球面内の極小部分多様体の曲率によるピンチング問題
Curvature pinching for minimal submanifolds of a sphere

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Let $S^{n+p}(c)$ be an $(n+p)$-dimensional Euclidean sphere of constant curvature c and let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Let A_ξ be the Weingarten endomorphism associated a normal vector field ξ and T the tensor defined by $T(\xi,\eta) = \mathrm{trace} A_\xi A_\eta$.

Recently, Montiel, Ros and Urbano [7] proved the following: Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Let σ be the second fundamental form of M in $S^{n+p}(c)$. If M is Einstein, $T = k\langle\ ,\ \rangle$ and

$$|\sigma|^2 \leq \frac{np(n+2)}{2(n+p+2)}c$$

then M is isotropic and has the parallel second fundamental form, where $\langle , \rangle$ is the Riemannian metric.

Xia[16] showed: Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then

$$S \geq (n-1)c - \frac{p(n+2)}{2(n+p+2)}c \quad \text{and} \quad T = k\langle\ ,\ \rangle$$

if and only if one of the following conditions is satisfied: A) $S = (n-1)c$ and M is totally geodesic, B) $S = (n-1)c - \frac{p(n+2)}{2(n+p+2)}c$ and M is isotropic and has the parallel second fundamental form.

Using the result of Sakamoto [13], we know that M which is isotropic with parallel second fundamental form is a compact rank one symmetric space. Hence if the immersion ψ of M into $S^{n+p}(c)$ is full, then ψ is one of the following standard ones (See §2): $S^n(c) \to S^n(c)$;$PR^2(\frac{1}{3}c) \to S^4(c)$;$S^2(\frac{1}{3}c) \to S^4(c)$;$CP^2(c) \to S^7(c)$;$QP^2(\frac{3}{4}c) \to S^{13}(c)$;$CP^2(\frac{4}{3}c) \to S^{25}(c)$.

Matsuyama [9] proved the following:

Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$ and ψ the immersion. Then

$$|\sigma(v,v)|^2 \leq \frac{p}{n+p+2}c \quad \text{and} \quad T = k\langle\ ,\ \rangle$$

if and only if one of the following conditions is satisfied:

(A) $|\sigma(v,v)|^2 \equiv 0$ and M is totally geodesic.

(B) $|\sigma(v,v)|^2 = \frac{p}{n+p+2}c$ and M is isotropic and has parallel second fundamental form. Hence if ψ is full, then ψ is one of the following standard ones: $S^n(c) \to S^n(c)$;$PR^2(\frac{1}{3}c) \to S^4(c)$;$S^2(\frac{1}{3}c) \to S^4(c)$;$CP^2(c) \to S^7(c)$;$QP^2(\frac{3}{4}c) \to S^{13}(c)$;$CP^2(\frac{4}{3}c) \to S^{25}(c)$.

Let M be a compact Riemannian manifold, UM its unit tangent bundle, and UM_x the fibre of UM over a point x of M. we suppose that M is isometrically immersed in an $(n+p)$-dimensional Riemannian manifold $\tilde{M}$. We define

$$T : T_x^{\perp}M \times T_x^{\perp}M \to R$$

by the expression

$$T(\xi,\eta) = \mathrm{trace} A_\xi A_\eta,$$

where $T_x^{\perp}M$ is the normal space to M at x. Then T is a symmetric bilinear map.

Let ∇ be the Riemannian connection. A and $\nabla^{\perp}$ are the Weingarten endomorphism and the normal connection. The first and the second covariant derivatives of the normal valued tensor σ are given by

$$(\nabla\sigma)(X,Y,Z) = \nabla_X^{\perp}(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

and

$$\begin{aligned}(\nabla^2\sigma)(X,Y,Z,W) &= \nabla_X^{\perp}((\nabla\sigma)(Y,Z,W)) - (\nabla\sigma)(\nabla_X Y, Z, W) \\ &\quad -(\nabla\sigma)(Y,\nabla_X Z, W) - (\nabla\sigma)(Y,Z,\nabla_X W),\end{aligned}$$

respectively, for any vector fields X, Y, Z and W tengent to M. Let R and $R^{\perp}$ denote the curvature tensor associated with ∇ and $\nabla^{\perp}$, respectively. Then σ and $\nabla\sigma$ are symmetric and for $\nabla^2\sigma$ we have the Ricci-identity

$$\begin{aligned}&(\nabla^2\sigma)(X,Y,Z,W) - (\nabla^2\sigma)(Y,X,Z,W) \\ &\quad = R^{\perp}(X,Y)\sigma(Z,W) - \sigma(R(X,Y)Z,W) - \sigma(Z,R(X,Y)W)\end{aligned} \tag{1}$$

If S and ρ is the Ricci tensor of M and the scalar curvature of M, respectively, since M is a minimal submanifold in $S^{n+p}(c)$, then from the Gauss equation we have

$$S(v,w) = (n-1)c\langle v,w\rangle - \sum_{i=1}^{n}\langle A_{\sigma(v,e_i)}e_i, w\rangle, \tag{2}$$

$$\rho = n(n-1)c - |\sigma|^2. \tag{3}$$

LEMMA. *Let M be an n-dimensional minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then for $v \in UM_x$ we have*

$$\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i,e_i,v) = \sum_{i=1}^{n}|(\nabla\sigma)(e_i,v,v)|^2 + nc|\sigma(v,v)|^2 \tag{4}$$

$$
\begin{aligned}
&+ \quad 2\sum_{i=1}^{n}\langle A_{\sigma(v,v)}e_i, A_{\sigma(e_i,v)}v\rangle - 2\sum_{i=1}^{n}\langle A_{\sigma(v,e_i)}e_i, A_{\sigma(v,v)}v\rangle \\
&- \quad \sum_{i=1}^{n}\langle A_{\sigma(v,v)}e_i, A_{\sigma(v,v)}e_i\rangle.
\end{aligned}
$$

The purpose of this paper is to prove the following:

THEOREM 1. *Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then*

$$|\sigma|^2 \leq \frac{np(p+2)}{2(n+p+2)}c \quad \text{and} \quad T = k\langle\ ,\ \rangle$$

if and only if one of the following conditions is satisfied:
(A) $|\sigma|^2 \equiv 0$ and M is totally geodesic.
(B) $|\sigma|^2 = \frac{np(p+2)}{2(n+p+2)}c$ and M is isotropic and has parallel second fundamental form. Hence if ψ is full, then ψ is one of the following standard ones: $S^n(c) \to S^n(c)$;$PR^2(\frac{1}{3}c) \to S^4(c)$;$S^2(\frac{1}{3}c) \to S^4(c)$;$CP^2(c) \to S^7(c)$;$QP^2(\frac{3}{4}c) \to S^{13}(c)$;$CP^2(\frac{4}{3}c) \to S^{25}(c)$.

Here,in order to prove the Theorem 2. we used the following generalized maximum principle due to Omori [11] and Yau [18].

Generalized Maximum Principle. (Omori [11] and Yau [18])*Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f \in C^2(M)$ a function bounded from above on M^n. Then, for any $\epsilon\rangle 0$, there exists a point $p \in M^n$ such that*

$$f(p) \geq \sup f - \epsilon, \qquad ||\text{grad } f||\langle\epsilon, \qquad \Delta f(p) < \epsilon.$$

THEOREM 2. *Let M be an n-dimensional complete minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then if $|\sigma|^2 \leq \frac{np(p+2)}{2(n+p+2)}c$ and $T = k\langle\ ,\ \rangle$, then the second fundamental form is parallel.*

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