# 実空間型内の一定な平均曲率をもつ部分多様体について
# On submanifolds with constant mean curvature in a real space form

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Let $\tilde{M}^{n+p}(c)$ be an $(n+p)$-dimensional complete, connected and simply connected Riemannian manifold with constant sectional curvature $c$. We call it a space form. A space form $\tilde{M}^{n+p}(c)$ is one of the following:

(i) If $c > 0$, then $\tilde{M}^{n+p}(c)$ is a Euclidean sphere $S^{n+p}(c)$;

(ii) If $c = 0$, then $\tilde{M}^{n+p}(c)$ is a Euclidean space $R^{n+p}$;

(iii) If $c < 0$, then $\tilde{M}^{n+p}(c)$ is a hyperbolic space $H^{n+p}(c)$.

Let $M^n$ be an $n$-dimensional, connected and orientable submanifold isometrically immersed in $\tilde{M}^{n+p}(c)$. Denote by $h^\alpha_{ij}$ the local component of the second fundamental form for each $i,\ j,\ \alpha$ $(1 \le i, j \le n,\ n+1 \le \alpha \le n+p)$. We set

$$S := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h^\alpha_{ij})^2 \quad \text{and} \quad H := \frac{1}{n}\sqrt{\sum_{\alpha=n+1}^{n+p}\Big(\sum_{i=1}^{n} h^\alpha_{ii}\Big)^2}$$

be the squared norm of the second fundamental form and the mean curvature of $M^n$ in $\tilde{M}^{n+p}(c)$, respectively. $M^n$ is called minimal if the mean curvature $H$ of $M^n$ is equal to zero.

Now, we denote by $A_\alpha$ the $n \times n$ matrix of $h^\alpha_{ij}$ with respect to indices $i,\ j$. Define linear maps $\phi_\alpha : T_xM \to T_xM$ by

$$\langle \phi_\alpha X, Y\rangle := \frac{1}{n}\,\mathrm{trace}\, A_\alpha \langle X, Y\rangle - \langle A_\alpha X, Y\rangle \quad \text{for} \ \ n+1 \le \alpha \le n+p,$$

where $\langle\ ,\ \rangle$ is the Riemannian metric of $M^n$. Moreover, we define the bilinear map $\phi : T_xM \times T_xM \to T_xM^\perp$ by

$$\phi(X, Y) = \sum_{\alpha=n+1}^{n+p} \langle \phi_\alpha X, Y\rangle e_\alpha,$$

where $\{e_{n+1}, \ldots, e_{n+p}\}$ denotes an orthonormal basis. It is easy to check that $\mathrm{trace}\,\phi = 0$ and that

$$|\phi|^2 := \sum_{\alpha=n+1}^{n+p} \mathrm{trace}\,\phi_\alpha^2 = S - nH^2.$$

Let

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c)$$

and

$$Q_H(x) = \frac{3}{2}x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(H^2 + c)$$

be the polynomials for each real number $H \in \boldsymbol{R}$. We put $A_H$ the square of the positive root of $P_H(x) = 0$ and $B_H$ one of $Q_H(x) = 0$.

Besides, in the case of $p = 1$, we denote by $h_{ij}$ the local component of the second fundamental form for each $i$, $j$ $(1 \leq i, j \leq n)$ and by $A$ the $n \times n$ matrix of $h_{ij}$ with respect to indices $i$, $j$. We choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. Then we have $H = \frac{1}{n}\Big|\sum_{i=1}^{n} \lambda_i\Big|$ and $S = \sum_{i=1}^{n} \lambda_i^2$. In the hypersurface we may put $\phi = \phi_{n+1}$. Then $\phi : T_xM \to T_xM$ satisfies

$$\langle \phi X, Y \rangle := \frac{1}{n} \operatorname{trace} A \langle X, Y \rangle - \langle AX, Y \rangle.$$

It easily check that trace $\phi = 0$ and that

$$|\phi|^2 := \operatorname{trace} \phi^2 = \frac{1}{2n} \sum_{i,j=1}^{n} (\lambda_i - \lambda_j)^2.$$

Hence we get that $|\phi|^2 = 0$ if and only if $M^n$ is totally umbilic.

We study generalizations of the results of the following theorems. Moreover, we also study in the case of $c = -1$.

**Theorem 0.1** (see Alencar and do Carmo [1]). *Let $M^n$ be a compact and orientable hypersurface with constant mean curvature $H$ in $S^{n+1}(1)$. Assume that $|\phi|^2 \leq A_H$ for all $x \in M$. Then*

(i) *either $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic or $|\phi|^2 \equiv A_H$.*

(ii) *$|\phi|^2 \equiv A_H$ if and only if*

(A) *$H = 0$ and $M^n$ is a Clifford torus in $S^{n+1}(1)$, i.e., $M^n$ is a product of spheres $S^{n_1}(r_1) \times S^{n_2}(r_2)$, $n_1 + n_2 = n$, of appropriate radii.*

(B) *$H \neq 0$, $n \geq 3$, and $M^n = S^{n-1}(1) \times S^1(\sqrt{1-r^2}) \subset S^{n+1}(1)$, where $r^2 < \frac{n-1}{n}$.*

(C) *$H \neq 0$, $n = 2$, and $M^2 = S^1(1) \times S^1(\sqrt{1-r^2}) \subset S^3(1)$, where $r^2 \neq \frac{1}{2}$.*

**Theorem 0.2** (see Uchida and Matsuyama [10]). *Let $M^n$ be a complete, connected and orientable submanifold with nonzero constant mean curvature $H$ in $S^{n+2}(c)$. If $|\phi|$ satisfies $|\phi|^2 \leq A_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic hypersurface $S^{n+1}(c)$ of $S^{n+2}(c)$ and*

(i) *either $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic or $|\phi|^2 \equiv A_H$.*

(ii) $|\phi|^2 \equiv A_H$ *if and only if*

(B) $n \geq 3$ *and* $M^n = S^{n-1}(r_1) \times S^1(r_2) \subset S^{n+1}(c)$, *where* $r_1^2 + r_2^2 = \frac{1}{c}$ *and* $r_1^2 < \frac{n-1}{nc}$.

(C) $n = 2$ *and* $M^2 = S^1(r_1) \times S^1(r_2) \subset S^3(c)$, *where* $r_1^2 + r_2^2 = \frac{1}{c}$ *and* $r_1^2 \neq \frac{1}{2c}$.

The purpose of this paper is to prove the following theorems:

**Theorem 1**. *Let $M^n$ be a complete, connected and orientable submanifold with nonzero constant mean curvature $H$ in $S^{n+p}(c)$ $(p \geq 3)$. If $|\phi|$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic submanifold $S^{n+1}(c)$ of $S^{n+p}(c)$, and $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic.*

**Theorem 2.1**. *Let $M^n$ be a complete, connected and orientable hypersurface with constant mean curvature $H > 1$ in $H^{n+1}(-1)$. Assume that $|\phi|^2 \leq A_H$ for all $x \in M^n$. Then*

(i) *either $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic or $|\phi|^2 \equiv A_H$.*

(ii) *$|\phi|^2 \equiv A_H$ if and only if $M^n$ is isometric to $S^{n-1}(r) \times H^1(-\frac{1}{r^2+1})$ for some $r > 0$.*

**Theorem 2.2**. *Let $M^n$ be a complete, connected and orientable submanifold with constant mean curvature $H > 1$ in $H^{n+2}(-1)$. If $|\phi|$ satisfies $|\phi|^2 \leq A_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic hypersurface $H^{n+1}(-1)$ of $H^{n+2}(-1)$ and*

(i) *either $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic or $|\phi|^2 \equiv A_H$.*

(ii) *$|\phi|^2 \equiv A_H$ if and only if $M^n$ is isometric to $S^{n-1}(r) \times H^1(-\frac{1}{r^2+1})$ for some $r > 0$.*

**Theorem 2.3**. *Let $M^n$ be a complete, connected and orientable submanifold with constant mean curvature $H > 1$ in $H^{n+p}(-1)$ $(p \geq 3)$. If $|\phi|$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then $M^n$ lies in a totally geodesic submanifold $H^{n+1}(-1)$ of $H^{n+p}(-1)$, and $|\phi|^2 \equiv 0$ and $M^n$ is totally umbilic.*

The following generalized maximum principle due to Omori [8] and Yau [11] will be used in order to prove our theorems:

**Generalized Maximum Principle** (see Omori [8] and Yau [11]). *Let $M^n$ be a complete Riemannian manifold whose Ricci curvature is bounded from the below and $f \in C^2(M)$ a function bounded from the above on $M^n$. Then, for any $\epsilon > 0$, there exists a point $p \in M^n$ such that*

$$f(p) \geq \sup f - \epsilon, \quad \|\text{grad } f\|(p) < \epsilon \quad \textit{and} \quad \Delta f(p) < \epsilon.$$

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