周期的準対称第二基本形式を持つケーラー超曲面について

On Kaehler hypersurfaces with cyclic semi-symmetric second fundamental form

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Let $\tilde{M}(c)$ be a complex $(n+1)$-dimensional complex space form of constant holomorphic sectional curvature c (i.e. complete, simply connected Kaehler manifold with constant holomorphic sectional curvature, say, c). If c is positive, then $\tilde{M}(c)$ is the complex projective space $P_{n+1}(c)$ with the Fubini-Study metric of constant holomorphic sectional curvature c. If c is negative, then $\tilde{M}(c)$ is the complex hyperbolic space $D_{n+1}(c)$ with the Bergman metiric of constant holomorphic sectional curvature c. If c is zero, then $\tilde{M}(c)$ is the complex Euculidean space C_{n+1} (See[6]).

Let M be a connected manifold of the complex dimension n ($\geq$2) isometrically and holomorphically immersed in $\tilde{M}(c)$. Then we call M a Kaehler hypersurface of $\tilde{M}(c)$. Let R and ∇ be the curvature tensor of M and the covariant differentiation in M, respectively. Futhermore, let A be the second fundamental form of M in $\tilde{M}(c)$ defined on a neighborhood of each point of M. Yumetaro MASHIKO, Satoshi KUROSU and Yoshio MATSUYAMA [2] classified these Kaehler hypersurfaces with regard to the semi-symmetric second fundamental form i.e.,

$$(R(X,Y)A)Z = 0$$

for any X, Y and Z tangent to M. Then M is *totally geodesic* in $\tilde{M}$.
The second fundamental form A is called the *cyclic semi-symmetric second fundametal form* if M satisfies the condition of

$$(R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0$$

for any X, Y and Z tangent to M.

The purpose of this paper is to classify Kaehler hypersurfaces with the cyclic semi-symmetric second fundamental form in a complex space form. We note that this condition is weaker than $(R(X,Y)A)Z = 0$. We prove the following theorem:

Theorem *Let M be a Kaehler hypersurface of the complex dimension $n \geq 2$ in a complex space form $\tilde{M}(c)$ with the cyclic semi-symmetric second fundamental form. Suppose A satisfies*
$(R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0$
on a neighborhood of each point of M for any X, Y and Z tangent to M. If,
(i)$c \neq 0$, then M is totally geodesic,
(ii)$c = 0$, then M is rank of $A \leq 1$.

The author would like to express the will of great thanks for indications and cooperations of Professor Y.Matsuyama on writing this paper.

Let M be a connected manifold of the complex dimension $n \geq 2$ isometrically and holomorphically immersed in a complex space form $\tilde{M}(c)$ of complex dimension $n+1$. Then we call M a Kaehler hypersurface of $\tilde{M}(c)$. A complex structure J and Kaehler metric g on M are induced the complex structure $\tilde{J}$ and the Kaehler metric $\tilde{g}$ of $\tilde{M}(c)$, respectively. Let ∇ (resp. $\tilde{\nabla}$) denote the covariant differentiation in M (resp. $\tilde{M}(c)$). Extend ξ to a normal vector field defined in a neighbourhood U of $x \in M$ and define $-AX$ to be the tangent component of $\tilde{\nabla}_X \xi$ for $X \in T_x M$. AX depends only on ξ at x and X, and we call A the second fundamental form. Let R (resp. $\tilde{R}$) be the curvature tensor of M (resp. $\tilde{M}(c)$). Let X,Y and Z be the tangent vectors on M. Then, we have the following relationships:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi + g(JAX, Y)J\xi, \tag{1}$$

$$\begin{aligned}
\tilde{\nabla}_X \xi &= -AX + s(X)J\xi, && (2)\\
AJ &= -JA, && (3)\\
R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, && (4)\\
R(X,Y)Z &= \tilde{R}(X,Y)Z + g(AY,Z)AX - g(AX,Z)AY \\
&\quad + g(JAY,Z)JAX - g(JAX,Z)JAY, && (5)\\
(\nabla_X A)Y - s(X)JAY &= (\nabla_Y A)X - s(Y)JAX, && (6)\\
\tilde{R}(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y \\
&\quad + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\}, && (7)
\end{aligned}$$

where we call (1) *Gauss' formula* , (2) *Weingarten's formula* , (5) *Gauss equation* and (6) *Codazzi equation*, respectively.

Next, we denote the (1, 1)-type Ricci tensor of M by S. For any point x of $U(x)$, S is defined by

$$SX = \sum_{i=1}^{n} R(X, e_i)e_i + \sum_{i=1}^{n} R(X, Je_i)Je_i,$$

where $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ is an orthonormal basis of the tangent space $T_x M$. Using (5) and (7), we obtain

$$SX = \frac{n+1}{2}cX - 2A^2 X$$

for any X tangent to M on $U(x)$. If A satisfies

$$(R(X,Y)A)Z = 0$$

for any X, Y and Z tangent to M, then the A is said to be the semi-symmetric second fundamental form. If A satisfies

$$(R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0$$

for any X, Y and Z tangent to M, then the A is said to be the cyclic semi-symmetric second fundamental form.

Here let A be the second fundamental form of M in $\tilde{M}(c)$ defined on a neighborfood of each point of M. M is called a Kaehler hypersurface with the recurrent second fundamental form (resp. the birecurrent second fundamental form) if there exists a 1-form α (resp. a coveriant tensor field α of order 2) such that the second fundamental form A of M satisfies $(\nabla_X A)Y = \alpha(X)AY$ (resp. $(\nabla^2_{X,Y}A)Z = (\nabla_X\nabla_Y A - \nabla_{\nabla_X Y}A)Z = \alpha(X,Y)AZ)$ for any X, Y and $Z \in T_xM$(See[7]).

Now, we prepare the following results without proof.

Theorem 1, *Let M be a Kaehler hypersurface in a complex space form $\tilde{M}(c)$. The following conditions are equivalent:*
(i) M has *the recurrent second fundamental form,*
(ii) M has *the birecurrent second fundamental form,*
(iii) $(R(X,Y)A)Z = 0$ *on a neighborhood of each point of M,*
(iv) *M is totally geodesic in $\tilde{M}(c)$.*

Theorem 2, *Let M be a Kaehler hypersurface of the complex dimension $n \geq 1$ in a complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c. If M is Einstein, then M is locally symmetric and either M is of constant holomorphic sectional curvature $\tilde{c}$ and totally geodesic in $\tilde{M}(c)$ or M is locally holomorphically isometric to the complex quadric Q_n in the complex projective space $P_{n+1}(c)$, the latter case arising only when* $c > 0$.

Theorem 3, *Let M be a Kaehler hypersurface of complex dimension $n \geq 2$ with the cyclic Ricci semi-symmetric tensor in a complex space form $\tilde{M}(c), c \neq 0$. Then M is Einstein.*

Theorem 4, *Let M be a Kaehler hypersurface of complex dimension $n \geq 2$ with the nearly parallel Ricci tensor in a complex space form $\tilde{M}(c)$. Then the Ricci tensor of M is parallel.*

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