivalence. In other words, an equivalent pair of AU Introspection and Symmetry is not equivalent to Negative Introspection even when Necessitation is not used.

Before proving this theorem, we show the following lemmas.

Lemma 6.4.1. If $\langle \Omega, K, U \rangle$ satisfies K2, then

K2* $K(E \cap F) \subseteq (K(E) \cap K(F))$.

Proof. Suppose that $\langle \Omega, K, U \rangle$ satisfies K2. It is evident that $(E \cap F) \subseteq E$ and $(E \cap F) \subseteq F$. By K2, $K(E \cap F) \subseteq K(E)$ and $K(E \cap F) \subseteq K(F)$. Hence, $K(E \cap F) \subseteq (K(E) \cap K(F))$. □

This property K2* is the relaxing Conjunction ($K(E \cap F) = K(E) \cap K(F)$), which is one of the standard properties of the knowledge operator. Theorem 6.4.1 needs K2*, not Conjunction. See proofs of Lemma 6.4.2 and 6.4.4.

As the following proof of Lemma 6.4.2 shows, K4 is not necessary.

Lemma 6.4.2. If $\langle \Omega, K, U \rangle$ satisfies K2–3 and U1, then U2 is satisfied.

Proof. Suppose that $\langle \Omega, K, U \rangle$ satisfies K2–3 and U1. Then,

$$KU(E) \overset{\text{K2,U1}}{\subseteq} K(\neg K(E) \cap \neg K \neg K(E))$$

$$\overset{\text{K2}^*}{\subseteq} K\neg K(E) \cap K \neg K \neg K(E)$$

$$\overset{\text{K3}}{\subseteq} K\neg K(E) \cap \neg K \neg K(E) = \emptyset.$$ □

Lemma 6.4.2 suggests that if a standard state-space model satisfies Monotonicity, Truth, and Plausibility, then KU Introspection is satisfied.

Lemma 6.4.3. If $\langle \Omega, K, U \rangle$ satisfies K2–3 and U0, then

U3* Reverse AU Introspection: $UU(E) \subseteq U(E)$.

Proof. Suppose that $\langle \Omega, K, U \rangle$ satisfies K2–3 and U0. By Lemma 6.4.2, U2 holds. Then, $UU(E) \overset{\text{U0}}{=} \neg KU(E) \cap \neg K \neg KU(E) \overset{\text{U2}}{=} \Omega \cap \neg K(\Omega) = \neg K(\Omega)$. Here, by K2, if $E \subseteq \Omega$, then $K(E) \subseteq K(\Omega)$, and if $\neg K(E) \subseteq \Omega$, then $K\neg K(E) \subseteq K(\Omega)$. Then, $\neg K(\Omega) \subseteq \neg K(E)$ and $\neg K(\Omega) \subseteq \neg K \neg K(E)$, that is, $UU(E) = \neg K(\Omega) \subseteq \neg K(E) \cap \neg K \neg K(E) = U(E)$. □

U3* is proposed by Fukuda (2021). By Lemma 6.4.3, the following property holds.

Remark 19. Suppose that Lemma 6.4.3. If U3 holds, then

U3** $U(E) = UU(E)$.

The following properties are shown by Modica and Rustichini (1994).[1]

Lemma 6.4.4 (Modica and Rustichini 1994)**.** If $\langle \Omega, K, U \rangle$ satisfies K2–4 and U0, then it also satisfies the following:

A1 AK-Self Reflection: $AK(E) = A(E)$;

A2 AA-Self Reflection: $AA(E) = A(E)$; and

A3 A-Introspection: $KA(E) = A(E)$.

Proof. Suppose that $\langle \Omega, K, U \rangle$ satisfies K2–4 and A1.

Proof of A1. $AK(E) = KK(E) \cup K\neg KK(E) \overset{\text{K4}}{=} K(E) \cup K \neg K(E) = A(E)$.

Proof of A3. First, given $K(E)$, by K2 and K4, because $K(E) \subseteq A(E)$, $K(E) = KK(E) \subseteq KA(E)$ $(*)$. Next, given $K\neg K(E)$, $K\neg K(E) \subseteq A(E)$

[1] Those properties are proposed in other literature. A1 and A2 are proved by Modica and Rustichini (1999) and Halpern (2001), respectively, and A3 is proved by Heifetz, Meier, and Schipper (2006).

and $K\neg K(E) \subseteq \neg K(E)$ by K3, that is, $K\neg K(E) \subseteq (\neg K(E) \cap A(E))$. Then, $K\neg K(E) \overset{\text{K4}}{=} KK\neg K(E) \overset{\text{K2}}{\subseteq} K(\neg K(E) \cap A(E))$. $K(\neg K(E) \cap A(E)) \overset{\text{K6}}{\subseteq} K\neg K(E) \cap KA(E)$. That is, $K\neg K(E) \subseteq KA(E)$. Then, $A(E) = K(E) \cup K\neg K(E) \subseteq K(E) \cup KA(E)$. Because $K(E) \subseteq KA(E)$ $(*)$, $K(E) \cup KA(E) = KA(E)$, that is, $A(E) \subseteq KA(E)$. By K3, because $KA(E) \subseteq A(E)$, $KA(E) = A(E)$.

Proof of A2. $AA(E) = KA(E) \cup K\neg KA(E) \overset{\text{A3}}{=} A(E) \cup K\neg A(E) = A(E) \cup KU(E) \overset{\text{U2}}{=} A(E) \cup \emptyset = A(E)$. □

Those properties can be proved in set-theoretical approaches as follows: In contrast to the proofs of Lemmas 6.4.1 and 6.4.2, The proof of Lemma 6.4.4 needs Positive Introspection, K4.

By the above lemmas, we can prove our main theorem.

Proof of Theorem 6.4.1. Suppose that $\langle \Omega, K, U \rangle$ satisfies K2–4 and U0.

First, assume U3; then, by Remark 19, $U(E) = UU(E)$. Next, by the definition of the awareness operator, for any $E \subseteq \Omega$, $A(E) \overset{\text{U3}^{**}}{=} AU(E) \overset{\text{U0}}{=} KU(E) \cup K\neg KU(E) \overset{\text{U2}}{=} \emptyset \cup K(\neg\emptyset) = \emptyset \cup K(\Omega) = K(\Omega)$. Because E is arbitrary, $A(E) = A(\neg E) = K(\Omega)$. Therefore, $U(E) = \neg A(E) = \neg A(\neg E) = U(\neg E)$.[2]

Next, assume U4, that is, $U(E) = U(\neg E)$. By Lemma 6.4.4, because A2, that is, $AA(E) = A(E)$, is satisfied, $UA(E) = U(E)$. By U4, $U(E) = UA(E) = UU(E)$. Hence, $U(E) \subseteq UU(E)$. □

By Theorem 6.4.1, we can generalize Theorem 6.3.4.

Proof of Theorem 6.3.4. Suppose that $\langle \Omega, K, U \rangle$ satisfies K1–4 and U0.

First, assume U4. By Theorem 6.4.1, U3 holds.

Next, assume U3. By Lemma 6.4.3 and Remark 19, $U(E) = UU(E)$ holds. Then, $U(E) \overset{\text{U3}^{**}}{=} UU(E) \overset{\text{U0}}{=} \neg KU(E) \cap \neg K\neg KU(E) \overset{\text{U2}}{=} \neg\emptyset \cap \neg K(\neg\emptyset) = \Omega \cap \neg K(\Omega) = \neg K(\Omega) \overset{\text{K1}}{=} \neg\Omega = \emptyset$. By Remark 18, K5 holds.

Finally, assume K5. By Remark 18, U5 holds, that is, $U(E) = \emptyset$ for any $E \subseteq \Omega$. Because E is arbitrary, $U(E) = U(\neg E) = \emptyset$. That is, U4 holds. □

Theorems 6.3.1, 6.3.2, and 6.3.3 are evident from Theorem 6.3.4.

Note that Theorem 6.4.1 generalizes Theorems 6.3.1 and 6.3.4, but not Theorems 6.3.2 and 6.3.3. Theorems 6.3.1 and 6.3.4 require K4, whereas Theorems 6.3.2 and 6.3.3 do not require K4.

The relationship between Theorems 6.3.4 and 6.4.1 implies the following corollary.

Corollary 6.4.1. In $\langle \Omega, K, U \rangle$, if K2–4 and U0 hold, but K1 does not hold, then K5 equivalent to neither U3 nor U4, but U3 and U4 are equivalent.

[2]If we use U1 and not U0, $AU(E) \supseteq KU(E) \cup K\neg KU(E)$. Then, Symmetry might not hold, because $A(E) = K(\Omega)$ might not hold. See Example 6.

E	$K(E)$	$\neg K(E)$	$K\neg K(E)$	$\neg K\neg K(E)$	$U(E)$	$UU(E)$	$U(\neg E)$
a	a	b,c	b	a,c	c	c	c
b	b	a,c	a	b,c	c	c	c
c	$\emptyset$	a,b,c	a,b	c	c	c	c
a,b	a,b	c	$\emptyset$	a,b,c	c	c	c
a,c	a	b,c	b	a,c	c	c	c
b,c	b	a,c	a	b,c	c	c	c
a,b,c	a,b	c	$\emptyset$	a,b,c	c	c	c
$\emptyset$	$\emptyset$	a,b,c	a,b	c	c	c	c

Table 6.1: Example 6

Example 6. Let us consider a state space $\Omega = \{a, b, c\}$. Suppose that the knowledge operator K satisfies the following: $K(\Omega) = \{a, b\}$, $K(\{a\}) = \{a\}$, $K(\{b\}) = \{b\}$, $K(\{c\}) = \emptyset$, $K(\{a, b\}) = \{a, b\}$, $K(\{a, c\}) = \{a\}$, $K(\{b, c\}) = \{b\}$, and $K(\emptyset) = \emptyset$. In this example, the knowledge operator satisfies Monotonicity (K2), Truth (K3), and Positive Introspection (K4), but only Necessitation (K1) does not hold. Based on this formulation, $\neg K(E)$, $K\neg K(E)$, $\neg K\neg K(E)$, and the unawareness operator U based on Modica and Rustichini's definition can be described as in Table 6.1. It is clear that Negative Introspection (K5) (or Triviality, U5) does not hold, but AU Introspection (U3) and Symmetry (U4) hold.

Here, let us use plausible unawareness $U^* : 2^\Omega \to 2^\Omega$, that is, U^* satisfies U1 but not U0, and suppose $U^*(\{a\}) = \emptyset$, $U^*(\{b, c\}) = \{c\}$, $U^*U^*(\{a\}) = \{c\}$, and $U^*U^*(\{b, c\}) = \{c\}$. Then, U4 holds, but U3 does not hold. In other words, plausible unawareness might not lead equivalence between AU Introspection and Symmetry. $\square$

6.5 Concluding Remarks

This chapter (i) shows that AU Introspection and Symmetry for unawareness are equivalent when relaxing Necessitation; and (ii) generalizes a proof of Theorem 6.3.4 proposed by Chen, Ely, and Luo (2012).

This study excludes Necessitation. Given $\langle \Omega, K, U \rangle$, let $\omega \in \Omega$ be a state. Then, Necessitation is redefined as follows: for any state $\omega \in \Omega$, $\omega \in K(\Omega)$. This means that in any state, the agent always knows the whole state space. When we relax Necessitation, there exists some state ω such that $\omega \notin K(\Omega)$; that is, in some state, the agent does not know the whole state space even if in other state $\omega' \in \Omega$, $\omega' \in K(\Omega)$. In other words, by excluding Necessitation, depending on the given state, the agent might or might not know it. This may seem contradictory. However, depending on the specific situation, it can be said that there is no contradiction. Let us consider the example of COVID-19 as follows. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. ω_1 is interpreted as "The agent is infected with COVID-19 and gets a fever," ω_2 is interpreted as "The agent gets a fever, but is not infected with COVID-19," ω_3 is interpreted as "The agent is infected with COVID-19 but does not get a fever," and ω_4 is interpreted as "The agent is not infected with COVID-19." Here, let $\omega_1 \in K(\Omega)$, $\omega_2 \in K(\Omega)$, $\omega_3 \notin K(\Omega)$, and $\omega_4 \notin K(\Omega)$. In other words, at ω_1 and ω_2, the agent knows the whole state space about her COVID-19 status, whereas she does not know it about it at ω_3 and ω_4. This formulation can be interpreted as follows. When an agent infected with COVID-19 develops a fever, she suspects that she is infected with COVID-19. However, if an agent is infected with COVID-19 but does not have a fever, she does not realize that she is infected with COVID-19 because it is the same condition as not being infected with COVID-19. Moreover, she might have forgotten what she even knew about COVID-19. This can be rephrased as follows. The agent is not usually "aware" of COVID-19 and forgets about its existence when she does not have a fever, but when the agent does have a fever, she recalls her knowledge of COVID-19 and considers the possibility of infection. In other words, by relaxing Necessitation, a more realistic situation can be modeled.

The content of this chapter is directly related to the content of Chapter 5. Chapter 5 showed that the unawareness operator based on the generalized knowledge operator does not lead to Symmetry if the subjective state space is not equivalent to the objective state space. In light of the result presented in Chapter 5, we must reconsider the characterization of Symmetry in the standard information structure. This chapter is one perspective of such a reconsideration.

This study has one limitation. We exclude only Necessitation, because our focus is on axioms of the knowledge operator. However, as well known, in standard information structures that may be non-partitional, the knowledge operator based on the standard information function or the standard possibility correspondence cannot exclude only Necessitation. In other words, equivalence of AU Introspection and Symmetry must be equivalent to "trivial" unawareness in standard information structures. Future work could aim to define a novel knowledge operator that excludes only Necessitation in standard information structures. Chapter 7 defines such a knowledge operator.

Recent studies on the present one include that by Fukuda (2021). Fukuda (2021) proposes generalized state-space models that nest both unawareness structures and non-partitional state-space models. He posits that AU Introspection is not consistent with Necessitation, relaxes AU Introspection, and replaces AU Introspection with Reverse AU Introspection.

Chapter 7

Unaware Non-Decision Makers

7.1 Introduction

In economics, decision theory or game theory is used to analyze an agent's decision making (i.e., economics focuses on decision makers). However, some of the economic problems that must be addressed include entities excluded from decision making such as externalities. There are also people who do not realize they are in the decision-making arena such as the informationally vulnerable. Since such agents are not considered to be decision-makers, their information structure has lacked suitable discussion. To bridge this gap, this chapter discusses the information structures of non-decision makers.

First, this thesis focuses on empty information sets. Standard discussions about information structures assume that all information sets are non-empty. This means that if any state occurs, then the agent must obtain some information set. By contrast, non-decision makers find it impossible to recognize any relevant information. For example, some people are infected with COVID-19, but might be asymptomatic. Then, they cannot recognize that they are infected with COVID-19. Moreover, they cannot actually distinguish whether they are infected with COVID-19.

We can examine such cases by relaxing some of the assumptions of the knowledge operator in standard information structures. Specifically, for some state, the information set is empty. However, the standard knowledge operator leads us to know the whole state space —even if information set is empty. This seems to be unrealistic. For example, asymptomatic infected people may not be aware of the whole state space, until some external factor forces them to think about it. Hence, the standard knowledge operator does not apply to their state of knowledge in such cases.

To address this issue, this chapter redefines the knowledge operator. In particular, it makes possible the interpretation that if an information set is empty, then the agent cannot know anything. This study's knowledge operator can therefore represent the case of COVID-19 or the information structures of general non-decision makers. Moreover, these knowledge operators have interesting features that distinguish them from those of previous studies. As is well known, the standard knowledge operator must lead to Necessitation and Monotonicity; that is, both properties are equivalent. By contrast, these two properties might not be equivalent in this chapter's knowledge operator. If an information set is empty, then Monotonicity holds, but Necessitation does not.[1]

Moreover, the knowledge operator and the unawareness operator in this study have two implications. The first relates to the relationships with Triviality. Dekel, Lipman, and Rustichini (1998) show that if the unawareness operator satisfies Plausibility, KU Introspection, and AU Introspection, then the following properties hold.

1. Triviality:

   If the knowledge operator satisfies Necessitation, then the agent must be aware of any event.

2. Unawareness Leads to Ignorance:

   Suppose that the knowledge operator satisfies Monotonicity. If there is some event of which the agent is unaware, then the agent does not know any event.

As described above, since the standard knowledge operator must satisfy Necessitation and Monotonicity, the above two properties must hold. By contrast, our knowledge operator satisfies Monotonicity and Unawareness Leads to Ignorance holds; while Triviality might not hold because Necessitation might not hold.

The other implication concerns the relationship with the equivalence of Symmetry and AU Introspection. Chen, Ely, and Luo (2012) refer to Modica and Rustichini (1994, 1999), showing that if Necessitation, Monotonicity, Truth, and Positive Introspection hold and the unawareness operator is defined by second-order ignorance, then Symmetry, AU Introspection, and Negative Introspection are equivalent. By contrast, Chapter 6 in this thesis shows the equivalence between Symmetry and AU Introspection even if Necessitation does not hold. Hence, Chapter 6 is based on this chapter's model in that the information functions assumed in this chapter lead to knowledge operator having such properties.

The remainder of this chapter is organized as follows. Section 7.2 provides the preliminaries and Section 7.3 redefines the non-decision maker's knowledge operator. Section 7.4 characterizes the unawareness operator. The final section concludes.

[1] Note that this chapter's knowledge operator is different to it in Chapter 5. When we use the knowledge operator in Chapter 5, an agent cannot know the complement set of her or his subjective state space. By contrast, when we use it in this chapter, she or he can know the whole state space if she or he obtains a nonempty information set.

7.2 Preliminaries

7.2.1 Information Structure

First, the information structure $\langle \Omega, P \rangle$ is defined. Let Ω be the objective state space, let a state be $\omega \in \Omega$, and let $P : \Omega \to 2^{\Omega}$ be the information function. Here, the agent's subjective state space $Z \subseteq \Omega$ is defined as

$$Z = \bigcup_{\omega \in \Omega} P(\omega).$$

The standard information structure usually makes the following assumptions.

P0 For any $\omega \in \Omega$, $P(\omega) \neq \emptyset$.

P1 For any $\omega \in Z$, $\omega \in P(\omega)$.

P2 For any $\omega, \omega' \in \Omega$, if $\omega' \in P(\omega)$, then $P(\omega') \subseteq P(\omega)$.

P3 For any $\omega, \omega' \in \Omega$, if $\omega' \in P(\omega)$, then $P(\omega') \supseteq P(\omega)$.

Remark 20. Suppose that the information function P satisfies P1. Then, P0 and $Z = \Omega$ if and only if

P1* For any $\omega \in \Omega$, $\omega \in P(\omega)$.

P is partitional if and only if P satisfies P1*, P2, and P3.

7.2.2 Standard Knowledge Operator

Next, the standard knowledge operator $K^* : 2^{\Omega} \to 2^{\Omega}$ is defined. Let $E \subseteq \Omega$ be an event. For any $\omega \in \Omega$ and $E \subseteq \Omega$, we define the standard knowledge operator K^* as follows:

$$\begin{cases} \omega \in K^*(E) \text{ if } P(\omega) \subseteq E; \text{ and} \\ \omega \notin K^*(E) \text{ otherwise.} \end{cases}$$

As is well known, the standard knowledge operator has the following properties.

Remark 21. Given the information structure $\langle \Omega, P \rangle$, K^* satisfies the following.

K1* Necessitation:

$K^*(\Omega) = \Omega$.

K2* Monotonicity:

$E \subseteq F \Longrightarrow K^*(E) \subseteq K^*(F)$.

K3* Conjunction:

$K^*(E \cap F) = K^*(E) \cap K^*(F)$.

K4* Truth:
If P1* holds, then $K^*(E) \subseteq E$.

K5* Positive Introspection:
If P2 holds, then $K^*(E) \subseteq K^*K^*(E)$.

K6* Negative Introspection:
If P3 holds, then $\neg K^*(E) \subseteq K^*\neg K^*(E)$.

In the information structure, by the definition of the standard knowledge operator, Necessitation holds even if P0 does not hold. However, it seems that the information set being empty means that the agent cannot obtain any information set. This can be interpreted as "the agent does not have any information." Nonetheless, it is strange that Necessitation holds because Necessitation means that given any state, the agent must know the whole state space —even if the information set is empty.

The next section defines the knowledge operator when if P0 does not hold, then Necessitation does not hold.

7.3 Knowledge Operator of Non-Decision Makers

Before defining the knowledge operator without P0, we provide additional assumptions of the information function that are relaxed from P1.

P4 For any $\omega \in \Omega$, if $P(\omega) \neq \emptyset$, then $\omega \in P(\omega)$.

P5 For any $\omega \in \Omega$, if there exists $\omega' \in \Omega$ such that $\omega \in P(\omega')$, then $\omega \in P(\omega)$.

P4 means that given any state, if that state leads to a non-empty information set, then the information set has the state. P5 means that given any state, if another state leads to a non-empty information set possessing the given state, then the given state leads to some information set possessing the given state.

Let us provide the novel knowledge operator $K : 2^\Omega \to 2^\Omega$ such that if P0 does not hold, then Necessitation does not hold. For any state $\omega \in \Omega$ and any event $E \subseteq \Omega$, K is defined as follows:

$$\begin{cases} \omega \in K(E) \text{ if } P(\omega) \subseteq E \text{ and } P(\omega) \neq \emptyset; \text{ and} \\ \omega \notin K(E) \text{ otherwise.} \end{cases}$$

Given any state and event, if the non-empty information set that the state leads to is a subset of the event, then the agent knows the event. By contrast, if an event does not provide an information set that is a subset of an event or the given state leads to the information set being empty, then the agent does not know the event. Then, this study's knowledge operator has the following properties.

Proposition 7.3.1. Given the information structure $\langle \Omega, P \rangle$, K satisfies the following.

K1 Necessitation:

P0 holds if and only if $K(\Omega) = \Omega$.

K2 Monotonicity:

$E \subseteq F \Longrightarrow K(E) \subseteq K(F)$.

K3 Conjunction:

$K(E \cap F) = K(E) \cap K(F)$.

K4 Truth:

If P4 holds, then $K(E) \subseteq E$.

K5 Positive Introspection:

If P2 and P5 hold, then $K(E) \subseteq KK(E)$.

K6 Negative Introspection:

P0 and P3 hold if and only if $\neg K(E) \subseteq K \neg K(E)$.

Proof.

K1 First, assume P0. $K(\Omega) \subseteq \Omega$ is obvious. Given any $\omega \in \Omega$. From P0, $P(\omega) \neq \emptyset$. Since $P(\omega) \subseteq \Omega$ is obvious, $\omega \in K(\Omega)$; that is, $K(\Omega) = \Omega$.

Next, assume that P0 does not hold; that is, there exists $\omega \in \Omega$ such that $P(\omega) = \emptyset$. Then, $\omega \notin K(\Omega)$ is obvious; that is, $\omega \in \neg K(\Omega)$. Then, from the definition of the knowledge operator K, $K(\Omega) \neq \Omega$.

K3 First, if $\omega \in K(E \cap F)$, $P(\omega) \subseteq E \cap F$ and $P(\omega) \neq \emptyset$. That is, $P(\omega) \subseteq E$ and $P(\omega) \subseteq F$. Hence, because $\omega \in K(E)$ and $\omega \in K(F)$, $\omega \in K(E) \cap K(F)$ and $K(E \cap F) \subseteq K(E) \cap K(F)$.

Next, given any $\omega \in K(E) \cap K(F)$, since $P(\omega) \subseteq E$, $P(\omega) \subseteq F$, and $P(\omega) \neq \emptyset$, $P(\omega) \subseteq E \cap F$ and $P(\omega) \neq \emptyset$; that is, $\omega \in K(E \cap F)$. Hence, $K(E \cap F) \supseteq K(E) \cap K(F)$. Therefore, $K(E \cap F) = K(E) \cap K(F)$.

K2 Pick any $E, F \subseteq \Omega$ with $E \subseteq F$. Then, from K3, $K(E) = K(E \cap F) = K(E) \cap K(F) \subseteq K(F)$.

K4 Suppose that the information function P satisfies P4. Given any $\omega \in K(E)$, $P(\omega) \subseteq E$ and $P(\omega) \neq \emptyset$. From P4, since $\omega \in P(\omega)$, $\omega \in E$. That is, $K(E) \subseteq E$.

K5 Suppose that P satisfies P2 and P5 and $\omega \in K(E)$. Then, $P(\omega) \subseteq E$ and $P(\omega) \neq \emptyset$. Here, given any $\omega' \in P(\omega)$, from P2, $P(\omega') \subseteq P(\omega)$; moreover, from P5, $\omega' \in P(\omega')$. Because $P(\omega') \subseteq E$, $\omega' \in E$, $\omega' \in K(E)$. Therefore, since $P(\omega) \subseteq K(E)$, then $\omega \in KK(E)$. Hence, $K(E) \subseteq KK(E)$.

K6 First, suppose P0 and P3. Given any $\omega \in \neg K(E)$, $\omega \notin K(E)$. Then, $P(\omega) \not\subseteq E$. Here, given any $\omega' \in P(\omega)$, from P3, since $P(\omega') \supseteq P(\omega)$, $P(\omega') \not\subseteq E$. From P0, since $P(\omega') \neq \emptyset$, $\omega' \notin K(E)$; that is, $\omega' \in \neg K(E)$. Then, since $P(\omega) \subseteq \neg K(E)$, $\omega \in K\neg K(E)$. Hence, $\neg K(E) \subseteq K\neg K(E)$.

Next, assume that P0 does not hold. That is, there exists $\omega \in \Omega$ such that $P(\omega) = \emptyset$. Then, for any event $E \subseteq \Omega$, $\omega \notin K(E)$; that is, $\omega \in \neg K(E)$. Here, since $\neg K(E)$ is an event, $\omega \notin K\neg K(E)$. Hence, $\neg K(E) \not\subseteq K\neg K(E)$.

□

Proposition 7.3.1 differs from Remark 21. The Necessitation and Negative Introspection of the standard knowledge operator do not need P0, while those of this study's novel knowledge operator need P0. Moreover, Truth does not need P1 but only needs P4. Furthermore, Positive Introspection requires not only P2 but also P5. In both cases, P1 is not required.

Let us suppose P4 and P5; Then, the following remark holds.

Remark 22. Suppose that P4 and P5 hold. Then, $K(Z) = K(\Omega) = Z$.

Proof. First, from Monotonicity, since $Z \subseteq \Omega$, $K(Z) \subseteq K(\Omega)$.

Next, given any $\omega \in K(\Omega)$, $P(\omega) \subseteq \Omega$ and $P(\omega) \neq \emptyset$. From P4 and the definition of Z, since $\omega \in P(\Omega)$, $\omega \in Z$. Therefore, $K(\Omega) \subseteq Z$.

Finally, given any $\omega \in Z$, there exists $\omega' \in \Omega$ such that $\omega \in P(\omega')$. Then, from P5, $\omega \in P(\omega)$. From the definition of Z, since $P(\omega) \subseteq Z$, $\omega \in K(Z)$. That is, $Z \subseteq K(Z)$.

Hence, $K(Z) = K(\Omega) = Z$. □

This remark shows that, under P4 and P5, at any state in the subjective state space, the agent knows the subjective state space as well as the objective state space ($Z \subseteq K(Z) = K(\Omega)$). Furthermore, the knowledge is true in the subjective state space ($K(Z) = K(\Omega) \subseteq Z$). This is interpreted as at any state outside the subjective state space, the agent cannot know all events.

Example 7. An example of COVID-19 can be formulated as follows. Given $\Omega = \{\omega_1, \omega_2, \omega_3\}$, we interpret ω_1 as "Alice is infected with COVID-19 and presents with symptoms," ω_2 as "Alice is infected with COVID-19, but does not present with symptoms," and ω_3 as "Alice is not infected with COVID-19." Let $E_1 = \{\omega_1\}$ and $E_2 = \{\omega_2, \omega_3\}$. Here, first, $P_A(\omega_1) = E_1$, whereas $P_A(\omega_2) = P_A(\omega_3) = \emptyset$. Next, let K_A^* be her standard knowledge operator, while K_A is her knowledge operator of our version. K_A^* satisfies the followings. $\omega_1 \in K_A^*(E_1)$ and $\omega_1 \in K_A^*(\Omega)$, that is, given ω_1, she knows that she is infected with COVID-19 and presents with symptoms; and $\omega_2, \omega_3 \in K_A^*(E_1)$, $\omega_2, \omega_3 \in K_A^*(E_2)$, and $\omega_2, \omega_3 \in K_A^*(\Omega)$; that is, given ω_2 or ω_3, she knows any event. By contrast, given our knowledge operator K_A, $\omega_1 \in K_A(E_1)$; that is, given ω_1, she knows that she is infected with COVID-19, whereas $\omega_2, \omega_3 \notin K_A(E_1)$, $\omega_2, \omega_3 \notin K_A(E_2)$, and $\omega_2, \omega_3 \notin K_A(\Omega)$, that is, at ω_2 and ω_3, she is not only ignorant of E_1 but also ignorant of E_2 and Ω. Then, $K_A(E_1) = K_A(\Omega) = E_1$ and $K_A(E_2) = \emptyset$. □

7.4 Relationship with the Unawareness Operator

This section discusses properties of the unawareness operator based on this study's knowledge operator. Let $U : 2^\Omega \to 2^\Omega$ be the unawareness operator, which is defined as follows:

$$U(E) = \neg K(E) \cap \neg K \neg K(E),$$

as in Modica and Rustichini (1994). Then, the awareness operator $A : 2^\Omega \to 2^\Omega$ is defined as $A(E) = \neg U(E)$.

Here, let us suppose P2, P4, and P5; then, from Proposition 7.3.1, Monotonicity, Conjunction, Truth, and Positive Introspection hold. Hence, the following properties hold.

Proposition 7.4.1. Given $\langle \Omega, P \rangle$, if P satisfies P2, P4, and P5, then U satisfies the following properties. Given any event $E \subseteq \Omega$:

U1 KU Introspection:
$KU(E) = \emptyset$.

U2* Reverse AU Introspection:
$U(E) \supseteq UU(E)$.

U8 AK-Self-Reflection:
$AK(E) = A(E)$.

U9 AA-Self-Reflection:
$AA(E) = A(E)$.

U10 A-Introspection:
$KA(E) = A(E)$.

Proof.

U1 From Conjunction, $KU(E) = K(\neg K(E) \cap \neg K \neg K(E)) = K \neg K(E) \cap K \neg K \neg K(E)$. From Truth, since $K \neg K \neg K(E) \subseteq \neg K \neg K(E)$, $K \neg K(E) \cap K \neg K \neg K(E) \subseteq K \neg K(E) \cap \neg K \neg K(E) = \emptyset$.

U2* From KU Introspection, $UU(E) = \neg KU(E) \cap \neg K \neg KU(E) = \neg K(\Omega)$. Here, from Monotonicity, for any $E \subseteq \Omega$, $K(E) \subseteq K(\Omega)$; that is, $\neg K(\Omega) \subseteq \neg K(E)$. Moreover, since $\neg K(E) \subseteq \Omega$, from Monotonicity, $K \neg K(E) \subseteq K(\Omega)$; that is, $\neg K(\Omega) \subseteq \neg K \neg K(E)$. Hence, $UU(E) = \neg K(\Omega) \subseteq \neg K(E) \cap \neg K \neg K(E) = U(E)$.

U8 From Positive Introspection, $AK(E) = KK(E) \cup K \neg KK(E) = K(E) \cup K \neg K(E) = A(E)$.

U10 Since $K(E) \subseteq A(E)$, from Monotonicity and Positive Introspection, $K(E) = KK(E) \subseteq KA(E)$. Moreover, $K\neg K(E) \subseteq A(E)$ and from Truth, $K\neg K(E) \subseteq \neg K(E)$; that is, $K\neg K(E) \subseteq A(E) \cap \neg K(E)$. Then, from Monotonicity and Positive Introspection, $K\neg K(E) = KK\neg K(E) \subseteq K(A(E) \cap \neg K(E))$. From Conjunction, $K(A(E) \cap \neg K(E)) = KA(E) \cap K\neg K(E)$; that is, $K\neg K(E) \subseteq KA(E)$. Then, $A(E) = K(E) \cup K\neg K(E) \subseteq K(E) \cup KA(E)$. Because $K(E) \subseteq KA(E)$, $K(E) \cup KA(E) = KA(E)$; that is, $A(E) \subseteq KA(E)$. From Truth, since $KA(E) \subseteq A(E)$, $KA(E) = A(E)$.

U9 From KU Introspection and A-Introspection, $AA(E) = KA(E) \cup K\neg KA(E) = A(E) \cup K\neg A(E) = A(E) \cup KU(E) = A(E)$.

□

U1 is provided by Dekel, Lipman, and Rustichini (1998); U2* by Fukuda (2021); U8 and U9 by Modica and Rustichini (1994, 1999) and Halpern (2001); and U10 by Heifetz, Meier, and Schipper (2006).

The following equivalence was proven in Chapter 6.

Lemma 7.4.1. Given $\langle \Omega, P \rangle$, suppose P2, P4, and P5. Then, the following properties are equivalent.

U2 AU Introspection:

$U(E) \subseteq UU(E)$.

U6 Symmetry:

$U(E) = U(\neg E)$.

Proof. See Chapter 6. □

Dekel, Lipman, and Rustichini (1998) provide U2 and Modica and Rustichini (1994) provide U6. Chen, Ely, and Luo (2012) were the first to prove the equivalence between AU Introspection and Symmetry. According to their results, each property is equivalent to Negative Introspection. In contrast to their work, Chapter 6 showed their equivalence under non-trivial unawareness. In other words, neither AU Introspection nor Symmetry is equivalent to Negative Introspection, although they are equivalent to each other.

Example 7 (Continued)**.** Let U_A be Alice's unawareness operator. Then, by the definition of the unawareness operator, her knowledge and unawareness can be depicted in Table 7.1. As shown in the table, her unawareness operator satisfies both AU Introspection and Symmetry. □

Finally, we suppose AU Introspection (or Symmetry) and show the following properties of unawareness.[2]

[2]No examples have been found where AU Introspection or Symmetry do not hold. Whether AU Introspection and Symmetry are always valid under P2, P4 and P5 is an interesting question. We relegate it to future research.

E	$K(E)$	$\neg K(E)$	$K\neg K(E)$	$\neg K\neg K(E)$	$U(E)$	$UU(E)$	$U(\neg E)$
ω_1 (E_1)	ω_1	ω_2, ω_3	$\emptyset$	Ω	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3
ω_2	$\emptyset$	Ω	ω_1	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3
ω_3	$\emptyset$	Ω	ω_1	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3
ω_1, ω_2	ω_1	ω_2, ω_3	$\emptyset$	Ω	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3
ω_1, ω_3	ω_1	ω_2, ω_3	$\emptyset$	Ω	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3
ω_2, ω_3 (E_2)	$\emptyset$	Ω	ω_1	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3
Ω	ω_1	ω_2, ω_3	$\emptyset$	Ω	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3
$\emptyset$	$\emptyset$	Ω	ω_1	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3	ω_2, ω_3

Table 7.1: Example 7

Proposition 7.4.2. Given $\langle \Omega, P \rangle$, suppose P2, P4, and P5. Here, let us assume AU Introspection. Then, given any event $E, F \subseteq \Omega$, the following properties hold.

Ua Triviality:

$\quad$ P0 if and only if $U(E) = \emptyset$.

Ub Unawareness Leads to Ignorance:

$\quad$ $U(E) \subseteq \neg K(F)$.

U3 Weak Necessitation:

$\quad$ $A(E) = K(Z)$.

U4 Strong Plausibility:

$\quad$ $U(E) = \bigcap_{n=1}^{\infty} (\neg K)^n (E)$.

U5 Weak Negative Introspection:

$\quad$ $\neg K(E) \cap A\neg K(E) = K\neg K(E)$.

U7 A-Conjunction:

$\quad$ $\cap_\lambda A(E_\lambda) = A(\cap_\lambda E_\lambda)$.

Proof. Suppose P2, P4, P5, and AU Introspection.

Ua First, suppose that P0 holds. From Proposition 7.4.1, P0 holds if and only if Necessitation holds; that is $K(\Omega) = \Omega$. Here, from AU introspection and Proposition 7.4.1, $U(E) = UU(E) = \neg KU(E) \cap \neg K \neg KU(E) = \neg\emptyset \cap \neg K(\neg\emptyset) = \Omega \cap \neg K(\Omega) = \neg K(\Omega)$. From Necessitation, since $\neg K(\Omega) = \neg\Omega = \emptyset$, $U(E) = \emptyset$.

Next, suppose that P0 does not hold; that is, $K(\Omega) \neq \Omega$. Then, $U(E) = \neg K(\Omega) \neq \emptyset$.

Ub From Ua, $U(E) = \neg K(\Omega)$. For any $F \subseteq \Omega$, from Monotonicity, $K(F) \subseteq K(\Omega)$; that is, $\neg K(\Omega) \subseteq K(F)$. Because $\neg K(\Omega) = U(E)$, $U(E) \subseteq K(F)$.

U3 From Ua, $U(E) = \neg K(\Omega)$; that is, $A(E) = K(\Omega)$. From Remark 22, since $K(\Omega) = K(Z)$, $A(E) = K(Z)$.

U4 From Proposition 7.4.1 and AU Introspection, $U(E) = \bigcap_{n=1}^{\infty}(U)^n(E) = \bigcap_{n=1}^{\infty}(\neg K)^{2n-1}(\neg K)^{2n}(E) = \bigcap_{n=1}^{\infty}(\neg K)^n(E)$.

U5 First, from Symmetry and AK-Self-Reflection, $A\neg K(E) = AK(E) = A(E)$. Hence, $\neg K(E) \cap A\neg K(E) = \neg K(E) \cap A(E) = \neg K(E) \cap (K(E) \cup K\neg K(E)) = (\neg K(E) \cap K(E)) \cup (\neg K(E) \cap K\neg K(E)) = \neg K(E) \cap K\neg K(E) \subseteq K\neg K(E)$; that is, $\neg K(E) \cap A\neg K(E) \subseteq K\neg K(E)$.

Here, from Monotonicity and Remark 22, since $\neg K(E) \subseteq \Omega$ is obvious, $K\neg K(E) \subseteq K(\Omega) = Z$. From Weak Necessitation, since $A\neg K(E) = K(Z) = Z$, $K\neg K(E) \subseteq A\neg K(E)$. Moreover, from Truth, $K\neg K(E) \subseteq \neg K(E)$; that is, $K\neg K(E) \subseteq \neg K(E) \cap A\neg K(E)$. Therefore, $\neg K(E) \cap A\neg K(E) = K\neg K(E)$.

U7 From Weak Necessitation, $A(\cap_\lambda E_\lambda) = K(Z) = \cap_\lambda K(Z) = \cap A(E_\lambda)$.

□

Ua, Ub, U3, and U4 are provided by Dekel, Lipman, and Rustichini (1998); U5 by Halpern (2001); and U7 by Modica and Rustichini (1999) and Halpern (2001). Unawareness Leads to Ignorance, Weak Necessitation, Strong Plausibility, Weak Negative Introspection, and A-Conjunction need only AU Introspection, whereas Triviality needs not only AU Introspection but also P0. Hence, without P0, we can discuss non-trivial unawareness using our knowledge operator.

Let us turn to Unawareness Leads to Ignorance. In our model under the assumption of AU Introspection, Unawareness Leads to Ignorance always holds.[3] Some may think that Unawareness Leads to Ignorance is an awful property, because unawareness of some event makes any knowledge impossible. However, such supposition may be due to the all-encompassing "universal" interpretation of the whole state space Ω.[4] When we turn to an example that permits "small world" interpretation of Ω, Unawareness Leads to Ignorance may have some realistic meanings.

Example 8. Bob, who is unemployed, is eligible to apply for unemployment insurance. If he applies, he can receive insurance benefits at any time. However, he is informationally vulnerable and has no access to information about the unemployment insurance system. Therefore, he has no knowledge about this system. In other words, he does not have any information necessary to apply for unemployment insurance, including whether he is eligible to apply, how to apply, or any other information about receiving unemployment insurance. Additionally, because he is unaware of this information, he has no knowledge

[3]In Example 7, the property holds. See Table 7.1.

[4]Dekel, Lipman, and Rustichini's (1998) view of Unawareness Leads to Ignorance might be because of this. They regard it as "trivial."

of the unemployment insurance system. This is an example where Unawareness Leads to Ignorance is meaningful.

Let us mathematically consider this situation. Let $\Omega = \{\omega_1, \omega_2\}$. We interpret ω_1 as "Bob can receive unemployment insurance" and ω_2 as "Bob cannot receive unemployment insurance." Each event is interpreted as follows.

$E_1 = \{\omega_1\}$: Bob can receive unemployment insurance.
$E_2 = \{\omega_2\}$: Bob cannot receive unemployment insurance.
$\Omega = \{\omega_1, \omega_2\}$: Bob's country has an unemployment insurance program.

Here, first, let P_B be his information function and $P_B(\omega_1) = P_B(\omega_2) = \emptyset$. Next, we have the knowledge operator of a non-decision maker K_B, then $\omega_1, \omega_2 \notin K_B(\emptyset) = K_B(E_1) = K_B(E_2) = K_B(\Omega) = \emptyset$, that is, $\omega_1, \omega_2 \in \neg K_B(\emptyset) = \neg K_B(E_1) = \neg K_B(E_2) = \neg K_B(\Omega) = \Omega$. That is, given any state, he cannot know any event.

Let us define his unawareness operator U_B. Then, his unawareness operator satisfies $\omega_1, \omega_2 \in U_B(\emptyset) = U_B(E_1) = U_B(E_2) = U_B(\Omega) = \Omega$. That is, for any event $E, F \subseteq \Omega$, $U_B(E) \subseteq \neg K_B(F)$. $\square$

7.5 Concluding Remarks

This chapter provides the non-decision maker's knowledge operator. It is built on the interpretation that an agent receiving an empty information set cannot know any event. Moreover, this chapter characterizes the properties of unawareness based on the knowledge operator. In our models, Triviality holds if and only if any information set is non-empty, whereas Unawareness Leads to Ignorance always holds. Based on the "small world" interpretation of the whole state space, this thesis provides an interpretation of the property that might be "non-trivial." This interpretation may lead to future applied research on unawareness. However, one issue remains to be addressed. This study's model assumes a single agent (i.e., interactive situations are not considered). Hence, future research could aim to address this issue.

Part IV

Conclusion Part

Chapter 8

Concluding Remarks

8.1 Conclusion

This thesis explores two topics. First, PART II discussed the discovery process in simultaneous-move games with unawareness. As noted in Chapters 1 and 2, players may revise their subjective games when they observe actions of their opponents that they were unaware of prior to the game. Schipper (2021) proposed discovery processes as such model updates. He demonstrated that any rationalizable discovery process converges to some extensive-form game with unawareness possessing a rationalizable self-confirming equilibrium. However, he did not demonstrate that players' decisions converge to a self-confirming equilibrium, or that their moves converge on a specific solution concept. Chapter 3 investigated whether it is possible for players' play to converge to a particular solution concept via a discovery process, focusing on the simultaneous-move case. Any myopic discovery process converges to a common realizable CURB set, which is an extension of the CURB concept to simultaneous-move games with unawareness. Additionally, the game's common realizable CURB set includes support for players' myopic best responses. Chapter 4 examined the discovery of actions in coordination games involving unawareness. In Schipper (2021) and Chapter 3, each player adds previously unnoticed opponents' actions to her or his revised subjective game through a process of discovery. In situations where successful coordination is crucial, such as in coordination games with unawareness, it is necessary that players not only discover the opponents' unnoticed actions and add them to the opponents' action sets, but also be able to imitate the opponents' actions and select the same actions themselves. This study models an imitative discovered game in which each player adds the opponents' actions to all players' action sets. Moreover, it demonstrates the existence of a successful-coordination equilibrium in which coordination is successful in the subsequent stage game.

Second, PART III revisits unawareness in the models with standard information structure for a single agent. As pointed out by Modica and Rustichini

(1994, 1999), and Dekel, Lipman, and Rustichini (1998), since unawareness may be trivial in standard information structures, models of unawareness structures have become the norm in the study of unawareness. However, as Ewerhart (2001) and Fukuda (2021) point out, Non-Triviality in standard information structures can be investigated by modifying the definitions and assumptions of knowledge and unawareness operators. Chapter 5 models unawareness similarly to Ewerhart (2001), and characterize the knowledge and unawareness operators. Chapter 6 reexamines the relationship among Symmetry, AU Introspection, and Negative Introspection. As Modica and Rustichini (1994), Dekel, Lipman, and Rustichini (1998), and Chen, Ely, and Luo (2012) point out, Symmetry and AU Introspection may be equivalent to Negative Introspection in conventional information structures. Nonetheless, if Necessitation is not true, the equivalence may not hold. This study relaxes Necessitation condition, focuses on the relationship between Symmetry and AU Introspection, and demonstrates the conditions under which the two are equivalent in the presence of non-trivial unawareness. Chapter 7 reexamines the knowledge operator's information function definition. The standard discussion about an information function assumes that any information set is not empty. However, Necessitation holds even if some information set is empty. That is, agents know the whole state space regardless of whether or not they obtain a set of information. This property is counterintuitive. Following is a redefinition of the knowledge operator: If the information set is empty, the agent has no knowledge. Additionally, this study reevaluates unawareness in standard information structures. Necessitation does not hold under non-trivial unawareness, whereas Monotonicity always holds. Chapter 7's knowledge operator is related to its Chapter 6 counterpart. Other aspects of unawareness have also been identified.

8.2 Further Research

In each chapter thus for, future technical research topics have been described. In addition to these, we would like to propose the following agenda: PART II have only analyzed simultaneous-move games and not their extensive-form counterparts. During the plays of extensive-form games with unawareness, a player may discover the unnoticed actions of their opponents. Then, players can update their subjective game and refine their decision-making strategy. Such a model must be constructed. PART III analyzes single-agent models of information structures, but not interactive situations. The subsequent phase of this research should consist of an examination of interactive situations. Furthermore, this thesis focuses exclusively on theoretical research. Possible directions for future research include the following.

Ma and Schipper (2017) conducted an experiment to determine whether risk preferences are invariant to awareness changes. Their research demonstrated that it is possible to conduct experiments involving decision-making under unawareness. Future studies may create an experimental model of the discovery process outlined in PART II and test how players' decision-making changes as their awareness shifts. Aoki (2001, 2011) and Takizawa (2017) mention the relationship between players' cognitive frameworks (or knowledge), and institutions. Incorporating unawareness into comparative institutional analysis could also be a direction for future research. The model of the unaware non-decider described in Chapter 7 may also be applicable to externalities if interpreted as a model of the knowledge of subjects who are removed from decision-making under particular circumstances.

These issues are the subject of future research.

Bibliography

Aoki, M. (2001). *Towards a Comparative Institutional Analysis*. MIT Press.

Aoki, M. (2011). Institutions as cognitive media between strategic interactions and individual beliefs. *Journal of Economic Behavior and Organization 79*, 20–34.

Asheim, G. B., M. Voorneveld, and J. W. Weibull (2016). Epistemically robust strategy subsets. *Games 7*.

Aumann, R. J. (1976). Agreeing to disagree. *The Annals of Statistics 4*(6), 1236–1239.

Aumann, R. J. (1987). Correlated equilibrium as an expression of Baysian rationality. *Econometrica 55*, 1–18.

Aumann, R. J. and A. Brandenburger (1995). Epistemic condition for Nash equilibrium. *Econometrica 63*(5), 1161–1180.

Auster, S. (2013). Asymmetric awareness and moral hazard. *Games and Economic Behavior 82*, 503–521.

Bacharach, M. (1985). Some extensions of a claim of Aumann in an axiomatic model of knowledge. *Journal of Economic Theory 37*, 167–190.

Basu, K. and J. W. Weibull (1991). Strategy subsets closed under rational behavior. *Economics Letters 36*, 141–146.

Benisch, M., G. B. Davis, and T. Sandholm (2010). Algorithms for closed under rational behavior (curb) sets. *Journal of Artificial Intelligence Research 38*, 513–534.

Bernheim, B. D. (1984). Rationalizable strategic behavior. *Econometrica 52*(4), 1007–1028.

Brandenburger, A. (1992). Knowledge and equilibrium in games. *Journal of Economic Perspectives 6*(4), 83–101.

Cave, J. A. K. (1983). Learning to agree. *Economics Letters 12*, 147–152.

Chen, Y.-C., J. C. Ely, and X. Luo (2012). Note on unawareness: Negative Introspection versus AU Introspection (and KU Introspection). *International Journal of Game Theory 41*, 325–329.

Čopič, J. and A. Galeotti (2006). Awareness as an equilibrium notion: Normal-form games. mimeo.

Dekel, E., B. L. Lipman, and A. Rustichini (1998). Standard state-space models preclude unawareness. *Econometrica 66*(1), 159–173.

Ewerhart, C. (2001). Heterogeneous awareness and the possibility of agreement. mimeo.

Fagin, D. and J. Y. Halpern (1988). Belief, awareness, and limited reasoning. *Artificial Intelligence 34*, 39–76.

Feinberg, Y. (2021). Games with unawareness. *The B.E. Journal of Theoretical Economics 21*(2), 433–488.

Filiz-Ozbay, E. (2012). Incorporating unawareness into contract theory. *Games and Economic Behavior 76*, 181–194.

Fudenberg, D. and D. K. Levine (1993). Self-confirming equilibrium. *Econometrica 61*(3), 523–543.

Fukuda, S. (2021). Unawareness without AU Introspection. *Journal of Mathematical Economics 94*, 102456.

Galanis, S. (2011). Syntactic foundations for unawareness of theorems. *Theory and Decision 71*, 593–614.

Galanis, S. (2013). Unawareness of theorems. *Economic Theory 52*, 41–73.

Galanis, S. (2018). Speculation under unawareness. *Games and Economic Behavior 109*, 598–615.

Galanis, S. and S. Kotronis (2021). Updating awareness and information aggregation. *The B.E. Journal of Theoretical Economics 21*(2), 613–635.

Geanakoplos, J. (1992). Common knowledge. *Journal of Economic Perspectives 6*, 53–82.

Geanakoplos, J. (1994). Common knowledge. In R. J. Aumann and S. Hart (Eds.), *Handbook of Game Theory with Economic Applications*, Volume 2, Chapter 40, pp. 1437–1496. Amsterdam: North Holland.

Geanakoplos, J. (2021). Game theory without partitions, and application to speculation and consensus. *The B.E. Journal of Theoretical Economics 21*, 361–394.

Geanakoplos, J. and H. Polemarchakis (1982). We can't disagree forever. *Journal of Economic Theory 28*, 192–200.

Grandjean, G., A. Mauleon, and V. Vannetelbosch (2017). Strongly rational sets for normal-form games. *Economic Theory Bulletin 5*, 35–46.

Grant, S. and J. Quiggin (2013). Inductive reasoning about unawareness. *Economic Theory 54*(3), 717–755.

Guarino, P. (2020). An epistemic analysis of dynamic games with unawareness. *Games and Economic Behavior 120*, 257–288.

Halpern, J. Y. (2001). Alternative semantics for unawareness. *Games and Economic Behavior 37*, 321–339.

Halpern, J. Y. and L. C. Rêgo (2014). Extensive games with possibly unaware players. *Mathematical Social Sciences 70*, 42–58.

Harsanyi, J. C. (1967). Games with incomplete information played 'Bayesian' player, Part I. *Management Science 14*, 159–182.

Heifetz, A., M. Meier, and B. C. Schipper (2006). Interactive unawareness. *Journal of Economic Theory 130*, 78–94.

Heifetz, A., M. Meier, and B. C. Schipper (2008). A canonical model for interactive unawareness. *Games and Economic Behavior 62*, 305–324.

Heifetz, A., M. Meier, and B. C. Schipper (2013a). Unawareness, beliefs, and speculative trade. *Games and Economic Behavior 77*, 100–121.

Heifetz, A., M. Meier, and B. C. Schipper (2013b). Dynamic unawareness and rationalizable behavior. *Games and Economic Behavior 81*, 50–68.

Heifetz, A., M. Meier, and B. C. Schipper (2021). Prudent rationalizability in generalized extensive-form games with unawareness. *The B.E. Journal of Theoretical Economics 21*(2), 525–556.

Heinsalu, S. (2012). Equivalence of the information structure with unawareness to the logic of awareness. *Journal of Economic Theory 147*, 2453–2468.

Heinsalu, S. (2014). Universal type structures with unawareness. *Games and Economic Behavior 83*, 255–266.

Hurkens, S. (1995). Learning by forgetful players. *Games and Economic Behavior 11*, 304–329.

Karni, E. and M.-L. Vierø(2013). "Reverse Bayesianism": A choice-based theory of growing awareness. *The American Economic Review 103*(7), 2790–2810.

Karni, E. and M.-L. Vierø(2017). Awareness of unawareness: A theory of decision making in the face of ignorance. *Journal of Economic Theory 168*, 301–328.

Kobayashi, N. and Y. Sasaki (2021). Rationalizable self-confirming equilibrium in normal-form games with unawareness. mimeo.

Lewis, D. K. (1969). *Convention.* Cambridge: Harvard University Press.

Li, J. (2009). Information structures with unawareness. *Journal of Economic Theory 144*, 977–993.

Ma, W. and B. C. Schipper (2017). Does exposure to unawareness affect risk preferences? A preliminary result. *Theory and Decision 83*(2), 245–257.

Meier, M. and B. C. Schipper (2014). Bayesian games with unawareness and unawareness perfection. *Economic Theory 56*, 219–249.

Milgrom, P. (1981). An axiomatic characterization of common knowledge. *Econometrica 49*, 219–222.

Milgrom, P. and N. Stokey (1982). Information, trade and common knowledge. *Journal of Economic Theory 26*, 17–27.

Modica, S. and A. Rustichini (1994). Awareness and partitional information structures. *Theory and Decision 37*, 107–124.

Modica, S. and A. Rustichini (1999). Unawareness and partitional information structures. *Games and Economic Behavior 27*, 265–298.

Myerson, R. and J. Weibull (2015). Tenable strategy blocks and settled equilibrium. *Econometrica 83*, 943–976.

Ostrovsky, M. (2012). Information aggregation in dynamic markets with strategic traders. *Econometrica 80*(6), 2595–2647.

Ozbay, E. Y. (2007). Unawareness and strategic announcements in games with uncertainty. In *In: Samet, D. (Ed.), Proceedomgs of the 11th Conference on Theoretical Aspects of Rationality and Knowledge. Presses Universitaires de Louvain*, pp. 231–238.

Pearce, D. G. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica 52*(4), 1029–1050.

Perea, A. (2022). Common belief in rationality in games with unawareness. *Mathematical Social Sciences 119*, 11–30.

Piermont, E. (2017). Introspective unawareness and observable choice. *Games and Economic Behavior 106*, 134–152.

Pruzhansky, V. (2003). On finding curb sets in extensive games. *International Journal of Game Theory 32*, 205–210.

Rêgo, L. C. and J. Y. Halpern (2012). Generalized solution concepts in games with possibly unaware players. *International Journal of Game Theory 41*, 131–155.

Rubinstein, A. (1989). The eretronic mail game: Strategic behavior under "Almost Common Knowledge". *The American Economic Review 79*(3), 385–391.

Sadzik, T. (2021). Knowledge, awareness and probabilistic beliefs. *The B.E. Journal of Theoretical Economics 21*(2), 489–524.

Samet, D. (1990). Ignoring ignorance and agreeing to disagree. *Journal of Economic Theory 52*(1), 190–207.

Sasaki, Y. (2017). Generalized Nash equilibrium with stable belief hierarchies in static games with unawareness. *Annals of Operations Research 256*(2), 271–284.

Sasaki, Y. (2022). Unawareness of decision criteria in multicriteria games. *Mathematical Social Sciences 119*, 31–40.

Schipper, B. C. (2013). Awareness-dependent subjective expected utility. *International Journal of Game Theory 42*, 725–753.

Schipper, B. C. (2014). Unawareness: A gentle introduction to both the literature and the special issue. *Mathematical Social Sciences 70*, 1–9.

Schipper, B. C. (2015). Awareness. In H. Ditmarsch, J. Y. Halpern, W. van der Hoek, and B. P. Kooi (Eds.), *Handbook of Epistemic Logic*. College Publications.

Schipper, B. C. (2018). Discovery and equilibrium in games with unawareness. Technical report, University of California, Davis.

Schipper, B. C. (2021). Discovery and equilibrium in games with unawareness. *Journal of Economic Theory 198*, 105365.

Sebenius, J. K. and J. Geanakoplos (1983). Don't bet on it: Contingent agreements with asymmetric information. *Journal of the American Statistical Association 78*(382), 424–426.

Shin, H. S. (1993). Logical structure of common knowledge. *Journal of Economic Theory 60*(1), 1–13.

Tada, Y. (2018). A study of an agent's epistemic revision for the games' structure and the other agents' perceptions: Formulation to replay of games with unawareness. Master's thesis, Chuo University. (in japanese).

Takizawa, H. (2017). Masahiko Aoki's conception of institutions. *Evolutionary and Institutional Economics Review 14*(2), 523–540.

Vierø, M.-L. (2021). An intertemporal model of growing awareness. *Journal of Economic Theory 197*, 105351.

Voorneveld, M., W. Kets, and H. Norde (2005). An axiomatization of minimal curb sets. *International Journal of Game Theory 33*, 479–490.

West, T. G. and G. S. West (1998). *Four Texts on Socrates: Plato's "Euthyphro", "Apology of Socrates", and "Crito" and Aristophanes' "Clouds"* (Revised edition ed.). Cornell University Press.

Young, H. P. (1993). The evolution of conventions. *Econometrica 61*, 57–84.

Young, H. P. (1998). *Individual Strategy and Social Structure*. Princeton University Press.