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Abstract

This paper aims to show that delay matters in continuous- and discrete-time framework. It constructs a simple dynamic model of a boundedly rational monopoly. First the existence of the unique equilibrium state is proved under general price and cost function forms. Conditions are derived for its local asymptotical stability with both continuous and discrete time scales. The global dynamic behavior of the systems is then numerically examined, demonstrating that the continuous system is globally asymptotically stable without delay and in the presense of delay if the delay is sufficiently small. Then stability of the continuous system is lost via Hopf bifurcation. In the discrete case without delay, the steady state is locally asymptotically stable if the speed of adjustment is small enough, then stability is lost via period–doubling bifurcation. If the delay is one or two steps, then stability loss occurs via Neimark-Sacker bifurcation.

Keywords: Gradient dynamics, Boundedly rational monopoly, Continuous and discrete dynamics, Time delay

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1 Introduction

It is well known that economic agents are not fully rational, and usually prefer to employ simple rules which were previously tested (Kahneman et al., 1986). Many different output adjustment schemes have been developed. Bischi et al. (2010) offer a collection of the most important schemes in the case of oligopolies. In earlier studies linear models were examined, where local asymptotical stability implies global asymptotical stability (Okuguchi and Szidarovszky, 1999). In the last two decades an increasing attention has been given to examine the asymptotical property of nonlinear economic systems including monopolies and duopolies (Bischi et al., 2010). It is also well known that economic dynamic systems inherently incorporate delays in their actions and delay is one of the essentials for economic dynamics. Nevertheless, little attention has been given to studies on delay dynamics of boundedly rational economic agents. The main purpose of this paper is to show how delay affects dynamics in continuous- as well as in discrete-time framework.

In this paper dynamics of boundedly rational monopolies are discussed with the most popular adjustment rule in which the firm adjusts output in proportion to its marginal profit. Such an adjustment scheme is known as gradient adjustment. In contrary to best response dynamics, only local information is needed for the adjustment process which is always available to the firm. Baumol and Quandt (1964) investigated cost free monopolies in both discrete and continuous time scales and the dependence of the profit on varying price. They developed a simple adjustment mechanism that converges to the profit maximizing output. Puu (1995) has revisited this model with discrete time setting and a cubic demand function. It is shown that complex dynamics can emerge if the price function has a reflection point. Naimzada and Ricchiuti (2008) reconsidered Puu's model with linear cost and cubic price function and exhibited the birth of chaos through the period-doubling bifurcation even if the price function does not have a reflection point. Their model was then generalized by Askar (2013) with more general price functions. In both studies local asymptotical stability was analytically examined and global dynamics by computer simulations.

We consider monopoly dynamics from three different points of view. First, we are concerned with gradient dynamics in continuous-time framework while most of recent studies considered discrete-time dynamics. Second, we will further generalize the model of Asker (2013) by introducing a more general class of cost functions and determine the non-negativity condition that prevents time-trajectories from being negative. Third, we devote a little more space to exhibit that "delay" discrete-time monopoly gives rise to complex dynamics via Neimark-Sacker bifurcation while "non-delay" monopoly goes to chaos through a period-doubling cascade.

The paper is organized as follows. In Section 2, the model is presented and the existence of the unique profit maximizing output level is proved. In Section 3, the dynamic system with the gradient adjustment is constructed in continuous time scale. Without delayed information the continuous model is always locally asymptotically stable, and the stability can be lost if only delayed revenue information is available. In Section 4, the continuous-time model is discretized. It is analytically shown and numerically confirmed that the discrete-time model can give rise to aperiodic oscillations through a period-doubling bifurcation or Neimark-Sacker bifurcation according to whether the model involves "delay" or not. In the final section, concluding remarks are given.

2 Continuous Model

A general inverse demand function as well as a general cost function are considered in a monopoly. Let $p(q) = a - bq^\alpha$ be the price function and $C(q) = cq^\beta$ be the cost function. The case of $\alpha = 3$ and $\beta = 1$ was examined by Naimzada and Ricchiuti (2008), the more general case with any $\alpha \geq 3$ and $\beta = 1$ was discussed by Askar (2013). We move one step forward from their studies and consider the case where both α and β are greater than 1.

Assumption 1. $\alpha > 1$ and $\beta > 2$.

The profit of the monopoly is given as

$$\pi(q) = (a - bq^\alpha)q - cq^\beta. \tag{1}$$

Notice that

$$\pi'(q) = a - b(\alpha+1)q^\alpha - c\beta q^{\beta-1} \tag{2}$$

and

$$\pi''(q) = -b\alpha(\alpha+1)q^{\alpha-1} - c\beta(\beta-1)q^{\beta-2} < 0, \tag{3}$$

so $\pi(q)$ is strictly concave in q, furthermore

$$\pi(0) = 0, \ \lim_{q\to\infty} \pi(q) = -\infty \text{ and } \pi'(0) = a > 0.$$

Therefore there is a unique profit maximizing output $\bar{q}$ which is the unique solution of equation

$$b(\alpha+1)q^\alpha + c\beta q^{\beta-1} = a. \tag{4}$$

The left hand side is zero at $q = 0$, converges to ∞ as q tends to infinity and is strictly increasing. So the value of $\bar{q}$ can be obtained by using simple numerical methods (see, for example, Szidarovszky and Yakowitz, 1978). The remaining of this section has two subsections. In Section 2.1, the condition of local asymptotical stability is derived. In Section 2.2, we examine the effects caused by changing values of α and β on stability.

2.1 Stability

Assuming gradient dynamics, the firm adjusts its output according to the following differential equation:

$$\dot{q}(t) = k\pi'(q(t))$$

or

$$\dot{q}(t) = k(a - b(\alpha+1)q(t)^{\alpha} - c\beta q(t)^{\beta-1}). \tag{5}$$

This is a nonlinear system with positive adjustment coefficient k. The unique steady state of this system is the profit maximizing output $\bar{q}$. Local asymptotic stability can be examined by linearization. The linearized equation can be written as

$$\dot{q}_{\varepsilon}(t) = k\pi'(\bar{q})q_{\varepsilon}(t)$$

or

$$\dot{q}_{\varepsilon}(t) = k(-b\alpha(\alpha+1)\bar{q}^{\alpha-1} - c\beta(\beta-1)\bar{q}^{\beta-2})q_{\varepsilon}(t) \tag{6}$$

where $q_{\varepsilon}(t) = q(t) - \bar{q}$. Since the multiplier of $q_{\varepsilon}(t)$ in the right hand side is negative, the steady state $\bar{q}$ is locally asymptotically stable. Let $r(q)$ denote the right hand side of equation (5). Notice that it strictly decreases in q and $r(\bar{q}) = 0$. So $r(q) < 0$ if $q > \bar{q}$ and $r(q) > 0$ if $q < \bar{q}$. Therefore if $q(0) < \bar{q}$, then $q(t)$ strictly increases and converges to $\bar{q}$, if $q(0) > \bar{q}$, then $q(t)$ strictly decreases and converges to $\bar{q}$, and if $q(0) = \bar{q}$, then $q(t) = \bar{q}$ for all $t \geq 0$. Hence system (5) is global asymptotically stable, and with $q(0) > 0$, the entire trajectory $q(t)$ remains positive. Although this result is well known, we formally state it as it is a benchmark of this study:

Theorem 1 *For any k, the non-delay continuous time model (5) is locally and globally asymptotically stable.*

Assume next that the monopoly receives marginal revenue information with a positive delay $\tau > 0$, which could be due to delay price information. Then system (5) becomes a delay differential equation,

$$\dot{q}(t) = k(a - b(\alpha+1)q(t-\tau)^{\alpha} - c\beta q(t)^{\beta-1}). \tag{7}$$

The derivatives of the right hand side with respect to $q(t-\tau)$ and $q(t)$ are

$$\frac{\partial \dot{q}(t)}{\partial q(t-\tau)} = -kb\alpha(\alpha+1)q(t-\tau)^{\alpha-1}$$

and

$$\frac{\partial \dot{q}(t)}{\partial q(t)} = -kc\beta(\beta-1)q(t)^{\beta-2},$$

so the linearized equation has the form

$$\dot{q}_{\varepsilon}(t) = -kAq_{\varepsilon}(t-\tau) - kBq_{\varepsilon}(t) \tag{8}$$

with

$$A = b\alpha(\alpha+1)\bar{q}^{\alpha-1} \text{ and } B = c\beta(\beta-1)\bar{q}^{\beta-2}.$$

We know that the system is locally and globally asymptotically stable for $\tau = 0$. In order to find stability switches with increasing value of τ, assume that $q_{\varepsilon}(t) = e^{\lambda t}u$, and substitute it into equation (8) to obtain:

$$\lambda e^{\lambda t} = -kAe^{\lambda(t-\tau)} - kBe^{\lambda t} \tag{9}$$

so the characteristic equation is a mixed polynomial-exponential equation

$$\lambda + kB + kAe^{-\lambda\tau} = 0. \tag{10}$$

With any stability switch, $\lambda = i\omega$ with $\omega > 0$, so

$$i\omega + kA(\cos\omega\tau - i\sin\omega\tau) + kB = 0 \tag{11}$$

and separating the real and imaginary parts we get two equations for two unknowns ω and τ:

$$\begin{aligned} kA\cos\omega\tau &= -kB \\ kA\sin\omega\tau &= \omega. \end{aligned} \tag{12}$$

By adding the squares of these equations we have

$$\omega^2 = k^2(A^2 - B^2). \tag{13}$$

Since there is no positive solution for ω if $A - B \leq 0$, no stability switch occurs for any $\tau > 0$. In such a case, delay is often called *harmless*. Positive solution exists if $A > B$. From (12) we see that $\cos\omega\tau < 0$ and $\sin\omega\tau > 0$, therefore from the second equation in (12),

$$\tau_n = \frac{1}{k\sqrt{A^2 - B^2}}\left(\pi - \sin^{-1}\left(\frac{\sqrt{A^2 - B^2}}{A}\right) + 2n\pi\right) \quad (n \geq 0). \tag{14}$$

In order to find the direction of the stability switches at these critical values, consider λ as a function of the bifurcation parameter τ, $\lambda = \lambda(\tau)$, and differentiate the characteristic equation (10) with respect to τ to have

$$\frac{d\lambda}{d\tau} + kAe^{-\lambda\tau}\left(-\frac{d\lambda}{d\tau}\tau - \lambda\right) = 0. \tag{15}$$

Using equation (10) yields

$$\frac{d\lambda}{d\tau} = \frac{-\lambda(\lambda + kB)}{1 + (\lambda + kB)\tau}. \tag{16}$$

With $\lambda = i\omega$, we have

$$\frac{d\lambda}{d\tau} = \frac{(\omega^2 - i\omega kB)(1 + kB\tau - i\omega\tau)}{(1 + kB\tau)^2 + (\omega\tau)^2} \tag{17}$$

and so

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right] = \frac{\omega^2}{(1 + kB\tau)^2 + (\omega\tau)^2} > 0. \tag{18}$$

The crossing of the imaginary axis is from left to right as τ increases. Consequently stability is lost at the smallest critical value

$$\tau_0 = \frac{1}{k\sqrt{A^2 - B^2}}\left(\pi - \sin^{-1}\left(\frac{\sqrt{A^2 - B^2}}{A}\right)\right) \tag{19}$$

and cannot be regained for any $\tau > \tau_0$. Thus we have the following result.

Theorem 2 *The steady state in the continuous system is locally asymptotically stable if $A \leq B$ or if $A > B$ and $0 \leq \tau < \tau_0$ whereas it is locally unstable if $A > B$ and $\tau > \tau_0$. If $\tau = \tau_0$, then Hopf bifurcation occurs giving the possibility of the birth of limit cycles.*

2.2 Effects of α and β on Stability

We derive conditions to guarantee that $A > B$.

(i) Assume first that $\alpha - 1 > \beta - 2$ (that is, $\beta < \alpha + 1$), then $A > B$ if and only if

$$b\alpha(\alpha+1)\bar{q}^{\alpha+1-\beta} > c\beta(\beta-1) \tag{20}$$

or

$$\bar{q} > q_1 := \left(\frac{c\beta(\beta-1)}{b\alpha(\alpha+1)}\right)^{\frac{1}{\alpha+1-\beta}} \tag{21}$$

which is the case if $f(q_1) < a$ where $f(q)$ denotes the left hand side of equation (4);

(ii) If $\beta = \alpha + 1$, then $A > B$ if and only if

$$b > c; \tag{22}$$

(iii) And finally, if $\beta > \alpha + 1$, then $A > B$ if and only if

$$b\alpha(\alpha+1) > c\beta(\beta-1)\bar{q}^{\beta-\alpha-1} \tag{23}$$

or

$$\bar{q} < q_1 = \left(\frac{b\alpha(\alpha+1)}{c\beta(\beta-1)}\right)^{\frac{1}{\beta-\alpha-1}} \tag{24}$$

which occurs if $f(q_1) > a$.

First, to see dynamic behavior generated by the delay differential equation (7), we assume the same parameter specifications as in Naimzada and Ricchiuti (2008) and Askar (2013).

Assumption 2. $a = 4$, $b = 3/5$ and $c = 1/2$.

Second, to describe these conditions in the (α, β) plane, we substitute q_1 into $f(q)$ to obtain the following form,

$$g(\alpha,\beta) = \frac{3}{5}(1+\alpha)\left(\frac{5}{6}\frac{\beta(\beta-1)}{\alpha(\alpha+1)}\right)^{\frac{\alpha}{1+\alpha-\beta}} + \frac{1}{2}\beta\left(\frac{5}{6}\frac{\beta(\beta-1)}{\alpha(\alpha+1)}\right)^{\frac{\beta-1}{1+\alpha-\beta}}$$

which is shown in Figure 1(A), where the red-coloured vertical plane is the surface of (α, β) satisfying $\beta = \alpha + 1$ and divides the 3D space into two subspace,

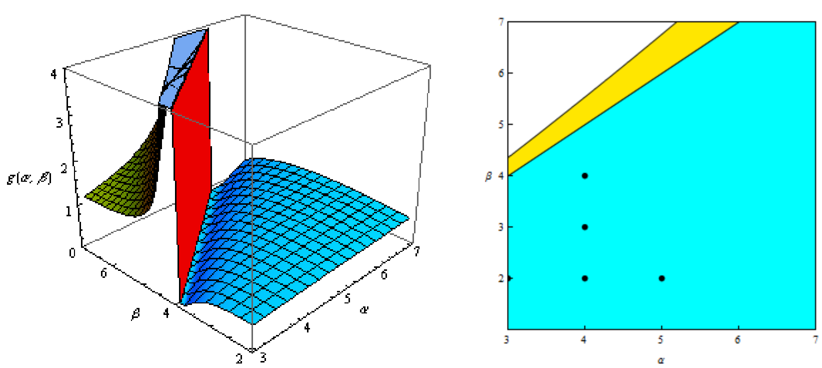
$\beta > \alpha + 1$ in one of them and $\beta < \alpha + 1$ in the other. The graph of $g(\alpha, \beta)$ in the former is colored in meshed yellow and the one in the latter is in meshed light blue. The top surface of the 3D space is the cross-section at the height being 4. Since $b > c$ is assumed, q_1 converges to zero when β approaches $\alpha + 1$ from below and converges to ∞ when β approaches $\alpha + 1$ from above, that is,

$$\lim_{\beta \to (\alpha+1)_-} \left(\frac{c\beta(\beta - 1)}{b\alpha(\alpha + 1)} \right)^{\frac{1}{\alpha+1-\beta}} = 0$$

and

$$\lim_{\beta \to (\alpha+1)_+} \left(\frac{c\beta(\beta - 1)}{b\alpha(\alpha + 1)} \right)^{\frac{1}{\alpha+1-\beta}} = \infty.$$

The graphical expression of the divergence of q_1 is given by the fact that the meshed yellow surface is asymptotic to the $\beta = \alpha + 1$ plane. It then follows that it crosses the top surface and its intersection is the locus described by $g(\alpha, \beta) = 4$ or $f(q_1) = a$. On the other hand, the graphical expression of the convergence of q_1 is that the meshed light blue surface crosses the bottom surface of the 3D space along the locus defined by $g(\alpha, \beta) = 0$ or $\beta = \alpha + 1$. It is apparent from Figure 1(A) that $g(\alpha, \beta) < 4$ or $f(q_1) < a$ when $\beta < \alpha + 1$. Figure 1(B) is a projection of Figure1(A) onto the (α, β) plane in which the upper positive-sloping curve describes $f(q_1) = a$ and the lower positive-sloping line represents the locus of $\beta = \alpha + 1$. Thus $A > B$ in the colored regions of Figure 1(B), more precisely, condition (iii) holds in the yellow region that is defined by $\{(\alpha, \beta) \mid f(q_1) > a \text{ and } \beta > \alpha + 1\}$, condition (i) holds in the light-blue region $\{(\alpha, \beta) \mid f(q_1) < a \text{ and } \beta < \alpha + 1\}$ and condition (ii) holds on the positive-sloping line $\beta = \alpha + 1$. We mention the meanings of the five dotted points in Figure 1(B) later.



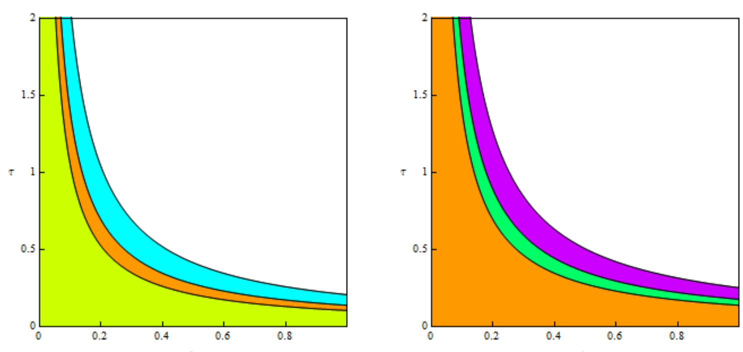
(A). 3D graph of $g(\alpha, \beta)$

(B). Division

Figure 1. The graphical representation of the conditions for $A > B$

In the case of $A > B$, equation (19) is the partition curve in the (k, τ) space that divides the parametric space into two subregions, the stable region below the curve in which the steady state is locally stable and the unstable region in which it is locally unstable. We numerically confirm how the values of α and β affect stability of the steady state. In the first example, we examine the effect caused by increasing α on stability. So α is increased from 3 to 5 by unity while β is fixed at 2 as denoted by three dotted points horizontally located at $\beta = 2$ in Figure 1(B). The stable regions with $\alpha = 3, 4, 5$ are colored in light blue, ocher and yellowish green, respectively and one is supersimposed on another in Figure 2(A). The boundary of each region corresponds to the partition curve with corresponding value of α. Comparison of these cases show that the partition curve shifts downward as the value of α increases. This implies that increasing α has a destabilizing effect by shrinking the stable region. In the second example, β is increased from 2 to 4 by unity while α is fixed at 4 as shown by three dotted points vertically located at $\alpha = 4$ in Figure 1(B). The stable regions with $\beta = 2, 3, 4$ are colored in ocher, green and purple, respectively and one is superimposed on another in Figure 2(B). The boundary of each region corresponds to the partition curve with corresponding value of β. We can see that the partition curve shifts upward as the value of β increases. This implies that increasing value of β has a stabilizing effect enlarging the stable region. We summarize these results as follows:

Proposition 1 *Increasing α destabilizes the steady state whereas increasing β stabilizes it.*



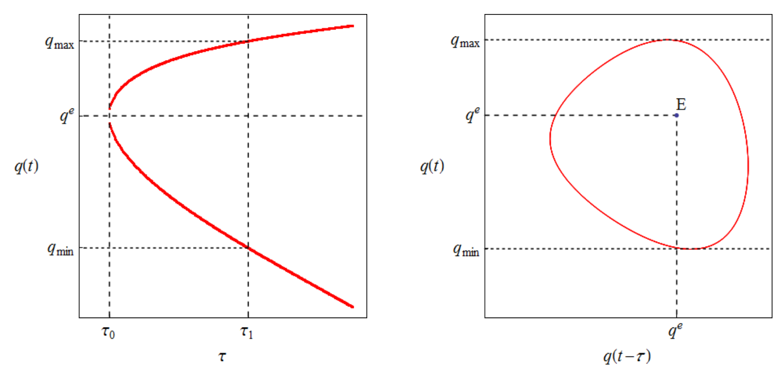
A. Effect of increasing α B. Effect of increasing β

Figure 2. The stabilizing effects caused by parameter changes

Finally we numerically confirm the birth of a limit cycle when the stability conditions are violated by increasing the length of delay. Specifying $\alpha = 3$, $\beta = 2$ and $k = 0.2$ and increasing the value of τ from $\tau \simeq 1.036$[1] to $\tau \simeq 1.386$

[1] For $\alpha = 3$ and $\beta = 2$, the critical value τ_0 is approximated as 1.036478.

with increment of 1/1000 , we simulate the dynamic equation (5) for $0 \leq t \leq 1000$ and plot the local maximum and minimum of the generated trajectory for $950 \leq t \leq 1000$ for each value of τ to obtain the bifurcation diagram with respect to τ as shown in Figure 3(A). This diagram implies that a limit cycle emerges for $\tau > \tau_0$ and its diameter (difference between the local maximum and minimum) becomes larger as τ increases. It also indicates that a much larger value of τ (approximately, larger than 1.39 in this example) generates a trajectory having a negative minimum and thus makes dynamics economically uninteresting. Taking $\tau = \tau_1 \simeq 1.236$, we depict a pair of $(q(t-\tau), q(t))$ for $950 \leq t \leq 1000$ in Figure 3(B) to show the birth of limit cycle. Notice that the difference between the upper intersection of the upper branch with the vertical dotted line at $\tau = \tau_1$ and the lower intersection of the lower branch in Figure 3(A) corresponds to the difference between the maximum and minimum shown in Figure 3(B). These numerical examples confirm Theorem 2.



A. Bifurcation diagram　　B. Phase diagram

Figure 3. Emergence of a limit cycle

3 Discrete Dynamics

Our concern in this section is on how the different choice of the time scale affects dynamics examined in the previous section. To this end, we discretize the delay differential equation (7) by replacing $\dot{q}(t)$ with $q(t+1) - q(t)$ to obtain

$$q(t+1) = q(t) + k(a - b(\alpha+1)q(t-\tau)^{\alpha} - c\beta q(t)^{\beta-1}) \tag{25}$$

and then reconsider local and global dynamics in discrete time. The steady state of this difference equation is the same as $\bar{q}$ of the differential equation. We mention that this discrete-time equation has a τ-step delay when $\tau \geq 1$.[2] The

[2] A sailent feature of a discrete-time equation is that the equation involves at least one difference or time-delay of the dependent variable. So we refere to the τ-step delay when τ is greater than unity.

remaining of this section is divided into two parts. In the first part, we examine the case of $\tau = 0$ and give a more precise account of the non-negativity condition. In the second part, we discuss the case of $\tau \geq 1$ in detail to concentrate on delay effects in the discrete-time framework.

3.1 No-delay: $\tau = 0$

If $\tau = 0$, then equation (25) becomes a nonlinear first-order difference equation

$$q(t+1) = q(t) + k(a - b(\alpha+1)q(t)^{\alpha} - c\beta q(t)^{\beta-1}) \tag{26}$$

where the right hand side is denoted by $R(q(t))$. The linearized equation of (26) has the form

$$q_{\varepsilon}(t+1) = \left[1 - k(b\alpha(\alpha+1)\bar{q}^{\alpha-1} + c\beta(\beta-1)\bar{q}^{\beta-2})\right] q_{\varepsilon}(t) \tag{27}$$

or

$$q_{\varepsilon}(t+1) = [1 - k(A+B)]\, q_{\varepsilon}(t). \tag{28}$$

The steady state is locally asymptotically stable if

$$|1 - k(A+B)| < 1 \tag{29}$$

or

$$k < k^S := \frac{2}{A+B}. \tag{30}$$

As expected, the discrete-time equation (27) has a more restrictive stability condition than the continuous-time equation (6).[3] Indeed, the former has a critical value of the speed of adjustment and is locally asymptotically stable if the speed of adjustment is smaller than the critical value and locally unstable if larger. On the other hand, the continuous-time equation is locally asymptotically stable for any value of the speed of adjustment.

We now proceed to numerical simulations on global behavior when the steady state is locally unstable. We first determine the non-negativity condition that guarantees the non-negativity of the trajectories for all $t \geq 0$ when $\alpha \geq 2$ and $\beta = 1$ and then numerically verify it in the case of $\alpha \geq 2$ and $\beta = 2$.[4] Solving equation (4) with $\beta = 1$ yields the explicit form of $\bar{q}$,

$$\bar{q} = \left(\frac{a-c}{b(\alpha+1)}\right)^{\frac{1}{\alpha}}. \tag{31}$$

In order to have positive solution of the first order condition we have to assume that $a > c$.

Lemma 1 *Given k, there is a unique $q_M(k)$ such that $R(q_M(k)) = 0$.*

[3] It is well-known that the choice of time scales matters.

[4] See Matsumoto and Szidarovszky (2013) for the non-negativity condition in the general case with $\alpha \geq 1$ and $\beta \geq 2$.

Proof. $R(q_M(k)) = 0$ implies that q_M is the solution of equation

$$b(\alpha+1)q^\alpha = (a-c) + \frac{1}{k}q \tag{32}$$

where the left hand side is denoted by $F(q)$. At $q = 0$, $F(0) = 0$ and is less than the right hand side. As q goes to infinity, $F(q)$ converges to ∞ faster than the right hand side. So there is at least one solution. Since $F(q)$ is strictly convex and the right hand side is linear, the solution is unique. ■

It can be verified, due to equation (32), that

$$\lim_{k\to 0} q_M(k) = \infty,$$

$$\lim_{k\to\infty} q_M(k) = \bar{q}$$

and

$$\frac{dq_M(k)}{dk} = \frac{q_M/k^2}{1/k - F'(q_M)} < 0$$

where $F'(q_M) > 1/k$ because the $F(q)$ curve intercepts the right-hand side of equation (32) from below at $q = q_M$.

In the same way, we have the following.

Lemma 2 *Given k, there is a unique $q_m(k)$ such that $R'(q_m(k)) = 0$.*

Proof. $R'(q_m(k)) = 0$ implies that $q_m(k)$ is the solution of

$$\alpha b(\alpha+1)q^{\alpha-1} = \frac{1}{k} \tag{33}$$

where the left hand side is zero at $q = 0$, increasing and converges to ∞ as $q \to \infty$ so the existence of the unique solution is clear. ■

Solving (33) gives the explicit form of $q_m(k)$,

$$q_m(k) = \frac{1}{D}k^{-\frac{1}{\alpha-1}}$$

with

$$D = (\alpha b(\alpha+1))^{\frac{1}{\alpha-1}} > 0.$$

It is then clear that

$$\lim_{k\to 0} q_m(k) = \infty$$

$$\lim_{k\to\infty} q_m(k) = 0$$

and

$$\frac{dq_m(k)}{dk} = -\frac{1}{D(\alpha-1)}k^{-\frac{\alpha}{\alpha-1}} < 0,$$

so $q_m(k)$ strictly decreases in k. By these properties of $q_m(k)$, the following results are clear.

Lemma 3 *There is a positive* $\bar{k}$ *such that* $q_m(\bar{k}) = \bar{q}$ *and*

$$q_m(k) \gtrless \bar{q} \text{ for } k \lessgtr \bar{k}.$$

The next lemma describes the relation between $q_M(k)$ and $q_m(k)$.

Lemma 4 $q_M(k) > q_m(k)$ *for any* $k > 0$.

Proof. Notice first that equation (33) can be rewritten as

$$\alpha b(\alpha+1)q^{\alpha} = \frac{1}{k}q. \tag{34}$$

Let $G(q)$ denote the left hand side. Then from (32) we have

$$F(q) = (a-c) + \frac{q}{k} \text{ at } q = q_M$$

and

$$G(q) = \frac{q}{k} \text{ at } q = q_m.$$

Under $\alpha > 1$, $G(q) > F(q)$ always for $q > 0$ and since $a > c$, the right hand side of the first equation is larger than that of the second equation. Therefore $q_M(k) > q_m(k)$ follows for any $k > 0$. ■

The last lemma confirms the shape of the $R(q_m(k))$ curve and the relation with the $q_m(k)$ curve. In order to clarify the dependency, we denote $R(q_m(k))$ as $R(q_m(k), k)$ and then introduce a new function, $\boldsymbol{R}(k) = R(q_m(k), k)$.

Lemma 5 $\boldsymbol{R}(k)$ *takes a U-shaped profile with respect to* k *and*

$$\boldsymbol{R}(k) \lesseqgtr q_m(k) \text{ according to } k \lesseqgtr \bar{k}.$$

Proof. Differentiating $\boldsymbol{R}(k)$ with respect to k yields

$$\frac{d\boldsymbol{R}}{dk} = \left.\frac{\partial R(q_m(k), k)}{\partial q}\right|_{q=q_m} \frac{dq_m}{dk} + \left.\frac{\partial R(q_m(k), k)}{\partial k}\right|_{q=q_m}$$

where the first term on the right hand side is zero at $q = q_m$. So

$$\begin{aligned} \frac{d\boldsymbol{R}}{dk} &= a - c - b(\alpha+1)q_m^{\alpha} \\ &= b(\alpha+1)\left(\bar{q}^{\alpha} - q_m^{\alpha}\right) \end{aligned} \tag{35}$$

where the equality $a - c = b(\alpha+1)\bar{q}^{\alpha}$ from (31) is used at the second step. It is then obtained that

$$\frac{d\boldsymbol{R}}{dk} \lesseqgtr 0 \text{ according to } \bar{q} \lesseqgtr q_m,$$

and by Lemma 3,

$$\bar{q} \lesseqgtr q_m \Longleftrightarrow k \lesseqgtr \bar{k}.$$

So $\boldsymbol{R}(k)$ takes a U-shaped profile and has its minimum at $k = \bar{k}$. Notice that from (25) and (35)

$$\frac{d\boldsymbol{R}}{dk} = \frac{1}{k}\left(R(q_m(k)) - q_m(k)\right),$$

so

$$\boldsymbol{R}(k) \lesseqgtr q_m(k) \text{ for } k \lesseqgtr \bar{k}$$

which means that the $q_m(k)$ curve passes the minimum point of $\boldsymbol{R}(k)$. ■

Since substituting $q_m(k)$ into $R(q)$ represents

$$R(q_m(k)) = q_m(k) + k(a - c - b(\alpha+1)(q_m(k))^\alpha)$$

and equation (33) can be written as

$$q_m(k) = \alpha k b(\alpha+1)(q_m(k))^\alpha,$$

we have

$$R(q_m(k)) = k(a-c) + (\alpha-1)b(\alpha+1)k \cdot (q_m(k))^\alpha$$

where

$$k \cdot (q_m(k))^\alpha = \frac{1}{D^\alpha}\left(\frac{1}{k}\right)^{\frac{1}{\alpha-1}}.$$

We then have

$$\lim_{k\to 0} R(q_m(k)) = \lim_{k\to 0} k(q_m(k))^\alpha = \infty$$

and

$$\lim_{k\to\infty} R(q_m(k)) = \lim_{k\to\infty} k(a-c) = \infty.$$

$R(q_m(k))$ is asymptotic to the vertical line when k approaches zero and to the $k(a-c)$ curve when k goes to infinity.

From Lemmas $1-4$, we have the following result.

Theorem 3 *There is a unique value k^N such that $R(q_m(k^N)) = q_M(k^N)$ and the non-negativity condition*

$$R(q_m(k)) \leq q_M(k)$$

holds for $k \leq \bar{k}$.

Proof. $q_M(k) > q_m(k)$ and $q_m(k) > R(q_m(k))$ for $k < \bar{k}$ imply that $R(q_m(k)) < q_M(k)$. For $k \geq \bar{k}$, $R(q_m(k)) \geq \bar{q}$ and increasingly converges to ∞ while $q_M(k)$ decreasingly converge to $\bar{q}$. Thus there is a unique value k^N such that $R(q_m(k^N)) = q_M(k^N)$ and $R(q_m(k)) \leq q_M(k)$ for $k \leq k^N$. ■

Solving $R(q_m(k^N)) = q_M(k^N)$ for k^N and using (25) and (34) we have

$$k^N = \frac{\alpha q_M + (1-\alpha) q_m}{\alpha(\alpha - c)}$$

Equation (30) with $\beta = 1$ and equation (31) yield

$$k^S = \frac{2}{\alpha b(\alpha+1)\bar{q}^{\alpha-1}}$$

and from (4)

$$b(\alpha+1)\bar{q}^{\alpha} = a - c$$

implying that

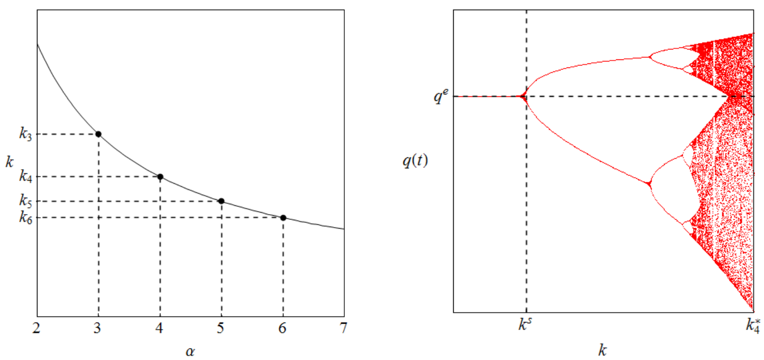
$$k^S = \frac{2\bar{q}}{\alpha(a-c)}.$$

Taking $\beta = 2$ and solving $R(q_m(k)) = q_M(k)$ with $\alpha = 3, 4, 5, 6$ yield the upper bounds of k,

$$k_3^N \simeq 0.298,\ k_4^N \simeq 0.228,\ k_5^N \simeq 0.188 \text{ and } k_6^N \simeq 0.162$$

where k_α^N is the critical value k^N for α. Figure 4(A) depicts a locus of k_α^N in the (α, k) plane and any trajectory generated by equation (26) with $k \in (0, k_\alpha^N)$ is non-negative for any $t \geq 0$. Figure 4(B) presents a bifurcation diagram with respect to k when $\alpha = 4$. It can be seen that the steady state is destabilized when $k = k^S$ ($\simeq 0.154$) and may be chaotic via the period-doubling bifurcation as k increases. It may not be possible to obtain an explicit functional form of k_α^N in terms of α and k for $\beta \geq 3$ as in the case of $\beta = 2$, however, it is possible to calculate a particular solution of $q_M(k)$ as well as $R(q_m(k))$ after specifying a value of β and thus to derive a bifurcation diagram as in Figure 4(B). Changing the value of α does not affect the qualitative aspects of the results obtained with $\alpha = 4$. The results in the case of $\tau = 0$ are summarized as follows:

Theorem 4 *Under Assumptions 1 and 2 with $\alpha = 4$ and $\beta = 2$, the monopoly equilibrium with $\tau = 0$ is locally asymptotically stable if $k < k^S$, loses stability if $k = k^S$ and arrives at complex dynamics involving chaos through a period-doubling bifurcation if $k^S < k \leq k_\alpha^N$.*



(A) k_α^N locus (B) Bifurcation diagram

Figure 4. Dynamic behavior with $\beta = 2$ and $\alpha = 3, 4, 5, 6$

3.2 Delay: $\tau \geq 1$

If $\tau = 1$, then equation (25) has one-step delay and then becomes a nonlinear second-order difference equation

$$q(t+1) = q(t) + k(a - b(\alpha+1)q(t-1)^{\alpha} - c\beta q(t)^{\beta-1}) \tag{36}$$

which can be converted to an equivalent 2D system of first-order difference equations,

$$\begin{aligned} x(t+1) &= q(t), \\ q(t+1) &= q(t) + k(a - b(\alpha+1)x(t)^{\alpha} - c\beta q(t)^{\beta-1}). \end{aligned} \tag{37}$$

The linearized system is

$$\begin{pmatrix} x(t+1) \\ q(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -kA & 1-kB \end{pmatrix} \begin{pmatrix} x(t) \\ q(t) \end{pmatrix}$$

and its characteristic equation is transformed into a quadratic equation,

$$\lambda^2 + (kB - 1)\lambda + kA = 0. \tag{38}$$

The sufficient and necessary condition that this quadratic equation has roots inside the unit cycle are given by the following three conditions,

$$\begin{aligned} &1 + (kB - 1) + kA > 0, \\ &1 - (kB - 1) + kA > 0, \\ &1 - kA > 0. \end{aligned} \tag{39}$$

The first condition of (39) is always satisfied and so is the remaining two conditions if and only if either

$$A - B \geq 0 \text{ and } k < \frac{1}{A}$$

or

$$A - B < 0 \text{ and } k < \min\left\{\frac{1}{A}, \frac{2}{B-A}\right\}.$$

As shown in Figure 1, we see the conditions under which pair (α, β) gives rise to $A - B > 0$. Henceforth in order to simplify the analysis, we confine our attention to the case where $A - B > 0$ and put the following assumption.

Assumption 3. $A - B > 0$.

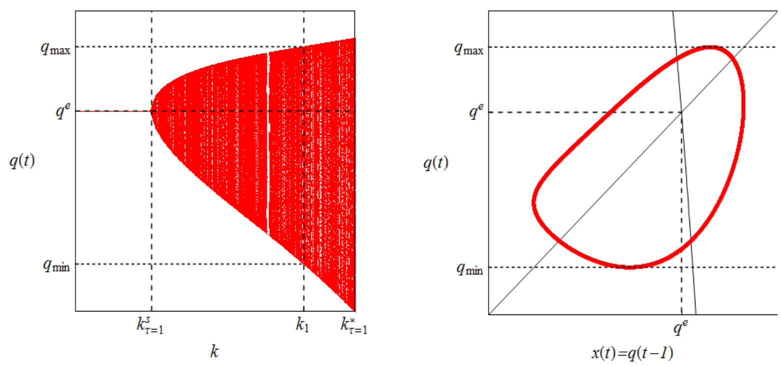
Destabilization of the monopoly steady state occurs only by violating the third condition from which the threshold value is defined as

$$k^S_{\tau=1} = \frac{1}{A}.$$

The monopoly equilibrium changes stability through a pair of complex conjugate roots. In particular, as k becomes larger than $k^S_{\tau=1}$, the steady state bifurcates to a periodic cycle, which is then replaced with a quasi-periodic cycle. Such a stability change is called Neimark-Sacker (NS henceforth) bifurcation. The results analytically obtained are the following:

Theorem 5 *Under Assumptions 1,2 and 3 with $\alpha = 4$ and $\beta = 2$, the monopoly equilibrium with $\tau = 1$ is locally asymptotically stable if $k < k^S_{\tau=1}$, loses stability if $k = k^S_{\tau=1}$ and generates aperiodic oscillations through a NS bifurcation if $k^S_{\tau=1} < k < k^N_{\tau=1}$.*

Taking $\alpha = 4$ and $\beta = 2$ and selecting k as the bifurcation parameter, we illustrate the bifurcation diagram in Figure 5(A) in which stability of the steady state is changed to instability at $k = k^S_{\tau=1}(= 1/12)$ and cyclic behavior emerges for $k > k^S_{\tau=1}$. When k arrives at $k^N_{\tau=1}(\simeq 0.119)$, the non-negativity condition is violated resulting in the birth of economically uninteresting behavior. In Figure 5(B), we take $k = k_1(= 0.11)$ and depict a phase diagram, that is, a cyclic behavior in the $(q(t-1), q(t))$ plane.



(A). Bifurcation diagram (B). Limit cycle

Figure 5. Dynamic behavior via Hopf bifurcation

We further extend our analysis to a two-step delay (i.e., $\tau = 2$) where the marginal revenue includes the delayed information obtained at period $t-2$. The dynamic equation (25) is now a third-order difference equation,

$$q(t+1) = q(t) + k(a - b(\alpha+1)q(t-2)^{\alpha} - c\beta q(t)^{\beta-1}). \tag{40}$$

This can be written as a 3D system of first-order difference equations

$$\begin{aligned} x(t+1) &= y(t) \\ y(t+1) &= q(t) \\ q(t+1) &= q(t) + k\left(a - b(\alpha+1)x(t)^{\alpha} - c\beta q(t)^{\beta-1}\right), \end{aligned} \tag{41}$$

where the steady state is $(\bar{x}, \bar{y}, \bar{q})$ with $\bar{x} = \bar{y} = \bar{q}$. Linear approximation of equation (41) yields the approximated system having the form

$$\begin{pmatrix} x(t+1) \\ y(t+1) \\ q(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -kA & 0 & 1-kB \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ q(t) \end{pmatrix} \tag{42}$$

and the corresponding characteristic equation is cubic

$$\lambda^3 + (kB-1)\lambda^2 + kA = 0. \tag{43}$$

The steady state is locally asymptotically stable if all eigenvalues of equation (43) are less than unity in absolute value. Farebrother (1973) and Okuguchi and Irie (1990) have proved that the most simplified form of the sufficient and necessary conditions for the cubic equation to have roots only inside the unit cycle are

$$
\begin{aligned}
&1 + a_1 + a_2 + a_3 > 0, \\
&1 - a_1 + a_2 - a_3 > 0, \\
&1 - a_2 + a_1 a_3 - a_3^2 > 0, \\
&a_2 < 3
\end{aligned}
\tag{44}
$$

where

$$a_1 = kB - 1,\ a_2 = 0 \text{ and } a_3 = kA.$$

It can be verified that the first and fourth conditions are always satisfied while the second and third condition holds if

$$k < k_2^S := \frac{2}{A+B} \text{ and } k < k_3^S := \frac{2}{A + \sqrt{A^2 + 4A(A-B)}}.$$

If we denote the left-hand side of the third condition in (44) by $g(k)$, it can be written as

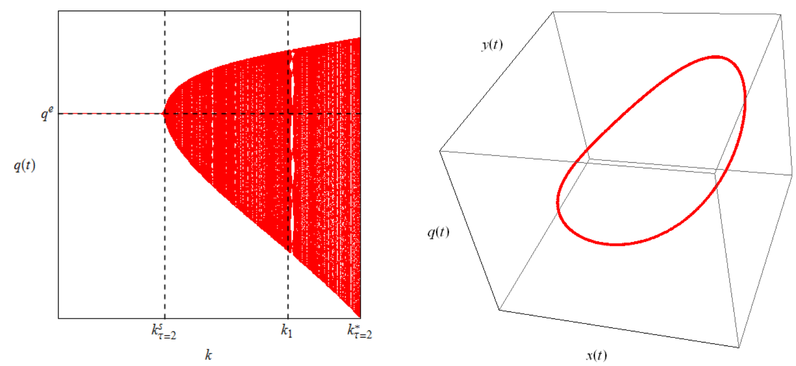
$$g(k) = A(B-A)k^2 - Ak + 1.$$

This quadratic polynomial has two real roots, one is negative and the other, k_3^S, is positive. Substituting k_2^S into $g(k)$ yields

$$g(k_2^S) = -\frac{(A-B)(5A+B)}{(A+B)^2} < 0.$$

This inequality implies that $k_3^S < k_2^S$. Numerical simulations are illustrated in Figure 6 in which the stability is lost at $k_3^S \simeq 0.053$, the non-negativity condition is violated at $k_{\tau=2}^N \simeq 0.053$. The phase diagram is illustrated for $k_1 = 0.0675$. Figure 6 is essentially similar to Figure 5. Therefore rewriting k_3^S as $k_{\tau=2}^S$ describes the results that have been made as follows:

Theorem 6 *Under Assumptions 1,2 and 3 with $\alpha = 4$ and $\beta = 2$, the monopoly equilibrium with $\tau = 2$ is locally asymptotically stable if $k < k_{\tau=2}^S$, loses stability if $k = k_{\tau=2}^S$ and generates aperiodic oscillations through a NS bifurcation if $k_{\tau=2}^S < k < k_{\tau=2}^N$.*



(A). Bifurcation diagram (B). Limit cyle

Figure 6. Dynamic behavior of system (41)

4 Conclusion

Static and dynamic monopolies were examined with general price and cost functions. The existence of a unique profit maximizing output was first proved which can be computed by solving a single-variable monotonic equation. Continuous time scales were then assumed and the local asymptotic stability of the resulting dynamic system was proved regardless of model parameter values. If time delay was assumed in obtaining revenue information, then the system is locally asymptotically stable with arbitrary length of the delay for $A \leq B$, and if $A > B$, then local asymptotic stability occurs if the delay is smaller than a given threshold. Notice that this threshold value decreases with increasing value of the speed of adjustment. Under discrete time scales the system is locally asymptotically stable if the speed of adjustment is below a certain threshold which is decreasing if the values of A and/or B increases. If the length of the delay is one step, then similar result holds where the threshold decreases in A but is independent of the value of B when $B < A$. If the length of the delay is two time steps, then the threshold depends on both A and B, increases as A decreases and/or B increases. It should be worthwhile to point out that in the discrete-time model the steady state loses stability via period-doubling bifurcation if $\tau = 0$ and via Neimark-Sacker bifurcation if $\tau \geq 1$.

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