小平の消滅定理について
On Kodaira's Vanishing Theorem

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1 Introduction

In this thesis, we discuss about Kodaira-Akizuki-Nakano vanishing theorem and its proof from the view point of differential geometry. We explain the notation, which we need to state the theorem. The thesis consists of 9 sections.

In Section 2, we discuss de Rham cohomologies and Dolbeault cohomologies and relationships between them. In Section 3, we treat connections. In Section 4, we introduce Kähler manifolds. In section 5, we discuss about Hodge identities. In Section 6, we introduce Hodge theorem which are used in the proof essentially. In Section 7, we give Kodaira-Akizuki-Nakano vanishing theorem and its proof.

2 Motivation

We introduce Kodaira-Akizuki-Nakano vanishing theorem. Its proof from the view point of algebraic geometry given by P.Deligne and L.Illusie. ([5]) In this thesis, its proof from the view point of differential geometry given by Y.Akizuki and S.Nakano. ([4]) Kodaira embedding theorem is provided as application of vanishing theorem.

3 Kodaira-Nakano's vanishing theorem

Let M be a Kähler manifold of dimension n, E a line bundle on M and $\mathbf{\Omega}^p(E)$ the sheaf of holomorphic p-forms on E.

Theorem 1. *(Kodaira-Nakano's vanishing theorem)*
If E is a positive line bundle,then

$$H^q(M, \mathbf{\Omega}^p(E)) = 0 \quad \textit{for } p + q > n.$$

Where $H^q(M, \mathbf{\Omega}^p(E))$ is a sheaf cohomology.

To proof the theorem, we use isomorphism between sheaf cohomology and the space of harmonic forms. By using the isomorphism, harmonic form $\eta = 0$ then we obtain $H^q(M, \mathbf{\Omega}^p(E)) = 0$. We introduce de Rham cohomologies, Dolbeault cohomorogies and the space of harmonic forms to state the isomorphism.

Let M be a complex manifold of dimension n, $A^p(M)$ the space of complex-valued p-forms on M, $d : A^p(M) \to A^{p+1}(M)$ a differential form, $Z^p(M)$ the subspace of closed complex-valued p-forms on M.

The quotient group

$$H^p_{DR}(M) := \frac{Z^p(M)}{dA^{p-1}(M)}$$

is called *the de Rham cohomorogy group of* M.

Let $z = (z_1, z_2, \ldots, z_n)$ be a point belonging to M.

$$T'^*_z(M) := \langle dz_1, dz_2, \ldots, dz_n \rangle,$$

and

$$T''^*_z(M) := \langle d\bar{z}_1, d\bar{z}_2, \ldots, d\bar{z}_n \rangle$$

the vector spaces spanned by $dz_i, d\bar{z}_i$. The k-forms are decomposed into (p,q)-forms:

$$A^k(M) = A^{k,0}(M) \oplus A^{k-1,1}(M) \oplus A^{k-2,2}(M) \oplus \cdots \oplus A^{0,k}(M) = \bigoplus_{p+q=k} A^{p,q}(M),$$

where

$$A^{p,q}(M) = \left\{ \varphi \in A^k(M); \varphi|_U \in \bigwedge^p T'^*_z(M) \otimes \bigwedge^q T''^*_z(M), z \in U \subset M \right\}.$$

In particular, $d \in A^{p,q}(M)$. We obtain $d\varphi \in A^{p+1,q}(M) \oplus A^{p,q+1}(M) \subset A^{p+q+1}(M)$.

We define operators

$$\begin{cases} \partial \,:\, A^{p,q}(M) \to A^{p+1,q}(M) \\ \bar{\partial} \,:\, A^{p,q}(M) \to A^{p,q+1}(M) \end{cases}$$

Let $Z^{p,q}_{\bar{\partial}}(M)$ a space of $\bar{\partial}$-closed forms of type (p,q). We define *the Dolbeault cohomology group* by

$$H^{p,q}_{\bar{\partial}}(M) \;:=\; \frac{Z^{p,q}_{\bar{\partial}}(M)}{\bar{\partial}(A^{p,q-1}(M))}$$

for each (p,q).

Theorem 2. *(The Dolbeault theorem)*
Let M be a complex manifold of dimension n, and $\mathbf{\Omega}^p$ a sheaf of holomorphic p-forms on M. Then we have

$$H^q(M, \mathbf{\Omega}^p) \cong H^{p,q}_{\bar{\partial}}(M).$$

From the inner product on $A^{p,q}(M)$, we define the adjoint operator $\bar{\partial}^* : A^{p,q}(M) \to A^{p,q-1}(M)$ of $\bar{\partial}$ by $(\bar{\partial}^*\psi, \eta) = (\psi, \bar{\partial}\eta)$ for all $\eta \in A^{p,q-1}(M)$. We define $\bar{\partial}$-Laplacian

$$\mathbf{\Delta}_{\bar{\partial}} : A^{p,q}(M) \to A^{p,q}(M)$$

by $\mathbf{\Delta}_{\bar{\partial}}\psi := (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\psi$ for ψ is differential forms. All defferential forms satisfying *the Laplace equation*

$$\mathbf{\Delta}_{\bar{\partial}}\psi = 0$$

are called *harmonic forms*: the space of harmonic forms of type (p,q) is denoted by $\mathcal{H}^{p,q}(M)$ and called *the harmonic space*.

By Hodge theorem, we obtain isomorphism:

$$\mathcal{H}^{p,q}(M) \cong H^{p,q}_{\bar{\partial}}(M)$$

By the Dolbeault theorem, we have the isomorphism

$$H^q(M, \mathbf{\Omega}^p) \cong \mathcal{H}^{p,q}(M),$$

where $H^q(M, \mathbf{\Omega}^p)$ is a sheaf cohomology. This isomorphism holds on line bundle. It plays a key role to prove Kodaira-Nakano vanishing theorem.

We treat Kähler metric and Kähler manifold in this thesis. Let M be a compact complex manifold of dimension n with a hermitian metric ds^2. Here we have the hermitian metric ds^2 locally:

$$ds^2 = \sum_{i,j=1}^{n} h_{ij} dz_i \otimes d\bar{z}_j = \sum_{i=1}^{n} \varphi_i \otimes \bar{\varphi}_i,$$

where $z = (z_1, \ldots, z_n)$ is a local coordinates on M, $(\varphi_1, \ldots, \varphi_n)$ is a coframe for the hermitian metric. We call ds^2 a Kähler matric if its associated $(1,1)$-form

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} \varphi_i \wedge \bar{\varphi}_i$$

is d-closed.

Definition. *A compact complex manifold M with a hermitian metric ds^2 is called a Kähler manifold if ds^2 is a Kähler metric.*

We discuss about Hodge identities on Kähler manifold. Let M be a Kähler manifold of dimension n ds^2 a Kähler metric associated $(1,1)$-form ω. We define an additional operator

$$L : A^{p,q}(M) \to A^{p+1,q+1}(M)$$

by

$$L(\eta) = \eta \wedge \omega$$

for each p, q. Let

$$\Lambda = L^* : A^{p,q}(M) \to A^{p-1,q-1}(M)$$

be its adjoint operator. Let $\{A^p(M), d\}_p$ be a complex. We have operators ∂ and $\bar{\partial}$ satisfying $d = \partial + \bar{\partial}$. We set $d^c = \dfrac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$. We get identities with the operators above, which are called the *Hodge identities.*

Lemma 1. *(The Hodge identities)*
Let Λ, L, d, d^c be as above. We have an identity

$$[\Lambda, d] = -4\pi d^{c^*},$$

where $[\Lambda, d] := \Lambda d - d\Lambda$. Equivalently we have

$$[L, d^*] = 4\pi d^c.$$

We regard the Kähler metric as the Euclidean metric locally. Therefore, it is enough to prove the assertion on $\mathbb{C}^n$.

We consider the commutator $[L, \Lambda]$ of the operator L and Λ. Hodge identities and $[L, \Lambda]$ are used in the proof of Kodaira vanishing theorem. By the straight computation, we have the equalities:

$$[L, \Lambda] = n - p - q.$$

In the proof of Kodaira vanishing theorem, we compute $\sqrt{-1}([\Lambda, \Theta]\eta, \eta)$, where η is harmonic form and Θ is curvature form. Then we obtain $\eta = 0$ when $p + q > n$. Which complete the proof.

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