平面代数曲線の二次変換と種数について

On the quadratic transformations and genera of algebraic plane curves

中央大学大学院理工学研究科 数学専攻 劉 彬

1 Introduction

Our aim is to compute the genera of plane curves with some singularities. For our purpose, the quadratic transformations is a key tool.

Afer we discuss the theory of quadratic transformations, we consider the algebraic curve defined by the polynomial $F = (X^2 + Y^2)^3 - 4X^2Y^2 = 0$.

2 Algebraic curves

Definition 1. *Let C be a plane curve. A point P on C is called a simple one if the local ring $\mathcal{O}_{C,P}$ is regular.*
Let F be an element of $k[x, y, z]$. For simplicity, we denote by F a plane curve $V(F) = C \subset \mathbb{P}^2 = Projk[x, y, z]$.

Definition 2. *Let C be a curve. A point P on C is called singular one if the local ring $\mathcal{O}_{C,P}$ is not regular.*

Definition 3. *Let C be a curve. C is called non-singular curve if every point on C is simple point.*

Definition 4. *Let F be a curve in $\mathbb{P}^2$, and we set $f = F(X, Y, 1)$. We let $P = (0, 0, 1)$ belong to F. Namely, there exists $m \geq 1$ such that $f = f_m + f_{m+1} + \cdots + f_n$, where f_i is homogenous polynomials of degree i in $k[X, Y]$, and $f_m \neq 0$. Then the lowest degree m is called a multiplicity of F at $P = (0, 0, 1)$, denoted by $m_p(F)$. If m =2, P is called a double point. If m =3, P is called a triple point.*

Definition 5. *Let $f_m = \prod {L_i}^{r_i}$, where L_i are distinct lines. L_i is called tangent lines to F at $P = (0, 0, 1)$. r_i is called multiplicity of the tangent. P is called ordinary multiple point if F has m distinct tangent lines at P. An ordinary double point is called a node.*

Definition 6. *Let F, G be plane curves. Let $P = (x, y) \in \mathbb{A}^2$. $I(P, F \cap G) :=$ $\mathrm{dim}(\mathcal{O}_\mathrm{P}(\mathbb{A}^2)/(\mathrm{F}, \mathrm{G}))$ is called the intersection number of F and G at P.*

Theorem 1. *(Bezout's Theorem)*
Let F, G be projective plane curves of degrees m and n respectively. Assume F and G have no common component. Then $\sum_P I(P, F \cap G) = mn$.

Theorem 2. *(Riemann's Theorem)*
Let D be a divisor on a curve X of genus g, $l(D) := dim_k H^0(X, \mathcal{L}(D))$. Then there is a constant g such that $l(D) \geq deg(D) + 1 - g$ for all divisors D. The smallest such g is called the genus of X. g is a non-negative integer.

Theorem 3. *Let C be a plane curve with only ordinary multiple points. Let n be the degree of $C, r_P = m_P(C)$. Then the genus g of C is given by the formula* $g = \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{r_P(r_P - 1)}{2}$.

Theorem 4. *Let C be a plane curve of degree $n, r_P = m_P(C), P \in C$,* $\frac{(n-1)(n-2)}{2} = \sum_{P \in C} \frac{r_P(r_P - 1)}{2}$*, then C is rational.*

Theorem 5. *(Riemann-Roch Theorem)*
Let D be a divisor on a curve X of genus g, $l(D) := dim_k H^0(X, \mathcal{L}(D))$, $K :=$ canonical divisor on X, $\mathcal{L}(D) :=$ an invertible sheaf on X. Then $l(D) - l(K - D) = degD + 1 - g$.

3 Quadratic Transformations

Definition 7. $P = (0:0:1), P' = (0:1:0), P'' = (1:0:0) \in \mathbb{P}^2$ *is called the fundamental points.*

Definition 8. $L = V(Z), L' = V(Y), L'' = V(X)$ *is called the exceptional lines.*
$Q : \mathbb{P}^2 - \{P, P', P''\} \longrightarrow \mathbb{P}^2, U := \mathbb{P}^2 - V(XYZ)$.
$(x, y, z) \longmapsto Q(x : y : z) := (yz : xz : xy)$

Definition 9. *Q is called standard quadratic transformation when $Q(X : Y : Z) := (YZ : XZ : XY)$, for $(X : Y : Z) \in U$.*

Definition 10. *Let $F \in k[X, Y, Z]$ be a equqtion of $C, n = degF$. F^Q is called algebraic transform when $F^Q := F(YZ, XZ, XY)$. $degF^Q = 2n$.*

Definition 11. *Let $m_P(C) = r, m_{P'}(C) = r', m_{P''}(C) = r''$. Then $F^Q = Z^r Y^{r'} X^{r''} F'$, where X, Y, Z do not divide F'. F' is called proper transformation of F.*

Definition 12. *C is called in good position if no exceptional line is tangent to C at a fundamental point.*

Definition 13. *C is called in excellent position if C is in good position, and $I(P, L \cap C) = n, I(P, L' \cap C) = n - r', I(P, L'' \cap C) = n - r''$.*

Proposition 1. *Let $C \in \mathbb{P}^2$ be an irreducible curve, $P \in C$. Then*

(1) If $m_P(C) = r$, then Z^r is the largest power of Z which divides F^Q.

(2) deg $F' = 2n - r - r' - r''$, $(F')' = F$. F' is irreducible, and $V(F') = C'$.

(3) $m_P(F') = n - r' - r'', m_{P'}(F') = n - r - r'', m_{P''}(F') = n - r - r'$.

(4) If C is in good position, C' is too.

4 Computation

$F = (X^2 + Y^2)^3 - 4X^2Y^2 = 0$. $V(F) \subseteq \mathbb{A}^2$.
Then F can be changed to $f(x, y, z) = (x^2 + y^2)^3 - 4x^2y^2z^2 = 0$.
The singular points are $(x : y : z) = (0 : 0 : 1)$ and $(\pm i : 1 : 0)$.

Case 1. When $P = (0 : 0 : 1)$, $r_{(0:0:1)} = 4$.

Case 2. When $P = (i : 1 : 0)$, $r_{(i:1:0)} = 2$.

Case 3. When $P = (-i : 1 : 0)$, $r_{(-i:1:0)} = 2$.

Let

$$\begin{cases} L'' := V(Y - X) = V(x - y) \\ L' := V(X + Y) = V(x + y) \\ L := V\left(X - \dfrac{1}{\sqrt{2}}\right) = V(z - \sqrt{2}x) \end{cases}$$

Let

$$\begin{cases} U := x + y \\ V := x - y \\ W := z - \sqrt{2}x \end{cases}$$

Then

$$\begin{cases} P(x, y, z) = (0 : 0 : 1) \\ P'(x, y, z) = \left(\dfrac{1}{\sqrt{2}} : -\dfrac{1}{\sqrt{2}} : 1\right) \\ P''(x, y, z) = \left(\dfrac{1}{\sqrt{2}} : \dfrac{1}{\sqrt{2}} : 1\right) \end{cases}$$

It can be changed to

$$\begin{cases} P(U, V, W) = (0 : 0 : 1) \\ P'(U, V, W) = (0 : 1 : 0) \\ P''(U, V, W) = (1 : 0 : 0) \end{cases}$$

Therefore $f(U,V,W) = \frac{1}{8}(U^2+V^2)^3 - \frac{1}{8}(U^2-V^2)^2(U+V+\sqrt{2}W)^2 = 0$.

Hence we put $f := (U^2+V^2)^3 - (U^2-V^2)^2(U+V+\sqrt{2}W)^2$.

When $P = (0:0:1)$, $m_{(0:0:1)}(C) = 4$.

When $P' = (0:1:0)$, $m_{(0:1:0)}(C) = 1$.

When $P'' = (1:0:0)$, $m_{(1:0:0)}(C) = 1$.

$f^Q(U,V,W) = 2W^4U^1V^1 \times \{W^2[2UV(U^2+V^2) - (U^2-V^2)^2] - 2\sqrt{2}W(U+V)(U^2-V^2)^2 - UV(U^2-V^2)^2\}$

Let $f'(U,V,W) = W^2\left\{2UV(U^2+V^2) - (U^2-V^2)^2\right\} - 2\sqrt{2}W(U+V)(U^2-V^2)^2 - UV(U^2-V^2)^2$.

The singular points are $(0:0:1)$ and $(-1:1:0)$ and $(1:1:0)$.

Case 1. When $(U:V:W) = (0:0:1)$, $r_{(0:0:1)} = 4$.

Case 2. When $(U:V:W) = (1:1:0)$, $r_{(1:1:0)} = 2$.

Case 3. When $(U:V:W) = (-1:1:0)$, $r_{(-1:1:0)} = 2$.

Case 4. When $(U:V:W) = (i+1:i-1:-\sqrt{2}i)$, $r_{(i+1:i-1:-\sqrt{2}i)} = 2$.

Case 5. When $(U:V:W) = (-i+1:i-1:\sqrt{2}i)$, $r_{(-i+1:i-1:\sqrt{2}i)} = 2$.

So $\frac{(n-1)(n-2)}{2} = \frac{(6-1)(6-2)}{2} = 10$.

$\frac{\Sigma r_p(r_p-1)}{2} = 2 \times \frac{2(2-1)}{2} + \frac{4(4-1)}{2} + 2 \times \frac{2(2-1)}{2} = 2+6+2 = 10$.

The genus is $g = \frac{(n-1)(n-2)}{2} - \frac{\Sigma r_p(r_p-1)}{2} = 10 - 10 = 0$.

Since $\frac{(n-1)(n-2)}{2} = \frac{\Sigma r_p(r_p-1)}{2}$, we can say $f(U,V,W)$ is rational.

References

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