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# Learning Monopolies with Delayed Feedback on Price Expectations

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**Abstract**

We call the intercept of the price function with the vertical axis the *maximum price* and the slope of the price function the *marginal price*. In this paper it is assumed that a monopolistic firm has full information about the marginal price and its own cost function but is uncertain on the maximum price. However, by repeated interaction with the market, the obtained price observations give a basis for an adaptive learning process of the maximum price. It is also assumed that the price observations have fixed delays, so the learning process can be described by a delayed differential equation. In the cases of one or two delays, the asymptotic behavior of the resulting dynamic process is examined, stability conditions are derived. Three main results are demonstrated in the two delay learning process. First, it is possible to stabilize the equilibrium which is unstable in the one delay model. Second, complex dynamics involving chaos, which is impossible in the one delay model, can emerge. Third, alternations of stability and instability (i.e., stability switches) occur repeatedly.

**Keywords**: Bounded rationallity, Monopoly dynamics, Fixed time delay, Adaptive learning, Hopf bifurcation

**JEL Classification** : C62, C63, D21, D42

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# 1 Introduction

This paper uses the familiar monopoly model in which there is only one firm having linear price and cost functions. Implicit in the text-book approach is an assumption of complete and instantaneous information availability on price and cost functions. In consequence, the textbook-monopoly can find its optimal choices of price and quantity to maximize profit with one shot. Thus the traditional monopoly model is *static* in nature. It is now well-known that the assumption of such a rational monopoly is questionable and unrealistic in real economies, since there are always uncertainty and a time delay in collecting information and determining optimal responses, and in addition, function relations such as the market price function cannot be determined exactly based on theoretical consideration and observed data. Getting closer to the real world and improving the monopoly theory, we replace this extreme but convenient assumption with the more plausible one. Indeed, the monopolistic firm is assumed, first, to have only limited knowledge on the price function and, second, to obtain it with time delay. As a natural consequence of these alternations, the firm gropes for its optimal choice by using delay data obtained through market experiences. The modified monopoly model becomes *dynamic* in nature.

In the recent literature, it has been demonstrated that a boundedly rational monopoly may exhibit simple as well as complex dynamic behavior. Nyarko (1991) solves the profit maximizing problems without knowing the slope and intercept of a linear demand and shows that using Bayesian updating leads to cyclic actions and beliefs if the market demand is mis-specified. Furthermore, in the framework with discrete-time scale, Puu (1995) shows that the boundedly rational monopolist behaves in an erratic way under cubic demand with a reflection point. In the similar setting, Naimzada and Ricchiuti (2008) represent that complex dynamics can arise even if cubic demand does not have a reflection point. Naimzada (2012) exhibits that delay monopoly dynamics can be described by the well-known logistic equation when the firm takes a special learning scheme. More recently, Matsumoto and Szidarovszky (2014a) demonstrate that the monopoly equilibrium undergoes to complex dynamics through either a period-doubling or a Neimark-Sacker bifurcation.

This paper considers monopoly dynamics in continuous time scale and presents a new characterization of a monopoly's learning process under a limited knowledge of the market demand. It is a continuation of Matsumoto and Szidarovszky (2012) (MS henceforth) where the monopolist does not know the price function and fixed time delays are introduced into the output adjustment process based on the gradient of the marginal expected profit. It also aims to complement Matsumoto and Szidarovszky (2014b,c) where uncertain delays are modeled by continuously distributed time delays when the firm wants to react to average past information instead of sudden market changes.[1] Under a circumstance in

[1] There are two different ways to model time delays in continuous-time scale: fixed time delay and continuously distributed time delay (fixed delay and continuous delay henceforth). The choice of the type of delay results in the use of different analytical tools. In the cases of fixed delay, dynamics is described by a delay differential equation whose characteristic equation

which price is uncertain and the price information is delayed, this paper examines the learning scheme in the cases of a single delay and two delays. It is an extended version of MS and thus has similarities and dissimilarities to MS. Its main purpose to show that cyclic and erratic behavior can emerge from quite simple economic structures when uncertainty, information delays and behavioral nonlinearities are present is the same. Gradient dynamics without optimal behavior in MS is replaced with the learning scheme with profit maximizing behavior. In spite of this behavioral difference, derived mathematical equations and their solutions are the same. However the ways to arrive at the solutions are very different; an elementary method applicable only to the very special form of the mathematics equation is used in MS while we apply a more general method developed by Gu *et al.* (2005). As a result, the stability/instability conditions are simplified and clarified. Since economics behind the mathematical equations are different, the results to be obtained have different economic implications.

This paper develops as follows. The basic mathematical model is formulated and a single delay equation is examined in Section 2. In Section 3, it is assumed that the firm formulates its price expectation based on two delayed observations by using a delay feedback. Complete stability analysis is given, the stability regions are determined and illustrated. The occurrence of Hopf bifurcation is shown when one of the two delays is selected as a bifurcation parameter. The last section offers conclusions and further research directions.

## 2 The Basic Model

Consider a single product monopolist that sells its product to a homogeneous market. Let $q$ denote the output of the firm, $p(q) = a - bq$ the price function and $C(q) = cq$ the cost function.[2] Since $p(0) = a$ and $|\partial p(q)/\partial q| = b$, we call $a$ the *maximum price* and $b$ the *marginal price*. There are many ways to introduce uncertainty into this framework. In this study, it is assumed that the firm knows the marginal price but does not know the maximum price. In consequence it has only an estimate of it, which is denoted by $a^e$. So the firm believes that its profit is

$$\pi^e = (a^e - bq)q - cq$$

and its best response is

$$q^e = \frac{a^e - c}{2b}.$$

is a mixed polynomial-exponential equation with infinitely many eigenvalues. Bellman and Cooke (1963) offer methodology of complete stability analysis in such models. On the other hand, in the cases of continuous delay, Volterra-type integro-differential equations are used to model the dynamics. The theory of continuous delays with applications in population dynamics is offered by Cushing (1977). Since Invernizzi and Medio (1991) have introduced continuous delays into mathematical economics, its methodology is used in analyzing many economic dynamic models.

[2] Linear functions are assumed only for the sake of simplicity. We can obtain a similar learning process to be defined even if both functions are nonlinear. It is also assumed for the sake of simplicity that the firm has perfect knowledge of production technology (i.e., cost function).

Further, the firm expects the market price to be

$$p^e = a^e - bq^e = \frac{a^e + c}{2}. \tag{1}$$

However, the actual market price is determined by the real price function

$$p^a = a - bq^e = \frac{2a - a^e + c}{2}. \tag{2}$$

Using these price data, the firm updates its estimate. The simplest way for adjusting the estimate is the following. If the actual price is higher than the expected price, then the firm shifts its believed price function by increasing the value of $a^e$, and if the actual price is the smaller, then the firm decreases the value of $a^e$. If the two prices are the same, then the firm wants to keep its estimate of the maximum price. This adjustment or learning process can be modeled by the differential equation

$$\dot{a}^e(t) = k\left[p^a(t) - p^e(t)\right],$$

where $k > 0$ is the speed of adjustment and $t$ denotes time. Substituting relations (1) and (2) reduces the adjustment equation to a linear differential equation with respect to $a^e$ as

$$\dot{a}^e(t) = k\left[a - a^e(t)\right]. \tag{3}$$

In another possible learning process, the firm revises the estimate in such a way that the growth rate of the estimate is proportional to the difference between the expected and actual prices. Replacing $\dot{a}^e(t)$ in equation (3) with $\dot{a}^e(t)/a^e(t)$ yields a different form of the adjustment process

$$\frac{\dot{a}^e(t)}{a^e(t)} = k\left[a - a^e(t)\right]$$

or multiplying both sides by $a^e(t)$ generates the logistic model

$$\dot{a}^e(t) = ka^e(t)\left[a - a^e(t)\right]. \tag{4}$$

This equation is proposed by Verhust (1845) and has been extensively studied, especially as a biological model of single species dynamics in the theoretical biology. It has already been shown that the positive equilibrium $a^e(t) = a$ of equation (4) is globally stable. We adopt this nonlinear equation as the basic learning process henceforth.[3]

If there is a time delay $\tau$ in the estimated price, then we can rewrite the estimated price and market price at time $t$ based on information available at time $t - \tau$ as

$$p^e(t; t-\tau) = a^e(t-\tau) - bq^e(t; t-\tau)$$

[3] The linear approximation of nonlinear equation (4) to be considered below is essentially equivalent to the linear equation (3).

and

$$p^a(t; t-\tau) = a - bq^e(t; t-\tau),$$

where $q^e(t; t-\tau)$ is the delay best reply determined at time $t$ depending on the expected maximum price at time $t-\tau$,

$$q^e(t; t-\tau) = \frac{a^e(t-\tau) - c}{2b}.$$

Then equation (4) have to be modified as

$$\dot{a}^e(t) = ka^e(t)\left[a - a^e(t-\tau)\right] \tag{5}$$

which is the delay logistic equation. It has two equilibria as well, $a^* = 0$ and $a^* = a > 0$. If $F(a^e(t), a^e(t-\tau))$ denotes the right hand side of equation (5), then the linear approximation in a neighborhood of an equilibrium $(a^*,\ a^*)$ is

$$\dot{a}^e_\delta(t) = \left.\frac{\partial F}{\partial a^e(t)}\right|_{(a^*,\ a^*)} a^e_\delta(t) + \left.\frac{\partial F}{\partial a^e(t-\tau)}\right|_{(a^*,\ a^*)} a^e_\delta(t-\tau),$$

where $a^e_\delta(t) = a^e(t) - a$. Small perturbations from the trivial equilibrium satisfy the linear equation,

$$\dot{a}^e_\delta(t) = aka^e_\delta(t)$$

which implies that $a^* = 0$ is locally unstable with exponential growth. We thus draw our attention only to the positive equilibrium The linearized version of equation (5) around the positive equilibrium is written as a linear delay differential equation,

$$\dot{a}^e_\delta(t) = -aka^e_\delta(t-\tau). \tag{6}$$

Introducing the new variable $z(t) = a^e_\delta(t)$ and the new parameter $\alpha = ak > 0$ reduce equation (6) to

$$\dot{z}(t) + \alpha z(t-\tau) = 0. \tag{7}$$

Apparently $z(t) = 0$ is the only equilibrium of the modified delay equation (7).

If there is no delay (i.e., $\tau = 0$), then equation (7) becomes an ordinary differential equation with characteristic polynomial $\lambda + \alpha$. So the only eigenvalue is negative implying the local asymptotic stability of the zero solution of equation (7). In consequence, the positive solution of equation (5) corresponds to the true value of the maximum price $a^* = a$ and the monopolist can learn the true demand function through a comparison with the real data.[4] We expect asymptotical stability for sufficiently small positive values of $\tau$ by continuity of variables with respect to $\tau$. However, as the length of delay changes largely, the stability of the zero solution may also change. Such phenomenon is referred to as *stability switch*. The essentially same equation as equation (7) is fully studied in MS. Applying their Theorem 1, we obtain the following results concerning dynamics of equation (5):

[4] This scheme of learning is now well-known. See, for example, Bischi, *et al.* (2010).

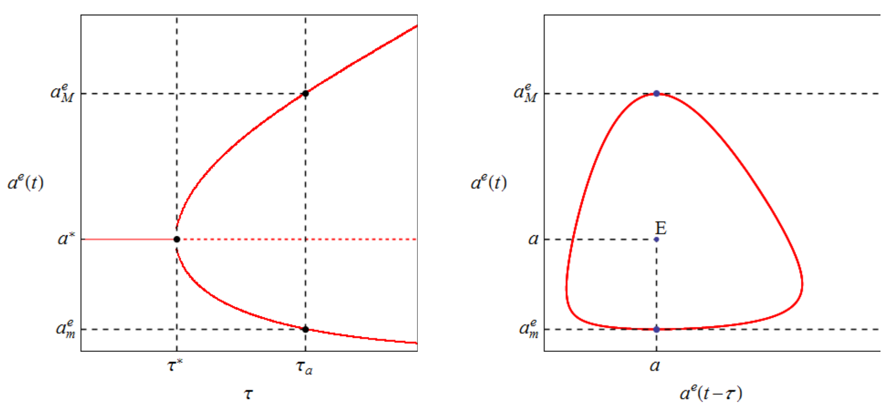
($R_1$) There is a critical value of $\tau$,

$$\tau^* = \frac{\pi}{2ak}. \tag{8}$$

($R_2$) The positive equilibrium $a^* = a$ is locally asymptotically stable for $\tau < \tau^*$, loses stability at $\tau = \tau^*$ and bifurcates to a limit cycle through Hopf bifurcation for $\tau > \tau^*$.

($R_3$) It is numerically confirmed that the Hopf bifurcation is supercritical.[5]

These results imply that the stability switch occurs at $\tau = \tau^*$ and a destabilized trajectory does not diverges globally but keeps fluctuating around the positive equilibrium. According to equation (8), the critical value $\tau^*$ becomes smaller as $a$ and/or $k$ increases. Since $a$ is the maximum demand and $k$ is the adjustment speed, larger demand and a high adjustment speed can be destabilizing factors for the positive equilibrium. Taking $a = 1$ and $k = 1$ that leads to $\tau^* = \pi/2 \simeq 1.57$, we perform simulations. Figure 1(A) is a bifurcation diagram with respect to $\tau \in [1.4,\ 2]$ and numerically confirms the first two analytical results above. Figure 1(B) illustrates a supercritical Hopf cycle for $\tau_a = 1.8$ under which the Hopf cycle has the maximum $a^e_M \simeq 2.3$ and the minimum $a^e_m \simeq 0.197$.[6] These two values are depicted as black dots on the upper and lower branches of the bifurcation diagram and also on the maximum and minimum points of the limit cycle.




(A) Bifurcation diagram (B) Hopf cycle

Figure 1. Dynamics of delay logistic equation (5)

[5] It may be possible to analytically show this by applying the normal form theory.
[6] These values are numerically determined by the rule of thumb.

## 3 Delay Feedback

Under the logistic dynamic process, learning takes place only for $\tau < \tau^*$ and becomes unsuccessful for $\tau > \tau^*$ under which cyclic behavior occurs. If the monopolist behaves along a periodic orbit, then he may realize that his expectations are systematically wrong. Under such circumstance, although he is assumed to be boundedly rational, the monopolist may change the way he forms expectations somehow. One possible and simply way is to use the information obtained in the past two different time, $t - \tau_1$ and $t - \tau_2$.[7] It is assumed that $\tau_1 < \tau_2$ and the expectations formed at time $t - \tau_1$ (i.e., $a^e(t - \tau_1)$) are the most recent information available to the monopolist at time $t$ due to the bounded rationality. The expectations formed at time $t - \tau_2$ (i.e., $a^e(t - \tau_2)$) are also known. The difference between $a^e(t - \tau_1)$ and $a^e(t - \tau_2)$ is referred to as *delay feedback*. The monopolist employs a different learning mechanism with taking account this delay feedback into

$$\frac{\dot{a}^e(t)}{a^e(t)} = k\left[a - a^e(t - \tau_1)\right] + \beta\left[a^e(t - \tau_1) - a^e(t - \tau_2)\right],$$

where $\beta$ is a coefficient of the feedback. The growth rate of the expectation adjustment is determined by two factors, the observed price difference and the delay feedback. This equation is equivalently written as

$$\dot{a}^e(t) = ka^e(t)\left[a - \omega a^e(t - \tau_1) - (1 - \omega)a^e(t - \tau_2)\right], \tag{9}$$

where the coefficient $\omega$ is defined by

$$\omega = 1 - \frac{\beta}{k}.$$

Notice that equation (9) is reduced to equation (5) if $\beta = 0$ or $\beta = k$. Thus $\beta \neq 0$ and $\beta \neq k$ are assumed. If $k > \beta > 0$, then $1 > \omega > 0$ under which the monopolist uses interpolation between the observations. Further, he puts more weight on the expectation at time $t - \tau_1$ if $\omega > 1/2$ and time $t - \tau_2$ if $\omega < 1/2$. If $\beta < 0$ or $\beta > k$, then $\omega > 1$ or $\omega < 0$ under which the monopolist uses extrapolation to predict the current price. It is natural to suppose that the monopolist uses interpolation and the more recent observation is more valuable (i.e., $1 > \omega \geq 1/2$), we make the following assumption:

**Assumption 1** $\beta > 0$ and $k \geq 2\beta$.

Equation (9) is the logistic equation with two delays and has two equilibria, the zero equilibrium, $a^* = 0$ and the positive equilibrium, $a^* = a > 0$, both of which are equilibria of the logistic equation with one delay (5). Let

[7] If the monopolist is supposed to use all the past data up to time $\tau_1$, it is appropriate to introduce continuously distributed time delay. See Matsumoto and Szidarovszky (2014 b, c) for detail.

$G(a^e(t), a^e(t-\tau_1), a^e(t-\tau_2))$ be the right hand side of equation (9). The linear approximation in a neighborhood of the equilibrium $\boldsymbol{a}^* = (a^*,\ a^*,\ a^*)$ is

$$\dot{a}^e_\delta(t) = \left.\frac{\partial G}{\partial a^e(t)}\right|_{\boldsymbol{a}^*} a^e_\delta(t) + \left.\frac{\partial G}{\partial a^e(t-\tau_1)}\right|_{\boldsymbol{a}^*} a^e_\delta(t-\tau_1) + \left.\frac{\partial G}{\partial a^e(t-\tau_2)}\right|_{\boldsymbol{a}^*} a^e_\delta(t-\tau_2).$$

Similarly to the one delay logistic equation, it can be shown that the zero equilibrium is locally unstable. On the other hand, the linear approximation in a neighborhood of the positive equilibrium $\boldsymbol{a}^* = (a,\ a,\ a)$ is

$$\dot{a}^e_\delta(t) = \alpha\left[-\omega a^e_\delta(t-\tau_1) - (1-\omega)a^e_\delta(t-\tau_2)\right] \tag{10}$$

where we can remember $\alpha = ak$.

To study the change of stability of equation (10) as the delays $\tau_1$ and $\tau_2$ vary, we follow the method developed by Gu *et al.* (2005).[8] If $a^e_\delta(t) = e^{\lambda t}u$, then the corresponding characteristic equation is

$$\lambda + \alpha\omega e^{-\lambda\tau_1} + \alpha(1-\omega)e^{-\lambda\tau_2} = 0. \tag{11}$$

We introduce the following functions,

$$p_0(\lambda) = \lambda,\ p_1(\lambda) = \alpha\omega \text{ and } p_2(\lambda) = \alpha(1-\omega)$$

and confine our analysis to the case where the following four conditions are satisfied:

(I) $\deg[p_0(\lambda)] \geq \max\left\{\deg\left[p_1(\lambda)\right], \deg\left[p_2(\lambda)\right]\right\}$.

(II) $p_0(0) + p_1(0) + p_2(0) \neq 0$.

(III) The polynomials $p_0(\lambda),\ p_1(\lambda)$ and $p_2(\lambda)$ do not have any common roots.

(IV) $\lim_{\lambda\to\infty}\left(\left|\frac{p_1(\lambda)}{p_0(\lambda)}\right| + \left|\frac{p_1(\lambda)}{p_0(\lambda)}\right|\right) < 1$.

Equation (11) satisfies these conditions. Since $\deg[p_0(\lambda)] = 1$ and $\deg\left[p_1(\lambda)\right] = \deg\left[p_2(\lambda)\right] = 0$,[9] condition (I) is satisfied. If this condition is violated, then the equilibrium cannot be stable for any positive delays (see, for example, Bellman and Cooke, 1963). Condition (II) is clearly satisfied as $p_0(0) + p_1(0) + p_2(0) = k \neq 0$. It prevents $\lambda = 0$ being a solution of equation (11) for which no asymptotical stability is possible. Condition (III) is natural. $p_i(\lambda)$ for $i = 0, 1, 2$ apparently have no common roots. If polynomials have a common root $\lambda_0$ such that $q_i(\lambda) = (\lambda - \lambda_0)p_i(\lambda)$ for $i = 0, 1, 2$, then equation (11) can be factored as

$$(\lambda - \lambda_0)(q_0(\lambda) + q_1(\lambda)e^{-\lambda\tau_1} + q_2(\lambda)e^{-\lambda\tau_2}) = 0$$

[8] In Matsumoto and Szidarovszky (2012), an elementary method is used to solve equation (10). It is applied only to a special form of the two delay equation. On the other hand, Gu's method is highly advanced and systematic. It is applicable to a more general form. In our special case, both lead us to the same solution in different forms.

[9] "deg" means the degree of polynomial.

where $q_i(\lambda)$ for $i = 0, 1, 2$ now satisfy condition (III). Since equation (10) is a delay differential equation of retarded type, the limit in our case is zero and thus condition (IV) is satisfied. Notice that condition (IV) is a restriction of condition (I). If condition (I) is violated, then the limit becomes infinity. Condition (IV) is a necessary condition for the continuity condition which states that if the delays $(\tau_1, \tau_2)$ continuously vary within the first quadrant of $R^2$, then the number of zero of equation

$$p_0(\lambda) + p_1(\lambda)e^{-\lambda\tau_1} + p_2(\lambda)e^{-\lambda\tau_2} = 0$$

with positive real parts can change only when a zero appears on or crosses the imaginary axis. This fact is the basis for identifying the stability switch curve with zeros on the imaginary axis.

We define

$$a_1(\lambda) = \frac{p_1(\lambda)}{p_0(\lambda)} = \frac{\alpha\omega}{\lambda} \text{ and } a_2(\lambda) = \frac{p_2(\lambda)}{p_0(\lambda)} = \frac{\alpha(1-\omega)}{\lambda}$$

and then rewrite equation (11) as

$$1 + a_1(\lambda)e^{-\lambda\tau_1} + a_2(\lambda)e^{-\lambda\tau_2} = 0 \tag{12}$$

We examine the stability switch of the positive of the non-trivial solution of dynamic equation (10) as the delays $(\tau_1, \tau_2)$ vary. The modified characteristic equation (12) must have a pair of pure conjugate imaginary roots at the critical delays for which the stability switch occurs. So let $\lambda = i\nu$ with $\nu > 0$ which is then substituted into equation (12) to obtain

$$1 + a_1(i\nu)e^{-i\nu\tau_1} + a_2(i\nu)e^{-i\nu\tau_2} = 0 \tag{13}$$

where

$$a_1(i\nu) = -i\frac{\alpha\omega}{\nu} \text{ and } a_2(i\nu) = -i\frac{\alpha(1-\omega)}{\nu}. \tag{14}$$

The absolute values of these pure imaginary roots are

$$|a_1(i\nu)| = \frac{\alpha\omega}{\nu} \text{ and } |a_2(i\nu)| = \frac{\alpha(1-\omega)}{\nu} \tag{15}$$

and the arguments are

$$\arg[a_1(i\nu)] = \frac{3\pi}{2} \text{ and } \arg[a_2(i\nu)] = \frac{3\pi}{2}. \tag{16}$$

We can now consider the three terms in the left hand side of equation (13) as three vectors in the complex plane with the magnitudes 1, $|a_1(i\nu)|$ and $|a_2(i\nu)|$. The right hand side is zero, which implies that if we put these vectors head to tail, then they form a triangle as illustrated in Figure 2. Since the sum of lengths of the two line segments is not shorter than that of the remaining line segment in a triangle, these absolute values satisfy the following three inequality conditions

$$1 \leq |a_1(i\nu)| + |a_2(i\nu)| \Longleftrightarrow \nu \leq \alpha,$$

$$|a_1(i\nu)| \leq 1 + |a_2(i\nu)| \Longleftrightarrow \alpha(2\omega - 1) \leq \nu$$

and

$$|a_2(i\nu)| \leq 1 + |a_1(i\nu)| \Longleftrightarrow -\nu \leq 0 \leq \alpha(2\omega - 1).$$

The third condition is always fulfilled under Assumption 1. So the first and second conditions determine the feasible domain of $\nu$,

$$\alpha(2\omega - 1) \leq \nu \leq \alpha. \tag{17}$$

Let $\theta_1$ and $\theta_2$ be the right hand and left hand internal angles of the triangle in Figure 2. They can be calculated by the law of cosine as

$$\theta_1(\nu) = \cos^{-1}\left(\frac{1 + |a_1(i\nu)| - |a_2(i\nu)|}{2\,|a_1(i\nu)|}\right) = \cos^{-1}\left(\frac{\nu^2 + 2\alpha^2\omega - \alpha^2}{2\alpha\nu\omega}\right) \tag{18}$$

and

$$\theta_2(\nu) = \cos^{-1}\left(\frac{1 + |a_2(i\nu)| - |a_1(i\nu)|}{2\,|a_2(i\nu)|}\right) = \cos^{-1}\left(\frac{\nu^2 - 2\alpha^2\omega + \alpha^2}{2\alpha\nu(1-\omega)}\right). \tag{19}$$

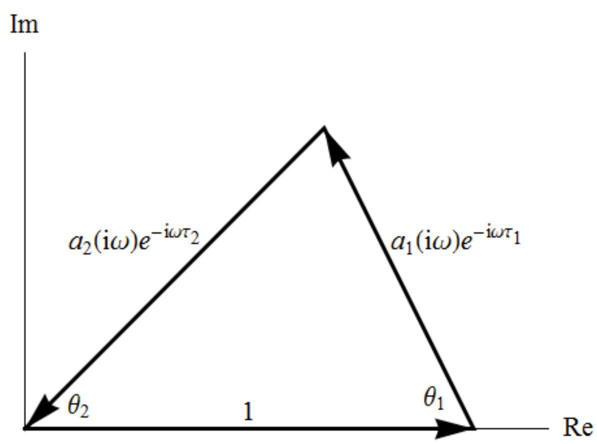



Figure 2. Triangle formed by 1, $|a_1(i\nu)|$ and $|a_2(i\nu)|$

The arguments in (16) and the internal angles just obtained in (18) and (19) satisfy the following relations

$$\pm\theta_1 = \pi - \left\{\arg\left[a_1(i\nu)e^{-i\nu\tau_1}\right] + 2m\pi\right\}$$

and

$$\pm\theta_2 = \pi - \left\{\arg\left[a_2(i\nu)e^{-i\nu\tau_2}\right] + 2n\pi\right\}.$$

Using the formula $\arg\left[a_2(i\nu)e^{-i\nu\tau_2}\right] = \arg\left[a_2(i\nu)\right] + \arg[e^{-i\nu\tau_2}]$ and (16), we can solve these equations for the delays $\tau_1$ and $\tau_2$ and obtain

$$\tau_1 = \frac{1}{\nu}\left(\frac{3\pi}{2} + (2m-1)\pi \pm \theta_1(\nu)\right),$$

and

$$\tau_2 = \frac{1}{\nu}\left(\frac{3\pi}{2} + (2n-1)\pi \mp \theta_2(\nu)\right).$$

We then have two sets of curves in the first quadrant of the region of $(\tau_1, \tau_2)$ and the characteristic equation has a pair of pure imaginary roots on these curves:

$$C^+(m,n)\colon \begin{cases} \tau_1^+ = \dfrac{1}{\nu}\left(\dfrac{3\pi}{2} + (2m-1)\pi + \theta_1(\nu)\right) \\[2ex] \tau_2^+ = \dfrac{1}{\nu}\left(\dfrac{3\pi}{2} + (2n-1)\pi - \theta_2(\nu)\right) \end{cases} \tag{20}$$

where

$$\begin{aligned} m &= m_0,\ m_0+1,\ m_0+2, \ldots \text{ such that } \tau_1^+ \geq 0, \\ n &= n_0,\ n_0+1,\ n_0+2, \ldots \text{ such that } \tau_2^+ \geq 0. \end{aligned}$$

and

$$C^-(m,n)\colon \begin{cases} \tau_1^- = \dfrac{1}{\nu}\left(\dfrac{3\pi}{2} + (2m-1)\pi - \theta_1(\nu)\right) \\[2ex] \tau_2^- = \dfrac{1}{\nu}\left(\dfrac{3\pi}{2} + (2n-1)\pi + \theta_2(\nu)\right) \end{cases} \tag{21}$$

where

$$\begin{aligned} m &= \bar{m}_0,\ \bar{m}_0+1,\ \bar{m}_0+2, \ldots \text{ such that } \tau_1^- \geq 0, \\ n &= \bar{n}_0,\ \bar{n}_0+1,\ \bar{n}_0+2, \ldots \text{ such that } \tau_2^- \geq 0. \end{aligned}$$

Notice that $m_0$ and $\bar{m}_0$ are the smallest positive integers so that $\tau_1 \geq 0$ and while $n_0$ and $\bar{n}_0$ are the smallest positive integer so that $\tau_2 \geq 0$. Given $(m,n)$, $C^{\pm}(m,n)$ constructs a segment of $(\tau_1^{\pm}, \tau_2^{\pm})$ for $\nu \in [\alpha(2\omega-1), \alpha]$. The next result shows that these segments are smoothly connected as one continuous curve. All the proofs of the theorems are collectively given in the Appendix.

**Theorem 1** *With fixed value of $m$, the segments of $C^+(m,n)$ and $C^-(m,n)$ form a continuous curve as $n$ increases.*

Figure 3 numerically confirms Theorem 1 and illustrates the segments $C^+(m,n)$ and $C^-(m,n)$ with the value of $\nu$ varying from $\alpha(2\omega-1)$ to $\alpha$ for $n=0,1,2$ and $m=0$. The parameter values are $\omega=0.8$, $a=1$ and $k=1$, the last two of which imply $\alpha=1$. The positive sloping line in the lower part of Figure 3 is the 45 degree line[10] so that the condition $\tau_1<\tau_2$ is violated in the gray color region, which will be eliminated from further considerations. The red curves are $C^+(m,n)$ and the blue curves are $C^-(m,n)$. The green dots are the initial points of the segments and the black dots are the end points. Notice that two curves are connected at these points, in particular, for the initial points, $I^+(0,n+1)=I^-(0,n)$ at for $n=0,1$ and for the end points, $E^+(0,n)=E^-(0,n)$ for $n=0,1,2$. The red and blue curves shift upward when $n$ increases (i.e., increments of the initial and end points are $2\pi/(\alpha(2\omega-1))$ and $2\pi/\alpha$, respectively) and move rightward when $m$ increases. In order to keep $\tau_2$ positive, $C^+(0,0)$ is defined only for $\tau<\tau_1^0\simeq 2.35$.[11] Further, $\tau_1^m\simeq 1.493$ is the minimum $\tau_1$-value of the segments $C^-(0,n)$ while $\tau_1^M\simeq 2.733$ is the maximum $\tau_1$-value of the segments $C^+(0,n)$. To determine these numerical values, we solve $d\tau_1^-/d\nu=0$ and $d\tau_1^+/d\nu=0$ for $\nu$ to obtain the minimizer $\nu_m\simeq 0.956$ and the maximizer $\nu_M\simeq 0.621$, both of which are substituted into $\tau_1^-$ and $\tau_1^+$, respectively, to obtain $\tau_1^m$ and $\tau_1^M$.

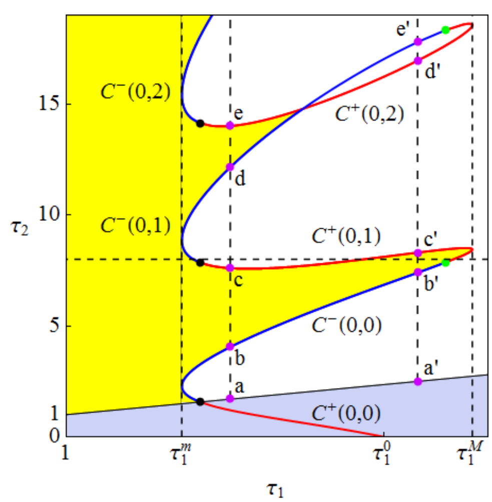



Figure 3. Partition curve in the $(\tau_1,\tau_2)$ plane

[10] The aspect ratio of Figure 3 is appropriately adjusted. In particular, $1\le\tau_1\le 2.8$ and $0\le\tau_2\le 19$.

[11] Taking $m=n=0$ and solving $\tau_2^+(\nu)=0$ yields $\nu=\alpha\sqrt{2\omega-1}$. Substituting it into $\tau_1^+(\nu)$ gives

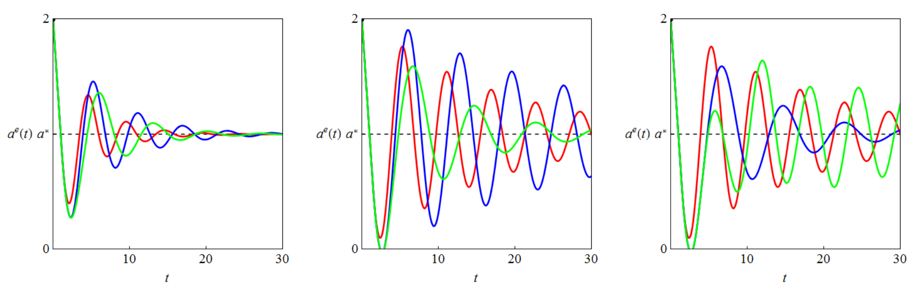
$$\frac{1}{\alpha\sqrt{2\omega-1}}\left(\frac{\pi}{2}+\cos^{-1}\left(\frac{2\omega-1}{\omega}\right)\right)$$

which is approximately 2.35 for $\alpha=1$ and $\omega=0.8$.

Concerning stability, the next result examines two cases, one with $\tau_1 = 0$ and the other with $0 < \tau_1 < \tau_1^m$ and show that the positive equilibrium is locally asymptotically stable. Those delays that do not affect stability are referred to *harmless.*

**Theorem 2** *The positive solution of equation (9) is locally stable (i) if* $\tau_1 = 0$ *and* $\tau_2 \geq 0$ *and (ii) if* $0 < \tau_1 < \tau_1^m$ *and* $\tau_2 \geq \tau_1$.

The positive equilibrium is locally asymptotically stable with harmless delays. However transient dynamic could be affected by the delays. Taking $\alpha = k = 1$ and $\omega = 4/5$, we perform several simulations with the same initial function and different values of the delays. Figure 4 geometrically summarizes the results. In Figure 4(A) $\tau_1 = 1.1$ and three different values of $\tau_2$ are $\tau_2^R = 1.1$, $\tau_2^B = 1.1 \times 2$ and $\tau_2^G = 1.1 \times 3$ where $R$, $B$ and $G$ stand for red, blue and green. The caption of Figure 4(A) means the sequence of the convergence speed, that is, the red trajectory with the smallest delay of $\tau_2$ converges first, then the green trajectory follows and finally the blue trajectory comes. The captions of another figures have the same meaning. In Figure 4(B) the values of the delays are changed to $\tau_1 = 1.4$, $\tau_2^R = 1.4$, $\tau_2^B = 1.4 \times 2$ and $\tau_2^G = 1.4 \times 3$. As a result, the green trajectory with the largest delay of $\tau_2$ converges first. In Figure 4(C), $\tau_1 = 1.4$ is kept and the multipliers are changed to 3 and to 6 leading to $\tau_2^B = 1.4 \times 3$ and $\tau_2^G = 1.4 \times 6$. In consequence, the blue trajectory with the medium delay of $\tau_2$ converges first. It is clear that different lengths of $\tau_2$ affect convergence speed but it is unclear whether a shorter delay speeds up convergence.




(A) $R > G > B$ (B) $G > R > B$ (C) $B > R > G$

Figure 4. Harmless delay affects convergence speed

We now turn to the general case where $\tau_1^m \leq \tau_1 \leq \tau_1^M$ and $\tau_2 > \tau_1$. The outer part of the red-blue connected curve in Figure 3 is called a *stability switch curve* since, as will be shown, a stability is switched on the curve from stability to instability (i.e., stability loss) or from instability to stability (i.e., stability gain).[12] Before proceeding to the stability analysis, we discuss the direction of

[12] This curve is often called a *partition curve* as it separates the positive region of the delays into two subregions.

stability switch in which the solution of equation (12) cross the imaginary axis as $(\tau_1, \tau_2)$ deviates from the partition curve. We first show the following result.

**Theorem 3** *The sign of $Q$ is negative on $C^+(m,n)$ and positive on $C^-(m,n)$ where $Q$ is defined as*

$$Q = \mathrm{Im}\left[a_1(i\nu)a_2(-i\nu)e^{i\nu(\tau_2-\tau_1)}\right].$$

As in Gu *et al.* (2005), we call the direction of any segment of the stability switch curve with increasing $\nu$ the *positive direction.* As we move along the curve in the positive direction, the region on the left hand side is called the *region on the left* which will be denoted as $\boldsymbol{L}$ and the region on the right hand side the *region on the right* denoted as $\boldsymbol{R}$. We now can state Proposition 6.1 of Gu *et al.* (2005) as follows.

**Theorem 4** *As $(\tau_1, \tau_2)$ moves from $\boldsymbol{R}$ to $\boldsymbol{L}$, a stability loss occurs if $Q > 0$ and so does a stability gain if $Q < 0$.*

Theorems 3 and 4 imply that the positive equilibrium of equation (9) is locally stable in the yellow region of Figure 3 and unstable in the white region. We numerically confirm the stability switch on the stability-switch curve. To this end we first fix $\tau_1^a = 1.7$, a little bit larger than $\tau_1^m$ and increase $\tau_2$ from 1.7 to 19 along the vertical dotted line starting at point $(\tau_1^a, \tau_1^a)$.[13] As is seen in Figure 3, the line crosses the stability-switch curve from below four times, and each intersection has a purple dot. Figure 5(A) is a bifurcation diagram with respect to $\tau_2$ and shows that the stability switch occurs at each intersection, stability to instability at points $c$ and $e$ and instability to stability at points $b$ and $d$. The equilibrium point bifurcates to a limit cycle that expands, shrinks and then merges to the equilibrium point as $\tau_2$ moves from points $a$ to $b$ or points $c$ to $d$. According to Theorem 4, we have the following analytical results which are coincide with the numerical results obtained above:

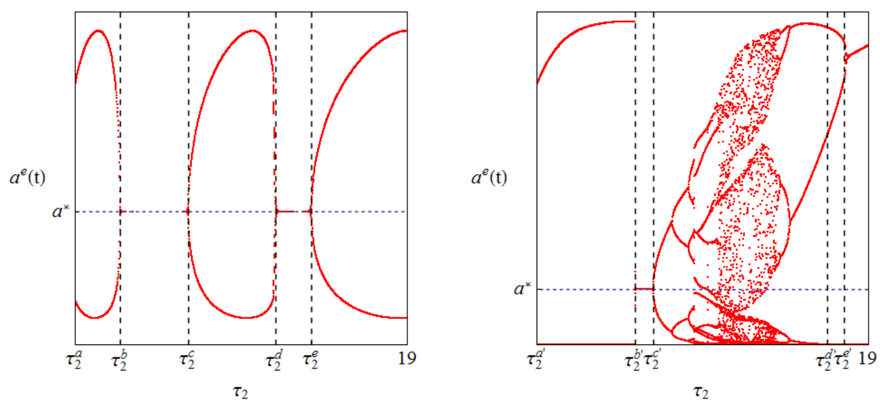
(i) stability is gained at points $b$ and $d$ since $(\tau_1, \tau_2)$ crosses the curve from $\boldsymbol{L}$ to $\boldsymbol{R}$ and $Q > 0$ on $C^-(0,0)$ and $C^-(0,1)$.

(ii) stability is lost at points $c$ and $e$ since $(\tau_1, \tau_2)$ crosses the curve from $\boldsymbol{L}$ to $\boldsymbol{R}$ and $Q < 0$ on $C^+(0,1)$ and $C^-(0,2)$.

Much more complex dynamics can arise when $\tau_1$ takes a larger value. In Figure 5(B) $\tau_1$ is changed to $\tau_1^{a'} = 2.5$ and then $\tau_2$ is increased from the point $(\tau_1^{a'}, \tau_1^{a'})$ on the 45 degree line to 19. As in the first example, stability is gained at point $b'$ and lost again at point $c'$. After losing stability, different from the first example, the positive equilibrium goes to complex dynamics via a period-doubling-like bifurcation and then merges to a limit cycle via a period

[13] Notice that point $(\tau_1^a, \tau_1^a)$ is on the 45 degree line and thus $\tau_2 > \tau_1$ holds on the vertical line.

halving-like bifurcation.[14] At point $d'$, the real part of another eigenvalue pair becomes positive. So two parts of eigenvalues have positive real part after point $d'$. After point $e'$, the equilibrium is still unstable because the real part of only one of the two eigenvalue pairs turns to be negative and thus there is still an eigenvalue pair with positive real part. Therefore the equilibrium is unstable after point $c'$ even though the vertical dotted line crosses the stability-switch curve. Let $L$ and $R$ be the the numbers of intercepts of the vertical line with $C^+(m,n)$ and $C^-(m,n)$, respectively. Then it is true that the equilibrium is stable with $(\tau_1, \tau_2)$ if $R > L$ and unstable otherwise.




(A) $\tau_1 = 1.7$ (B) $\tau_1 = 2.5$

Figure 5. Bifurcation diagrams

The end points of $C^+(0,0)$ and $C^-(0,0)$ coincide and are denoted as the black dots at the intersection of the 45 degree line and the stability-switch curve. If $\tau_1 = \tau_2$, then the two delay equation (9) is identical with the one delay equation (5). So along the 45 degree line in Figure 3, the positive equilibrium is stable if $\tau_1 = \tau_2 < \tau^*$ and unstable if $\tau_1 = \tau_2 > \tau^*$where $\tau^* = 2\pi/ak$. For $\tau < \tau_1^m$, both the two delay equation and the one delay equation are stable. The second delay is harmless in this case. For $\tau_1^m < \tau_1 < \tau^*$, the one delay equation is still stable and deviating $(\tau_1, \tau_2)$ from the 45 degree line destabilizes the positive equilibrium of the two delay equation. The second delay is a destabilizing factor. Lastly for $\tau_1 > \tau^*$, both equations are unstable. Increasing the value of $\tau_2$ can make the equilibrium stable. So the second delay is a stabilizing factor in this case.

Comparing these stability/instability results and returning to the original spirit of equation (9), we numerically confirms the roles of the second delay $\tau_2$ in the learning process which are summarized as follows:

$(S_1)$ $\tau_2$ does not affect the learning process as far as $\tau_1 < \tau_1^m$.

[14] Although it is not depicted in Figure 5(B), the period doubling-halving bifurcation is repeated as $\tau_2$ increases further.

($S_2$) The delay feedback can help to realize the true value of the maximum demand if an appropriate value of $\tau_2$ is selected. In other word, the delay feedback can stabilize the equilibrium that is unstable in the one delay equation.

($S_3$) Since the stability region becomes smaller as $\tau_1$ increases, learning become more difficult for a larger value of $\tau_1$.

($S_4$) When alternations of stability and instability take place several times along the vertical line segment connecting points $(\tau_1^0, \tau_1^0)$ and $(\tau_1^0, \tau_2)$ with $\tau_2 > \tau_1^0$, the stabilizing and destabilizing effects of $\tau_2$ alternately affect the positive equilibrium.

($S_5$) Various dynamics ranging from periodic cycles to chaos can emerge depending on the choice of $\tau_2$ when the equilibrium is unstable.

Stability switches can occur for $\tau_1^m < \tau_1 < \tau_1^M$. These threshold values, $\tau_1^m$ and $\tau_1^M$, are $\omega$-dependent and Figure 6 illustrates how they depend on $\omega$. There, $\omega$ is increased from 0.55 to 0.99 with an increment of 0.01.[15] For each value of $\omega$, $\tau_1^m$ and $\tau_1^M$ are calculated from the first equations of (20) and (21) in the same way as explained above. Connecting these $\tau_1^m$s and $\tau_1^M$s forms downward-sloping and slightly upward-sloping curves. Painting the region surrounded by these curves in yellow yields Figure 6. As can be seen, $\tau_1^m$ increases and $\tau_1^M$ decreases as $\omega$ increases. When $\omega = 1$, the two delay equation (9) becomes the one delay equation (5) having the critical value $\tau^* = \pi/2 \simeq 1.57$ for $\alpha = 1$. So, both of $\tau_1^m$ and $\tau_1^M$ converge to $\tau^*$ as $\omega$ approaches unity.

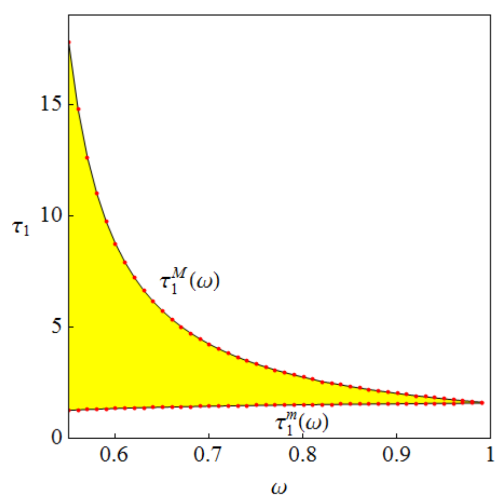



Figure 6. Interval $[\tau_1^m(\nu), \tau_1^M(\nu)]$

[15] Since $\tau_1^M$ becomes very large for $\omega$ close to 1/2, we start from 0.55.

Finally we examine whether the roots $\lambda = i\nu$ are single or not.

**Theorem 5** *The roots $\lambda = i\nu$ are single.*

So at each point where stability is lost, only one pair of roots change the sign of their real part from negative to positive, and at any point where stability might be regained, only one pair of roots change the sign of their real part from positive to negative.

# 4 Conclusion

This paper proposed a learning process of the monopolistic firm that knows its cost function and the marginal price but has uncertainty about the maximum price. It is able to update repeatedly its belief of the maximum price by comparing the actual and predicted market prices. The paper focuses on the case in which the firm's prediction is obtained by interpolation based on two delay price data and examines the asymptotical stability of the delay learning process. When the process is asymptotically stable, learning is successful because the firm can arrive at the true value of the maximum price. Local stability conditions are derived, the stability regions are determined and illustrated. Furthermore, the global behavior of the trajectory is numerically examined.

Three main results are demonstrated in the two delay model. First, it is possible to stabilize the equilibrium which is unstable in the one delay model by adopting the delay feedback of the past price information in the two delay model. Second, complex dynamics involving chaos, which is impossible in the one delay model, can emerge in the two delay model, especially when the delays are large enough. Consequently it can be mentioned that the delay feedback is a double-edged sword. On one hand, the belief of the firm about the maximum price might converge to the true value under a simple adaptive rule. On the other hand, the firm might suffer from unsuccessful learning process. The new aspect shown in the present paper is the repeated alternations of stability and instability (i.e., the stability switches) and this is the third result. Introducing delay allows us to dispense with the restrictive assumption on instantaneous availability of price information which in turn ensures that a delay model is applicable to a large class of dynamics models in which uncertainty and time delays are often observed.

The dynamic model (3) is linear, when local asymptotical stability implies global asymptotical stability. However (4) and (5) are nonlinear, where only local asymptotic stability can be guaranteed under the derived conditions. The learning processes (3) and (4) can be generalized as

$$\dot{a}^e(t) = g(a - a^e(t))$$

where function $g$ is sign preserving, that is, for all $\delta \neq 0$,

$$\delta g(\delta) > 0.$$

This extension carries over to two delay models like (9). In our future research, different types of such nonlinear learning schemes will be introduced in our model and we will investigate the asymptotical behavior of the resulted dynamics. Uncertainty, time delay and learning of other model parameters will be additional subjects of our study.

# Appendix

In this appendix, the proofs of the Theorems given in this paper are presented.

**Proof of Theorem 1**:

As is seen in (17), the domain of $\nu$ is the interval $[\alpha(2\omega-1),\alpha]$. At the initial point $\nu=\alpha(2\omega-1)$, we have

$$\frac{\nu^2+2\alpha^2\omega-\alpha^2}{2\alpha\nu\omega}=1 \text{ and } \frac{\nu^2-2\alpha^2\omega+\alpha^2}{2\alpha\nu(1-\omega)}=-1$$

implying

$$\theta_1(\alpha(2\omega-1))=0 \text{ and } \theta_2(\alpha(2\omega-1))=-\pi$$

and at the end point $\nu=\alpha$, we have

$$\frac{\nu^2+2\alpha^2\omega-\alpha^2}{2\alpha\nu\omega}=1 \text{ and } \frac{\nu^2-2\alpha^2\omega+\alpha^2}{2\alpha\nu(1-\omega)}=1$$

implying

$$\theta_1(\alpha)=0 \text{ and } \theta_2(\alpha)=0.$$

Therefore the initial and end points of $C^+(m,n)$ are

$$I^+(m,n)=\left(\frac{1}{\alpha(2\omega-1)}\left(\frac{3\pi}{2}+2(m-1)\pi\right),\ \frac{1}{\alpha(2\omega-1)}\left(\frac{3\pi}{2}+(2n-1)\pi-\pi\right)\right)$$

and

$$E^+(m,n)=\left(\frac{1}{\alpha}\left(\frac{3\pi}{2}+(2m-1)\pi\right),\ \frac{1}{\alpha}\left(\frac{3\pi}{2}+(2n-1)\pi\right)\right).$$

Similarly, the initial and end points of $C^-(m,n)$ are

$$I^-(m,n)=\left(\frac{1}{\alpha(2\omega-1)}\left(\frac{3\pi}{2}+(2m-1)\pi\right),\ \frac{1}{\alpha(2\omega-1)}\left(\frac{3\pi}{2}+(2n-1)\pi+\pi\right)\right)$$

and

$$E^-(m,n)=\left(\frac{1}{\alpha}\left(\frac{3\pi}{2}+(2m-1)\pi\right),\ \frac{1}{\alpha}\left(\frac{3\pi}{2}+(2n-1)\pi\right)\right).$$

Notice that $E^+(m,n)=E^-(m,n)$ and $I^+(m,n+1)=I^-(m,n)$, that is, $C^+(m,n)$ and $C^-(m,n)$ have the same end points and $C^+(m,n{+}1)$ and $C^-(m,n)$ have the same initial points. So with fixed value of $n$, these curves from a continuous curve when the segments of $C^+(m,n)$ and $C^-(m,n)$ are attached to each other at the initial and endpoints.

**Proof of Theorem 2**.

(i) If $\tau_1=0$, then the characteristic polynomial (11) has the form

$$\lambda+\alpha\omega+\alpha(1-\omega)e^{-\lambda\tau_2}=0,$$

and if $\lambda = i\nu$ with $\nu > 0$, then

$$i\nu + \alpha\omega + \alpha(1-\omega)(\cos\tau_2\nu - i\sin\tau_2\nu) = 0.$$

Separation of the real and imaginary parts gives

$$\alpha(1-\omega)\cos\tau_2\nu = -\alpha\omega$$

and

$$\alpha(1-\omega)\sin\tau_2\nu = \nu.$$

By adding the squares of these equations

$$\alpha^2(1-\omega)^2 = \alpha^2\omega^2 + \nu^2$$

or

$$\nu^2 = \alpha^2(1-2\omega)$$

which is impossible since $\omega \geq 1/2$. If in addition $\tau_2 = 0$, then the positive solution is also stable, since the characteristic equation becomes

$$\lambda + \alpha\omega + \alpha(1-\omega) = 0$$

implying a negative eigenvalue, $\lambda = -\alpha$.

(ii) if $\tau_1 = \tau_2$, then the two delay equation (10) is reduced to the one delay equation (5) which is stable for $\tau < \tau^*$. As $\tau_1^m < \tau^*$, equations (10) and (5) are stable as far as $(\tau_1, \tau_2)$ moves along the 45 degree line. Any other $(\tau_1, \tau_2)$ with $0 < \tau_1 < \tau_1^m$ and $\tau_2 > \tau_1$, does not solve the characteristic equation (13) implying that no stability switch occurs. Thus the positive equilibrium is locally stable.

**Proof of Theorem 3.**
From (14) we have

$$\begin{aligned} Q &= \operatorname{Im}\left[\frac{\alpha\omega}{i\nu}\left(\frac{\alpha(1-\omega)}{-i\nu}\right)(\cos\nu(\tau_2-\tau_1) + i\sin\nu(\tau_2-\tau_1))\right] \\ &= \frac{\alpha^2\omega(1-\omega)}{\nu^2}\sin\nu(\tau_2-\tau_1). \end{aligned}$$

We consider first $C^+(m,n)$. From (20)

$$\sin\nu(\tau_2-\tau_1) = \sin\left(-\cos^{-1}A - \cos^{-1}B\right)$$

where

$$A = \frac{\nu^2 + 2\alpha^2\omega - \alpha^2}{2\alpha\nu\omega} > 0 \text{ and } B = \frac{\nu^2 - 2\alpha^2\omega + \alpha^2}{2\alpha\nu(1-\omega)}.$$

Notice that $B \geq 0$ if $\nu \geq \alpha\sqrt{2\omega - 1}$ and $B < 0$ as $\nu < \alpha\sqrt{2\omega-1}$. Furthermore, both $\cos^{-1}A$ and $\cos^{-1}B$ are between 0 and $\pi$. So

$$\begin{aligned} \sin\nu(\tau_2-\tau_1) &= -\sin(\cos^{-1}A)\cos(\cos^{-1}B) - \cos(\cos^{-1}A)\sin(\cos^{-1}B) \\ &= -A\sqrt{1-B^2} - B\sqrt{1-A^2} \end{aligned}$$

which can be positive only when $B < 0$ and

$$-B\sqrt{1-A^2} > A\sqrt{1-B^2}$$

that is, $-B > A$. It can be written as

$$-\frac{\nu^2 - 2\alpha^2\omega + \alpha^2}{2\alpha\nu(1-\omega)} > \frac{\nu^2 + 2\alpha^2\omega - \alpha^2}{2\alpha\nu\omega}$$

or equivalently

$$\nu < \alpha(2\omega - 1)$$

which cannot occur. So at every point of $C^+(m,n)$, $Q < 0$.

We next consider $C^-(m,n)$. From (21),

$$\begin{aligned}
\sin\nu(\tau_2 - \tau_1) &= \sin\left(\cos^{-1} A + \cos^{-1} B\right) \\
&= \sin(\cos^{-1} A)\cos(\cos^{-1} B) + \cos(\cos^{-1} A)\sin(\cos^{-1} B) \\
&= B\sqrt{1-A^2} + A\sqrt{1-B^2}
\end{aligned}$$

which is positive if $B \geq 0$ or if $B < 0$ and

$$A\sqrt{1-B^2} > B\sqrt{1-A^2}.$$

The last inequality can be written as $A > -B$ or $\nu \geq \alpha(2\omega - 1)$ which is always the case. So at every point of $C^-(m,n)$, $Q > 0$.

**Proof of Theorem 4.**
It is given as Proposition 6.1 in Gu *et al.* (2005).

**Proof of Theorem 5.**
If a root $\lambda$ is multiple, the it solves two equations,

$$\lambda + \alpha\omega e^{-\lambda\tau_1} + \alpha(1-\omega)e^{-\lambda\tau_2} = 0 \tag{A-1}$$

and

$$1 - \tau_1\alpha\omega e^{-\lambda\tau_1} - \tau_2\alpha(1-\omega)e^{-\lambda\tau_2} = 0, \tag{A-2}$$

from which we have

$$e^{-\lambda\tau_1} = \frac{1+\tau_2\lambda}{\alpha\omega(\tau_1 - \tau_2)} \text{ and } e^{-\lambda\tau_2} = \frac{1+\tau_1\lambda}{\alpha(1-\omega)(\tau_2 - \tau_1)}. \tag{A-3}$$

From (A-3) we see that

$$\cos\nu\tau_1 - i\sin\nu\tau_1 = \frac{1+i\nu\tau_2}{\alpha\omega(\tau_1 - \tau_2)}$$

implying that

$$\sin \nu\tau_1 = -\nu\tau_2 \cos \nu\tau_1 \tag{A-4}$$

and from (A-3) we also have

$$\sin \nu\tau_2 = -\nu\tau_1 \cos \nu\tau_2. \tag{A-5}$$

However comparing the imaginary parts of (A-2),

$$\alpha\omega\tau_1 \sin \nu\tau_1 + \alpha(1-\omega)\tau_2 \sin \nu\tau_2 = 0. \tag{A-6}$$

By using (20), we have

$$\frac{\partial \tau_1^+}{\partial \nu} = -\frac{1}{\nu^2}\left(\nu\tau_1^+\right) + \frac{1}{\nu}\left(-\frac{1}{\sqrt{1-A^2}}\frac{\partial A}{\partial \nu}\right)$$

where

$$\sqrt{1-A^2} = \sqrt{1-\cos^2\left(\nu\tau_1^+ - \frac{\pi}{2}\right)} = \sqrt{1-\sin^2\left(\nu\tau_1^+\right)} = \left|\cos \nu\tau_1^+\right| = -\cos \nu\tau_1^+,$$

since $\theta_1(\nu) \in (0,\pi)$ and so $\nu\tau_1^+ \in (\pi/2 + 2m\pi, 3\pi/2 + 2m\pi)$. Furthermore,

$$\frac{\partial A}{\partial \nu} = \frac{\nu^2 + \alpha^2 - 2\alpha^2\omega}{2\alpha\omega\nu^2} = \frac{B(1-\omega)}{\omega\nu}$$

implying that

$$\frac{\partial \tau_1^+}{\partial \nu} = -\frac{1}{\nu^2}\left[\nu\tau_1^+ - \frac{B(1-\omega)}{\omega \cos \nu\tau_1^+}\right]. \tag{A-7}$$

Notice that from the second equation of (20),

$$B = \cos\left(\frac{\pi}{2} - \nu\tau_2^+\right) = \sin \nu\tau_2^+$$

and by using (A-4) and (A-6), the derivative (A-7) can be further simplified:

$$\begin{aligned} -\frac{1}{\nu^2}\left[\nu\tau_1^+ - \frac{\sin \nu\tau_2^+(1-\omega)}{\omega \cos \nu\tau_1^+}\right] &= -\frac{1}{\nu^2}\left[\nu\tau_1^+ + \frac{\frac{\tau_1^+}{\tau_2^+}\sin \nu\tau_1^+}{\cos \nu\tau_1^+}\right] \\ &= -\frac{\tau_1^+}{\nu^2\tau_2^+ \cos \nu\tau_1^+}\left[\nu\tau_2^+ \cos \nu\tau_1^+ + \sin \nu\tau_2^+\right] \\ &= 0 \end{aligned}$$

meaning that any multiple root is a stationary point of $\tau_1^+$.

Consider next $\tau_2^+$ as function of $\nu$. Notice first that

$$\frac{\partial \tau_2^+}{\partial \nu} = -\frac{1}{\nu^2}\left(\nu\tau_2^+\right) + \frac{1}{\nu}\left(-\frac{1}{\sqrt{1-B^2}}\frac{\partial B}{\partial \nu}\right)$$

where from the second equation of (20),

$$\sqrt{1-B^2} = \sqrt{1-\cos^2\left(\frac{\pi}{2}-\nu\tau_2^+\right)} = \sqrt{1-\sin^2\left(\nu\tau_2^+\right)} = \left|\cos\nu\tau_2^+\right| = \cos\nu\tau_2^+,$$

since $\cos^{-1} B \in (0,\pi)$ and so $\nu\tau_2^+ \in (-\pi/2+2n\pi, \pi/2+2n\pi)$. Furthermore,

$$\frac{\partial B}{\partial \nu} = \frac{\nu^2-\alpha^2+2\alpha^2\omega}{2\alpha(1-\omega)\nu^2} = \frac{A(1-\omega)}{(1-\omega)\nu}$$

implying that

$$\frac{\partial \tau_2^+}{\partial \nu} = -\frac{1}{\nu^2}\left[\nu\tau_2^+ - \frac{A\omega}{(1-\omega)\cos\nu\tau_2^+}\right]. \tag{A-8}$$

Notice that from the second equation of (20),

$$A = \cos\left(\nu\tau_1^+ - \frac{\pi}{2}\right) = \sin\nu\tau_1^+$$

and by using (A-5) and (A-6), the derivative (A-8) can be further simplified:

$$\begin{aligned} -\frac{1}{\nu^2}\left[\nu\tau_2^+ - \frac{\sin\nu\tau_1^+\omega}{(1-\omega)\cos\nu\tau_2^+}\right] &= -\frac{1}{\nu^2}\left[\nu\tau_2^+ + \frac{\frac{\tau_2^+}{\tau_1^+}\sin\nu\tau_2^+}{\cos\nu\tau_2^+}\right] \\ &= -\frac{\tau_2^+}{\nu^2\tau_1^+\cos\nu\tau_2^+}\left[\nu\tau_1^+\cos\nu\tau_2^+ + \sin\nu\tau_2^+\right] \\ &= 0 \end{aligned}$$

meaning that any multiple root is a stationary point of both $\tau_1^+$ and $\tau_2^+$, which is impossible, since we cannot have both horizontal and vertical tangent lines simultaneously at any point of the stability switch curve.

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