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# Nonlinear Multiplier-Accelerator Model with Investment and Consumption Delays

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# Nonlinear Multiplier-Accelerator Model with Investment and Consumption Delays*

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**Abstract**

This paper shows how cyclic dynamics of national income can emerge in the multiplier-accelerator model with continuous time scale when delays in investment and consumption are presented. An $S$-shaped functional form of investment and a linear consumption function are adopted to illustrate the phenomenon and to compute the stability-switching curves on which a stability gain or loss occurs. Assuming that the equilibrium national income is locally stable if there are no delays, it is demonstrated that one delay is harmless and with two delays, the system can produce limit cycles and the stability switch repeatedly occurs when one of the delays increases and the other is kept to be positive constant.

**Keywords**: Nonlinear multiplier accelerator model, Business cycle, Investment delay, Consumption delay, Stability switch

**JEL Classifications**: C63, E12, E32

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## 1 Introduction

Macroeconomic variables such as national income, employment rate, interest rate, exchange rate, etc. exhibit persistent and irregular fluctuations. Although linear models may be aperiodic if exogenous shocks are appropriately introduced, it has been a main interest in studying macroeconomic dynamics to detect endogenous sources of such irregularity. "Delay" and "nonlinearity" were once thought to be two of the main ingredients for endogenous cyclic behavior. If we look back at the classic literature in the 1930's – 1950's, we find jewelry papers on macro dynamics, Kalecki (1935) assuming a gestation lag of investment, Samuelson (1939) building a multiplier-accelerator model, Kaldor (1940) adopting nonlinear investment based on the profit principle, Hicks (1950) extending Samuelson's model with the floor and ceiling, Goodwin (1951) assembling Samuelson's model in a different way with a nonlinear delay investment based on the acceleration principle. However, maybe due to mathematical difficulties to deal with delay differential equations or nonlinear difference equations, both of which are main tools for analyzing economic dynamics, macro dynamic studies with delay and/or nonlinearity gradually disappear from the main stream.

In the early 1980's, chaos theory sheds light on the roles of nonlinearity in deterministic processes to explain various complex dynamic behaviors of macro economic variables. A lot of efforts, since then, have been devoted to investigate the applicability of the chaos theory to dynamic economic analysis and to reconsider the various existing economic models. Samuelsonian multiplier-accelerator models that we draw much attention to in this study are also affected and extended in various directions. The original model is linear and can generate only damped or explosive oscillations. Westerhoff (2006) modifies the induced investment to depend on a nonlinear mix of extrapolative and regressive expectation formations. Further, Lines and Westerhoff (2006) use a weighted average of extrapolative and reverting expectations formations. Direct consequence of these alternations is that the adjustment process of national income becomes nonlinear and the birth of complex output fluctuations are numerically confirmed. Concerning the discrete-time Hicks' version, Hommes (1995) focuses on consumption and investment delays distributed over several periods and shows that strange-chaotic attractor can occur. The distinguished feature of the Hicks' version is an introduction of an income ceiling and an investment floor to the Samuelson version. These exogenous factors are endogenized through a capital formation theory. The floor is related to the capital stock through a depreciation rate in Puu *et al.* (2005) and the ceiling to the capital stock through the income-capital ratio in Puu (2007). Both are taken into account simultaneously in Sushko *et al.* (2010). It is shown that periodic, quasi periodic and aperiodic cycles can emerge in these models. Concerning the continuous-time Hicks' version, Goodwin (1951) replaces the piecewise linearity in an investment function with a smooth nonlinearity. Puu (2000) chooses truncated Taylor expansions of a nonlinear investment function. Matsumoto and Szidarovszky (2010) invoke continuously distributed time delays in consumption and investment and make it likely that the modified model possesses a piecewisely connected limit cycle.

This paper purposes to reconsider the lost roles of delays for the emergence of persistent fluctuations. To this end, we extend continuous time Goodwin's nonlinear multiplier-accelerator model by explicitly dealing with investment delay and consumption delay. Goodwin's model is augmented with nonlinear accelerator and investment delay. In demonstrating emergence of persistently cyclic behavior, the role of nonlinearities has been highlighted whereas the role of the delays have been made implicit. This is mainly because Goodwin's delay differential equation that describes dynamics of the national income is approximated in a neighborhood of zero-delay to obtain a second-order differential equation. As a natural consequence, considerations on delays lie outside of the scope of the main discussion. Recently the role of investment delay is discussed in Matsumoto and Szidarovszky (2014a) in which Goodwin's original model with one delay in investment is reexamined. In this paper, we further add consumption delay to it and investigate whether two delays in investment and consumption are responsible for various macro dynamic fluctuations. Although this has the similar structure to the model considered in Matsumoto and Szidarovszky (2014b) in which two delays in investment and consumption under Goodwin setting is investigated, there is an essential and important difference. The main tool used there is not applicable to our model and thus we use a completely different method developed by Gu *et al.* (2005) to analyze it. Further, the two models have only a small difference in model construction, the results obtained are very much different.

The paper is organized as follows. In Section 2, the basic elements of the multiplier-accelerator model are recapitulated. In Section 3 the one delay model is considered as a benchmark. In Section 4, the two delay model is analytically investigated and the stability switch is rigorously considered. In Section 5, some numerical simulations are presented. Section 6 contains some concluding remarks.

# 2 Multiplier-Accelerator Model

We recapitulate the main points of the multiplier-accelerator model of business cycle. Samuelson (1939) constructs a linear model that combines the multiplier theory with the acceleration principle and explains the cyclic nature of the ups and downs in business cycle. The model is cast in discrete-time and based on the fact that national income at time $t$ is the sum of consumption, $C_t$, and induced investment, $I_t$ (i.e., $Y_t = C_t + I_t$). The model has two ingredients. One is that consumption is a fixed fraction determined by the marginal propensity to consume, $\alpha$, of national income with a one-period delay,

$$C_t = \alpha Y_{t-1},\ 0 < \alpha < 1$$

and the other is that induced investment is proportional to changes in consumption in period $t$ that are proportional to changes in national income in period $t-1$ (i.e., the acceleration principle),

$$I_t = \beta(C_t - C_{t-1}) = \alpha\beta(Y_{t-1} - Y_{t-2}).$$

Simple substitutions yield a second-order difference equation of national income,

$$Y_t - \alpha(1+\beta)Y_{t-1} + \alpha\beta Y_{t-2} = 0. \tag{1}$$

With no autonomous expenditure, the zero solution of equation (1) is the equilibrium national income that is determined by the multiplier. It has been demonstrated that depending on the parameter values, various dynamics including cyclical oscillations can emerge.[1] Since equation (1) is linear, oscillations if any are mainly damped or explosive but can have constant amplitude under very special conditions. Hicks (1950) reformulates Samuelson's model as nonlinear (i.e., piecewise linear) by introducing the investment floor due to the depreciation of the existing capital stock and the output ceiling due to the full employment level of output. Hicks's model is also in discrete time and its possible formulation is

$$\begin{aligned} C_t &= \alpha Y_{t-1}, \\ I_t &= \max\left[\beta(Y_{t-1} - Y_{t-2}),\ -I_L\right], \\ Y_t &= \min\left[C_t + I_t,\ Y^c\right] \end{aligned} \tag{2}$$

where $-I_L$ denotes the investment floor and $Y^c$ is the income ceiling. These exogenous bounds prevent the unstable fluctuates from being infinitely explosive. Goodwin (1951) casts the discrete-time model in continuous time scale and replaces the piecewise linear investment function with the smooth nonlinear function. Five different versions of his model are presented and its second version is described by the following two-dimensional system:

$$\begin{aligned} \varepsilon\dot{Y}(t) &= \dot{K}(t) - (1-a)Y(t) \\ \dot{K}(t) &= \varphi(\dot{Y}(t)) \end{aligned} \tag{3}$$

where $\varphi(\dot{Y})$ denotes the induced investment with $\varphi'(\dot{Y}) > 0$ and $\varphi''(\dot{Y}) \neq 0$. Delay $\delta$ is introduced in the investment function in the third version,

$$\dot{K}(t) = \varphi(\dot{Y}(t-\delta))$$

which is substituted into the first equation in (3) to obtain a delay differential equation of neutral type,

$$\varepsilon\dot{Y}(t) - \varphi(\dot{Y}(t-\delta)) + (1-\alpha)Y(t) = 0. \tag{4}$$

Goodwin (1951) does not deal with equation (4) but considers its linear approximation with respect to $\delta$,

$$\varepsilon\delta\ddot{Y}(t)(t) + \left[\varepsilon + (1-\alpha)\delta\right]\dot{Y}(t) - \varphi(\dot{Y}(t)) + (1-\alpha)Y(t) = 0.$$

The delay differential equation turns to be a second-order nonlinear differential equation, which is the fourth version of Goodwin's model. Under the instability

[1] For a qualitative analysis, see, for example, Gandolfo (2009).

condition, $\varepsilon + (1-\alpha)\delta > \varphi'(0)$, and the approximation condition that $\delta$ is sufficiently small, it is shown that Goodwin's differential equation can have a cyclic solution. Matsumoto and Szidarovszky (2014a) rigorously investigate dynamics generated by equation (4) with larger values of $\delta$ and demonstrate that equation (4) can produce not only smooth cyclic oscillations but also sawtooth oscillations.

We now move one more step forward. A delay in consumption in the discrete-time model is explicitly taken into account in *a la* Goodwinian continuous-time model. In particular, we recast the nonlinear discrete-time multiplier-accelerator model in continuous time scale with the following modifications,

$$\begin{aligned} C(t) &= \alpha Y(t-\eta), \\ I(t) &= \varphi(\dot{Y}(t-\delta)), \\ Y(t) &= \int_0^t \frac{1}{\varepsilon} e^{-\frac{t-\tau}{\varepsilon}} E(\tau) d\tau \end{aligned} \tag{5}$$

where $E(\tau) = C(\tau) + I(\tau)$ is the total expenditure, $\eta > 0$ is the consumption delay. The last equation indicates that national income lags behind the expenditure and this delay is of exponential form. The basic dynamic structure of equation (5) is similar to that of Philips (1954) in which three delays in investment, consumption and production are analyzed in continuous-time scale. Notice that all the delays are continuously distributed and have exponential forms in Philips' model whereas consumption and investment delays are fixed in system (5). Differentiating the last equation in (5) with respect to $t$ and substituting delayed consumption and investment into the resultant expression presents a differential equation with two fixed delays,

$$\varepsilon\dot{Y}(t) - \varphi(\dot{Y}(t-\delta)) + Y(t) - \alpha Y(t-\eta) = 0. \tag{6}$$

This is the dynamic model we will analyze. Similarity to equation (4) is clear. It preserves the four main features of the discrete-time multiplier-accelerator model, the multiplier, the acceleration principle, the nonlinear investment function and the delays in consumption and investment.[2] Its linearly approximated version is

$$\varepsilon\dot{Y}(t) + Y(t) - \nu\dot{Y}(t-\delta) - \alpha Y(t-\eta) = 0 \tag{7}$$

where $\nu = \varphi'(0)$. With the notation

$$a = \frac{1}{\varepsilon},\ b = \frac{\nu}{\varepsilon} \text{ and } c = \frac{\alpha}{\varepsilon}$$

equation (7) becomes

$$\dot{Y}(t) + aY(t) - b\dot{Y}(t-\delta) - cY(t-\eta) = 0. \tag{8}$$

[2] Professor Anna Antonova indicates this form of the consumption delay in the private correspondence. See Matsumoto and Szidarovszky (2014b) for another form of the consumption delay in which the last two terms, $Y(t) - \alpha Y(t-\eta)$, are replaced with $(1-\alpha)Y(t-\eta)$.

The corresponding characteristic equation is obtained by substituting an exponential solution, $Y(t) = e^{\lambda t}u$,

$$\lambda + a - b\lambda e^{-\delta\lambda} - ce^{-\eta\lambda} = 0. \tag{9}$$

Stability of equation (8) can be examined by finding the locations of the eigenvalues of equation (9). Before proceeding, we make the following assumption since, under many circumstances, it is natural to suppose that the investment delay is longer than the consumption delay.

**Assumption 1**. $\delta > \eta$

As a benchmark for the stability analysis, we first consider the continuous-time model without delays,

$$\varepsilon\dot{Y}(t) - \varphi(\dot{Y}(t)) + (1-\alpha)Y(t) = 0 \tag{10}$$

which is obtained by substituting the second equation in (3) into the first or taking away $\delta$ and $\eta$ from equation (6). The local asymptotical stability of equation (10) can be examined by linearization around the steady state $\bar{Y} = 0$:

$$\varepsilon\dot{Y}(t) - \nu\dot{Y}(t) + (1-\alpha)Y(t) = 0.$$

If $\varepsilon = \nu$, then $Y(t) = 0$ for all $t \geq 0$ is a solution, which is uninteresting. It is also checked that the steady state is locally asymptotically stable if $\varepsilon > \nu$ and locally unstable if $\varepsilon < \nu$. In the existing literature, local instability is usually assumed for cyclic study.[3] Since we will consider the delay effects on the stable steady state, we impose the following condition in which the steady state of the non-delay equation (10) is locally asymptotically stable:

**Assumption 2**. $\varepsilon > \nu$

## 3 The Single-Delay Case

In this section we briefly examine the single delay effect on the output dynamics, taking away Assumption 1. Assume first that $\delta = 0$ and $\eta > 0$, so equation (9) becomes

$$(1-b)\lambda + a - ce^{-\eta\lambda} = 0. \tag{11}$$

At $\eta = 0$ the eigenvalue is $(\alpha - 1)/(\varepsilon - \nu)$, so equation (11) is stable since $\nu < \varepsilon$ and $0 < \alpha < 1$. At any stability switch $\lambda = i\omega$, where we can assume that $\omega > 0$, since the conjugate of any eigenvalue is also an eigenvalue. Substituting it into equation (11) transforms equation (11) to

$$i(1-b)\omega + a - c(\cos\eta\omega - i\sin\eta\omega) = 0$$

[3] Matsumoto (2009) assumes local stability of the steady state and exhibits that a Goodwinean multiplier-acclerator model can have multiple limit cycles.

and then separating the real and imaginary parts gives two equations for two unknowns $\omega$ and $\eta$ as

$$a - c\cos\eta\omega = 0$$

$$(1-b)\omega + c\sin\eta\omega = 0.$$

Moving $a$ and $(1-b)\omega$ to the right hand sides and adding the squared equations yield

$$(1-b)^2\omega^2 = c^2 - a^2 < 0.$$

This inequality means that there is no positive $\omega$ satisfying the last equation. Thus we conclude that no stability switch occurs and the steady state is locally asymptotically stable for all $\eta \geq 0$. That is, consumption delay is *harmless*.

We now examine the opposite case in which $\delta > 0$ and $\eta = 0$, so equation (9) becomes

$$\lambda + a - c - b\lambda e^{-\delta\lambda} = 0. \tag{12}$$

It can be checked that the steady state $\bar{Y}$ solves equation (8) with $\eta = 0$ and is locally asymptotically stable for $\delta = 0$. Assuming that $\lambda = i\omega$ with $\omega > 0$ is a solution of equation (12) and substituting it into equation (12) present

$$a - c - b\omega\sin\delta\omega = 0,$$

$$\omega - b\omega\cos\delta\omega = 0.$$

Moving $a - c$ and $\omega$ to the right hand sides, squaring them and adding them together give

$$(a-c)^2 + (1-b^2)\omega^2 = 0.$$

Since $1 - b^2 > 0$ by Assumption 2, there is no positive $\omega$ solving the above equation. Thus no stability switch occurs for all $\delta > 0$ and the investment delay is also harmless under Assumption 2. Lastly we consider one more special case in which the delays are identical, $\delta = \eta$. The characteristic equation is

$$\lambda + a - (b\lambda + c)e^{-\delta\lambda} = 0 \tag{13}$$

which, with $\lambda = \imath\omega$, can be divided into two equations,

$$c\cos\delta\omega + b\omega\sin\delta\omega = a$$

$$-c\sin\delta\omega + b\omega\cos\delta\omega = \omega$$

from which we have

$$(1-b^2)\omega^2 = c^2 - a^2 < 0.$$

This inequality implies that there is no positive $\omega$ solving the above equation. Thus the steady state is locally asymptotically stable for all $\delta = \eta \geq 0$. In other words, the single delay does not affect asymptotic behavior. However, it matters in transient behavior.[4] To sum up, we have the following:

[4] See Matsumoto and Szidarovszky (2014a) for some numerical results in which the delay generates oscillations, causes slower convergence and makes time trajectories kinked while the time trajectory with no delay monotonically converges to the zero solution.

**Theorem 1** *Given Assumption 2, the zero solution of one-delay differential equation (8) with $\delta = 0$, $\eta = 0$ or $\delta = \eta$ is locally asymptotically stable for all $\eta > 0$, $\delta > 0$ or $\delta = \eta > 0$.*

## 4 The Two-Delay Case

Characteristic equation (9) is now investigated with applying Gu's method (Gu, *et al.* (2005)) developed to analyze a two delay differential equation.[5] Dividing its both sides by $a + \lambda$ and introducing the new functions,

$$a_1(\lambda) = -\frac{b\lambda}{a+\lambda} \text{ and } a_2(\lambda) = -\frac{c}{a+\lambda}$$

simplify equation (9),

$$a(\lambda) = 1 + a_1(\lambda)e^{-\delta\lambda} + a_2(\lambda)e^{-\eta\lambda} = 0. \tag{14}$$

Suppose that $\lambda = i\omega$ with $\omega > 0$.

$$a_1(i\omega) = -\frac{b\omega^2}{a^2+\omega^2} - i\frac{ab\omega}{a^2+\omega^2} \tag{15}$$

and

$$a_2(i\omega) = -\frac{ac}{a^2+\omega^2} + i\frac{c\omega}{a^2+\omega^2}. \tag{16}$$

Their absolute values are

$$|a_1(i\omega)| = \frac{b\omega}{\sqrt{a^2+\omega^2}} \text{ and } |a_2(i\omega)| = \frac{c}{\sqrt{a^2+\omega^2}} \tag{17}$$

and their arguments are

$$\arg(a_1(i\omega)) = \tan^{-1}\left(\frac{a}{\omega}\right) + \pi \text{ and } \arg(a_2(i\omega)) = \pi - \tan^{-1}\left(\frac{\omega}{a}\right). \tag{18}$$

We can consider the three terms in $a(\lambda)$ as three vectors in the complex plane with the magnitudes 1, $|a_1(i\omega)|$ and $|a_2(i\omega)|$. The solutions of $a(\lambda)$ means that if we put these vectors head to tail, they form two triangles in two different ways. One triangle is illustrated in Figure 1 and the other is obtained by turning it

[5] In Matsumoto and Szidarovszky (2014b), the similar two delay model is analysed with an elementary method, which is unapplicable to analyse equation (9) due to the existence of the constant term, $a$.

over with respect to the axis of abscissa.

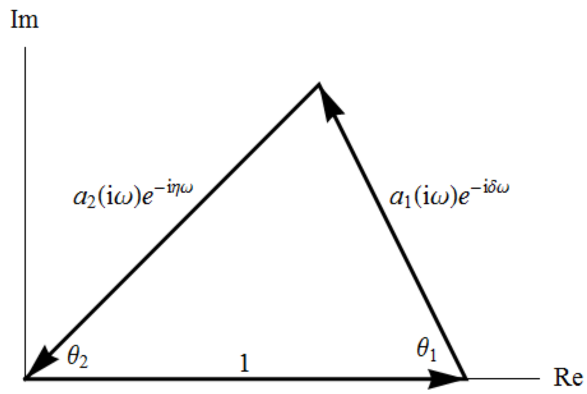


Figure 1. Triangle formed by 1, $|a_1(i\omega)|$ and $|a_2(i\omega)|$

In a triangle consisting of three line segments, the length of the sum of any two adjacent line segments is not shorter than the length of the remaining line segment,

$$1 \leq |a_1(i\omega)| + |a_2(i\omega)|,$$

$$|a_1(i\omega)| \leq 1 + |a_2(i\omega)|,$$

and

$$|a_2(i\omega)| \leq 1 + |a_1(i\omega)|.$$

Substituting the absolute values in (17) renders these three conditions to the following two conditions,

$$f(\omega) = (1 - b^2)\omega^2 - 2bc\omega + a^2 - c^2 \leq 0$$

and

$$g(\omega) = (1 - b^2)\omega^2 + 2bc\omega + a^2 - c^2 \geq 0.$$

Both $f(\omega)$ and $g(\omega)$ have the same discriminant,

$$D = 4[c^2 - a^2(1 - b^2)].$$

In the following we draw attention to the case of $D > 0$,[6] otherwise $f(\omega) > 0$ for all $\omega$ implying no stability switch. Solving $g(\omega) = 0$ gives the solutions

$$\omega_1 = \frac{-bc - \sqrt{c^2 - a^2(1 - b^2)}}{1 - b^2} \text{ and } \omega_2 = \frac{-bc + \sqrt{c^2 - a^2(1 - b^2)}}{1 - b^2}$$

[6] Under Assumption 1, $D > 0$ when $\alpha > \sqrt{1 - (\nu/\varepsilon)^2}$

and so does solving $f(\omega) = 0$,

$$\omega_3 = \frac{bc - \sqrt{c^2 - a^2(1-b^2)}}{1-b^2} \text{ and } \omega_4 = \frac{bc + \sqrt{c^2 - a^2(1-b^2)}}{1-b^2}.$$

Since both $\omega_1$ and $\omega_2$ are negative and both $\omega_3$ and $\omega_4$ are positive, the two conditions, $f(\omega) \leq 0$ and $g(\omega) \geq 0$, are satisfied when $\omega$ is in interval $[\omega_3, \omega_4]$.

We will next find all the pairs of $(\delta, \eta)$ satisfying $a(i\omega) = 0$. The internal angles, $\theta_1$ and $\theta_2$, of the triangle in Figure 1 can be calculated by the law of cosine as

$$\theta_1(\omega) \quad = \cos^{-1}\left(\frac{a^2 + (1+b^2)\omega^2 - c^2}{2b\omega\sqrt{a^2+\omega^2}}\right) \tag{19}$$

and

$$\theta_2(\omega) \quad = \cos^{-1}\left(\frac{a^2 + (1-b^2)\omega^2 + c^2}{2c\sqrt{a^2+\omega^2}}\right). \tag{20}$$

Solving two equations for $\delta$ and $\eta$

$$\left\{\arg\left[a_1(i\omega)e^{-i\delta\omega}\right] + 2m\pi\right\} \pm \theta_1(\omega) = \pi$$

and

$$\left\{\arg\left[a_2(i\omega)e^{-i\eta\omega}\right] + 2n\pi\right\} \mp \theta_2(\omega) = \pi$$

yield

$$\delta = \frac{1}{\omega}\left[\tan^{-1}\left(\frac{a}{\omega}\right) + \pi + (2m-1)\pi \pm \theta_1(\omega)\right].$$

and

$$\eta = \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{a}\right) + \pi + (2n-1)\pi \mp \theta_2(\omega)\right]$$

where arguments defined in (18) are used. Then for any $\omega$ satisfying the two conditions, $f(\omega) \leq 0$ and $g(\omega) \geq 0$, we can find the pairs of $(\delta, \eta)$ constructing the partition curves for $\omega_3 \leq \omega \leq \omega_4$:

$$C_1(m,n) = \{\delta_1(\omega,m), \eta_1(\omega,n)\} \tag{21}$$

where

$$\begin{aligned} \delta_1(\omega,m) &= \frac{1}{\omega}\left[\tan^{-1}\left(\frac{a}{\omega}\right) + 2m\pi + \theta_1(\omega)\right] \\ \eta_1(\omega,n) &= \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{a}\right) + 2n\pi - \theta_2(\omega)\right] \end{aligned} \tag{22}$$

and

$$C_2(m,n) = \{\delta_2(\omega,m), \eta_2(\omega,n)\} \tag{23}$$

where

$$\begin{aligned} \delta_2(\omega,m) &= \frac{1}{\omega}\left[\tan^{-1}\left(\frac{a}{\omega}\right) + 2m\pi - \theta_1(\omega)\right] \\ \eta_2(\omega,n) &= \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{a}\right) + 2n\pi + \theta_2(\omega)\right] \end{aligned} \tag{24}$$

with $m, n = 0, 1, 2, ...$ Notice that $m$ and $n$ are selected to be nonnegative integers so that $\delta > 0$ and $\eta > 0$.

Notice first that $f(\omega_3) = 0$ and $f(\omega_4) = 0$. Then at $\omega = \omega_k$ for $k = 3, 4$,

$$\frac{a^2 + (1 + b^2)\omega_k^2 - c^2}{2b\omega_k\sqrt{a^2 + \omega_k^2}} = 1 \text{ and } \frac{a^2 + (1 - b^2)\omega_k^2 + c^2}{2c\sqrt{a^2 + \omega_k^2}} = 1$$

implying that $\theta_1(\omega_k) = \theta_2(\omega_k) = 0$. Then the initial and end points of $C_1(m, n)$ are

$$\delta_1(\omega_k, m) = \frac{1}{\omega_k}\left[\tan^{-1}\left(\frac{a}{\omega_k}\right) + 2m\pi\right]$$

and

$$\eta_1(\omega_k, n) = \frac{1}{\omega_k}\left[-\tan^{-1}\left(\frac{\omega_k}{a}\right) + 2n\pi\right]$$

with $k = 3$ and $k = 4$, respectively. In the same way, these of $C_2(m, n)$ are

$$\delta_2(\omega_k, m) = \frac{1}{\omega_k}\left[\tan^{-1}\left(\frac{a}{\omega_k}\right) + 2m\pi\right]$$

and

$$\eta_2(\omega_k, n) = \frac{1}{\omega_k}\left[-\tan^{-1}\left(\frac{\omega_k}{a}\right) + 2n\pi\right]$$

with $k = 3$ and $k = 4$, respectively. Clearly $C_1(m, n)$ and $C_2(m, n)$ have the same initial points and the same end points. Figure 2 depicts the connecting curves for $m = 1$ and $n = 1$. The lower red curve is the segment $C_1(1, 1)$ and the upper blue curve is the segment $C_2(1, 1)$. Both segments start at the same initial point $S = (\delta^S, \eta^S)$ and arrive at the same end points$E = (\delta^E, \eta^E)$ as $\omega$ increases from $\omega_3$ to $\omega_4$.[7] The connecting curves takes a cigar-shaped profile. These curves are shifted to the right by increasing the values of $m$ and up by increasing the values of $n$.

We show that stability is lost when increasing $\delta$ crosses the $C_2(m, n)$ curve from the left while stability is gained when it crosses $C_1(m, n)$ from the left. We discuss the direction of stability switch in which the solution of equation (14) cross the imaginary axis as $(\delta, \eta)$ deviates from the partition curve. We first show the following result.

**Theorem 2** *The sign of $Q(\delta, \eta)$ is negative for $(\delta, \eta)$ on $C_1(m, n)$ and positive on $C_2(m, n)$ where $Q(\delta, \eta)$ is defined as*

$$Q(\delta, \eta) = \text{Im}\left[a_1(i\omega)a_2(-i\omega)e^{i\omega(\eta - \delta)}\right].$$

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[7] More precisely,

$$\delta^S = \delta_1(\omega_3, 1) = \delta_2(\omega_3, 1) \text{ and } \delta^E = \delta_1(\omega_4, 1) = \delta_2(\omega_4, 1)$$

and

$$\eta^S = \eta_1(\omega_3, 1) = \eta_2(\omega_3, 1) \text{ and } \eta^E = \eta_1(\omega_4, 1) = \eta_2(\omega_4, 1).$$

**Proof.** Direction of stability switch depends on the sign of $Q$. The term $a_1(i\omega)a_2(-i\omega)e^{\omega(\eta-\delta)}$ is written as

$$\left(-\frac{b\omega^2}{a^2+\omega^2} - i\frac{ab\omega}{a^2+\omega^2}\right)\left(-\frac{ac}{a^2+\omega^2} - i\frac{c\omega}{a^2+\omega^2}\right)e^{i\omega(\eta-\delta)}$$

which has the same sign as imaginary part of

$$(-\omega - ia)\,(-a - i\omega)e^{i\omega(\eta-\delta)} = i(\alpha^2+\omega^2)\,(\cos\omega(\eta-\delta) + i\sin\omega(\eta-\delta))$$

which has the same sign as $\cos\omega(\eta-\delta)$. Notice that in (19) and (20), both arguments are positive, so

$$\theta_1(\omega),\ \theta_2(\omega) \in \left[0, \frac{\pi}{2}\right].$$

Notice also that for $(\delta_1, \eta_1)$ on $C_1(m,n)$,

$$\omega(\eta_1 - \delta_1) = -\left(\tan^{-1}\left(\frac{a}{\omega}\right) + \tan^{-1}\left(\frac{\omega}{a}\right) + \theta_1(\omega) + \theta_2(\omega)\right)$$

Since

$$\tan^{-1}\left(\frac{a}{\omega}\right) + \tan^{-1}\left(\frac{\omega}{a}\right) = \frac{\pi}{2},$$

we then have

$$\omega(\eta_1 - \delta_1) = -\left(\frac{\pi}{2} + \theta_1(\omega) + \theta_2(\omega)\right) \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$$

implying that

$$\cos[\omega(\eta_1 - \delta_1)] < 0.$$

Therefore,

$$R_1 : Q(\delta,\eta) < 0 \text{ for } (\delta,\eta) \text{ on } C_1(m,n).$$

Similarly, for $(\delta_2, \eta_2)$ on $C_2(m,n)$,

$$\begin{aligned}\omega(\eta_2 - \delta_2) &= -\left(\tan^{-1}\left(\frac{a}{\omega}\right) + \tan^{-1}\left(\frac{\omega}{a}\right) - \theta_1(\omega) - \theta_2(\omega)\right) \\ &= -\left(\frac{\pi}{2} - \theta_1(\omega) - \theta_2(\omega)\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\end{aligned}$$

implying that

$$\cos[\omega(\eta_2 - \delta_2)] > 0.$$

Therefore,

$$R_2 : Q(\delta,\eta) > 0 \text{ for } (\delta,\eta) \text{ on } C_2(m,n).$$

The results, $R_1$ and $R_2$, complete the proof. ■

As in Gu *et al.* (2005), we call the direction of any segment of the stability switch curve with increasing $\omega$ the *positive direction*. As we move along the curve in the positive direction, the region on the left hand side is called the *region on the left* which will be denoted as $\boldsymbol{L}$ and the region on the right hand side the *region on the right* denoted as $\boldsymbol{R}$. We now can state Proposition 6.1 of Gu *et al.* (2005) as follows.

**Theorem 3** *As $(\delta, \eta)$ moves from $\boldsymbol{R}$ to $\boldsymbol{L}$, a stability loss occurs if $Q > 0$ and so does a stability gain if $Q < 0$.*

Notice in Figure 2 that $Q(\delta^A, \eta^0) > 0$ as point $A$, $(\delta^A, \eta^0)$, is on $C_2(1,1)$ and $Q(\delta^B, \eta^0) < 0$ as point $B$, $(\delta^B, \eta^0)$, is on $C_1(1,1)$. If we increase the value of $\delta$ along the dotted horizontal line starting at $\eta^0$, $\delta$ crosses the $C_2(1,1)$ curve from $\boldsymbol{R}$ to $\boldsymbol{L}$ at point $A$ and also crosses the $C_1(1,1)$ curve from $\boldsymbol{R}$ to $\boldsymbol{L}$ at point $B$. Thus according to Theorem 3, stability is lost at point $A$ and gained at point $B$. The zero solution of equation (14) loses stability by entering the cigar-shaped domain and regains stability by leaving it. As we will see later, the cigar-shaped domains defined for any other values of $m$ and $n$ are overlapped each other, distorting the shape of the instability region. However the basic principle of the stability switch is not changed.

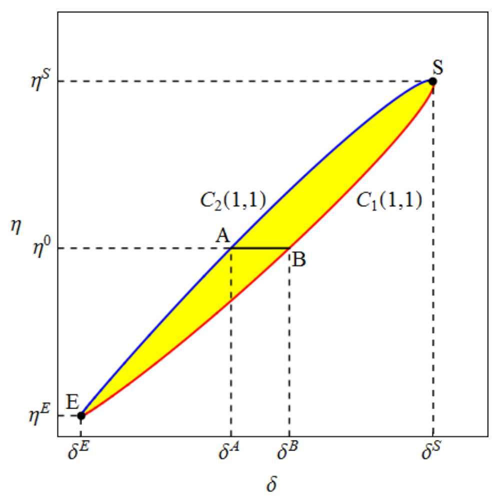



Figure 2. Partition curve with $m = 1$ and $n = 1$

# 5 Simulations

We perform four simulations to examine global dynamics generated by delay equation (6) and occurrence of the stability switch on the partition curve. To this end we take $\alpha = 0.6$ and $\varepsilon = 0.8$ and use the $S$-shaped investment function,

$$\varphi(x) = a_2 \left( \frac{a_1 + a_2}{a_1 e^{-x} + a_2} - 1 \right)$$

with $a_1 = 3$ and $a_2 = 1$.[8] It is checked that

$$\varphi'(x) > 0, \ \lim_{x \to \infty} \varphi(x) = a_1 \text{ and } \lim_{x \to -\infty} \varphi(x) = -a_2$$

the last two results of which implies that the upper bound of $\varphi(x)$ is three times larger than the lower bound.[9] It is also checked that Assumption 2 is satisfied and the discriminants of functions $f(\omega)$ and $g(\omega)$ are positive. We investigate how the dynamics of equation (6) depends on the delays, $\delta$ and $\eta$. In the first simulation, we fix $\eta = 1$ and increase $\delta$ from 1 to 4 along the dotted horizontal line at $\eta = 1$ in Figure 3(A). The six cigar-shaped domains are obtained for $m = 0, 1, 2, 3, 4, 5$ and $n = 1$ and the lower parts of these domains are illustrated as yellow regions. In order to simplify the exposition, the domains constructed by the curves $C_i(m, n)$ for $n \geq 2$ are omitted. The red and blue boundary curves are described by $C_1(m, 1)$ and $C_2(m, 1)$, respectively. Remember that the domain shifts to the right as the value of $m$ increases. As denoted by black dots, the horizontal line crosses the boundaries of these domains seven times at

$$\delta_1 \simeq 0.02, \ \delta_2 \simeq 0.11, \ \delta_3 \simeq 1.12, \ \delta_4 \simeq 1.62, \ \delta_5 \simeq 2.21, \ \delta_6 \simeq 3.14, \ \delta_7 \simeq 3.31.$$

It is already confirmed that the zero solution is locally stable along the axes of abscissa and ordinate on which one of the two delays is zero and also along the positive sloping black curve which is the 45 degree line and $\delta = \eta$ holds.[10] According to Theorem 3, stability is lost at $\delta_3$, $\delta_5$ and $\delta_7$ while it is gained at $\delta_4$ and $\delta_5$ since the followings hold:

(i) $\delta$ crosses the blue curve from $\boldsymbol{R}$ to $\boldsymbol{L}$ and $Q > 0$ on $C_2(1, 1), C_2(2, 1)$ and $C_2(3, 1)$;

(ii) $\delta$ crosses the red curve from $\boldsymbol{R}$ to $\boldsymbol{L}$ and $Q < 0$ on $C_1(1, 1)$ and $C_1(2, 1)$.

The bifurcation diagram in Figure 3(B) plots the local maximum and minimum of the trajectory against $\delta$ and presents the numerical results concerning the dynamics when the local stability is lost. The value of $\delta$ is selected to be greater than unity due to Assumption 1. It is observed that the equilibrium point bifurcates to a limit cycle that expands, shrinks and then merges to the

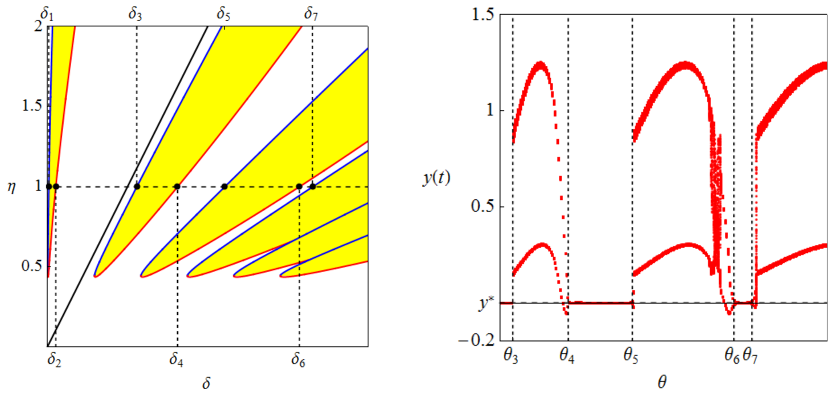
[8] Goodwin (1951) uses $\alpha = 0.6$ and $\varepsilon = 0.5$ in his simulations. Since we make Assumption 2, we change the value of $\varepsilon$ to 0.8 which is larger than $\nu$,

$$\nu = \frac{a_1 a_2}{a_1 + a_2} = 0.75.$$

[9] Goodwin (1951) imposes this asymmetric condition on his investment function.

[10] Notice that the ratio of the horizontal axes to the vertial axes is appropriately adjusted in Figure 3(A).

equilibrium point and the basic pattern of the birth of the cycle, growth and extinction is repeated for even larger values of $\delta$.



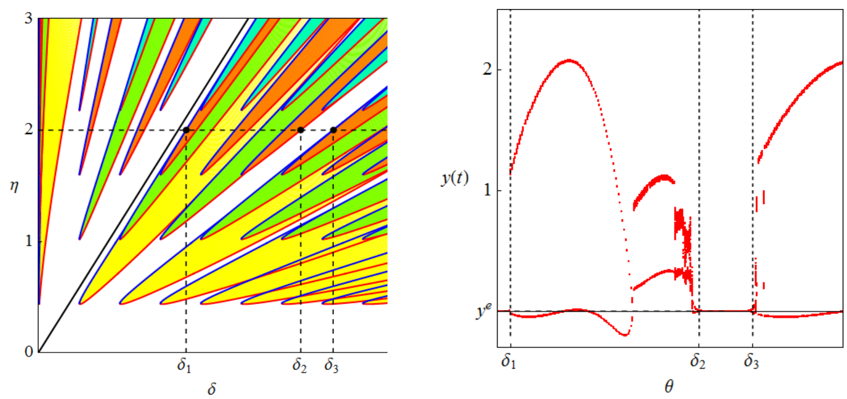

(A) Partition curves (B) Bifurcation diagram

Figure 3. Stability switches with $\eta = 1$

In the second simulation, the value of $\eta$ is changed to 2 and $\delta$ is increased from 2 to 5 along the dotted horizontal line in Figure 4(A). The red and blue curves of $C_1(m,n)$ and $C_2(m,n)$ for $m = 0, 1, ..., 8$ and $n = 1, 2, 3, 4$ are illustrated. The yellow regions are surrounded by the segments of $C_1(m,1)$ and $C_2(m,1)$ and are the same as the yellow regions in Figure 3(A). The green regions are surrounded by $C_1(m,2)$ and $C_2(m,2)$. Notice that the regions shift upward when $n$ increases to 2 from 1. Similarly, the orange regions and the light blue regions are surrounded by $C_1(m,3)$ and $C_2(m,3)$ and $C_1(m,4)$ and $C_2(m,4)$, respectively. The horizontal dotted line crosses the partition curves many times, however the stability switch takes place three times only at

$$\delta_1 \simeq 2, 11, \ \ \delta_2 \simeq 3.75 \text{ and } \delta_3 \simeq 4.22.$$

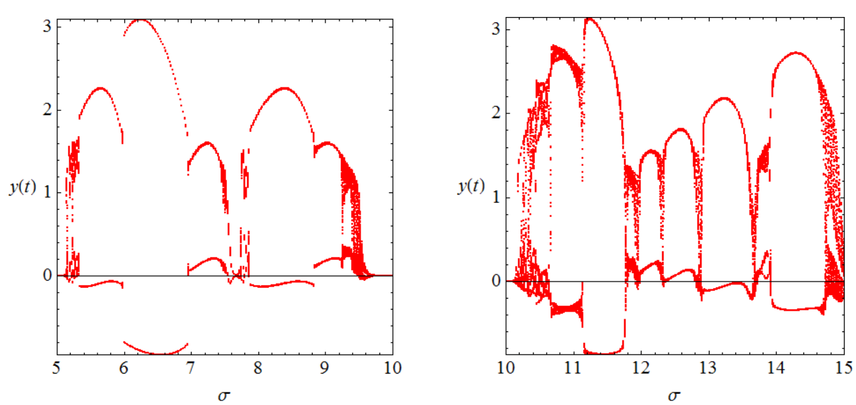
Stability is lost at $\delta_1$ and $\delta_3$ since the horizontal line enters the colored regions from the left to the right while stability is gained at $\delta_2$ since the line leaves the colored regions and enters the white region. As we have shown in Section 3, the zero solution is stable when $\delta > 0$ and $\eta$ along the axis of abscissa, $\eta > 0$ and $\delta = 0$ along the axis of ordinate and $\delta = \eta > 0$ along the 45 degree line denoted as the black upward sloping line. It is observed that the zero solution is still stable, due to continuity, in a neighborhood of the 45 degree line. Comparing Figure 3(A) with Figure 4(A), we can see that the stability region (i.e., the white region) becomes smaller as the value of the consumption delay $\eta$ becomes larger. Further the bifurcation diagram in Figure 4(B) indicates that the basic pattern of the cyclic oscillations are more distorted as $\eta$ increases.




(A) (B)

Figure 4. Stability switches with $\eta = 2$

In the third and fourth simulations, the value of $\eta$ is increased to 5 and 10, respectively. The divisions of the parameter space $(\delta, \eta)$ becomes very messy and thus are omitted. The corresponding bifurcation diagrams are given in Figure 5(A) and 5(B) in which delay equation (6) gives rise to more cyclic dynamics as the lengths of delays become larger.




(A) $\eta = 5$ (B) $\eta = 10$

Figure 5. Bifurcation diagrams with respect to $\delta$

# 6 Conclusions

An extension of Goodwin's continuous time scale multiplier-accelerator model is examined in which delayed consumption and investment are assumed. Conditions are given first to the local asymptotical stability of the steady state without delays, and then three special one-delay cases are investigated. The local asymptotical stability of these models are proved when the same holds for the non-delay model. In the two-delay case the stability-switching curves are determined on which stability gain and loss occur repeatedly when one of the delays increases while the other kept to be constant. An $S$-shaped functional form of investment and a linear consumption function are selected, and the simulation study shows how cyclic dynamics of national income can emerge. In the existing literature, cyclic dynamics is obtained when the steady state is locally unstable. It is thus worthwhile to emphasize that in this study, the steady state is locally asymptotically stable under no delays. It would be interesting to see what dynamics the delay model can generate if its steady state is locally unstable.

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