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Discrete and Continuous Dynamics in Nonlinear Monopolies*

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Abstract

Dynamic monopolies are investigated with discrete and continuous time scales by assuming general forms of the price and cost functions. The existence of the unique profit maximizing output level is proved. The discrete model is then constructed with gradient adjustment. It is shown that the steady state is locally asymptotically stable if the speed of adjustment is small enough and it goes to chaos through period-doubling cascade as the speed becomes larger. The non-negativity condition that prevents time trajectories from being negative is derived. The discrete model is converted into the continuous model augmented with time delay and inertia. It is then demonstrated that stability can be switched to instability and complex dynamics emerges as the length of the delay increases and that instability can be switched to stability as the inertia coefficient becomes larger. Therefore the delay has the destabilizing effect while the inertia has the stabilizing effect.

Keywords: Time delay, Inertia, Gradient dynamics, Continuous and discrete dynamics, Bounded rationality, Monopoly

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1 Introduction

The literature on monopolies and oligopolies plays a central role in mathematical economics. The existence and uniqueness of the equilibrium were the research issues in early stage and then dynamic extensions become the main topic of researchers. Linear models were first examined, where local asymptotical stability implies global stability. Each model is based on a particular output adjustment scheme. In applying best response dynamics, global information is needed about the profit function while in the case of gradient adjustments only local information is needed to assess the marginal profit. The early results on static and dynamic oligopolies are summarized in Okuguchi (1976) and their multiproduct generalization are discussed in Okuguchi and Szidarovszky (1999). During the last two decades an increasing attention has been given to nonlinear dynamics. Bischi et al. (2010) gives a comprehensive summary of the newer development. As a tractable case many authors have examined dynamic monopolies. Baumol and Quandt (1964) have investigated cost-free monopolies, the dynamic extensions of which were examined in both discrete and continuous time scales. Their adjustment scheme resulted in convergent processes. Puu (1995) assumed cubic price and linear cost functions and considered only discrete time scales. Naimzada and Ricchiuti (2008) assumed cubic price and linear cost functions, and their model was generalized by Askar (2013) with a general from of the price function keeping the linearity of the cost function. It is shown in these studies that chaotic dynamics arises via period doubling bifurcation.

In this paper, we reconsider monopoly dynamics in three different points of view. First, we generalize the model of Askar (2013) by introducing more general types of the cost function and study local and global stability with the non-negativity condition that prevents time-trajectories from being negative. The discrete-time model is converted into the continuous-time model by introducing a time delay and an inertia in the direction of output change. Second, after examining the stability with continuous time scale, we reveal that the discrete model is less stable than the continuous model. Lastly, we numerically and analytically show that the continuous model gives rise to complex dynamics involving chaos due to delay and inertia.

The paper is organized as follows. In Section 2, the discrete-time model is presented, conditions are derived for local asymptotic stability. In Section 3, the continuous-time model is constructed from the discrete-time model. Stability with respect to time delay and inertia is considered. In the final section, concluding remarks are given.

2 Discrete Time Model

In this section we construct a discrete time dynamic model of a monopoly, after determining the profit maximizing output. Consider a monopoly, where q is its output, $p(q) = a - bq^{\alpha}$ the price function ($a,\ b > 0,\ \alpha \geq 1$) and $C(q) = cq^{\beta}$ ($c > 0,\ \beta \geq 2$) its cost function. The profit of the monopoly is given as

$$\pi(q) = (a - bq^{\alpha})q - cq^{\beta}.$$

By differentiation

$$\pi'(q) = a - b(\alpha + 1)q^{\alpha} - c\beta q^{\beta-1} \tag{1}$$

and

$$\pi''(q) = -b\alpha(\alpha + 1)q^{\alpha-1} - c\beta(\beta - 1)q^{\beta-2} < 0.$$

So $\pi(q)$ is strictly concave in q, and since $\pi'(0) = a > 0$ and $\lim_{q\to\infty} \pi'(q) = -\infty$, there is a unique positive profit maximizing output, $\bar{q}$, which is the solution of the first-order condition

$$b(\alpha + 1)q^{\alpha} + c\beta q^{\beta-1} = a. \tag{2}$$

The left hand side is strictly increasing, its value is 0 at $q = 0$ and converges to infinity as $q \to \infty$, therefore the unique positive solution can be obtained by simple computer methods (see, for example, Szidarovszky and Yakowitz 1978). In the numerical considerations to be done below, we always use the following specification of the parameters,

$$a = 4,\ b = 3/5 \text{ and } c = 1/2$$

which are also used by Naimzada and Ricchiuti (2008) and Askar (2013). Equation (2) is depicted by the roughly-meshed surface shown in Figure 1 where the black dots on the surface represent the values of $\bar{q}$ for integer values of α and β in interval $[2, 8]$. It can be seen that the profit maximizing output is decreasing in α and β.

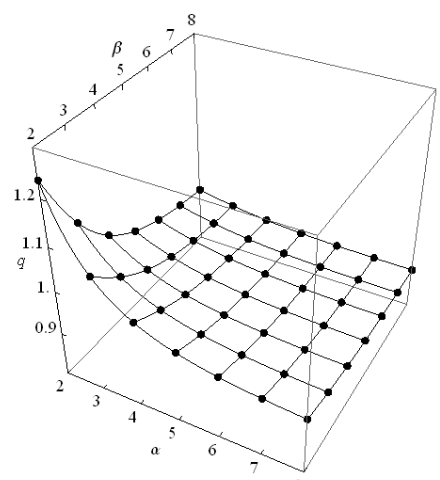


Figure 1. Determination of the optimal q

In applying gradient dynamics it is assumed that the monopoly adjusts its output level in proportion to the marginal profit,

$$q(t+1) - q(t) = k\pi'(q(t))$$

where $k > 0$ is the speed of adjustment. Substituting equation (1) results in the following adjustment equation of output:

$$q(t+1) = q(t) + k\left(a - b(\alpha+1)q(t)^{\alpha} - c\beta q(t)^{\beta-1}\right). \tag{3}$$

If $\alpha = 1$ and $\beta = 2$, then this equation is linear with uninteresting global dynamics. So will assume that either $\alpha > 1$ or $\beta > 2$ or both occur. So under this assumption, equation (3) is a nonlinear difference equation, the local asymptotic stability of which can be examined by linearization. Equations (2) and (3) imply that the steady state of this system is the profit maximizing output $\bar{q}$. Linearization of the right hand side around $\bar{q}$ results in the linear equation

$$q_\delta(t+1) = \left[1 - k\left(b\alpha(\alpha+1)\bar{q}^{\alpha-1} + c\beta(\beta-1)\bar{q}^{\beta-2}\right)\right] q_\delta(t)$$

with $q_\delta(t) = q(t) - \bar{q}$. Let

$$A = b\alpha(\alpha+1)\bar{q}^{\alpha-1} \text{ and } B = c\beta(\beta-1)\bar{q}^{\beta-2},$$

then we have

$$q_\delta(t+1) = (1 - k(A+B))\, q_\delta(t).$$

The steady state of (3) is locally asymptotically stable if

$$|1 - k(A+B)| < 1$$

and locally unstable if

$$|1 - k(A+B)| > 1.$$

Thus we have the following result on local stability:

Theorem 1 *The steady state of system (3) is locally asymptotically stable if $k < k^S$ and locally unstable if $k > k^S$ where*

$$k^S = \frac{2}{A+B}.$$

If we write difference equation (3) in function iteration form, then it becomes

$$q(t+1) = \varphi(q(t))$$

where

$$\varphi(q) = q + k\left(a - b(\alpha+1)q^{\alpha} - c\beta q^{\beta-1}\right)$$

with

$$\varphi'(0) = \begin{cases} 1 & \text{if } \alpha > 1 \text{ and } \beta > 2, \\ \\ 1 - kc\beta(\beta - 1) & \text{if } \alpha > 1 \text{ and } \beta = 2, \\ \\ 1 - kb(\alpha + 1)\alpha & \text{if } \alpha = 1 \text{ and } \beta > 2 \end{cases}$$

and

$$\lim_{q \to \infty} \varphi'(q) = -\infty \text{ and } \varphi''(q) < 0.$$

Since the right hand side of difference equation (3) takes a unimodal profile, the steady state could proceed to chaotic fluctuations via the period-doubling bifurcation if $\varphi(q)$ has strong nonlinearities and its steady state is locally unstable. Naimzada and Ricchiuti (2008) numerically confirm the generation of complex dynamics under $\alpha = 3$ and $\beta = 1$. Later Askar (2013) extends their analysis to the case with $\alpha \geq 3$. We also conduct numerical simulations on global behavior in the more general case of $\alpha \geq 1$ and $\beta \geq 2$. Figure 2 is a bifurcation diagram with $\alpha = 4$ and $\beta = 2$ in which the steady state loses stability at $k = k^S (\simeq 0.154)$. It is seen that the trajectories exhibit periodic and aperiodic fluctuations and are economically feasible (i.e., non-negative) for $k < k^N (\simeq 0.228)$. In Appendix we consider the non-negativity condition that guarantees the non-negativity of the trajectories for all $t \geq 0$ and determine the value of k^N. The main result obtained there is summarized as follows:

Theorem 2 *There is a positive k^N such that $R(q_m(k^N), k^N) = q_M(k^N)$ and the non-negativity condition*

$$R(q_m(k), k) \leq q_M(k)$$

holds for $k \leq k^N$ where, given k, $R(q, k) = \varphi(q)$, $q_m(k)$ maximizes $\varphi(q)$ for $q \geq 0$ and $q_M(k)$ solves $\varphi(q) = 0$.

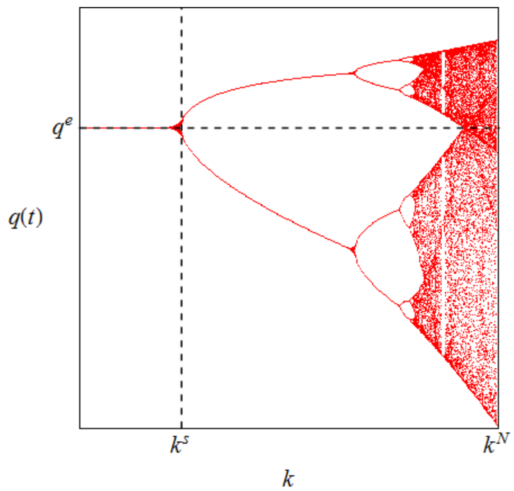


Figure 2. Bifurcation diagram of (3) with respect to k

3 Continuous Time Model

In this section we consider continuous-time dynamics based on the discrete-time model. Notice first that the discrete system can be rewritten as

$$[q(t+1) - q(t)] + q(t) = \varphi(q(t)).$$

There are many ways to transform a discrete-time model into a continuous-time model. One of the simplest ways is to replace the difference, $q(t+1) - q(t)$, with the derivative $\dot{q}(t)$,

$$\dot{q}(t) = -q(t) + \varphi(q(t))$$

or

$$\dot{q}(t) = k\pi'(q(t)).$$

Since $\pi''(q) < 0$, this continuous-time model is always locally asymptotically stable. Furthermore, its trajectory cannot be negative.[1] Another simple way in obtaining a corresponding continuous model is the following. We first assume delayed adjustment (i.e., $\varphi(q(t-\tau))$) and second, an inertia in the direction of output change (i.e., $q(t+1) - q(t) = \sigma\dot{q}(t)$) to get the delay differential equation

$$\sigma\dot{q}(t) = -q(t) + \varphi(q(t-\tau))$$

[1]When it goes to negative from positive, the trajectory has to go through zero where $\dot{q}(t) = ka > 0$. So the trajectory turns back to the positive region.

or

$$\sigma \dot{q}(t) = -q(t) + q(t-\tau) + k\left[a - b(\alpha+1)q(t-\tau)^{\alpha} - c\beta q(t-\tau)^{\beta-1}\right] \quad (4)$$

where $\sigma > 0$ is a constant and $\tau > 0$ is the delay.[2] It is clearly seen that the delay differential equation (4) reduces to the difference equation (3) in discrete time step τ if $\sigma = 0$. This suggests that the evolution in discrete time might describe well the evolution in continuous time if σ is small enough. In the following, we pursue a possibility that the delay contiuous-time model gives rise to complex dynamics when its discrete version also generates complex dynamics and shed light on the condition under which this is a case.

The linearization with respect to $q(t-\tau)$ results in the linear equation

$$\sigma \dot{q}_{\delta}(t) = -q_{\delta}(t) + (1 - k(A+B))\, q_{\delta}(t-\tau).$$

In order to get the characteristic equation, assume that $q(t) = e^{\lambda t}u$, then

$$\sigma\lambda + 1 - (1 - k(A+B))\, e^{-\lambda\tau} = 0. \quad (5)$$

We first show that the roots of the characteristic equation (5) are single. On the contrary, suppose that λ is not single. Then λ is a root of its derivative

$$\sigma + (k(A+B) - 1)e^{-\lambda\tau}(-\tau) = 0$$

from which

$$(k(A+B) - 1)e^{-\lambda\tau} = \frac{\sigma}{\tau}.$$

Substituting this relation into equation (5) gives

$$\sigma\lambda + 1 + \frac{\sigma}{\tau} = 0$$

where the only multiple eigenvalue can be

$$\lambda = -\frac{1}{\tau} - \frac{1}{\sigma}.$$

Consequently all pure complex roots are single.

If $\tau = 0$, then

$$\lambda = -\frac{k}{\sigma}(A+B) < 0.$$

Hence the steady state is locally asymptotically stable if there is no delay (i.e., $\tau = 0$) regardless of the value of the speed of adjustment. If $\sigma = 0$, then the continuous system is reduced to the discrete system and its stability depends on the value of the speed of adjustment as confirmed in Theorem 1. Our main concern in this section is placed on two issues: one is whether a positive delay destabilizes the stable steady state and the other is whether a positive inertia

[2] For this transformation, see Berezowski (2001) where the logistic map is connected with some physical process of definite inertia.

stabilizes the unstable steady state. The remaining part of this section is divided into two parts, the first issue is considered in the first part and so is the second issue in the second part. Before proceeding, Figure 3 depicts a bifurcation diagram of the delay differential equation (4) with respect to k similarly to Figure 2. For each value of k the equation is simulated for 2000 times where $\tau = 2$ and $\sigma = 0.1$. The data for $1900 < t < 2000$ are plotted vertically above the value of k. There are similarities and dissimilarities between this figure and Figure 2. For relatively smaller values of k, the continuous system is stable and thus trajectories converge to the stationary state. For values greater than the threshold value of k_1^S, a limit cycle appears with period-doubling and then bifurcates to a period-8 limit cycle with two-times doubling at the second threshold value of k_2^S. For even larger values of k, dynamic behavior of output is very aperiodic. So we will look more carefully into how the delay affects this bifurcation process.

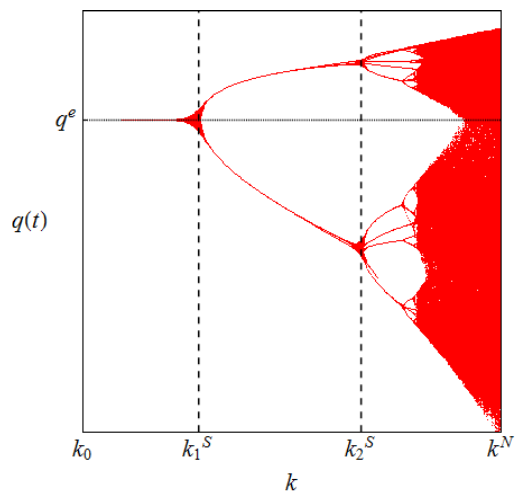


Figure 3. Bifurcation diagram of (4) with respect to k

3.1 Delay Effect

In order to find stability switches in the case of $\tau > 0$, we assume that $\lambda = i\omega$ with $\omega > 0$. Substituting it into equation (5) yields

$$\sigma i\omega + 1 - (1 - k(A + B)) (\cos \omega\tau - i \sin \omega\tau) = 0.$$

By separating the real and imaginary parts, we get two equations for the two unknowns τ and ω:

$$1 - [1 - k(A + B)] \cos \omega\tau = 0,$$

$$\sigma\omega + [1 - k(A + B)] \sin \omega\tau = 0,$$

which can be written as

$$
\begin{aligned}
\cos \omega\tau &= \frac{1}{1 - k(A + B)}, \\
\sin \omega\tau &= -\frac{\sigma\omega}{1 - k(A + B)}.
\end{aligned}
\tag{6}
$$

Adding the squares of the two equations and arranging the terms, we have

$$[1 - k(A + B)]^2 = 1 + (\sigma\omega)^2$$

or

$$\omega^2 = \frac{[1 - k(A + B)]^2 - 1}{\sigma^2}. \tag{7}$$

If $|1 - k(A + B)| < 1$, then there is no positive solution of ω, therefore there is no stability switch, that is, the steady state is locally asymptotically stable with arbitrary length of the delay. This is the case, when

$$k < k^S = \frac{2}{A + B}.$$

Notice that this is the stability condition in the discrete model. This follows the general understanding that a continuous-time model is more stable than a discrete-time model in the sense that the former stability region in the parameter space is larger than the latter's.

Lemma 1 *If the discrete-time model (3) is locally asymptotically stable, then so is the continuous-time model (4).*

Assume next that $k > k^S$ or $1 - k(A + B) < -1$. Then from (7) the solution for ω is

$$\bar{\omega} = \frac{\sqrt{[1 - k(A + B)]^2 - 1}}{\sigma}. \tag{8}$$

Since $1 - k(A + B)$ is negative, equations (6) imply that $\sin \omega\tau$ is positive and $\cos \omega\tau$ is negative, so $\pi/2 + 2n\pi \leq \bar{\omega}\tau < \pi + 2n\pi$ and

$$\tau_n = \frac{1}{\bar{\omega}} \left(\pi - \sin^{-1} \left(\frac{\sigma\bar{\omega}}{k(A + B) - 1} \right) + 2n\pi \right) \text{ for } n = 0, 1, 2, ... \tag{9}$$

We will next examine the direction of the stability switches. By selecting τ as the bifurcation parameter, we verify how the change in the length of the delay affects the real parts of the roots of the characteristic equations. Considering λ as the function of τ, $\lambda = \lambda(\tau)$, and implicitly differentiating the characteristic equation (6), we have

$$\sigma \frac{d\lambda}{d\tau} - (1 - k(A + B))e^{-\lambda\tau} \left(-\frac{d\lambda}{d\tau}\tau - \lambda \right) = 0.$$

For the sake of analytical convenience, we solve for $(d\lambda/d\sigma)^{-1}$,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{\sigma}{\lambda(\sigma\lambda+1)} - \frac{\tau}{\lambda} \tag{10}$$

where the relation $\sigma\lambda + 1 = (1 - k(A+B))\, e^{-\lambda\tau}$ from (5) is used to obtain the form at the right hand side. Substituting $\lambda = i\omega$ with $\omega > 0$ into equation (10) and then taking the real part represent

$$\begin{aligned}\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right] &= \mathrm{Re}\left[\frac{\sigma(\sigma\omega+i)}{\omega(\sigma\omega-i)(\sigma\omega+i)}\right] \\ &= \frac{\sigma^2}{(\sigma\omega)^2+1} > 0.\end{aligned}$$

The last inequality implies that increasing τ induces the crossing of the imaginary axis from left to right. So stability is lost at the critical delay with $n = 0$ in equation (9),

$$\tau_0 = \frac{\sigma}{\sqrt{[k(A+B)-1]^2 - 1}}\left(\pi - \sin^{-1}\left(\sqrt{1 - \frac{1}{[k(A+B)-1]^2}}\right)\right). \tag{11}$$

and stability cannot be regained later. In short, we summarize the results as follows:

Lemma 2 *In the case of $k > k^S$, the continuos model (4) is locally asymptotically stable if $\tau < \tau_0$.*

Lemmas 1 and 2 together imply the following:

Theorem 3 *Given $k > 0$, the delay differential equation (4) is locally asymptotically stable and delay is harmless if either*

$$k < k^S$$

or

$$k > k^S \text{ and } \tau < \tau_0.$$

We now numerically examine global dynamics when the stability of the steady state is lost. Figure 4(A) depicts the downward-sloping partition curve described by equation (11) with $\alpha = 3$ and $\beta = 1$.[3] The curve divides the (k, τ) plane into two regions. The steady state is locally unstable in the region above the curve and locally asymptotically stable in the yellow region below the curve. This stable region is further divided into two by the vertical dotted line

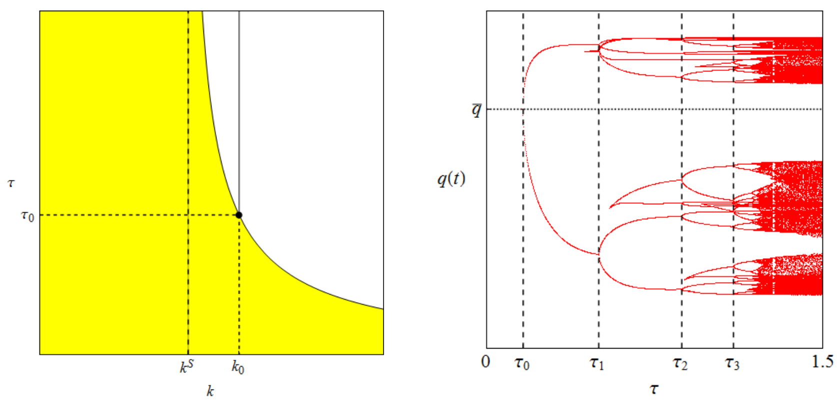
[3] Needless to say, specified values of α and β do not affect the qualitative aspects of the results to be obtained below.

at $k = k^S$. In the region left to the line, $k < k^S$ holds and thus the steady state is locally stable regardless of the length of delay. Such a delay is often called *harmless*. In the region right where $k > k^S$, we select the value k_0 $(= 0.29)$ and depict the vertical dotted-real line at $k = k_0$ crossing the partition curve at $\tau = \tau_0$. Lemma 2 implies that the steady state is locally asymptotically stable for $\tau < \tau_0$. On the partition curve, the real part of one eigenvalue is zero and its derivative with respect to τ is positive. If τ becomes larger than τ_0 by crossing the dotted vertical line at τ_0 from left to right, then one pair of complex eigenvalues changes the sign of their real part from negative to positive. Thus increasing $\tau > \tau_0$ destabilizes the model and the steady state bifurcates to a limit cycle as shown in Figure 4(B). At $\tau = \tau_1$, another pair of eigenvalues does the same. In consequence, a new limit cycle emerges from the existing limit cycle with doubling the period of the cycle.[4] So period-doubling bifurcation occurs. In addition to this, carefully observing Figure 4(B) indicates the emergence of another new limit cycle for $\tau_1 < \tau < \tau_2$.[5] Thus the period of the limit cycle is more than double. At the next critical value, $\tau = \tau_2$, a new pair of eigenvalues changes real part from negative to positive again and the existing cycle bifurcates to a new limit cycle with doubling period and another new limit cycles are born. In this way the bifurcation proceeds. If $\tau_{k-1} < \tau < \tau_k$, then exactly k pairs have positive real parts and if $\tau \to \infty$, then $k \to \infty$, so by increasing the values of τ, the dynamics of the system becomes more and more complex generating complex dynamics involving chaos. This is well illustrated in the bifurcation diagram in Figure 4(B) in which we call such bifurcation *quasi* period-doubling because the period is increased to more than double. Global behavior with respect to τ is summarized as follows:

Proposition 1 *Given k, the steady state loses stability at $\tau = \tau_0$ and bifurcates to chaos via the quasi period-doubling cascade as τ increases.*

[4]The values of τ_i for $i = 1, 2, ..., 5$ are determined by a rule of thumb.

[5]The same phenomeon is often observed in a delay differential equation. However, it is not clear yet why the new cycles arise suddenly.



(A) Partition curve (B) Bifurcation diagram

Figure 4. Delay effect

3.2 Inertia Effect

In the same way as in the previous section, we can detect the effect caused by a change in σ on dynamics. We first deal with the benchmark case of $\sigma = 0$. The characteristic equation is

$$1 + (k(A+B) - 1)\, e^{-\lambda\tau} = 0.$$

If $\lambda = \alpha + i\beta$ with $\beta > 0$, then it is written as

$$1 + (k(A+B) - 1)\, e^{-\alpha\tau}(\cos\beta\tau - i\sin\beta\tau) = 0.$$

Notice that stability depends on the sign of α. Separating real and imaginary parts yields two equations

$$1 + (k(A+B) - 1)\, e^{-\alpha\tau}\cos\beta\tau = 0 \tag{12}$$

and

$$(k(A+B) - 1)\, e^{-\alpha\tau}\sin\beta\tau = 0. \tag{13}$$

$\sin\beta\tau = 0$ is definitely determined from (13). On the other hand, the sign of $\cos\beta\tau$ in (12) is not determined unless the sign of $k(A+B)-1$ is specified,

$$\text{if } k(A+B) - 1 < 0, \text{ then } \cos\beta\tau > 0 \text{ implying } \cos\beta\tau = 1$$

and

$$\text{if } k(A+B) - 1 > 0, \text{ then } \cos\beta\tau < 0 \text{ implying } \cos\beta\tau = -1.$$

So from (12),

$$1 + (k(A + B) - 1)\, e^{-\alpha\tau}(+1) = 0 \quad \text{if } k(A + B) - 1 < 0,$$

$$1 + (k(A + B) - 1)\, e^{-\alpha\tau}(-1) = 0 \quad \text{if } k(A + B) - 1 > 0.$$

Then

$$e^{-\alpha\tau} = \pm\frac{1}{k(A + B) - 1} \text{ according to } \text{sign}[k(A + B) - 1] = \pm$$

Since double-sign is in the same order, $\alpha > 0$ if and only if either

$$(i)\ k(A + B) - 1 < 0 \text{ and } k(A + B) - 1 < -1$$

or

$$(ii)\ k(A + B) - 1 > 0 \text{ and } k(A + B) - 1 > 1.$$

Condition (i) cannot occur as $A > 0$ and $B > 0$. It is therefore shown that the real part α is negative if $k < k^S$ and positive if $k > k^S$. Hence, the stability condition in the continuous model with $\sigma = 0$ is the same as the one in the discrete model:

Theorem 4 *The dynamic equation (4) with $\sigma = 0$ is locally asymptotically stable if $k < k^S$ and locally unstable if $k.k^S$.*

Now we proceed to the case of $\sigma > 0$. If τ is given, then the critical values of σ can be obtained from equations (8) and (9),

$$\sigma_n = \frac{\tau\sqrt{[k(A + B) - 1]^2 - 1}}{(2n + 1)\pi - \sin^{-1}\left(\sqrt{1 - \frac{1}{[k(A+B)-1]^2}}\right)}. \tag{14}$$

They are the partition curves in the (σ, τ) space. Largest σ_n occurs at $n = 0$. σ_n decreases in n and converges to zero as $n \to \infty$. To verify whether stability switch takes place on the partition curve, we suppose that $\lambda = \lambda(\sigma)$. Implicitly differentiating the characteristic equation yields

$$\sigma\frac{d\lambda}{d\sigma} + \lambda - (1 - k(A + B))e^{-\lambda\tau}\left(-\frac{d\lambda}{d\sigma}\tau\right) = 0$$

implying that

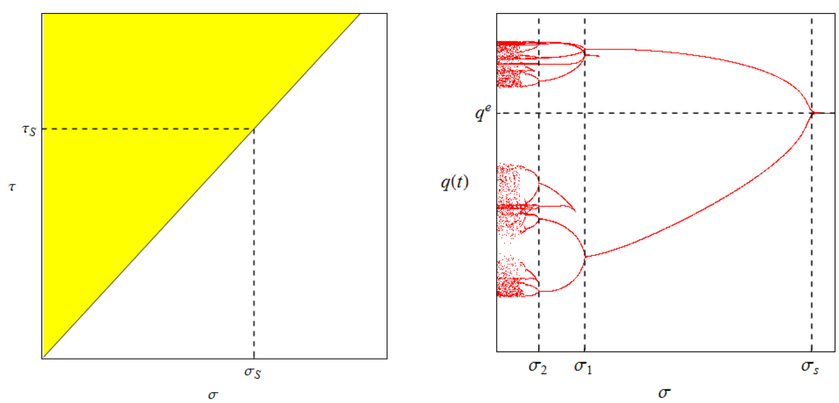
$$\left(\frac{d\lambda}{d\sigma}\right)^{-1} = -\frac{\tau(\sigma\lambda + 1)}{\lambda} - \frac{\sigma}{\lambda}.$$

If $\lambda = i\omega$ with $\omega > 0$, then the real part is

$$\begin{aligned}\text{Re}\left[\left(\frac{d\lambda}{d\sigma}\right)^{-1}\right] &= \text{Re}\left[-\frac{\tau(\sigma\omega^2 - i\omega)}{\omega^2}\right] \\ &= -\sigma\tau < 0.\end{aligned}$$

That is, when the value of σ crosses a critical value from left to right, then one pair of complex eigenvalues changes sign of real part from positive to negative. Let $\tau_0(\sigma)$ and $\sigma_0(\tau)$ denote the right hand sides of equations (9) and (14) with $n = 0$. Both functions are strictly increasing. Assume that the value of τ is fixed and assume that $\sigma > \sigma_0 = \sigma_0(\tau)$. Then $\tau = \tau_0(\sigma_0) < \tau_0(\sigma)$, so Lemma 2 implies the local asymptotic stability of the continuous model with parameters τ and σ. When the value of σ crosses σ_0 from right to left, then one pair of eigenvalues changes real part from negative to positive. Then at σ_1, another pair does the same, at σ_2 a new pair changes the sign of their real part from negative to positive, and so on. At each σ_n the number of eigenvalues with positive real parts increases by two when the value of σ crosses σ_n from right to left. So if $\sigma_n < \sigma < \sigma_{n-1}$, then exactly n pairs of eigenvalues have positive real parts, that is, at any $\sigma > 0$ there are only finitely many such eigenvalues and as σ decreases, their number increases. Furthermore this number tends to ∞ as $\sigma \to 0$. This is the reason of the "chaotic" behavior for small σ values as shown in the bifurcation diagrams of Figure 5(B). As σ increases, the number of eigenvalues with positive real parts decreases making the dynamics of the model more simple. This is also illustrated in Figure 5(B).

Proposition 2 *Given τ, dynamics becomes simple from complex via quasi-period halving cascade as σ increases to σ_0 where stability is gained and never lost for $\sigma > \sigma_0$.*



(A) Partition curve (B) Bifurcation diagram

Figure 5. Inertia effect

In summary we have the following result:

Theorem 5 *The steady state of the continuous model (4) is locally asymptotically stable when the inertia does not affect stability if either*

$$k < k^S$$

or

$$k > k^S \text{ and } \sigma > \sigma_0.$$

4 Concluding Remarks

A discrete and corresponding continuous dynamics of a monopoly were examined with general price and cost functions. First the existence of the unique profit maximizing output was proved and then stability conditions were derived for the dynamic extensions. The discrete system is locally asymptotically stable if the speed of adjustment is sufficiently small, in which case the continuous system is also locally asymptotically stable with any length of the delay. It was shown that the continuous system may become stable even in cases when the discrete system is unstable with selecting sufficiently small length of the delay.

In examining global behavior we have seen a similarity between the dependence from the delay τ and the inertia coefficient σ. The same quasi period doubling phenomenon occurs with increasing values of τ and with decreasing values of σ. This interesting fact was analytically proved and illustrated with numerical studies.

Appendix

In this Appendix, we consider the non-negativity condition for the trajectories. It is assumed that $\alpha \geq 1$ and $\beta \geq 2$ where $\alpha > 1$ or $\beta > 2$ or both. The dynamic equation is given by

$$q(t+1) = R(q(t), k)$$

where

$$R(q,k) = q + k(a - b(\alpha+1)q^{\alpha} - c\beta q^{\beta-1}) \tag{A-1}$$

with

$$R(0,k) = ka \text{ and } \lim_{q\to\infty} R(q,k) = -\infty.$$

Furthermore

$$R'(q) = 1 - k(b(\alpha+1)\alpha q^{\alpha-1} + c\beta(\beta-1)q^{\beta-2}). \tag{A-2}$$

Notice that $R(q)$ is strictly concave.

Result 1 *Given $k > 0$, there is a unique $q_M(k)$ such that $R(q_M(k)) = 0$.*

Proof. Equation (A-1) implies that q_M is the solution of the equation

$$b(\alpha+1)q^{\alpha} + c\beta q^{\beta-1} = a + \frac{1}{k}q \tag{A-3}$$

where the left hand side is denoted by $f(q)$. At $q = 0$, the left hand side of equation (A-3) is $f(0) = 0$ and is less than the right hand side. As q goes to infinity, $f(q)$ converges to ∞ faster than the right hand side. So there is at least one solution. Since $f(q)$ is strictly convex and the right hand side is linear, the solution is unique. Furthermore $R(q,k) > 0$ if $q \in (0, q_M)$ and $R(q,k) < 0$ if $q > q_M$. ■

It is verified from (A-3) that

$$\lim_{k\to 0} q_M(k) = \infty, \ \lim_{k\to\infty} q_M(k) = \bar{q} \tag{A-4}$$

and

$$q_M'(k) = -\frac{q/k^2}{f'(q_M) - 1/k} > 0$$

where the inequality is due to the fact that at $q = q_M$, the left hand side crosses the right hand side from below. In the same way we have the following.

Result 2 *Given k, there is a unique $q_m(k)$ that maximizes $R(q,k)$ for $q \geq 0$.*

Proof. At any interior maximum,

$$b(\alpha+1)\alpha q^{\alpha-1} + c\beta(\beta-1)q^{\beta-2} = \frac{1}{k}. \tag{A-5}$$

Let $g(q)$ denote the left hand side, which strictly increase in q and tends to ∞ as $q \to \infty$. Notice that

$$g(0) = \begin{cases} b(\alpha+1)\alpha & \text{if } \alpha = 1 \text{ and } \beta > 2, \\ c\beta(\beta-1) & \text{if } \alpha > 1 \text{ and } \beta = 2, \\ 0 & \text{otherwise.} \end{cases}$$

If $g(0) < 1/k$, then there is a unique solution $q_m(k)$ of equation (A-5). Furthermore, $g(q) < 1/k$ as $q < q_m(k)$ and $g(q) > 1/k$ as $q > q_m(k)$, so $q_m(k)$ is the unique maximizer. If $g(0) \geq 1/k$, then $g(q) > 1/k$ for all $q > 0$ implying that $\partial R(q,k)/\partial q < 0$, therefore $q_m(k) = 0$ is the unique maximizer. ■

Notice that $q_m(k)$ satisfies the following limit relations:

$$\lim_{k \to 0} q_m(k) = \infty.$$

This is clear if $g(0) < 1/k$ from equation (A-5), which becomes the case if k is sufficiently small. Furthermore

$$\lim_{k \to \infty} q_m(k) = 0$$

which is also a simple consequence of equation (A-5) for interior maximum, otherwise $q_m(k) = 0$ for large enough values of k. Notice also that $q_m(k)$ strictly decreases in k if it is positive and therefore solution of equation (A-5).

Result 3 *$q_M(k) > q_m(k)$ for any $k > 0$.*

Proof. Since $R(0,k) > 0$ and $R(q,k) \leq 0$ as $q \geq q_M$, the maximizer q_m has to be less than q_M, so we have $q_M(k) > q_m(k)$ for all $k > 0$. ■

If $g(0) > 0$, then there is a k_1 such that $g(0) = 1/k_1$ and if $k < k_1$, then $g(0) < 1/k$, and if $k \geq k_1$, then $g(0) \geq 1/k$. So for $k \geq k_1$, $q_m(k) = 0$ and if $k < k_1$, then $q_m(k)$ is interior. If $g(0) = 0$, then $q_m(k)$ is interior for all $k > 0$, so we may select $k_1 = \infty$ in this case.

Result 4 *$R(q_m(k), k)$ takes a U-shaped profile with respect to k and the $q_m(k)$ curve passes through its minimum point.*

Proof. Assume $k < k_1$, then $q_m(k)$ is interior. Differentiating $R(q_m(k), k)$ with respect to k gives

$$\frac{dR}{dk} = \left.\frac{\partial R}{\partial q}\right|_{q=q_m} \frac{dq_m}{dk} + \left.\frac{\partial R}{\partial k}\right|_{q=q_m}$$

where the first term on the right hand side is zero at $q = q_m$. So

$$\begin{aligned}\frac{dR}{dk} &= a - b(\alpha+1)\left(q_m(k)\right)^{\alpha} - c\beta\left(q_m(k)\right)^{\beta-1} \\ &= \frac{1}{k}\left(R(q_m(k)) - q_m(k)\right)\end{aligned}$$

where equation (A-1) evaluated at $q = q_m(k)$ is used at the second step. The profit maximizing output $\bar{q}$ is determined by $f(\bar{q}) = a$ and independent from k. Since $q_m(k)$ is strictly decreasing in k while positive, $\lim_{k\to 0} q_m(k) = \infty$ and $\lim_{k\to\infty} q_m(k) = 0$, there is a unique threshold value $\bar{k}$ such that $q_m(\bar{k}) = \bar{q}$. In consequence, since $q_m(k) > \bar{q}$ for $k < \bar{k}$, we have $R(q_m(k), k) < q_m(k)$ that then leads to $dR/dk < 0$. In the same way, since $q_m(k) < \bar{q}$ for $k > \bar{k}$, we have $R(q_m(k), k) > q_m(k)$ that then leads to $dR/dk > 0$. Therefore the $R(q_m(k), k)$ curve takes the U-shaped profile for $k < k_1$ and the $q_m(k)$ curve passes the minimum value. ■

From Results 1-4, we have the following condition for the non-negativity of the trajectory which is given earlier in Theorem 2.

Result 5 *There is a positive k^N such that $R(q_m(k^N), k^N) = q_M(k^N)$ and the non-negativity condition*

$$R(q_m(k), k) \leq q_M(k)$$

holds for $k \leq k^N$.

Proof. For $k < \bar{k}$,

$$R(q_m(k), k) < q_m(k) < q_M(k).$$

For $k > \bar{k}$, $R(q_m(k), k)$ is increasing in k and converges to ∞ as $k \to \infty$ while $q_M(k)$ is decreasing. Thus there is a unique $k^N > \bar{k}$ where $R(q_m(k^N), k^N) = q_M(k^N)$ such that

$$R(q_m(k), k) \leq q_m(k) < q_M(k) \text{ for } k \leq \bar{k}$$

$$R(q_m(k), k) < q_M(k) \text{ for } \bar{k} < k < k^N$$

and

$$R(q_m(k), k) > q_M(k) \text{ for } k > k^N.$$

■

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