Asymptotic behavior of energy to Klein-Gordon equations with time dependent potentials

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§1. Scattering for Klein-Gordon equations

We consider the Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u + V(x,t)u = 0, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}, \tag{P}$$

where $u_{tt} = \partial^2 u/\partial t^2$, Δ is the n-dimensional Laplacian, $n \geq 3$, $m = m(x) > 0$ and $V(x,t)$ is a potential which is bounded and continuous in $\mathbf{R}^n \times \mathbf{R}$. Now, we define the energy of the solution of this equation at time t:

$$\begin{cases} \|u(t)\|_E^2 = \dfrac{1}{2}\displaystyle\int_{\mathbf{R}^n} \{u_t(x,t)^2 + |\nabla u(x,t)|^2 + m^2 u(x,t)^2\} dx, \\ \|u(t)\|_{E_V}^2 = \dfrac{1}{2}\displaystyle\int_{\mathbf{R}^n} \{u_t(x,t)^2 + |\nabla u(x,t)|^2 + m^2 u(x,t)^2 + V(x,t)u(x,t)^2\} dx. \end{cases}$$

where E is the Hilbert space with the above energy norms.
In this paper, under suitable condition on $V(x,t)$, we shall treat a nondecay of the energy, and develop a scattering theory.

As is easily seen, we have

$$\|u(t)\|_{E_V}^2 - \frac{1}{2}\int_s^t \int_{\mathbf{R}^n} V_\tau(x,\tau) u^2 dx d\tau = \|u(s)\|_{E_V}^2$$

for any $t > s$. If $V_t(x,t) \leq 0$, then $\|u(t)\|_{E_V}$ is decreasing with t, and a question rises whether it decays or not as t goes to infinity. We conjecture that the perturbed solutions behave like the free solutions asymptotically as $t \to \pm\infty$, that is, we want to find unique $f_0^\pm \in E$ such that

$$\|U(t,0)f - U_0(t) f_0^\pm\|_E \to 0, \quad (t \to \pm\infty)$$

where $U_0(t)$ is the unitary group in E which represents the solution of the free equation $(V = 0)$ and $U(t,s)$ is the evolution operator which maps the solutions at time t to those at time s:

$$u(t) = U(t,s)u(s).$$

In order to state our results, we introduce the following conditions.

(A0) $V(x,t)$ satisfies

$$-\delta\left\{\ m^2+\left(\frac{n-2}{2r}\right)^2\ \right\}\le V(x,t)\le C_1\left\{\ m^2+\left(\frac{n-2}{2r}\right)^2\ \right\},$$

where $0<\delta<1\le C_1$.

(A1) $V(x,t)$ satisfies

$$|V_t(x,t)|\le \eta_1(t)\left\{\ m^2+\left(\frac{n-2}{2r}\right)^2\ \right\},$$

where $\eta_1(t)$ is a positive L^1-function of $t\ge 0$.

(A2) $m(x)$ and $V(x,t)$ satisfy

$$(m^2\psi)_r\le 0,$$

$$(V\psi)_r\le\{\varepsilon\psi'(r)+\eta_2(t)\}\left(\frac{n-2}{2r}\right)^2,$$

where $\varepsilon>0$ is sufficinetly small, $\eta_2(t)$ is a positive L^1-function of $t\ge 0$ and $\psi(r)$ is bounded and monotone increasing function of $r=|x|$ such that

$$0<\psi'(r)<r^{-1}\psi(r),\quad \psi''(r)\le 0.$$

(A3) $V(x,t)$ satisfies

$$|V(x,t)|\le\{C_2\psi'(r)+\eta_3(t)\}\left(\frac{n-2}{2r}\right),$$

where $\eta_3(t)$ is a positive L^1-function of $t\ge 0$.

The following lemma plays an important role in our results.

<u>Lemma 1</u> (Space-time weighted estimate)
Under $(A2)$, there exists $C_3>0$ such that for any $t>s$

$$\int_s^t\int_{\mathbf{R}^n}\left\{\frac{1}{2}\psi'(r)(u_\tau^2+|\nabla u|^2)-\psi''(r)\frac{n-1}{4r}u^2\right\}dxd\tau<C_3\|u(s)\|_E^2.$$

By using this lemma, we shall show that energy nondecay and scattering occur.

<u>Theorem 2</u> (Nondecay)
Assume $(A0)\sim(A3)$,
If $u^s(t)$ is the solution of the initial value probrem

$$\begin{cases}\partial_t^2u^s-\Delta u^s+m^2u^s+Vu^s=0, & t\ge s\\ u^s(s)=u_0(s)\end{cases}$$

and s is sufficiently large, then its energy $\|u^s(t)\|_E$ never decays as $t \to \infty$.

Theorem 3 (Scattering)
Assume $(A0) \sim (A3)$. Then

(i) For every $f \in E$ there exists $f_0^{\pm} \in E$ such that

$$\|U(t,s)f - U_0(t-s)f_0^{\pm}\|_E^2 \to 0, \quad (t \to \pm\infty)$$

we put

$$Z^{\pm} = s - \lim_{t \to \pm\infty} U_0(-t)U(t,0)$$

Then $Z^{\pm}$ defines a nontrivial bounded operator on E.

(ii) If C_2 in $(A3)$ is sufficiently small, then $Z^{\pm}$ gives a bijection on E. Thus, the scattering operator

$$S = Z^+(Z^-)^{-1} : f_0^- \to f_0^+$$

is well defined and also gives a bijection on E.

§2. Strichartz estimates and its applications

We introduce the Strichartz estimates for the Klein-Gordon equations and as its application, we shall show the unique existence of the solutions of the integral equations corresponded to (P). Define

$$\Phi[f](t) = \int_s^t e^{iA(t-\tau)} f(\tau) d\tau.$$

Lemma 4 (Strichartz estimates for Klein-Gordon equations)
Let $n \geq 3$, $\dfrac{n-2}{2n} < \dfrac{1}{p} \leq \dfrac{1}{2}$ and $\dfrac{1}{r} = \dfrac{n}{2}\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$. Then for any time interval I and for any $s \in \bar{I}$ the following estimates are true

$$\|\Phi[f]\|_{L^r(I;L^p)} \leq C_4 \|f\|_{L^{r'}(I;H^{2\mu}_{p'})},$$

$$\|\Phi[f]\|_{L^\infty(I;L^2)} \leq C_4^{1/2} \|f\|_{L^{r'}(I;H^{\mu}_{p'})},$$

and

$$\|e^{iAt}\phi\|_{L^r(I;L^p)} \leq C_4^{1/2} \|\phi\|_{H^\mu},$$

where $\dfrac{1}{r} + \dfrac{1}{r'} = 1$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $\mu = \left(1 + \dfrac{n}{2}\right)\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$ and $A = (m^2 - \Delta)^{1/2}$.

(B0) $V(x,t)$ satisfies

$$V(x,t) \in L^{\nu}(\mathbf{R}; L^q),$$

where

$$0 \le \frac{1}{q} < \frac{2}{n} \quad \text{and} \quad \frac{1}{\nu} = 1 - \frac{n}{2q}.$$

Moreover, $V(x,t)$ satisfies the smallness condition

$$C_4 \|V\|_{L^{\nu}(\mathbf{R};L^q)} < 1$$

where C_4 is a constant given in Lemma 4.

Since this lemma, we obtain the following theorem.

Theorem 5 (Unique existence and space-time L^p-L^q estimates)
Let $n \ge 3$ and assume $(B0)$. Then for each $f = {}^t(f_1, f_2) \in E$,

(i) the integral equation

$$u(t) = \dot{K}(t-s)f_1 + K(t-s)f_2 + \int_s^t K(t-\tau)V(\tau)u(\tau)d\tau$$

has a unique solution in $u(t) \in L^r(\mathbf{R}; H_p^{1/2})$. Here $K(t) = A^{-1}\sin(At)$.

(ii) This solution belongs to $C(\mathbf{R}; H^1)$ and coincides with $U(t,s)f$. Moreover, we have

$$\|u\|_{L^r(\mathbf{R};H_p^{1/2})} \le \frac{2C_4^{1/2}}{1 - C_4\|V\|_{L^{\nu}(\mathbf{R};L^q)}} \|f\|_E$$

and

$$\left\| \int_s^t K(t-\tau)V(\tau)u(\tau)d\tau \right\|_{H^1} \le \frac{2C_4\|V\|_{L^{\nu}(\mathbf{R};L^q)}}{1 - C_4\|V\|_{L^{\nu}(\mathbf{R};L^q)}} \|f\|_E.$$

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