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Abstract

We construct à la Cournot duopoly model in a continuous-time framework and consider dynamic behavior when the firms are heterogenous in determining their output decision: one firm has an information delay in the competitor's output as well as an implementation delay in its own output while the other firm does not have any delays. Two main results are obtained. One is that the information delay does not affect dynamics of the commodities and the other is that the implementation delay can destabilize the otherwise stable equilibrium. Furthermore, it is numerically confirmed that the two delay duopoly model can generate a wide spectrum of dynamics ranging from cyclic dynamics to chaos.

Keywords: Delay duopoly dynamics, Bounded rationality, Continuous-time system, Stability switch.

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1 Introduction

This paper presents a new characterization of price formation of a single commodity by which regular and irregular price oscillations can be constructed in a continuous-time framework with fixed time delays. In the existing literature of macro dynamics, it has been well-known since the 1930s that a delay in investment or production is one of the key factors to generate cyclic dynamics in national income, capital accumulation, employment rate, etc. See the pioneering papers, Kalecki (1935) and Goodwin (1951). On the other hand, in the existing literature of micro dynamics such as price and commodity oscillations, it has been less-known that production delays are also able to generate economic fluctuations of a continuous-time unstable economic system.[1] As early as in the 1930s, Haldane (1933) had already shown that cyclic behavior can arise in a simple price dynamics model with production delay, coaxing an idea from theoretical biology.[2] Larson (1964) constructs a continuous-time cobweb model for the pork market in which pork is inelastically supplied with 12 month delay and gives rise to cyclic behavior. Howroyd and Russel (1984) are the first to consider delay dynamics in a N-firm oligopoly model. On the other hand, Mackey (1989) formulates the price dynamics of a single commodity market as a nonlinear delay differential equation and rigorously derives the stability conditions and the birth of a cyclic oscillation via Hopf bifurcation. Recently Matsumoto and Szidarovszky (2014) reconsider delay dynamics in various monopoly models and emphasize that delay may explain various dynamic behavior of micro economic variables. In the existing literature, however, analysis is mostly confined to cases of a single dimensional model with one delay or multiple delay or a multiple dimensional model with one delay. In this paper we construct a multiple dimensional model with multiple delays and examine cyclic behavior in the model.

This paper revisits the delay duopoly game investigated by Elsadany and Matouk (2014) in order to reconsider the delay effect on stability on the Nash equilibrium in a continuous-time framework. Their game is built in a discrete-time framework and its distinguished feature is the heterogeneity of the duopolists. In predicting competitor's output, one duopolist uses a combination of the current and delayed information while the other adopts naive expectation. The main result is that the delay has a stabilizing effect: it enlarges the stability region of the Nash equilibrium. Generally speaking, a continuous-time model has a larger stability region than a discrete-time model (see, for example, Matsumoto and Szidarovszky (2015)). Further, the delay in a continuous-time model often exhibits stability switching from stability to instability if it becomes larger than some threshold value. However, it is not known yet whether such stability switch occurs in the heterogenous duopoly game. Hence in this paper, we retain the heterogneity of the discrete-time dynamic system and consider its effect in

[1] Since Cobweb dynamics examined by Ezekiel (1933), the discrete-time economic models have a long history in establishing cyclic behavior of the price and the commodity.

[2] Surprisingly, it was mentioned in the postscript of the paper in which his theory was completed in 1924.

a delay continuous-time framework . The followings are the main results:

(I) When only one firm has information delays in the competitor's output, such a heterogeneous expectation formation has no destabilizing effect;

(II) When the firm has a delay in implementing information about its own output, the otherwise stable equilibrium can be destabilized and give rise to simple as well as to complex fluctuations when the delay is sufficiently long.

This paper is organized as follows. In Section 2, the basic elements of El-sadany and Matouk's model are recapitulated. In Section 3 the corresponding continuous-time duopoly model is constructed and the delay effects due to heterogenous expectation formation are examined. In Section 4, the destabilizing effect caused by the implementation delay is analytically and numerically considered. In the final section some concluding remarks are given and further research directions are outlined.

2 Discrete-time Duopoly Model

A duopoly game with heterogenous bounded rationality is formulated in discrete-time framework. Two firms produce a homogenous good. Firm x produces the quantity x with marginal cost c_x, while firm y produces the quantity y at marginal cost c_y. The market demand is linear and depends on the total output of the industry,

$$p = a - b(x + y)$$

where a and b are positive constants. The profit of firm $z(= x, y)$ is

$$\pi_z = [a - b(x + y)]\, z - c_z z.$$

Bischi and Naimzada (2000) assume bounded rational firms and introduce gradient dynamics in which the firms adjust production levels according to their marginal profits in such a way that a firm increases output if the marginal profit is positive, decreases it if negative and does not change if zero. The gradient dynamics with boundedly rational firm z can be described by

$$z(t + 1) = z(t) + \alpha_z z(t) \frac{\partial \pi_z}{\partial z}$$

where $\alpha_z > 0$ is an adjustment coefficient. Hence in the duopoly game, the output adjustment process with gradient method is presented by a two dimensional system of difference equations,

$$\begin{aligned} x(t + 1) &= x(t) + \alpha_x x(t) \left[a - c_x - 2bx(t) - by^e(t + 1)\right], \\ y(t + 1) &= y(t) + \beta_y y(t) \left[a - c_y - bx^e(t + 1) - 2by(t)\right], \end{aligned} \tag{1}$$

where $x^e(t+1)$ and $y^e(t+1)$ are expected outputs at period $t+1$. The positive stationary outputs, $x^* = x^e(t) = x(t) = x(t+1)$ and $y^* = y(t) = y^e(t) = y(t+1)$, are given by

$$x^* = \frac{a - 2c_x + c_y}{3b} \text{ and } y^* = \frac{a - 2c_y + c_x}{3b} \tag{2}$$

with the assumption

$$a > \max[2c_x - c_y,\ 2c_y - c_x]$$

to ensure the positivity of the outputs. In the literature, traditional expectation formations such as naive expectation and adaptive expectation are adopted in dynamic games with homogeneous firms. Elsadany and Matouk (2014) construct a dynamic game with *heterogenous* expectation formation, one firm makes output decision based on delayed information on the competitor's output while the other firm makes its output decision on current information.[3] In particular, it is assumed that

$$y^e(t+1) = \eta y(t) + (1-\eta)y(t-1) \text{ and } x^e(t+1) = x(t) \tag{3}$$

with the weight η being positive and less than unity. It is demonstrated that dynamic system (1) with (3) enlarges the stability region and generates complex dynamics via period doubling cascade when stability is lost.

3 Continuous-time Duopoly Models

We modify three issues of the discrete-time model (1) in order to consider "delay dynamics" in a continuous-time framework. The first one is to replace $z(t+1) - z(t)$ with $\dot{z}(t)$, the second is to replace $y^e(t+1)$ with $y^e(t)$ and the last one is to substitute one unit discrete time delay by τ continuous time delay. Then the discrete-time model can be converted to a continuous-time model,

$$\begin{aligned} \dot{x}(t) &= \alpha_x x(t)\left[a - c_x - 2bx(t) - by^e(t)\right], \\ \dot{y}(t) &= \beta_y y(t)\left[a - c_y - bx(t) - 2by(t)\right], \end{aligned} \tag{4}$$

where $y^e(t)$ is the expected output formed at time t. We introduce the following four different expectation formations and then consider how the different formations affect dynamics in the continuous-time framework. Forming the expectation on the competitor's output, $(E1)$ uses the realized output at time $t-\tau$, which is similar to the naive expectation in a discrete-time model, $(E2)$ uses the weighted average of two past outputs at times $t-\tau_1$ and $t-\tau_2$, $(E3)$ is an extension of $(E2)$ and employs three past outputs at times $t-\tau_1$, $t-\tau_2$ and $t-\tau_3$ to obtain the weighted average and finally $(E4)$ generalizes the weighted average using all past outputs from time 0 to t and the weight is exponentially

[3] Yassen and Agiza (2003) and Hassan (2004) assume *homogeneous* expectation formations in which both firms use past production data to determine their current outputs.

declining with the most weight given to the most current output:

$$
\begin{aligned}
&(E1)\ y^e(t) = y(t-\tau); \\
&(E2)\ y^e(t) = \textstyle\sum_{k=1}^{2} \eta_k y(t-\tau_k),\ \sum_{k=1}^{2} \eta_k = 1; \\
&(E3)\ y^e(t) = \textstyle\sum_{k=1}^{3} \eta_k y(t-\tau_k),\ \sum_{k=0}^{3} \eta_k = 1; \\
&(E4)\ y^e(t) = \int_0^t \frac{1}{T} e^{-\frac{\tau}{T}} y(t-\tau) d\tau = \int_0^t \frac{1}{T} e^{-\frac{t-s}{T}} y(s) ds.
\end{aligned} \tag{5}
$$

3.1 One Fixed Delay

The point (x^*, y^*) is also the stationary point of the continuous-time model. Assuming formation $(E1)$ in (5) and introducing new notation, $x_\delta(t) = x(t) - x^*$ and $y_\delta(t) = y(t) - y^*$, we obtain a two dimensional system of linearized equations

$$
\begin{aligned}
\dot{x}_\delta(t) &= \alpha \left[-2bx_\delta(t) - by_\delta(t-\tau) \right], \\
\dot{y}_\delta(t) &= \beta \left[-bx_\delta(t) - 2by_\delta(t) \right]
\end{aligned} \tag{6}
$$

with

$$\alpha = \alpha_x x^* \text{ and } \beta = \beta_y y^*.$$

Substituting exponential solutions

$$x_\delta(t) = e^{\lambda t} u \text{ and } y_\delta(t) = e^{\lambda t} v$$

into the linearized system (6) and arranging terms yield an alternative linear system,

$$
\begin{pmatrix} \lambda + 2b\alpha & b\alpha e^{-\lambda\tau} \\ b\beta & \lambda + 2b\beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Excluding the trivial solutions (i.e., $u = v = 0$), we obtain non-trivial solutions by solving the characteristic equation

$$(\lambda + 2b\alpha)(\lambda + 2b\beta) - b^2 \alpha\beta e^{-\lambda\tau} = 0. \tag{7}$$

For $\tau = 0$, the characteristic equation is reduced to

$$\lambda^2 + 2b(\alpha + \beta)\lambda + 3\alpha\beta b^2 = 0.$$

Since, by assumption, $2b(\alpha+\beta) > 0$ and $3\alpha\beta b^2 > 0$, the non-delay characteristic equation has roots with negative real parts, implying that the stationary point is locally stable if there is no delay. It is true that the stationary state preserves stability as far as the delay is positive but sufficiently small. We are concerned with a threshold value (if exists) of the delay for which the stationary point is just destabilized. It is well known that if the stability of the stationary point switches at $\tau = \bar{\tau}$, then the characteristic equation must have a pair of pure conjugate imaginary roots there. To examine the stability switches of the dynamic system (6), we determine this threshold value $\bar{\tau}$. For this purpose, we substitute a purely imaginary solution $\lambda = i\omega$ with $\omega > 0$ into equation (7) that is separated to real and imaginary parts,

$$-\omega^2 + 4b^2\alpha\beta = b^2\alpha\beta \cos\tau\omega,$$

$$-2b(\alpha+\beta)\omega = b^2\alpha\beta \sin\tau\omega.$$

Squaring both sides of these equations and adding the resultant expressions yield, after arranging terms, a quartic equation in ω,

$$\omega^4 + 4b^2(\alpha^2+\beta^2)\omega^2 + 15\left(b^2\alpha\beta\right)^2 = 0. \tag{8}$$

Since the left hand side is positive for all real values of ω, stability switch cannot occur and therefore the steady state is always stable for any $\tau > 0$. In other words, time delay is *harmless*.

Theorem 1 *Under the expectation formation* $(E1)$, *the continuous-time dynamic system (4) is locally asymptotically stable for any value of* $\tau > 0$.

3.2 Two Fixed Delays

Assuming $(E2)$ as the output expectation formation and substituting the exponential forms, $x(t) = e^{\lambda t}u$ and $y(t) = e^{\lambda t}v$ into the linearized equation (4) yields

$$\begin{pmatrix} \lambda + 2b\alpha & b\alpha\left(\eta_1 e^{-\lambda\tau_1} + \eta_2 e^{-\lambda\tau_2}\right) \\ b\beta & \lambda + 2b\beta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Assuming nontrivial solution gives the corresponding characteristic equation,

$$(\lambda + 2b\alpha)(\lambda + 2b\beta) - b^2\alpha\beta\left(\eta_1 e^{-\lambda\tau_1} + \eta_2 e^{-\lambda\tau_2}\right) = 0.$$

This is equivalent with equation

$$a(\lambda, \tau_1, \tau_2) = 1 + a_1(\lambda)e^{-\lambda\tau_1} + a_2(\lambda)e^{-\lambda\tau_2} \tag{9}$$

where

$$a_k(\lambda) = \frac{-b^2\alpha\beta}{(\lambda+2b\alpha)(\lambda+2b\beta)}\eta_k \text{ for } k = 1, 2.$$

Suppose $\lambda = i\omega$ with $\omega > 0$, then

$$a_k(i\omega) = \frac{-\alpha\beta b^2\left[4b^2\alpha\beta - \omega^2 - i2b(\alpha+\beta)\omega\right]}{\left[4b^2\alpha\beta - \omega^2\right]^2 + \left[2b(\alpha+\beta)\omega\right]^2}\eta_k \text{ for } k = 1, 2$$

and their absolute values are

$$|a_k(i\omega)| = \frac{b^2\alpha\beta}{\sqrt{\left[4b^2\alpha\beta - \omega^2\right]^2 + \left[2b(\alpha+\beta)\omega\right]^2}}\eta_k \text{ for } k = 1, 2. \tag{10}$$

Gu *et al.* (2005) geometrically investigate the two delay equation based on the idea that solving $a(i\omega, \tau_1, \tau_2) = 0$ is equivalent to forming a triangle by treating three terms in (9) as three vectors in the complex plane and then placing them head to tail. Since the triangle consists of three line segments, it is a necessary and sufficient condition for the existence of a positive solution $\omega > 0$ of equation (9) that the sum of the lengths of any two adjacent line segments is not shorter than the length of the remaining line segment.

Theorem 2 *A purely imaginary root $\lambda = i\omega$ with $\omega > 0$ is a solution of $a(i\omega, \tau_1, \tau_2) = 0$ if and only if the following three inequalities hold:*

$$|a_1(i\omega)| + |a_2(i\omega)| \geq 1,$$

$$|a_1(i\omega)| + 1 \geq |a_2(i\omega)|,$$

$$|a_2(i\omega)| + 1 \geq |a_1(i\omega)|.$$

Proof. Let a, b and c be three line segments. The end points of a and b with a common starting point can be connected with a segment of length c if and only if

$$|a - b| \leq c \leq a + b.$$

The second inequality gives one condition. The first inequality can be rewritten as

$$-c \leq a - b \leq c,$$

which means that

$$b \leq a + c$$

and

$$a \leq b + c.$$

Three inequality conditions are obtained, which completes the proof. ■

We check whether the first condition of Theorem 2 holds. Substituting (10) into it presents the inequality condition in parametric terms,

$$\omega^4 + 4b^2(\alpha^2 + \beta^2)\omega^2 + 15\left(b^2\alpha\beta\right)^2 \leq 0. \tag{11}$$

Notice that the left hand side is positive with all real values of ω implying that stability switch cannot occur for $\tau_1 > 0$ and $\tau_2 > 0$.

Theorem 3 *Under the expectation formation ($E2$), the continuous-time dynamic system (4) is locally asymptotically stable for any values of $\tau_1 > 0$ and $\tau_2 > 0$.*

3.3 Three Fixed Delays

In the similar way, we can show that the expectation formation ($E3$) cannot destabilize the stationary state as well. As before, sustituting the exponential forms, $x(t) = e^{\lambda t}u$ and $y(t) = e^{\lambda t}v$ into the linearized system under ($E3$) yields

$$\begin{pmatrix} \lambda + 2b\alpha & b\alpha\left(\eta_1 e^{-\lambda\tau_1} + \eta_2 e^{-\lambda\tau_2} + \eta_3 e^{-\lambda\tau_3}\right) \\ b\beta & \lambda + 2b\beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Nontrivial solution exists if and only if

$$(\lambda + 2b\alpha)(\lambda + 2b\beta) - b^2\alpha\beta\left(\eta_1 e^{-\lambda\tau_1} + \eta_2 e^{-\lambda\tau_2} + \eta_3 e^{-\lambda\tau_3}\right) = 0.$$

As before, this equation can be rewritten as

$$a(\lambda, \tau_1, \tau_2, \tau_3) = 1 + a_1(\lambda)e^{-\lambda\tau_1} + a_2(\lambda)e^{-\lambda\tau_2} + a_3(\lambda)e^{-\lambda\tau_3} \tag{12}$$

where

$$a_k(\lambda) = \frac{-b^2\alpha\beta}{(\lambda + 2b\alpha)(\lambda + 2b\beta)}\eta_k \text{ for } k = 1, 2, 3,$$

Suppose $\lambda = i\omega$ with $\omega > 0$, then

$$a_k(i\omega) = \frac{-b^2\alpha\beta\left[4b^2\alpha\beta - \omega^2 - i2b(\alpha+\beta)\omega\right]}{\left[4b^2\alpha\beta - \omega^2\right]^2 + \left[2b(\alpha+\beta)\omega\right]^2}\eta_k \text{ for } k = 1, 2, 3,$$

and their absolute values are

$$|a_k(i\omega)| = \frac{b^2\alpha\beta}{\sqrt{\left[4b^2\alpha\beta - \omega^2\right]^2 + \left[2b(\alpha+\beta)\omega\right]^2}}\eta_k \text{ for } k = 1, 2, 3. \tag{13}$$

Almodaresi and Bozorg (2009) give a straightforward extension of the two delay case considered by Gu *et al.* (2005) to the more general case of three delays and obtain the following result:

Theorem 4 *Pure imaginary root $\lambda = i\omega$ with $\omega > 0$ is a solution of $a(i\omega, \tau_1, \tau_2, \tau_3) = 0$ if and only if the following four inequalities hold:*

$$|a_1(i\omega)| + |a_2(i\omega)| + |a_3(i\omega)| \geq 1,$$

$$|a_2(i\omega)| + |a_3(i\omega)| + 1 \geq |a_1(i\omega)|,$$

$$|a_1(i\omega)| + |a_3(i\omega)| + 1 \geq |a_2(i\omega)|,$$

$$|a_1(i\omega)| + |a_2(i\omega)| + 1 \geq |a_3(i\omega)|.$$

We again check the first condition, which, after substituting (13), can be written as

$$\omega^4 + 4b^2(\alpha^2 + \beta^2)^2\omega^2 + 15\left(b^2\alpha\beta\right)^2 \leq 0.$$

Similerly to the two delay case, the left hand side is positive with any real value of ω, implying that stability switch cannot occur for any positive delays.

Theorem 5 *Under the expectation formation* $(E3)$*, the continuous-time dynamic system (4) is locally asymptotically stable for any values of* $\tau_1 > 0,\ \tau_2 > 0$ *and* $\tau_3 > 0$.

3.4 Continuously Distributed Delays

The expectation formation $(E4)$ assumes that the expected output is a weighted average of all past outputs from time 0 to time t. It is a generalization of $(E2)$ and $(E3)$ with infinitely many continuously distributed delays. Adding the time derivative of the expectation $(E4)$ to linearized system (4) yields a three dimensional system of differential equations,

$$\dot{x}(t) = \alpha\left[-2bx(t) - by^e(t)\right],$$

$$\dot{y}(t) = \beta\left[-bx(t) - 2by(t)\right],$$

$$\dot{y}^e(t) = \frac{1}{T}\left[y(t) - y^e(t)\right].$$

The corresponding characteristic equation is given by

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$a_1 = \frac{1 + 2(\alpha+\beta)bT}{T} > 0,$$

$$a_2 = \frac{2(\alpha+\beta)b + 4\alpha\beta b^2 T}{T} > 0$$

and

$$a_3 = \frac{3\alpha\beta b^2}{T} > 0.$$

A set of necessary and sufficient conditions for all roots of the cubic characteristic equation to have negative real parts is

$$a_i > 0 \text{ for } i = 1, 2, 3 \text{ and } a_1a_2 - a_3 > 0,$$

which is a special case of the Routh-Hurwitz criterion. Since all coefficients are positive, we need to check whether the last condition is satisfied or not. Clearly

$$a_1a_2 - a_3 = \frac{b}{T^2}\left[8b^2\alpha\beta(\alpha+\beta)T^2 + b\left(4\alpha^2 + 9\alpha\beta + 4\beta^2\right)T + 2(\alpha+\beta)\right] \quad (14)$$

is always positive for any $T > 0$. Hence, the stationary state of the three dimensional system is locally stable.

Theorem 6 *Under the expectation formation* $(E4)$*, the continuous-time dynamic system (4) is locally asymptotically stable for any values of* $T > 0$.

4 Own and Competitor's Delays

In this section we examine the effects caused by the two delays on dynamics: one delay in the competitor's output and the other delay in the firm's own output. Thus the dynamic system is modified in the following way,

$$\begin{aligned} \dot{x}(t) &= \alpha_x x(t) \left[a - c_x - 2bx(t-\tau_x) - by(t-\tau_y) \right], \\ \dot{y}(t) &= \beta_y y(t) \left[a - c_y - bx(t) - 2by(t) \right] \end{aligned} \tag{15}$$

where $\tau_x > 0$ is the implementation delay and $\tau_y > 0$ is the competitor's information delay. The stationary point is the same as given in (2). The linearized system is

$$\begin{aligned} \dot{x}_\delta(t) &= \alpha \left[-2bx_\delta(t-\tau_x) - by_\delta(t-\tau_y) \right], \\ \dot{y}_\delta(t) &= \beta \left[-bx_\delta(t) - 2by_\delta(t) \right]. \end{aligned} \tag{16}$$

Substituting exponential solutions $x_\delta(t) = e^{\lambda t}u$ and $y_\delta(t) = e^{\lambda t}v$ reduces the linear system to an alternative form,

$$\begin{pmatrix} \lambda + 2\alpha b e^{-\lambda\tau_x} & b\alpha e^{-\lambda\tau_y} \\ b\beta & \lambda + 2b\beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Nontrivial solution exists if and only if

$$\lambda^2 + 2b\beta\lambda + 2b\alpha(\lambda + 2b\beta)e^{-\lambda\tau_x} - b^2\alpha\beta e^{-\lambda\tau_y} = 0. \tag{17}$$

Before turning to a closer examination of equation (17), a few remarks should be made concerning the effect caused by the implementation delay. Assuming that $\tau_y = 0$ and $\lambda = i\omega$ with $\omega > 0$, we can separate equation (17) into real and imaginary parts,

$$\begin{aligned} 2b\alpha(2b\beta\cos\tau_x\omega + \omega\sin\tau_x\omega) &= \omega^2 + b^2\alpha\beta, \\ 2b\alpha(\omega\cos\tau_x\omega - 2b\beta\sin\tau_x\omega) &= -2b\beta\omega. \end{aligned} \tag{18}$$

Adding the squares of these equations yields a fourth-order polynomial equation in ω,

$$\omega^4 + 2b^2\left[\alpha\beta + 2\left(\beta^2 - \alpha^2\right)\right]\omega^2 - 15b^4\alpha^2\beta^2 = 0$$

that can be solved for ω^2 and the positive solution is

$$\omega_+^2 = b^2\left\{-[\alpha\beta + 2\left(\beta^2 - \alpha^2\right)] + \sqrt{4\beta^2 + 4\alpha\beta^3 + 7\alpha^2\beta^2 + \left(2\alpha^2 - \alpha\beta\right)^2}\right\} > 0$$

Solving (18) for $\cos\tau_x\omega$ and $\sin\tau_x\omega$, substituting ω_+ into $\cos\tau_x\omega$ and then solving it for τ give the threshold value of the delay,

$$\tau_x^n = \frac{1}{\omega_+}\left[\cos^{-1}\left(\frac{b^2\beta^2}{\omega_+^2 + 4b^2\beta^2}\right) + 2n\pi\right], \text{ for } n = 0, 1, 2, ...$$

A routine bifurcation analysis shows that stability is lost at the smallest threshold value,

$$\tau_x^0 = \frac{1}{\omega_+} \cos^{-1}\left(\frac{b^2\beta^2}{\omega_+^2 + 4b^2\beta^2} \right). \tag{19}$$

and stability cannot be regained later.

Theorem 7 *The stationary point of (15) with $\tau_y = 0$ is locally asymptotically stable for $\tau_x < \tau_x^0$, loses stability at $\tau_x = \tau_x^0$ and bifurcates to a limit cycle via Hopf bifurcation for $\tau_x > \tau_x^0$.*

We now investigate the characteristic equation with two positive delays by applying Gu's method (Gu *et al.* 2005). Dividing both sides by $\lambda^2 + 2b\beta\lambda$ and introducing new functions

$$a_1(\lambda) = \frac{2b\alpha}{\lambda} \text{ and } a_2(\lambda) = \frac{-b^2\alpha\beta}{\lambda(\lambda + 2b\beta)}$$

simplify equation (17) as

$$a(\lambda, \tau_x, \tau_y) = 1 + a_1(\lambda)e^{-\lambda\tau_x} + a_2(\lambda)e^{-\lambda\tau_y} = 0. \tag{20}$$

Suppose that $\lambda = i\omega$ with $\omega > 0$. Substituting it into $a_i(\lambda)$ yields

$$a_1(i\omega) = -i\frac{2b\alpha}{\omega}$$

and

$$a_2(i\omega) = \frac{b^2\alpha\beta\omega}{\omega(\omega^2 + 4b^2\beta^2)} + i\frac{2b^3\alpha\beta^2}{\omega(\omega^2 + 4b^2\beta^2)}.$$

Their absolute values are

$$|a_1(i\omega)| = \frac{2b\alpha}{\omega}$$

and

$$|a_2(i\omega)| = \frac{b^2\alpha\beta}{\omega\sqrt{\omega^2 + 4b^2\beta^2}}.$$

We check whether the three conditions of Theorem 2 are satisfied. Substituting the absolute values reduces the three conditions to the following two conditions,

$$f(\omega) = \omega^4 - 4b\alpha\omega^3 + 4b^2(\alpha^2 + \beta^2)\omega^2 - 16b^3\alpha\beta^2\omega + 15b^4\alpha^2\beta^2 \leq 0$$

and

$$g(\omega) = \omega^4 + 4b\alpha\omega^3 + 4b^2(\alpha^2 + \beta^2)\omega^2 + 16b^3\alpha\beta^2\omega + 15b^4\alpha^2\beta^2 \geq 0.$$

Notice that $g(\omega) \geq 0$ is always true for $\omega \geq 0$. However, it is ambiguous whether $f(\omega) \leq 0$ holds or not. Using the variable transformation

$$\omega = x - \left(-\frac{4b\alpha}{4}\right) = x + b\alpha,$$

we rewrite $f(\omega)$ as

$$F(x) = x^4 - 2b^2(\alpha^2 - 2\beta^2)x^2 - 8b^3\alpha\beta^2 x + b^4\alpha^2(\alpha^2 + 3\beta^2).$$

The sequence of the signs of the polynomial coefficients are $(+,-,-,+)$ if $\alpha^2 > 2\beta^2$, $(+,+,-,+)$ if $\alpha^2 < 2\beta^2$, and $(+,-,+)$ if $\alpha^2 = 2\beta^2$. In any sequence, the number of sign changes is two. According to Descartes' rule of sign, this polynomial has either two or zero positive roots. If is clear that $F(0) > 0$ and $F(\infty) = \infty$. Substituting $x = \alpha b$ yields

$$F(\alpha b) = \alpha^2 b^4(-\beta^2) < 0.$$

The inequality implies that $F(x) = 0$ has two positive solutions, x_1 and x_2 where $0 < x_1 < \alpha b$ and $x_2 > \alpha b$. Hence $f(\omega) = 0$ has two positive solutions

$$\omega_1 = x_1 + \alpha b > 0 \text{ and } \omega_2 = x_2 + \alpha b > 0$$

and $f(\omega) \leq 0$ for $\omega \in [\omega_1, \omega_2]$, and a routin investigation can show that $f(\omega) > 0$ otherwise.[4]

We will next find all pairs of (τ_1, τ_2) satisfying $a(i\omega, \tau_1, \tau_2) = 0$. The three terms in (20) are three vectors in the complex plane that construct a triangle. Let us suppose that 1 is its base and let us denote the angle between 1 and $\left|a_1(i\omega)e^{-i\omega\tau_x}\right|$ by θ_1 and an angle between 1 and $\left|a_2(i\omega)e^{-i\omega\tau_y}\right|$ by θ_2. Since, $\left|e^{-i\omega\tau_x}\right| = \left|e^{-i\omega\tau_x}\right| = 1$, we have, by the law of cosine,

$$\begin{aligned} \theta_1(\omega) &= \cos^{-1}\left(\frac{1^2 + |a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2 \cdot 1 \cdot |a_1(i\omega)|}\right) \\ &= \cos^{-1}\left(\frac{\omega^4 + 4b^2(\alpha^2 + \beta^2)\omega^2 + 15b^4\alpha^2\beta^2}{4b\alpha\omega\left(\omega^2 + 4b^2\beta^2\right)}\right) \end{aligned}$$

and

$$\begin{aligned} \theta_2(\omega) &= \cos^{-1}\left(\frac{1^2 + |a_2(i\omega)|^2 - |a_1(i\omega)|^2}{2 \cdot 1 \cdot |a_2(i\omega)|}\right) \\ &= \cos^{-1}\left(\frac{\omega^4 - 4b^2(\alpha^2 - \beta^2)\omega^2 - 15b^4\alpha^2\beta^2}{2b^2\alpha\omega\sqrt{\omega^2 + 4b^2\beta^2}}\right). \end{aligned}$$

Since the triangle may be located above and also under the horizontal axis, we get two equations as

$$\left\{\arg\left[a_1(i\omega)e^{-i\tau_x\omega}\right] + 2m\pi\right\} \pm \theta_1(\omega) = \pi$$

and

$$\left\{\arg\left[a_2(i\omega)e^{-i\tau_y\omega}\right] + 2n\pi\right\} \mp \theta_2(\omega) = \pi$$

[4] A formal proof is given in the Appendix.

which yield the threshold values of the delays:

$$\tau_x^{\pm}(\omega, n) = \frac{1}{\omega}\left[\frac{3\pi}{2} + (2m-1)\pi \pm \theta_1(\omega)\right] \tag{21}$$

and

$$\tau_y^{\mp}(\omega, n) = \frac{1}{\omega}\left[\tan^{-1}\left(\frac{2b\beta}{\omega}\right) + (2n-1)\pi \mp \theta_2(\omega)\right]. \tag{22}$$

Let Ω be the set of ω for which $f(\omega) \leq 0$ holds. Then we can find all pairs of (τ_1, τ_2) constructing the stability switching curves for $\omega \in \Omega$ which consists of two sets of parametric segments,

$$L_1(m, n) = \{\tau_x^+(\omega, m), \tau_y^-(\omega, n)\} \tag{23}$$

and

$$L_2(m, n) = \{\tau_x^-(\omega, m), \tau_y^+(\omega, n)\}. \tag{24}$$

To illustrate the stability switching curves we first specify the parameter values:

Assumption: $\alpha = \beta = b = 1$.

Under this Assumption, we have

$$f(\omega) = \omega^4 - 4\omega^3 + 8\omega^2 - 16\omega + 15.$$

It is not difficult to show that $f(\omega) = 0$ has two real and positive roots, ω_s and ω_e, both of which are, without a loss of generality, supposed to satisfy $\omega_s < \omega_e$ where

$$\omega_s = 1 + \frac{1}{2\sqrt{3/K}} - \frac{1}{2}\sqrt{-\frac{1}{3}(12+K) + 16\sqrt{3/K}} \simeq 1.611$$

$$\omega_e = 1 + \frac{1}{2\sqrt{3/K}} - \frac{1}{2}\sqrt{-\frac{1}{3}(12+K) + 16\sqrt{3/K}} \simeq 2.326$$

with

$$K = -4 + \left(584 - 48\sqrt{87}\right)^{\frac{1}{3}} + 2\left(73 + 6\sqrt{87}\right)^{\frac{1}{3}}.$$

The domain of ω is the interval $[\omega_s, \omega_e](= \Omega)$. So the segment (23) starts at point $\left(\tau_x^+(\omega_s, m), \tau_y^-(\omega_s, n)\right)$ and terminate at point $\left(\tau_x^+(\omega_e, m), \tau_y^-(\omega_e, n)\right)$ and so does the segment (24) at points $\left(\tau_x^-(\omega_s, m), \tau_y^+(\omega_s, n)\right)$ and $\left(\tau_x^-(\omega_e, m), \tau_y^+(\omega_e, n)\right)$. Furthermore the angles of the triangle are

$$\theta_1(\omega) = \cos^{-1}\left(\frac{\omega^4 + 8\omega^2 + 15}{4\omega(\omega^2+4)}\right)$$

and

$$\theta_2(\omega) = \cos^{-1}\left(\frac{\omega^4 - 15}{2\omega\sqrt{\omega^2+4}}\right).$$

It is easily verified that

$$\frac{\omega_s^4 + 8\omega_s^2 + 15}{4\omega_s(\omega_s^2 + 4)} = 1, \ \frac{\omega_e^4 + 8\omega_e^2 + 15}{4\omega_e(\omega_e^2 + 4)} = 1$$

and

$$\frac{\omega_s^4 - 15}{2\omega_s\sqrt{\omega_s^2 + 4}} = -1, \ \frac{\omega_e^4 - 15}{2\omega_e\sqrt{\omega_e^2 + 4}} = 1$$

which then imply that

$$\theta_1(\omega_s) = 0, \ \theta_1(\omega_e) = 0$$

and

$$\theta_2(\omega_s) = \pi, \ \theta_2(\omega_e) = 0.$$

We then have the following results concerning the locations of the curve segments:

Theorem 8 *Given m and n, the segments $L_1(m, n+1)$ and $L_2(m, n)$ have the same starting point whereas the segments $L_1(m, n)$ and $L_2(m, n)$ have the same end point.*

Proof. For analytical simplicity we assume $m = 0$. The starting points of $L_1(0, n)$ and $L_2(0, n)$ are

$$L_1^s(0, n) = \{\tau_x^+(\omega_s, 0), \tau_y^-(\omega_s, n)\}$$

with

$$\tau_x^+(\omega_s, 0) = \frac{1}{\omega_s}\frac{\pi}{2} \text{ and } \tau_y^-(\omega_s, n) = \frac{1}{\omega_s}\left[\tan^{-1}\left(\frac{2}{\omega_s}\right) + 2(n-1)\pi\right],$$

and

$$L_2^s(0, n) = \{\tau_x^-(\omega_s, 0), \tau_y^+(\omega_s, n)\}$$

where

$$\tau_x^-(\omega_s, 0) = \frac{1}{\omega_s}\frac{\pi}{2} \text{ and } \tau_y^+(\omega_s, n) = \frac{1}{\omega_s}\left[\tan^{-1}\left(\frac{2}{\omega_s}\right) + 2n\pi\right].$$

Hence we have

$$L_1^s(0, n+1) = L_2^s(0, n).$$

In the same way, the end points of $L_1(0, n)$ and $L_2(0, n)$ are

$$L_1^e(0, n) = \{\tau_x^+(\omega_e, 0), \tau_y^-(\omega_e, n)\}$$

with

$$\tau_x^+(\omega_e, 0) = \frac{1}{\omega_e}\frac{\pi}{2} \text{ and } \tau_y^-(\omega_e, n) = \frac{1}{\omega_e}\left[\tan^{-1}\left(\frac{2}{\omega_e}\right) + (2n-1)\pi\right]$$

and

$$L_2^e(0,n) = \{\tau_x^-(\omega_e, 0), \tau_y^+(\omega_e, n)\}$$

where

$$\tau_x^-(\omega_e, 0) = \frac{1}{\omega_e}\frac{\pi}{2} \text{ and } \tau_y^+(\omega_e, n) = \frac{1}{\omega_e}\left[\tan^{-1}\left(\frac{2}{\omega_e}\right) + (2n-1)\pi\right].$$

Hence we have

$$L_1^e(0,n) = L_2^e(0,n).$$

The same result is obtained for any integer $m > 0$. ■

This result is numerically confirmed in Figure 1 where $m = 0$. The stability switching curve consists of the red and blue segments that correspond to $L_1(0,n)$ and $L_2(0,n)$ for $n = 0, 1, 2$. The red and blue segments shift upward when n increases and to the right when m increases. The upward-sloping blue segment crossing the horizontal axis at $\tau_x = \tau_x^0$ is the $L_2(0,0)$ segment.[5] It connects to the red $L_1(0,1)$ segment at a point where $L_1^s(0,1) = L_2^s(0,0)$ holds. As before, the upper script "s" means the starting (i.e, initial) point of the segment and

$$L_1^s(0,1) = (\tau_x^+(\omega_s, 1), \tau_y^-(\omega_s, 1)) \text{ and } L_2^s(0,0) = (\tau_x^-(\omega_s, 0), \tau_y^+(\omega_s, 0)).$$

This $L_1(0,1)$ segment then connects to the blue $L_2(0,1)$ segment at a point where $L_1^e(0,1) = L_2^e(0,1)$ holds. The upper script "e" means again the end point of the segment and

$$L_1^e(0,1) = (\tau_x^+(\omega_e, 1), \tau_y^-(\omega_e, 1)) \text{ and } L_2^e(0,1) = (\tau_x^-(\omega_e, 1), \tau_y^+(\omega_e, 1)).$$

As n increases, the two segments are connected in the same way to construct the continuous stability switching curve. The eigenvalues are purely imaginary on this curve. The stability switching curves $L_1(0,n)$ and $L_2(0,n)$ divide the first quadrant of the (τ_1, τ_2) plane into two parts. One contains the origin and its every point can be reached from the origin via continuous curve not crossing the stability switching curve. At the points in this region the real parts of the eigenvalues are negative, so the system is locally asymptotically stable. The points of the complement of this region except the curves give points when the system is unstable. We again notice that the blue $L_2(0,0)$ segment crosses the horizontal axis at $\tau_x = \tau_x^0 (\simeq 0.824)$. This indicates that stability is preserved for $\tau_x < \tau_x^0$ and lost for $\tau_x > \tau_x^0$ when $\tau_y = 0$. In other words, this is the threshold value of τ_x in the case of one delay. We already discussed the one delay case and derived the threshold value in (19), which is equal to 0.824 under Assumption. So we could confirm this threshold value in two different ways. To investigate how stability changes in case of two positive delays, we perform numerical simulations.

[5] The $L_1(0,0)$ segment is located in the fourth quadrant so it is not illustrated.

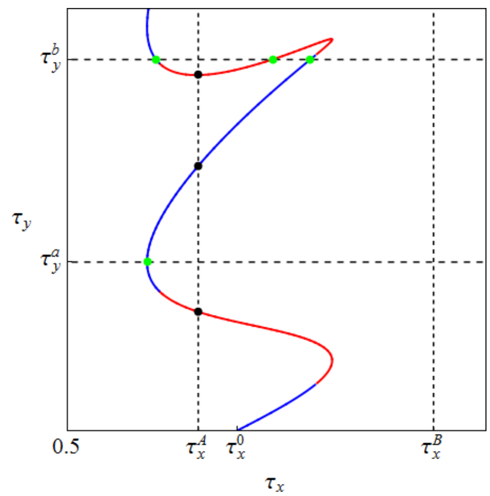
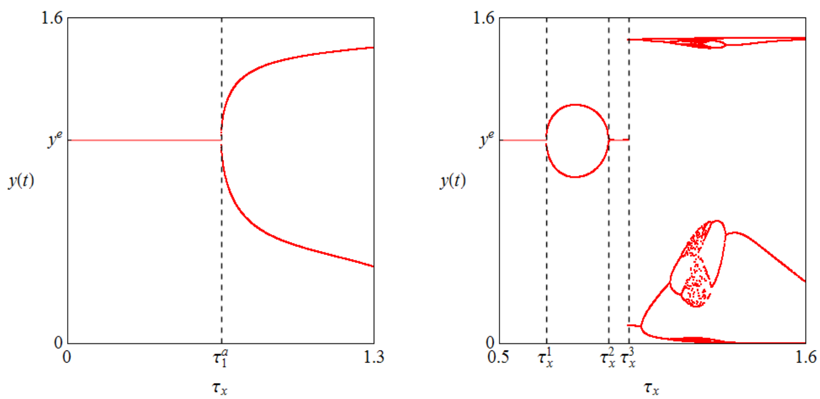


Figure 1. Stability switching curve

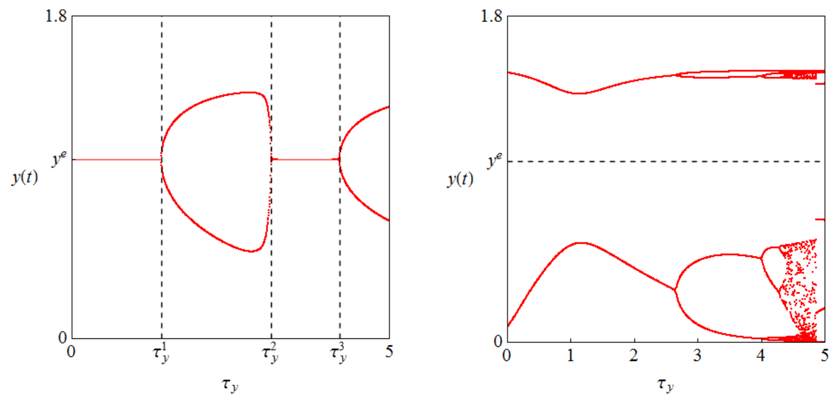
We now examine the effects caused by changing the value of τ_x, keeping the value of τ_y at some positive fixed value. In Figure 2(A) we increase the value of τ_x from 0 to 1.3 in 0.001 increments along the horizontal dotted line at $\tau_y = \tau_y^a (= 2)$. For each value of τ_x, the dynamic system runs for $0 \leq t \leq 1000$ and only the data for $950 \leq t \leq 1000$ are used to get rid of the transients. The local maxima and minima are plotted against each value of τ_x. If the bifurcation diagram has one point against τ_x, then the maximum point is identical with the minimum point, implying that the stationary state is locally stable. If it has two points, then a limit cycle having one maximum and one minimum emerges and if many points, then a limit cycle exhibits many ups and downs. The horizontal dotted line crosses the blue $L_2(0, 1)$ segment at $\tau_x^a \simeq 0.652$ denoted by the green dot in Figure 1. It is seen that the stationary state is locally asymptotically stable for $\tau_x < \tau_x^a$ and becomes unstable for $\tau_x > \tau_x^a$ as shown in Figure 2(A). When stability is lost at $\tau_x = \tau_x^a$, then the stationary point bifurcates to a limit cycle and does not regain stability for larger values of τ_x. We move to the second simulation in which the fixed value of τ_y is increased to $\tau_y^b = 4.4$ and the same procedure is repeated. As is seen in Figure 1, the horizontal dotted line at τ_y^b crosses the stability switching curve three times as denoted by three green dots. As described in Figure 2(B), both stability regain and stability loss occur in this example. In particular, stability is first lost at the first crossing point, the left most green dot denoted by $\tau_x^1 (\simeq 0.699)$ in Figure 2(B). The limit cycle emerges for τ_x in the interval $[\tau_x^1, \tau_x^2]$ where $\tau_x^2 (\simeq 0.892)$ corresponds to the middle green point. If the τ_x-value of the most right green point is denoted by $\tau_x^3 (\simeq 0.964)$, then stability is regained for $\tau_x \in [\tau_x^2, \tau_x^3]$. The stationary state again loses its stability for $\tau_x > \tau_x^3$. The bifurcation diagram shows that a trajectory gradually exhibits more complex dynamics through a quasi period-doubling process as τ_x become larger and then returns to the simple limit cycle through a quasi period-halving process as τ_x further increases.



(A) $\tau_y^a = 2$ (B) $\tau_y^b = 4.4$

Figure 2. Bifurcation diagrams with respect to τ_x, given τ_y

We now turn attention to the delay effect caused by changing values of τ_y, keeping the value of τ_x at some positive value. We also perform two simulations. In Figure 1, τ_x is fixed at $\tau_x^A = 0.75$ and τ_y increases along the vertical dotted line crossing the stability switching curve three times denoted by three black dotes. Although they are not labelled on the vertical axis to avoid confusion in Figure 1, we denote their τ_y-values by $\tau_y^1(\simeq 1.410)$, $\tau_y^2(\simeq 3.138)$ and $\tau_y^3(\simeq 4.215)$ with ascending order in Figure 3(A). The bifurcation diagram indicates that stability is lost at $\tau_y = \tau_y^1$, a limit cycle emerges for larger values than τ_y^1 and then regains stability at $\tau_y = \tau_y^2$. Stability is preserved for $\tau_y < \tau_y^3$ and is lost again for $\tau_y = \tau_y^3$. The diagram implies the similar dynamic cascade in which a process of stability loss, birth of limit cycle and regain of stability is repeated as the value of τ_y increases. In Figure 3(B) the fixed value of τ_x is increased to $\tau_x^B = 1.2$ from $\tau_x^A = 0.75$ and the value of τ_y increases along the vertical dotted line at $\tau_x^B = 1.2$ located to the right of the stability switching curve in Figure 1. Even for $\tau_y = 0$, the dynamic system is unstable as $\tau_x^B > \tau_x^0$ and generates a limit cycle with one maximum and one minimum. As the value of τ_y increases, the bifurcation diagram implies that the diameter of the limit cycle varies and then more complex dynamics appears through a period-doubling like process.



(A) $\tau_x^A = 0.75$ (B) $\tau_x^B = 1.2$

Figure 3. Bifurcation diagrams with respect to τ_y, given τ_x

5 Concluding Remarks

Delay dynamics of heterogeneous duopolies in a continuous-time framework was investigated. This is an extension of heterogeneous duopolies in a discrete-time framework considered by Elsadany and Matouk (2014). They investigated a heterogenous duopoly market in which one firm forms its expectation on the competitor's output only with delayed information and the other firm uses the current information. Following their spirit, we assumed that one firm has a delay in obtaining information about the competitor's output (i.e., information delay) in four different ways. Applying the recently developed method to examine stability of delay differential equations, we analytically demonstrated that the information delay is harmless to stability in the continuous-time duopoly model. In other words, no stability switch occurs regardless of the length of the delay. This shows a sharp difference from the main result of the discrete case that a delay in the competitor's production can have a stabilizing effect. Furthermore, we assumed that in addition to the information delay, the firm has a delay in implementing information about its own output (i.e., implementation delay) while the other firm can make its decision without any delays. It was demonstrated that the implementation delay matters. Constructing the stability switching curve on which stability is lost, we confirmed that stability can be switched to instability when the length of the delay takes over some threshold value. It is numerically verified that various dynamics ranging from simple to complex can emerge according to the values of the delays.

There are many directions for future studies. In this study, we assumed heterogenous duopolies in which only one firm has delayed information formulated in (15). One possible extension is given below in which duopolies are heterogenous and one firm has an implementation delay and the other has an information delay.

$$\dot{x}(t) = \alpha_x x(t) \left[a - c_x - 2bx(t - \tau_x) - by(t)\right],$$

$$\dot{y}(t) = \beta_y y(t) \left[a - c_y - bx(t - \tau_y) - 2by(t)\right].$$

However, the characteristic equation derived from this dynamic system is identical with equation (17). It may be interesting to consider duopoly models with different economic backgrounds generating exactly the same dynamics. The second extension concerns the transformation to a continuous-time model from a discrete-time model. We make two assumptions: one is replacing the discrete time difference $z(t+1) - z(t)$ with time derivative $\dot{z}(t)$ and the other is introducing continuous-time expectation formations concerning $y^e(t)$ instead of the discrete-time formation. The delay discrete-time formation was given as

$$y^e(t+1) = \eta y(t) + (1 - \eta) y(t-1)$$

which can be written as

$$\begin{aligned} y^e(t+1) &= \eta \left[y(t) - y(t-1)\right] + y(t-1) \\ &= \eta \dot{y}(t) + y(t-1). \end{aligned}$$

If the unit time difference is replace with delay τ, then we have the following form of a continuous-time system:

$$\dot{x}(t) = \alpha_x x(t) \left[a - c_x - 2bx(t) - b\left\{\eta \dot{y}(t) + y(t - \tau_x)\right\}\right],$$

$$\dot{y}(t) = \beta_y y(t) \left[a - c_y - bx(t) - 2by(t)\right].$$

This can be solved for $\dot{x}(t)$ and $\dot{y}(t)$ to derive a dynamic system of explicit delay differential equations. Further it is possible to introduce an implementation delay on the firm's own output although analysis will be more complicated. The third extension is to adopt the modelling method proposed by Berezowski (2001) and applied by Matsumoto and Szidarovszky (2014) in considering delay monopoly dynamics. The continuous-time system is described by the following delay equations,

$$\sigma_x \dot{x}(t) + x(t) = x(t - \tau_x) + \alpha_x x(t - \tau_x) \left[a - c_x - 2bx(t - \tau_x) - by^e(t)\right],$$

$$\sigma_y \dot{y}(t) + y(t) = y(t - \tau_x) + \beta_y y(t - \tau_y) \left[a - c_y - bx^e(t) - 2by(t - \tau_y)\right],$$

in which $\sigma_x \geq 0$ and $\sigma_y \geq 0$ denote the inertias (or frictions) in the production process. It should be noticed that this system can be reduced to system (1) if $\sigma_x = \sigma_y = 0$ and $\tau_x = \tau_y = 1$. So when σ_x and σ_y take smaller values, generated dynamics is similar to the discrete-time case (1). On the other hand, when σ_x and σ_y take larger values, it may generate different dynamics.

Appendix

In this Appendix, we will prove that equation

$$x^4 - 2b^2(\alpha^2 - 2\beta^2)x^2 - 8b^3\alpha\beta^2 x + b^4\alpha^2(\alpha^2 + 3\beta^2) = 0$$

has two distinct positive roots and no negative root exists. Let $x = \alpha b z$, then

$$\alpha^4 b^4 z^4 - 2b^2(\alpha^2 - 2\beta^2)\alpha^2 b^2 z^2 - 8b^3\alpha\beta^2\alpha b z + b^4\alpha^2(\alpha^2 + 3\beta^2) = 0$$

which can be simplifed as

$$h(z) = \alpha^2 z^4 - 2(\alpha^2 - 2\beta^2)z^2 - 8\beta^2 z + (\alpha^2 + 3\beta^2) = 0.$$

Notice that

$$h(0) = \alpha^2 + 3\beta^2 > 0, \ \lim_{z\to\pm\infty} h(z) = \infty \text{ and } h(1) = -\beta^2 < 0.$$

The derivative of $h(z)$ has the form

$$h'(z) = 4(z-1)\left(\alpha^2 z^2 + \alpha^2 z + 2\beta^2\right).$$

The sign of this derivative depends on the sign of the quadratic function

$$g(z) = \alpha^2 z^2 + \alpha^2 z + 2\beta^2$$

with the discriminant

$$D = 16\alpha^2(\alpha^2 - 8\beta^2).$$

If $\alpha^2 = 8\beta^2$, then it is positive except its negative vertex, so $h'(z) < 0$ as $z < 1$ and differes from the vertex, $h'(z) > 0$ as $z > 1$ and $h'(z) = 0$ as $z = 1$ or the vertex. Therefore $h(z)$ strictly decreases as $z < 1$ and strictly increases as $z > 1$. If $\alpha^2 < 8\beta^2$, then $h'(z) > 0$ as $z > 1$ and $h'(z) < 0$ as $z < 1$. Sience $h(0) > 0$ and both limits at $\pm\infty$ are positive there are two positive roots if $h(z)$ and no negative root exists. Assume next that $\alpha^2 > 8\beta^2$. Then the quadtratic function $g(z)$ has two negative roots:

$$z_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{\alpha^2 - 8\beta^2}}{2\alpha} \ (z_1 < z_2).$$

At these roots

$$z^2 = -z - \frac{2\beta^2}{\alpha^2},$$

so

$$
\begin{aligned}
h(z) &= \alpha^2\left(-z-\frac{2\beta^2}{\alpha^2}\right)^2 - 2(\alpha^2-2\beta^2)\left(-z-\frac{2\beta^2}{\alpha^2}\right) - 8\alpha^2\beta^2 z + (\alpha^2+3\beta^2) \\
&= \left(\alpha^2-8\beta^2\right)z + \frac{1}{\alpha^2}\left(\alpha^4+5\alpha^2\beta^2-4\beta^4\right).
\end{aligned}
$$

Since z_1 and z_2 are negative and their average value is $-1/2$, both are larger than -1. Since $\alpha^2 > 8\beta^2$,

$$\left(\alpha^2-8\beta^2\right)z > 8\beta^2-\alpha^2.$$

Hence

$$
\begin{aligned}
h(z) &> 8\beta^2-\alpha^2+\frac{1}{\alpha^2}\left(\alpha^4+5\alpha^2\beta^2-4\beta^4\right) \\
&> 8\beta^2+\frac{\beta^2}{\alpha^2}\left(5\alpha^2-4\beta^4\right) \\
&> 8\beta^2+\frac{\beta^2}{\alpha^2}\left(40\beta^2-4\beta^4\right) > 0.
\end{aligned}
$$

Since at both negative starting points the function is negative, $h(z)$ decreases from the ∞ limit as $z \to \infty$ to $h(z_1) > 0$, then increases to $h(z_2) > 0$ and decreases again until $h(1) < 0$ and then increases and tends to ∞ as $z \to \infty$. Since $h(0) > 0$, there are no negative roots, only two distinct positive roots.

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