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Analysis of Optimal Macroeconomic Policy Using Dynamic Optimization

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This paper analyzes policy maker's optimal macroeconomic policy path that minimizes Taylor-rule like social loss function of inflation gap and output gap, using a dynamic optimization structure. Under the assumption that policy maker can dynamically control the real output, this paper solves dynamic optimization equations and draw optimal path of output and inflation expectation variables, with the analysis of its stability. Three different cases of optimization are tested (forward-looking, backward-looking, and mixed expectation formation scenario), and the phase diagrams show that each system obtains a converging path, namely a saddle path, which ultimately reaches a stable and unique equilibrium point. The policy implications may seem to be quite paradoxical, although it could still stand and avoid inflation / deflation divergence under certain assumptions.

1. Introduction

For central bank/government policy makers, what is their definition of an optimal macroeconomic policy? The Fed economists Khan et al. (2003) described from a central bank standpoint: "Optimal monetary policy maximizes the welfare of a representative agent, given frictions in the economic environment."[1)]

Monetary policy wouldn't have to be a sole channel of optimizing social welfare. Davig and Gürkaynak (2015) point out that many of the past literatures focusing on optimal policy refer them as a monetary policy, instead of mixing fiscal policy into their welfare function. They point out that monetary policy which single-handedly maneuvers interest rates cannot address all inefficiencies in the economy, especially in a world of multiple policymakers (such as fiscal policymakers) with different objectives.

Approaches on its study of optimal policy differs among literatures. Rotemberg and Woodford (1997) analyzed monetary policy optimization through econometric approach,

1) They challenged to incorporate each of the mainstream views of Fisher, Keynes, and Friedman, and came to one of their conclusions that economy under a small deflation is a partial factor of an optimal monetary policy to stabilize the price level.

using vector autoregression models. As a variation of optimal monetary policy, for instance, Acharya et al. (2020) takes into account consumption inequality in their model, by assuming a heterogeneous agent, instead of a representative agent model. Also, Brayton et al. (2014) tackled this problem using the FRB/US Model to figure what they call "optimal-control policy".

In this paper, as many literatures do, a welfare loss function is formulated by inducing a simple Taylor-rule function in the model. Taylor-rule, which was elaborated by Taylor (1993) is not explicitly adopted as a policy instrument to determine a policy rate, yet it is still a very influential notion widely among policy makers. For example, in the Federal Reserve Bank (Fed) semiannual Monetary Policy Report[2], they provide Taylor Rule with several different versions since 2017 as a reference to theoretically show the appropriate policy rate. The current Fed chairman Jerome Powell pointed out on Taylor Rule that it never has been used as a strict policy instrument, though acknowledged it is still a useful reference to signal an appropriate interest rate[3].

In the Monetary Policy Report which was published in June 2022, all the versions of Taylor Rule are implying that the Federal Funds rate should have been higher than what it was at that moment[4]. The core personal consumption expenditure inflation as of May 2022 was 4.7% year on year, which is substantially higher than the Fed's target of 2%. The unemployment rate as of May 2022 was 3.6%, which is well below the Fed's longer-run projection of 4%[5]. The Fed intends to raise the policy rate expeditiously until the inflation rate heads back to the target, acknowledging the risk of facing higher unemployment rate. However, from its significant gap from the target, a concern of over-killing the economy has been arising and the Fed's handling of the policy rate setting seems more and more challenging. It would be beneficial if the Fed / government owns a certain instrument which guides an appropriate policy path to achieve the policy goals.

We analyze consolidated government (central bank and government)'s optimal policy path that minimizes social loss function using a dynamic optimization structure. The loss function here is defined by the Taylor Rule-like output/inflation gap combination, and the dynamic optimization is solved subject to the inflation expectation function constraint. Under the assumption that policy maker can dynamically control the real output, this paper solves the dynamic optimization equation and draw an optimal path of the output

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2) For example, the Monetary Policy Report (June 17, 2022) https://www.federalreserve.gov/monetarypolicy/2022-06-mpr-summary.htm

3) Chair Powell's quote on Taylor Rule mentioned in John Taylor's blog https://economicsone.com/2022/06/25/play-by-the-rules/

4) Monetary Policy Report (June 17, 2022).

5) June 15, 2022 FOMC Projection materials, https://www.federalreserve.gov/monetarypolicy/fomcprojtabl20220615.htm

and inflation expectation variable, with the analysis of its stability. Similar approach were taken by Asada (2010) and Semmler and Zhang (2004) to derive the optimal monetary policy and its characteristics at the equilibrium point, but this paper takes further steps, by switching the parameter of the inflation expectation function, which represents people's stance on their inflation expectation. Three different cases of optimization are tested : forward-looking, backward-looking, and mixed expectation formation scenarios.

2. Formulation of the Model

Eq(1) is the conventional linear "expectations-augmented Phillips curve". Variable π represents inflation, parameter ε is a reaction parameter from the output gap, Y is the real output, and $\bar{Y}$ represents natural output, or real output target. Eq(2) is the dynamic inflation expectation formation, which combines forward-looking and backward-looking expectations[6].

$$\pi = \varepsilon(Y - \bar{Y}) + \pi^e \; ; \; \varepsilon > 0 \tag{1}$$

$$\dot{\pi}^e = \alpha\{\xi(\pi - \pi^e) + (1 - \xi)(\bar{\pi} - \pi^e)\} \; ; \; \alpha > 0, 0 \leqq \xi \leqq 1 \tag{2}$$

PParameter ξ represents the weight of people's stance on their inflation expectation. If $\xi = 1$, the equation of motion for $\dot{\pi}^e$ is dependent on the inflation gap, which here is represented by the difference between actual inflation π and inflation expectation π^e. This shows that people have adaptive behavior on inflation. On the other hand, if $\xi = 0$, the equation turns out to express people's behavior as forward-looking, since $\dot{\pi}^e$ is now dependent on the gap between government's inflation target $\bar{\pi}$ and π^e. If $0 < \xi < 1$, then people's behavior on inflation is mixed between forward and backward-looking. The parameter α is the reaction parameter to the inflation expectation dynamics.

Eq(3) is derived by substituting Eq(1) into Eq(2), and now the equation of motion for $\dot{\pi}^e$ has real output as one of the variables.

$$\dot{\pi}^e = \alpha\{\xi\varepsilon(Y - \bar{Y}) + (1 - \xi)(\bar{\pi} - \pi^e)\} \tag{3}$$

Case of $\xi = 0$

Based on the model mentioned above, here, the case of $\xi = 0$ is considered. As previously explained, this transforms the $\dot{\pi}^e$ equation into a forward-looking system. Eq(3) now becomes Eq(4).

$$\dot{\pi}^e = \alpha\,(\bar{\pi} - \pi^e) \; ; \; \alpha > 0 \tag{4}$$

6) This paper sets real output Y as a control variable, whereas Asada (2010) puts nominal interest rate as a control variable.

The social loss function is defined as below. This transformation is similar to the method used in Taylor (1989), Chiang (1992), Woodford (2001) and Asada (2010)[7].

$$V = \theta\ (Y - \bar{Y})^2 + (1 - \theta)\ (\pi - \bar{\pi})^2 = V\ (Y, \pi)\ ;\ 0 < \theta < 1 \tag{5}$$

The parameter θ is the positive parameter which represents the weight of how much the policy maker prioritize between real output and inflation gap.

Eq(1) is now substituted into Eq(5), and in order to minimize the loss function, Eq(5) is turned into negative.

$$-V = -[\theta\ (Y - \bar{Y})^2 + (1 - \theta)\ \{\varepsilon\ (Y - \bar{Y}) + \pi^e - \bar{\pi}\}^2\ ] = W(Y, \pi^e) \tag{6}$$
$$0 < \theta < 1$$

Combining Eq(4) and Eq(6) gives the following dynamic optimization problem subject to the constraint equation.

$$\max_{Y} \int_0^{\infty} W\ (Y, \pi^e)\ e^{-\rho t}\ dt \tag{7}$$
$$s.\ t.\ \dot{\pi}^e = \alpha\ (\bar{\pi} - \pi^e)$$

In order to derive the optimal macroeconomic policy path, a current-value Hamiltonian equation is formalized as below, where λ is the co-state variable.

$$H = -[\theta\ (Y - \bar{Y})^2 + (1 - \theta)\ \{\varepsilon(Y - \bar{Y}) + \pi^e - \bar{\pi}\}^2] + \lambda\alpha\ (\bar{\pi} - \pi^e) \tag{8}$$

As explained in Chiang (1992), Pontryagin's maximum principle conditions are given as follows[8].

$$\max_{Y} H\ (Y, \pi^e, \lambda)\ \textit{for all } t \in [0, \infty] \tag{9}$$
$$\dot{\pi}^e = \frac{\partial H}{\partial \lambda} \qquad [\textit{equation of motion for } \dot{\pi}^e]$$
$$\dot{\lambda} = -\frac{\partial H}{\partial \pi^e} + \rho\lambda \qquad [\textit{equation of motion for } \lambda]$$
$$\lim_{t \to \infty} \lambda e^{-\rho t} = 0 \qquad [\textit{transversality condition}]$$

This Hamiltonian system here is solved with respect to the real output, Y. This implies that policy maker is controlling real output to minimize (optimize) the social loss function previously proposed.

A first order condition is required to show that the optimal control of Y will be an inte-

7) Consideration in monetary policy tradeoffs, welfare loss function, and problem solving of optimal monetary policy are also conducted by Galí (2015) and Woodford (2002).

8) Explanation on Pontryagin's maximum principle conditions and dynamic optimization are also available from Chiang and Wainwright (2005).

rior solution.

$$\frac{\partial H}{\partial Y} = -[2\theta\ (Y-\bar{Y}) + 2\varepsilon\ (1-\theta)\{\varepsilon(Y-\bar{Y}) + \pi^e - \bar{\pi}\}] = 0 \tag{10}$$

Further differentiation of Eq (10) with the result of negative shows that the control variable Y does maximize the Hamiltonian system.

$$\frac{\partial^2 H}{\partial Y^2} = -[2\theta + 2\varepsilon^2(1-\theta)] \tag{11}$$

$$= -2\ [\theta + \varepsilon^2(1-\theta)] < 0$$

Equation of motion for $\dot{\pi}^e$ is defined as below.

$$\dot{\pi}^e = \frac{\partial H}{\partial \lambda} = \alpha\ (\bar{\pi} - \pi^e) = F_1^1\ (\pi^e) \tag{12}$$

Equation of motion for λ, which is a costate variable, is described below.

$$\dot{\lambda} = -\frac{\partial H}{\partial \dot{\pi}^e} + \rho\lambda = [2\ (1-\theta)\{\varepsilon\ (Y-\bar{Y}) + \pi^e - \bar{\pi}\}] + \lambda\ (\alpha + \rho) \tag{13}$$

Real output Y of Eq (10) can be transformed as below.

$$Y = \frac{\bar{Y}\{\varepsilon^2(1-\theta) + \theta\} + \varepsilon\ (1-\theta)(\bar{\pi} - \pi^e)}{\theta + \varepsilon^2\ (1-\theta)} = Y(\pi^e) \tag{14}$$

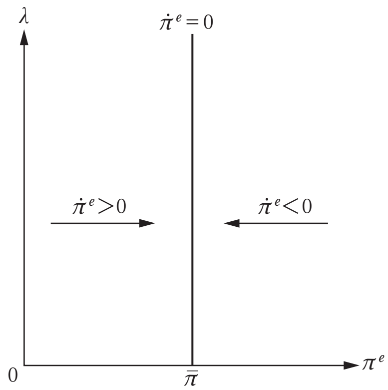
Eq (14) can be substituted into Eq (13) as below.

$$\dot{\lambda} = [2\ (1-\theta)\{\varepsilon\ (Y\ (\pi^e) - \bar{Y}) + \pi^e - \bar{\pi}\}] + \lambda\ (\alpha + \rho) = F_2^1\ (\pi^e, \lambda) \tag{15}$$

Phase diagram in case of $\xi = 0$

From the equilibrium point of $\dot{\pi}^e$ in Eq (12), the solution $\bar{\pi} = \pi^e$ can be derived. Then, we can derive Figure 1. It is assumed here that the policy maker would indirectly

Figure 1



maneuver the costate variable of λ by controlling the real output Y, which can be termed as a marginal increase (or decrease) in inflation expectation.

Assuming equilibrium $\dot{\lambda}=0$ in Eq(13), equation of λ can be derived as below.

$$\lambda = \frac{-2\,(1-\theta)\{\varepsilon\,(Y(\pi^e)-\bar{Y})+\pi^e-\bar{\pi}\}}{\alpha+\rho} \tag{16}$$

Since

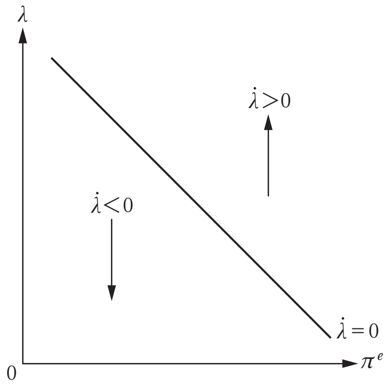
$$\frac{dY}{d\pi^e} = -\frac{\varepsilon\,(1-\theta)}{\theta+\varepsilon^2(1-\theta)} < 0,$$

We have the following relationship from Eq(16).

$$\left.\frac{d\lambda}{d\pi^e}\right|_{\dot{\lambda}=0} = \frac{-2\,(1-\theta)\left\{\varepsilon\frac{dY}{d\pi^e}+1\right\}}{\alpha+\rho} < 0$$

Therefore, we obtain Figure 2, since $\frac{\partial\dot{\lambda}}{\partial\pi^e} > 0$ from Eq(15).

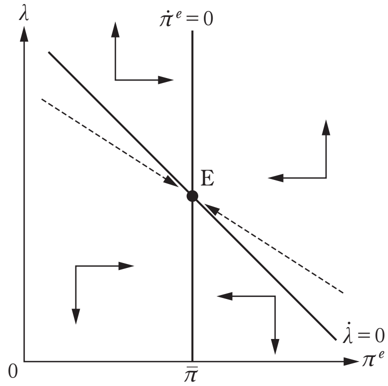
Figure 2



By combining Figure 1 and Figure 2, the phase diagram is complete.

As depicted in Figure 3, an implication drawn under the rule $\xi=0$ is that there is only one policy path which enables policy maker to achieve full employment and the inflation target goal, and this unique path toward the equilibrium is the saddle path. Only the saddle path satisfies the transversality condition in Eq(9). From a given initial state variable $\pi^e(0)$, an inflation expectation, policy maker needs to select the initial point of λ and control it following the saddle path, by indirectly maneuvering Y. Or else, the inflation expectation and costate variable λ would not end up at the equilibrium point of E.

Figure 3



If the starting point of inflation expectation exceeds the inflation target, the policy maker would have to raise Y to lower inflation expectation while increasing the costate variable, λ, since $\frac{dY}{d\pi^e} < 0$. On the other hand, if the initial point of inflation expectation is below the target, Y would have to decrease in order to lower λ. This is quite a paradoxical result, since in general, output and inflation expectation have a positive correlation, and increase in output would lead to a further rise in inflation expectation. The only way to achieve the policy target is to follow the saddle path, and if any other path is to be chosen, the output and inflation expectation would spread out indefinitely.

By taking into account Eq(12) and Eq(15), the characteristic equation of this system at the equilibrium point is described below, where λ_1. and λ_2 are two characteristic roots[9].

$$|\lambda I - J_1| = \lambda^2 - (traceJ)\ \lambda + (detJ) \tag{17}$$
$$= (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

Where,

$$J_1 = \begin{bmatrix} -\alpha & 0 \\ F_{21}^1 & F_{22}^1 \end{bmatrix}$$

$$F_{21} = \frac{dF_2}{d\pi^e} = 2\ (1-\theta)\left\{\frac{\theta}{\theta + \varepsilon^2(1-\theta)}\right\} > 0$$

$$F_{22} = \frac{dF_2}{d\lambda} = \alpha + \rho > 0$$

$$trace\, J = \lambda_1 + \lambda_2 = -\alpha + F_{22}$$

$$detJ = \lambda_1\lambda_2 = -\alpha F_{22} < 0.$$

9) Evaluating the dynamical properties of interest rate and policy tool is also done by Drumond et al. (2022).

This shows that the equilibrium point is a saddle point because the characteristic equation Eq(17) has one positive real root and one negative real root.

Case of $\xi = 1$

Now, this time, a backward-looking type of inflation expectation, is considered. This is the case of $\xi = 1$ in Eq(3).

The equation of motion for $\dot{\pi}^e$ now becomes a function of real output Y.

$$\dot{\pi}^e = \alpha\varepsilon(Y - \bar{Y})\ ;\ \alpha > 0 \tag{18}$$

Eq(18) becomes a constraint function, and the dynamic optimization problem is formulated as below.

$$\max_Y \int_0^\infty W(Y, \pi^e)\, e^{-\rho t}\, dt$$

$$s.t.\ \dot{\pi}^e = \alpha\varepsilon(Y - \bar{Y})$$

As similar to Eq(8), a current value Hamiltonian becomes as follows.

$$H = -[\theta(Y - \bar{Y})^2 + (1-\theta)\{\varepsilon(Y - \bar{Y}) + \pi^e - \bar{\pi}\}^2] + \lambda\alpha\varepsilon(Y - \bar{Y}) \tag{19}$$

where λ is the co-state variable.

First order condition and second derivative are described below.

$$\frac{\partial H}{\partial Y} = -[2\theta\,(Y - \bar{Y}) + 2\varepsilon(1-\theta)\{\varepsilon(Y - \bar{Y}) + \pi^e - \bar{\pi}\}] + \lambda\alpha\varepsilon = 0 \tag{20}$$

$$\frac{\partial^2 H}{\partial Y^2} = -[2\theta + 2\varepsilon^2(1-\theta)]$$

$$= -2\,[\theta + \varepsilon^2\,(1-\theta)] < 0$$

Equation of motion for the inflation expectation $\dot{\pi}^e$ is given below.

$$\dot{\pi}^e = \frac{\partial H}{\partial \lambda} = \alpha\varepsilon(Y - \bar{Y}) = F_1^2\,(Y) \tag{21}$$

From the first order condition in Eq(20), the equation of λ can be obtained.

$$\lambda = \frac{2}{\alpha}\left[(Y - \bar{Y})\left\{\frac{1}{\varepsilon}\theta + \varepsilon(1-\theta)\right\} + (1-\theta)(\pi^e - \bar{\pi})\right] \tag{22}$$

Differentiation of Eq(22) with respect to time t gives Eq(23).

$$\dot{\lambda} = \frac{2}{\alpha}\left[\dot{Y}\left\{\frac{1}{\varepsilon}\theta + \varepsilon(1-\theta)\right\} + \alpha\varepsilon(1-\theta)(Y - \bar{Y})\right] \tag{23}$$

Additionally, from the one of the maximum principal conditions, the motion for λ is obtained below.

$$\dot{\lambda} = -\frac{\partial H}{\partial \pi^e} + \rho\lambda = 2(1-\theta)\{\varepsilon(Y-\bar{Y}) + \pi^e - \bar{\pi}\} + \rho\lambda(Y, \pi^e) \tag{24}$$

Transversality condition is termed as follows.

$$\lim_{t\to\infty} \lambda\, e^{-\rho t} = 0 \tag{25}$$

By combining Eq(23) and (24) and bringing $\dot{Y}$ to the left side, the equation of motion for real output Y can be described as below.

$$\dot{Y} = \frac{1}{A}\,[2\,(1-\theta)(\pi^e - \bar{\pi}) + \rho\lambda(Y, \pi^e)] = F_2\,(Y, \pi^e) \tag{26}$$

Phase diagram in case of ξ =1

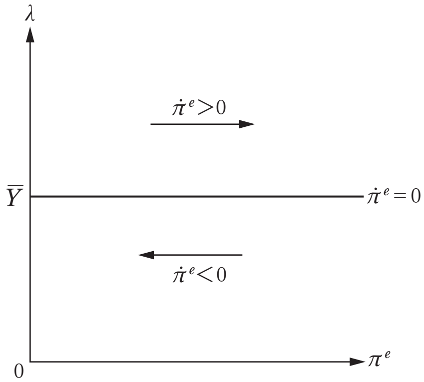
The locus for $\dot{\pi}^e = 0$ is derived from Eq(21), where

$$0 = \alpha\varepsilon(Y - \bar{Y})$$

The $\dot{\pi}^e = 0$ curve becomes horizontal, as sketched below. The level of Y at the equilibrium point, $\bar{Y}$ is given below.

$$Y = \bar{Y}$$

Figure 4



The $\dot{Y} = 0$ line is depicted by solving the total derivation of Eq(26), as shown below.

$$\frac{dY}{d\pi^e} = -\frac{\frac{\partial F_2}{\partial \pi^e}}{\frac{\partial F_2}{\partial Y}} < 0 \tag{27}$$

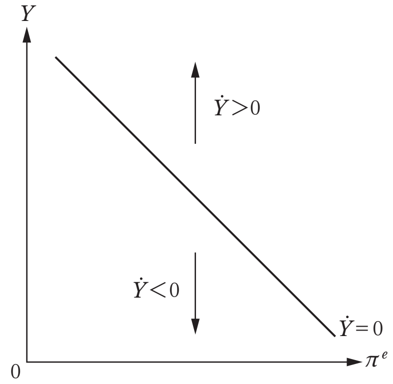
Each partial derivations are solved as below.

$$\frac{\partial F_2^2}{\partial \pi^e}=F_{21}^2=\frac{1}{A}\{2(1-\theta)+\rho(1-\theta)\}>0$$

$$\frac{\partial F_2^2}{\partial Y}=F_{22}^2=\frac{1}{A}\rho\frac{2}{\alpha}\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\}>0$$

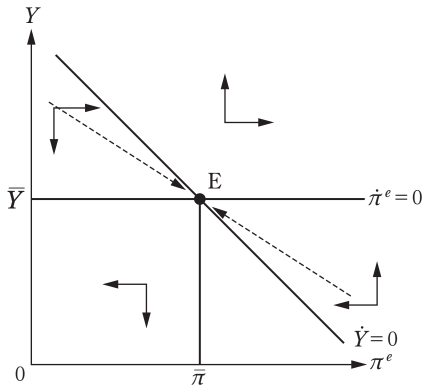
From the result of the total derivation, it is clear that $\dot{Y}=0$ line is negatively inclined.

Figure 5



By combining both lines, the phase diagram is depicted as below. It is now clear that the system with $\xi=1$ also has an unique path which converges into a saddle point E. Again, as like the case with $\xi=0$, the conclusion is quite paradoxical. If the starting point of inflation expectation is below the inflation target $\bar{\pi}$, the policy maker needs to decrease the output, Y. On the other hand, if the inflation expectation is exceeding the target, the policy maker now needs to increase the output to suppress the inflation expectation.

Figure 6



The evaluation of the dynamical properties of the model using the Jacobian matrix is described below, which shows that the equilibrium point becomes saddle point.

$$J_2=\begin{bmatrix} 0 & \alpha\varepsilon \\ F^2_{21} & F^2_{22} \end{bmatrix} \tag{28}$$

$$F^2_{21}=\frac{1}{A}\{2(1-\theta)+\rho(1-\theta)\}>0$$

$$F^2_{22}=\frac{1}{A}\rho\frac{2}{\alpha}\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\}>0$$

$$detJ = \lambda_1\lambda_2=0-\alpha\varepsilon F_{21}<0$$

Case of 0< ξ <1

Thirdly, the system with $0<\xi<1$ is tested to figure out its dynamic behavior. The loss function of policy maker is same as the previous cases. The dynamic optimization problem in this model is described as below.

$$\max_{Y}\int_0^\infty W(Y,\pi^e)\, e^{-\rho t}\, dt \tag{29}$$

$$s.\,t.\ \dot{\pi}^e=\alpha\{\xi\varepsilon(Y-\bar{Y})+(1-\xi)(\bar{\pi}-\pi^e)\}$$

Thus, current-value Hamiltonian is formatted using the new constraint function.

$$H=-[\theta(Y-\bar{Y})^2+(1-\theta)\{\varepsilon(Y-\bar{Y})+\pi^e-\bar{\pi}\}^2]+\lambda\alpha\{\xi\varepsilon(Y-\bar{Y}) + (1-\xi)(\bar{\pi}-\pi^e)\} \tag{30}$$

First order condition and second derivative become as follows.

$$\frac{\partial H}{\partial Y}=-[2\theta(Y-\bar{Y})+2\varepsilon(1-\theta)\{\varepsilon(Y-\bar{Y})+\pi^e-\bar{\pi}\}]+\lambda\alpha\xi\varepsilon=0 \tag{31}$$

$$\frac{\partial^2 H}{\partial Y^2}=-[2\theta+2\varepsilon^2(1-\theta)]<0$$

We can derive the following expression of λ by transforming the result from the first derivative equation above.

$$\lambda=\frac{2}{\alpha\varepsilon}\left[(Y-\bar{Y})\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\}+(1-\theta)(\pi^e-\bar{\pi})\right]=\lambda(Y,\pi^e) \tag{32}$$

Equation of motion for $\dot{\pi}^e$ is obtained below.

$$\dot{\pi}^e=\frac{\partial H}{\partial \lambda}=\alpha\{\xi\varepsilon(Y-\bar{Y})+(1-\xi)(\bar{\pi}-\pi^e)\}=F^3_1(Y,\pi^e) \tag{33}$$

Differentiation of Eq(32) with respect to time gives an equations of $\dot{\lambda}$.

$$\dot{\lambda}=\frac{2}{\alpha\xi}\left[\dot{Y}\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\}+\alpha(1-\theta)\{\xi\varepsilon(Y-\bar{Y})+(1-\xi)(\bar{\pi}-\pi^e)\}\right] \tag{34}$$

Another equation of $\dot{\lambda}$, which is part of the maximum principle, is described below.

$$\dot{\lambda} = -\frac{\partial H}{\partial \pi^e} + \rho\lambda = 2(1-\theta)\{\varepsilon(Y-\bar{Y}) + \pi^e - \bar{\pi}\} + \lambda(Y, \pi^e)\{\alpha(1-\xi)+\rho\} \tag{35}$$

By combining Eq(34) and (35), and bringing $\dot{Y}$ to the left side, the equation of motion for real output Y is available.

$$\dot{Y} = \frac{1}{A}\left[2(1-\theta)\left\{\pi^e - \bar{\pi} - \frac{1}{\xi}(1-\xi)(\bar{\pi}-\pi^e)\right\} + \lambda(Y, \pi^e)\{\alpha(1-\xi)+\rho\}\right] \tag{36}$$
$$= F_2^3(Y, \pi^e)$$

Phase Diagram in case of 0<ξ<1

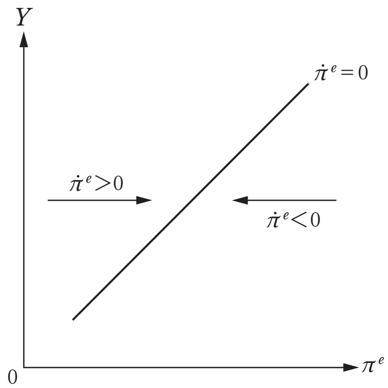
Phase diagram of this system can be depicted from Eq(33) and Eq(36).
It is clear that the slope of $\dot{\pi}^e = 0$ line is positive.

$$\left.\frac{dY}{d\pi^e}\right|_{\dot{\pi}^e=0} = -\frac{\dfrac{\partial F_1^3}{\partial \pi^e}}{\dfrac{\partial F_1^3}{\partial Y}} > 0 \tag{38}$$

Where,

$$\frac{\partial F_1^3}{\partial \pi^e} = F_{11} = -\alpha(1-\xi) < 0$$
$$\frac{dF_1^3}{dY} = F_{12} = \alpha\xi\varepsilon > 0$$

Figure 7



The slope of $\dot{Y} = 0$ line is negative, as described below.

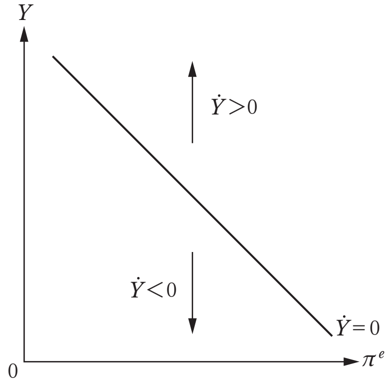
$$\left.\frac{dY}{d\pi^e}\right|_{\dot{Y}=0} = -\frac{\dfrac{\partial F_2^3}{\partial \pi^e}}{\dfrac{\partial F_2^3}{\partial Y}} < 0$$

Where,

$$\frac{\partial F_2^3}{\partial \pi^e} = F_{21}^3 = \frac{1}{A}\left[2(1-\theta)\left\{1-\frac{1}{\varepsilon}(1-\xi)\right\}\frac{2}{\alpha\xi}(1-\theta)\{\alpha(1-\xi)+\rho\}\right] > 0$$

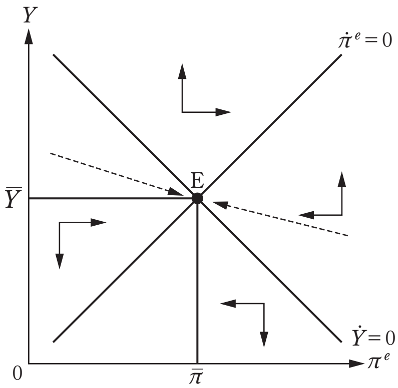
$$\frac{dF_2}{dY} = F_{22}^3 = \frac{1}{A}\left[\frac{2}{\alpha\xi}\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\}\{\alpha(1-\varepsilon)+\rho\}\right] > 0$$

Figure 8



The combination of Figure 7 and 8 gives the phase diagram, as depicted in Figure 9.

Figure 9



The system is stable as it converges to the equilibrium point, as proven in the Jacobian matrix below.

$$detJ_3 = det\begin{bmatrix} -\alpha(1-\xi) & \alpha\xi\varepsilon \\ F^3_{21} & F^3_{22} \end{bmatrix} < 0$$

Where again,

$$F^3_{21} = \frac{1}{A}\left[2(1-\theta)\left\{1-\frac{1}{\varepsilon}(1-\xi)\right\}\frac{2}{\alpha\xi}(1-\theta)\{\alpha(1-\xi)+\rho\}\right] > 0$$

$$F^3_{22} = \frac{1}{A}\left[\frac{2}{\alpha\xi}\left\{\frac{1}{\varepsilon}\theta+\varepsilon(1-\theta)\right\}\{\alpha(1-\varepsilon)+\rho\}\right] > 0$$

As discovered previously in the other cases, the system with the case of $0<\xi<1$ also owns a saddle path which converges to the equilibrium point of E.

3. Policy Implications

Here, we consider the policy implication under different scenarios. First is the case which inflation (inflation expectation) is above the target. In this case, the initial point of real output Y would have to be placed below the target, $\bar{Y}$. From this starting point, the appropriate path converging to the equilibrium point of $\bar{Y}$ and $\bar{\pi}$ would be to increase the real output. This initial condition of high inflation and low output is commonly called as stagflation. The process of increasing the output while lowering inflation rate may be possible through the increase in investment to enlarge its supply capacity.

The opposite case is when inflation rate is below the target, a situation which is called as a deflationary economy. Under this scenario, the initial point of real output needs to be above the target. From there, the appropriate path for the policy maker to reach the inflation target is to continuously lower the real output. This is somewhat counterintuitive, though one possible case is when policy maker stimulates the economy at the starting point and locates the initial point of real output substantially higher than the target. Inflation generally reacts with a lag, so the inflation rate could possibly move higher while the real output is contracting to the equilibrium point.

Our findings imply that the optimal policy solutions for the policy makers are more or less the same, regardless of people's forward/backward-looking stance on their inflation expectation. As a contrast, although Williams (2003) uses FRB/US macroeconomic model in its analysis, the author concluded that the performance of efficient monetary policy differs under rational and backward-looking models. Carlstrom and Fuerst (2000) argues that aggressive and backward-looking stance are necessary for the monetary authority to ensure determinacy. Benhabib et al. (2003) tested the stabilization behavior of the backward-looking interest-rate rules, which they figured that parameters distinguish the stabilization, and not all backward-looking feed-back rules guarantee the uniqueness of equilibrium.

The implications above are the policy paths which converge to the stable equilibrium point, but we need to consider here the cases when the policy actions were unsuccessful. The output/inflation expectation paths would then diverge, letting the two variables increase/decrease indefinitely. One consequence is the upward divergence of real output and inflation expectations, what is called as an inflation spiral, and the other is the downward divergence, which is a deflation spiral. The former case is frequently observable especially among the emerging economies, Developed economies also faced somewhat similar phenomenon in the 1970s that are commonly attributed to oil embargo factor, but at the same time to a wage-price spiral. The Fed's response under then-chairman Paul Volcker was the rampant hike of the policy rate, which consequently led to a deep recession afterwards[10]. Since then, the Fed's important objective upon achieving price stability is to first stabilize people's inflation expectation close their target, and it has seemingly been successful until the Covid-19 inflation. The opposite case, – deflationary spiral – is relatively rare, and its movement of economic variables haven't been as dramatic. One of the few examples is Japan, particularly since the beginning of the 1990s. The sluggish economy of Japan has experienced negative rate of year-on-year core inflation for nearly 20 years, which should be long enough to categorize Japan as the country which faced deflationary spiral.

We think that it is reasonable for the policy makers to aim the optimal policy path which converges to the target, considering its detrimental consequences of both cases of inflationary/deflationary spiral. As Bernanke (2002) made an important argument on Japan's stagnating economy: "Japan's deflation problem is real and serious; but, in my view, political constraints, rather than a lack of policy instruments, explain why its deflation has persisted for as long as it has", the responsibility is on policy maker's hands, and they should have the tools and policy space to deal with the problem.

4. Conclusion

In this paper, by applying Hamiltonian equations, several optimal policy paths were derived under different scenarios of inflation expectation behavior. First was a forward-looking inflation expectation scenario, second was the backward-looking scenario, and the third was the mixed behavior of the two. The phase diagrams showed that each system

10) Detail on the history of the Fed's battle against great inflation: Federal Reserve History "The Great Inflation 1965–1982" (https://www.federalreservehistory.org/essays/great-inflation).

obtains a unique converging path, namely a saddle path, which ultimately reaches a stable and unique equilibrium point.

The dynamic relationship between the real output and inflation expectation derived here implies that if inflation expectation is above the inflation target, the optimal option for policy maker is to select an initial point of real output that is below the target to position itself at the appropriate trajectory (saddle path). Then the optimal policy is to increase the output and follow the saddle path until the expected inflation converges to the equilibrium, or the inflation target. On the other hand, if the expected inflation is below the target, the optimal policy is to lower the real output by controlling it over the saddle path. In either case, if the policy outcome fails to follow the optimal path, inflation expectation and real output would dissipate and not end up reaching the equilibrium point. These implications are somewhat paradoxical, since an orthodox policy to stabilize the economy is to undertake anti-cyclical measures.

Similar antitheoretical observations are pointed by Asada (2010) and Mankiw (2001), against the New Keynsian dynamic model which can be found in the literature by Galí (2015). They referred to their findings as a "sign-reversal" problem. "Sign-reversal" problem is a paradoxical properties of the "New Keynesian" dynamic model that shows a counterfactual relationship between inflation and real output. According to Asada (2010), the New Keynsian dynamic model implies that "the rate of inflation accelerates whenever the actual output level is below the natural output level, and it decelerates whenever the actual output level is above the natural output level". The author pointed out the same problem for the New Keynsian IS model as well.

Lastly, we want to conclude by pointing out some caveats on this paper's analysis. The model assumes that policy maker is omniscient and have the capacity to start from wherever they prefer, which is referred to as a "jump variable" by Asada (2010) and Asada (2013). Also, it assumes that the real output is controllable so it can follow the optimal path without deviation. However, this is quite an unrealistic assumption, which similar argument has been made by Asada (2010). The dynamically optimal solution which is derived in this study should be considered as a "reference" or "yardstick" to observe how the real-life macroeconomic policy is deviating from the optimal path.

In the real-life policy setting, governors can only impact the real output Y indirectly. A derived form of the equilibrium condition for the goods market is usually formed as below.

$$Y = Y(r-\pi^e, G)\ ;\ \frac{\partial Y}{\partial (r-\pi^e)} < 0,\quad \frac{\partial Y}{\partial G} > 0$$

Here, r = nominal interest rate and G = real government expenditure. Normally, nominal interest rate is controlled by a central bank's monetary policy, and real government expenditure is decided by a government's fiscal policy. As such, the real output Y is not a variable which policy maker can directly control, instead is an indirect control variable which comes from a monetary and fiscal mixed policy. Optimal macroeconomic policy model which sets these policy tools as control variables may be the next step for the future study.

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