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Two Time Lags in the Public Sector: Macroeconomic Stability and Complex Behaviors

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Abstract

This study develops a macroeconomic model that considers two time lags in the public sector—a government expenditure lag and a tax collection lag—and examines the effects of these lags on local stability of the steady state. According to previous studies, a sufficiently large expenditure lag causes economic instability. However, we show that a tax collection lag can have a stabilizing effect on the steady state. In addition, we develop an analysis of global dynamics to demonstrate that an increase in a tax collection lag can yield complex behaviors.

JEL Classification: E12; E30; E62

Keywords: Keynesian macrodynamic model, fiscal policy lag, delay differential equations, stability analysis

1 Introduction

Recently, many studies have examined the effects of time lags on macroeconomic stability using traditional Keynesian models. For instance, Sportelli and Cesare (2005) introduce a tax collection lag into the dynamic IS-LM model developed by Schinasi (1981) and Sasakura (1994), which is a traditional Keynesian model, and examine the local and global dynamics of the system. The standard dynamic IS-LM model with no policy lag comprises three equations that represent the goods market, monetary market, and budget constraints of the consolidated government. These equations form an ODE (ordinary

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differential equations) system. The introduction of a time lag transforms this system from an ODE to a DDE (delay differential equations) system.[1]

Generally, models with a time lag can be categorized into two types: fixed lag model and distributed lag model. Fanti and Manfredi (2007) develop a dynamic IS-LM model with a distributed tax collection lag, whereas Sportelli and Cesare (2005) analyze the case of a fixed lag. Both these studies demonstrate that a time lag evidently causes complex behaviors, including chaos, and that a traditional fiscal policy is likely to be ineffective. Moreover, Matsumoto and Szidarovszky (2013) compare the case of a fixed lag with that of a distributed lag in tax collections. They demonstrate that a larger stable region can be established in the case of a fixed lag compared with a distributed lag.

Another type of traditional Keynesian macrodynamic model that incorporates a capital accumulation equation in place of the disequilibrium adjustment function of the monetary market, which is often termed the Kaldorian model, has been proposed. This model originated from Kaldor (1940) and its primary characteristic is found in the assumption of an S-shaped configuration of the investment function. Chang and Smyth (1971) reconstruct the Kaldorian model to form an ODE system. Asada and Yoshida (2001) introduce a fixed government expenditure lag into the model proposed by Chang and Smyth (1971) and show that an increase in the responsiveness of a fiscal policy could lead to economic instability.

Further, Gabisch and Lorenz (1989) propose a hybrid model of the standard dynamic IS-LM model and the Kaldorian model, which involves both functions of capital accumulation and disequilibrium adjustment in the monetary market. Cai (2005) and Neamţu, Opriş, and Chilarescu (2007) introduce a fixed capital accumulation lag and a fixed tax collection lag, respectively, into this hybrid model and comprehensively discuss the occurrence of a Hopf bifurcation.

Moreover, Zhou and Li (2009) and Sportelli, Cesare, and Binetti (2014) propose macrodynamic models with two fixed time lags. Zhou and Li (2009) develop Cai's (2005) model to include two capital accumulation lags. In addition, Sportelli, Cesare, and Binetti (2014) present a dynamic IS-LM model with two time lags in the public sector: a government expenditure lag and a tax collection lag. These studies demonstrate that the steady states fluctuate between stability and instability as a certain lag increases.

[1] Schinasi (1981) does not consider disequilibrium of the monetary market. Sasakura (1994) develops Schinasi's (1981) model by introducing a disequilibrium adjustment function of the monetary market. Sasakura's (1994) model is now used as a benchmark of the dynamic IS-LM model.

Unlike in Sportelli, Cesare, and Binetti (2014), this study uses the Kaldorian macrodynamic model to investigate the interaction of two time lags in the public sector. Therefore, our model can be considered as introducing a tax collection lag into Asada and Yoshida's (2001) model. We examine two cases where a fiscal policy is active and where it is passive. An active fiscal policy strongly responds to the national income, whereas a passive fiscal policy is less responsive to the national income. In addition, we perform a stability analysis employing a mathematical method developed by Gu, Niculescu, and Chen (2005). This method enables us to present an exact figure of a stability crossing curve—a curve that separates stable and unstable regions on a parameter plane. Few studies have employed this method for economic analysis.[2]

This study proceeds as follows: Section 2 presents a dynamic system that represents a model economy. Section 3 examines the local dynamics around the steady state. Subsequently, Section 4 examines the global dynamics. Section 5 presents our conclusion.

[2]We shall refer other Keynesian macrodynamic models that consider a time lag as follows. The time-to-build model developed by Kalecki (1935) is the basis of economic models with a fixed time lag. Szydlowski (2002, 2003) develops this model into models with economic growth. Moreover, Yoshida and Asada (2007) examine the effects of a lag in government expenditure (where they examine both distributed and fixed lags) using the so-called Keynes–Goodwin model. Further, Asada and Matsumoto (2014) introduce a distributed lag of monetary policy implementation into the Keynesian equilibrium model proposed by Asada (2010). Asada's (2010) model comprises a monetary policy rule and an expectation adjustment function. A fixed lag version of this model is proposed by Tsuduki (2015). Furthermore, Matsumoto and Szidarovszky (2014) develop a nonlinear multiplier-accelerator model with investment and consumption lags. Finally, Bellman and Cooke (1963) provide a helpful introductory textbook of delay differential equations (i.e., differential-difference equations).

2 The model

2.1 Dynamic system

The model economy comprises the following equations:

$$\dot{Y}(t) = \alpha[C(t) + I(t) + G(t) - Y(t)];\ \alpha > 0, \tag{1}$$

$$C(t) = c[Y(t) - T(t)] + \bar{C};\ 0 < c < 1;\ \bar{C} > 0, \tag{2}$$

$$T(t) = \tau Y(t - \theta_2) - \bar{T};\ 0 < \tau < 1;\ \bar{T} \geq 0, \tag{3}$$

$$I(t) = I(Y(t), K(t), r(t));\ I_Y > 0;\ I_K < 0;\ I_r < 0, \tag{4}$$

$$\dot{K}(t) = I(Y(t), K(t), r(t)), \tag{5}$$

$$G(t) = \beta[\bar{Y} - Y(t - \theta_1)] + \bar{G};\ \beta > 0;\ \bar{Y} > 0;\ \bar{G} > 0, \tag{6}$$

$$M(t)/P(t) = L(Y(t), r(t));\ L_Y > 0;\ L_r < 0, \tag{7}$$

$$M(t) = \gamma[\bar{Y} - Y(t)] + \bar{M};\ \gamma > 0;\ \bar{M} > 0, \tag{8}$$

$$P(t) = P(Y(t));\ P_Y > 0, \tag{9}$$

where Y = real national income (output); C = real private consumption; I = real private investment; G = real government expenditure; T = real income tax; K = real capital stock; M = nominal money supply; P = price level; r = nominal interest rate; α = adjustment speed of the goods market; c = marginal propensity to consume; $\bar{C}$ = base consumption; τ = marginal tax rate; $\bar{T}$ = real subsidy; β = responsiveness of government expenditure to national income (i.e., activeness level of the fiscal policy); $\bar{Y}$ = target level of real national income; $\bar{G}$ = target level of real government expenditure; γ = responsiveness of nominal money supply to national income (i.e., activeness level of the monetary policy); $\bar{M}$ = target level of nominal money supply; t = time; θ_1 = government expenditure lag; and θ_2 = tax collection lag.

Equations (1) and (2) represent a disequilibrium adjustment function of the goods market and a consumption function, respectively. Equation (3) is a tax collection function that represents income tax T as a function of past national income $Y(t - \theta_2)$. It may be more general to formulate T as a function not only of a past income but also of the present income denoted by $Y(t)$. However, this change does not affect the nature of our argument; hence, we simply assume that T is a function only of $Y(t - \theta_2)$. Equations (4) and (5) represent an investment function and a capital accumulation function, respectively. For simplicity, we assume that capital depreciation does not exist. Equation (6) represents a fiscal policy reaction function with a government expenditure lag. Equation (7) represents the monetary market equilibrium condition, where the left-hand side denotes real money balance and the right-hand side denotes a demand function for money. In this study, we ensure that the adjustment of the monetary market is rapid, and therefore, the balance of demand and supply of this market is always maintained. Equation (8) represents a monetary policy reaction function. Finally, Equation (9) represents an aggregate supply function, by which the price level is determined.

In the case of no tax collection lag (i.e., $\theta_2 = 0$), the system compounded from Equations (1)–(9) essentially becomes similar to that of Asada and Yoshida (2001). However, the existence of a positive θ_2 significantly complicates the dynamic property of the system, thereby resulting in a major change in the economic implication of time lags.

2.2 Summarizing the equations

In this section, we summarize Equations (1)–(9) in a two-dimensional dynamic system. Substituting Equations (8) and (9) into Equation (7) and solving for r, we obtain

$$r(t) = r(Y(t)), \tag{10}$$

where $r_Y = -(\gamma P + P_Y M + P^2 L_Y)/P^2 L_r > 0$.

Substituting Equation (3) into Equation (2) and substituting Equation (10) into Equation (4), we obtain

$$C(t) = cY(t) - c\tau Y(t - \theta_2) + \bar{C} + c\bar{T}, \tag{11}$$

$$I(t) = I(Y(t), K(t), r(Y(t))). \tag{12}$$

Finally, substituting Equations (6), (11), and (12) into Equation (1) and substituting Equation (12) into Equation (5) yields the following system of differential equations with two time lags:

$$\begin{aligned} \dot{Y}(t) &= \alpha[I(Y(t), K(t), r(Y(t))) - (1-c)Y(t) - \beta Y(t-\theta_1) - c\tau Y(t-\theta_2) \\ &\qquad + \bar{C} + c\bar{T} + \beta\bar{Y} + \bar{G}], \\ \dot{K}(t) &= I(Y(t), K(t), r(Y(t))). \end{aligned} \tag{13}$$

2.3 Linearization

To analyze the local dynamics of System (13), we linearize the system around the steady state (Y^*, K^*) and obtain

$$\begin{aligned} \dot{\hat{Y}}(t) &= \alpha[\{A_1 - (1-c)\}\hat{Y}(t) - \beta\hat{Y}(t-\theta_1) - c\tau\hat{Y}(t-\theta_2) + I_K\hat{K}(t)], \\ \dot{\hat{K}}(t) &= A_1\hat{Y}(t) + I_K\hat{K}(t), \end{aligned} \tag{14}$$

where $\hat{Y}(t) = Y(t) - Y^*$, $\hat{K}(t) = K(t) - K^*$, and $A_1 = I_Y + I_r r_Y$. By necessity, the coefficients of these equations are evaluated at the steady state.

Assuming the exponential functions $\hat{Y}(t) = C_1 e^{\lambda t}$ and $\hat{K}(t) = C_2 e^{\lambda t}$ (where C_1 and C_2 are arbitrary constants, and λ denotes the eigenvalue) as the solutions of the above system and substituting these into System (14), we obtain

$$\begin{bmatrix} \lambda - \alpha\{A_1 - (1-c)\} + \alpha\beta e^{-\theta_1\lambda} + \alpha c\tau e^{-\theta_2\lambda} & -\alpha I_K \\ -A_1 & \lambda - I_K \end{bmatrix} \begin{bmatrix} \hat{Y}(t) \\ \hat{K}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For non-trivial solutions to exist for this system, the determinant of the left-hand side matrix, denoted by $\Delta(\lambda)$, must equal zero; i.e.,

$$\begin{aligned} \Delta(\lambda) &= \lambda^2 - [I_K + \alpha\{A_1 - (1-c)\}]\lambda - \alpha(1-c)I_K \\ &\quad + \alpha\beta(\lambda - I_K)e^{-\theta_1\lambda} + \alpha c\tau(\lambda - I_K)e^{-\theta_2\lambda} = 0 \\ &= p_0(\lambda) + p_1(\lambda)e^{-\theta_1\lambda} + p_2(\lambda)e^{-\theta_2\lambda} = 0, \end{aligned} \tag{15}$$

where

$$\begin{aligned} p_0(\lambda) &= \lambda^2 + b_1\lambda + b_2, \\ b_1 &= -[I_K + \alpha\{A_1 - (1-c)\}], \\ b_2 &= -\alpha(1-c)I_K, \\ p_1(\lambda) &= \alpha\beta(\lambda - I_K), \\ p_2(\lambda) &= \alpha c\tau(\lambda - I_K). \end{aligned}$$

Equation (15) is a characteristic equation of System (14). The significant feature of this equation is the existence of the exponential terms ($e^{-\theta_1\lambda}$ and $e^{-\theta_2\lambda}$).

First, we examine the case with no time lags. When $\theta_1 = \theta_2 = 0$, Equation (15) can be rewritten as follows:

$$\Delta(\lambda) = \lambda^2 + (b_1 + \alpha(\beta + c\tau))\lambda + b_2 - \alpha I_K(\beta + c\tau) = 0, \tag{16}$$

which is an ordinary quadratic equation of λ.

Thus, we can state that if $b_1 + \alpha(\beta + c\tau) > 0$ (i.e., the coefficient of λ from Equation 16 is positive), the real parts of the roots of Equation (16) are negative.[3] In contrast, if $b_1+\alpha(\beta+c\tau) < 0$, then the real parts of the roots are positive. Therefore, if $b_1+\alpha(\beta+c\tau) > 0$, the steady state is locally stable, and if $b_1 + \alpha(\beta + c\tau) < 0$, it is unstable.

In the discussion below, we assume the following condition:

Assumption 2.1 $b_1 + \alpha(\beta + c\tau) > 0$.

This assumption implies that if a lag does not exist in the public sector, an economy is stable. Under this assumption, we analyze the effects of the lags (θ_1, θ_2) on local stability.

3 Local dynamics

The following analysis is performed based on the technique developed by Gu, Niculescu, and Chen (2005).

3.1 Preconditions

First, to apply the technique of Gu, Niculescu, and Chen (2005), some preconditions should be checked. According to their study, Equation (15) should satisfy the following conditions:

(I) $\deg(p_0(\lambda)) \geq \max\{\deg(p_1(\lambda)), \deg(p_2(\lambda))\}$;

(II) $\Delta(0) \neq 0$;

(III) a solution common to all three polynomials $p_0(\lambda) = 0$, $p_1(\lambda) = 0$, and $p_2(\lambda) = 0$ does not exist;

(IV) $\lim_{\lambda\to\infty}(|p_1(\lambda)/p_0(\lambda)| + |p_2(\lambda)/p_0(\lambda)|) < 1$.

In our system, Condition (I) is satisfied by $2 > \max\{1, 1\}$. Condition (II) is also satisfied by $\Delta(0) = \alpha I_K[-(1 - c) - \beta - c\tau] > 0$. Concerning Condition (III), we can check as follows: substituting I_K into $p_1(\lambda)$ and $p_2(\lambda)$, we obtain $p_1(I_K) = p_2(I_K) = 0$.

[3] See Chapter 18 in Gandolfo (2010) for details of the relationship between the roots and coefficients of a quadric equation.

However, $p_0(I_K) = -\alpha A_1 I_K \neq 0$. Hence, Condition (III) is satisfied. Finally, Condition (IV) is satisfied by $\lim_{\lambda\to\infty}(|p_1(\lambda)/p_0(\lambda)| + |p_2(\lambda)/p_0(\lambda)|) = 0$.

Now, we examine the effects of lags (θ_1, θ_2) on the stability of the steady state. The analysis proceeds as follows:

(1) We characterize the points at which the local dynamics can change, i.e., the points at which the pure imaginary roots appear.[4] These points are referred to as the crossing points.

(2) We depict the sets of the crossing points (which we refer to as the crossing curves) on the θ_1-θ_2 plane by using numerical simulation.

(3) We reveal the directions of changes in the signs of the real parts that occur when lags (θ_1, θ_2) cross the crossing curves.

3.2 Crossing points

Dividing Equation (15) by $p_0(\lambda)$, we obtain

$$1 + a_1(\lambda)e^{-\theta_1\lambda} + a_2(\lambda)e^{-\theta_2\lambda} = 0, \tag{17}$$

where

$$a_1(\lambda) = \frac{p_1(\lambda)}{p_0(\lambda)} = \frac{\alpha\beta(\lambda - I_K)}{\lambda^2 + b_1\lambda + b_2}, \tag{18}$$

$$a_2(\lambda) = \frac{p_2(\lambda)}{p_0(\lambda)} = \frac{\alpha c\tau(\lambda - I_K)}{\lambda^2 + b_1\lambda + b_2}. \tag{19}$$

Moreover, we denote a pure imaginary root as $\lambda = vi$ (where $v =$ imaginary part $\neq 0$ and $i = \sqrt{-1}$). Then, the values of v that satisfy Equation (17) can be characterized by the following lemma:

Lemma 3.1 (Gu, Niculescu, and Chen 2005, Proposition 3.1) For each v satisfying $p_0(vi) \neq 0$, $\lambda = vi$ is a solution of $\Delta(\lambda) = 0$ for some $(\theta_1, \theta_2) \in \mathbb{R}^2_+$ if and only if

$$|a_1(iv)| + |a_2(iv)| \geq 1, \tag{20}$$

$$-1 \leq |a_1(iv)| - |a_2(iv)| \leq 1. \tag{21}$$

[4] It is ensured from precondition (III) that a zero real root cannot be a root.

We denote the set of $v > 0$ that satisfy conditions (20) and (21) as Ω, which is termed as the crossing set.[5] For any given $v \in \Omega$, the sets (θ_1, θ_2) satisfying Equation (17) (each of which corresponds to a crossing point) must satisfy the following relationships (Figure 1).

$$\mp\delta_1 = \arg(a_1(iv)e^{-iv\theta_1}) + 2m\pi; \quad m = 0, 1, 2, \cdots, \tag{22}$$
$$\pm\delta_2 = \arg(a_2(iv)e^{-iv\theta_2}) + 2n\pi; \quad n = 0, 1, 2, \cdots, \tag{23}$$

where $\delta_1, \delta_2 \in [0, \pi]$.

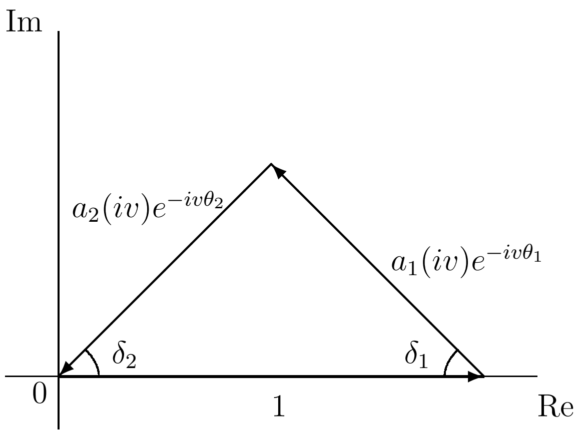

Figure 1: Triangle formed by 1, $|a_1(iv)|$, and $|a_2(iv)|$ on the complex plane

Incidentally, on the complex plane, a multiplication of amplitudes becomes a sum of parts; therefore, we obtain

$$\arg(a_1(iv)e^{-iv\theta_1}) = \arg(a_1(iv)) - v\theta_1, \tag{24}$$
$$\arg(a_2(iv)e^{-iv\theta_2}) = \arg(a_2(iv)) - v\theta_2. \tag{25}$$

Figure 1 also demonstrates that the following relationships hold:

$$\arg(a_1(iv)) = \tan^{-1}\left(\frac{\mathrm{Im}(a_1(iv))}{\mathrm{Re}(a_1(iv))}\right), \tag{26}$$
$$\arg(a_2(iv)) = \tan^{-1}\left(\frac{\mathrm{Im}(a_2(iv))}{\mathrm{Re}(a_2(iv))}\right). \tag{27}$$

[5]Pure imaginary roots are always conjugated. Therefore, we can assume $v > 0$ without a loss of generality.

Moreover, after some manipulation, Equations (18) and (19) derive the following expression:

$$\frac{\mathrm{Im}(a_1(iv))}{\mathrm{Re}(a_1(iv))} = \frac{\mathrm{Im}(a_2(iv))}{\mathrm{Re}(a_2(iv))} = \frac{b_1 v I_K + v(b_2 - v^2)}{b_1 v^2 - I_K(b_2 - v^2)}. \tag{28}$$

Thus, using Equations (24)–(28), Equations (22) and (23) can be rewritten as follows:

$$\theta_1 = \frac{\tan^{-1}(\frac{b_1 v I_K + v(b_2-v^2)}{b_1 v^2 - I_K(b_2-v^2)}) \pm \delta_1 + 2m\pi}{v}, \tag{29}$$

$$\theta_2 = \frac{\tan^{-1}(\frac{b_1 v I_K + v(b_2-v^2)}{b_1 v^2 - I_K(b_2-v^2)}) \mp \delta_2 + 2n\pi}{v}, \tag{30}$$

where the interior angles of the triangle denoted by δ_1 and δ_2 are given by the cosine theorem as follows:

$$\begin{aligned}
\delta_1 &= \cos^{-1}\left(\frac{1 + |a_1(iv)|^2 - |a_2(iv)|^2}{2|a_1(iv)|}\right) \\
&= \cos^{-1}\left(\frac{(b_2 - v^2)^2 + (b_1 v)^2 + (\alpha\beta I_K)^2 + (\alpha\beta v)^2 - (\alpha c\tau I_K)^2 - (\alpha c\tau v)^2}{2\sqrt{(\alpha\beta I_K)^2 + (\alpha\beta v)^2}\sqrt{(b_2 - v^2)^2 + (b_1 v)^2}}\right), \\
\delta_2 &= \cos^{-1}\left(\frac{1 + |a_2(iv)|^2 - |a_1(iv)|^2}{2|a_2(iv)|}\right) \\
&= \cos^{-1}\left(\frac{(b_2 - v^2)^2 + (b_1 v)^2 - (\alpha\beta I_K)^2 - (\alpha\beta v)^2 + (\alpha c\tau I_K)^2 + (\alpha c\tau v)^2}{2\sqrt{(\alpha c\tau I_K)^2 + (\alpha c\tau v)^2}\sqrt{(b_2 - v^2)^2 + (b_1 v)^2}}\right).
\end{aligned}$$

Equations (29) and (30) characterize the sets of the crossing points $(\theta_1, \theta_2) \in \mathbb{R}^2_+$. Depending on the signs of δ_1 and δ_2, we can define two types of crossing points, denoted by $L_1(m,n)$ and $L_2(m,n)$, as follows:

$$L_1(m,n): \quad \begin{aligned}
\theta_1 &= \frac{\tan^{-1}(\frac{b_1 v I_K + v(b_2-v^2)}{b_1 v^2 - I_K(b_2-v^2)}) + \delta_1 + 2m\pi}{v}, \\
\theta_2 &= \frac{\tan^{-1}(\frac{b_1 v I_K + v(b_2-v^2)}{b_1 v^2 - I_K(b_2-v^2)}) - \delta_2 + 2n\pi}{v},
\end{aligned}$$

$$L_2(m,n): \quad \begin{aligned}
\theta_1 &= \frac{\tan^{-1}(\frac{b_1 v I_K + v(b_2-v^2)}{b_1 v^2 - I_K(b_2-v^2)}) - \delta_1 + 2m\pi}{v}, \\
\theta_2 &= \frac{\tan^{-1}(\frac{b_1 v I_K + v(b_2-v^2)}{b_1 v^2 - I_K(b_2-v^2)}) + \delta_2 + 2n\pi}{v}.
\end{aligned}$$

In the next section, based on the study of Asada and Yoshida (2001), we illustrate the examples of $L_1(m,n)$ and $L_2(m,n)$ by using numerical simulations.

3.3 Numerical simulations

Following Asada and Yoshida's (2001) study, we assume the investment function as follows:

$$I(Y(t), K(t), r(Y(t))) = \frac{400}{1 + 12e^{-0.1(Y(t)-400)}} - 0.01\sqrt{Y(t)} - 0.5K(t) - 10\gamma(\sqrt{Y(t)} - \sqrt{\bar{Y}}).$$

Further, we set the parameter values as follows: $\alpha = 0.9$; $c = 0.625$; $\tau = 0.2$; $\bar{Y} = 400$; $\bar{C} + c\bar{T} + \bar{G} = 200$; and $\gamma = 8.6$. Under these specifications, the steady-state values of System (13) are given by $(Y^*, K^*) = (400, 61.138)$.

In the following discussion, we compare two cases: the case of an active fiscal policy with that of a passive fiscal policy.

3.3.1 Example 1

When $\beta = 4.1$, which represents a relatively active fiscal policy, the crossing set Ω is given by $v \in (3.6506, 3.8716)$ (Figure 2). For $v \in \Omega$, we can depict $L_1(m, n)$ and $L_2(m, n)$ as shown in Figure 3, where $m = 0, 1, 2$ and $n = 0, 1, 2$. The dotted curves represent $L_1(m, n)$, and the solid curves represent $L_2(m, n)$. These curves are referred to as the crossing curves.

3.3.2 Example 2

When $\beta = 0.1$, which represents a passive fiscal policy, the crossing set Ω is given by $v \in (0.2636, 0.5120)$ (Figure 4). In this case, the crossing curves $L_1(m, n)$ and $L_2(m, n)$ can be depicted for $v \in \Omega$ as shown in Figure 5. The starting points of both curves $L_1(m, n)$ and $L_2(m, n)$ (i.e., the points corresponding to $v = 0.2636$) are given by the upper connecting points of the circles.

Next, we examine how the real parts of the roots change when lags (θ_1, θ_2) cross the crossing curves.

3.4 Direction of crossing

We reveal the direction in which the roots cross the imaginary axis when the value of θ_1 increases. It is determined by the sign of $d\mathrm{Re}\lambda/d\theta_1|_{\lambda=iv\ (where\ v\in\Omega)}$. If $d\mathrm{Re}\lambda/d\theta_1|_{\lambda=iv} > 0$, the roots cross the imaginary axis from left to right with an increase in θ_1 (which indicates destabilization). In contrast, if $d\mathrm{Re}\lambda/d\theta_1|_{\lambda=iv} < 0$, the roots cross the imaginary axis

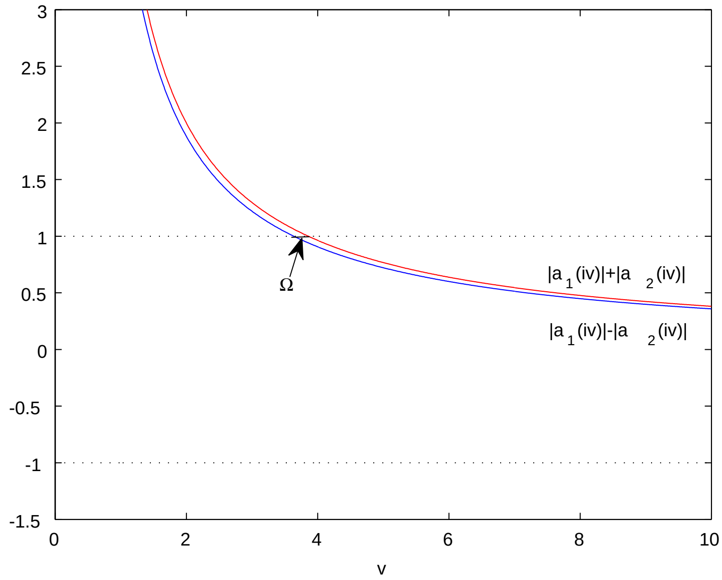

Figure 2: Crossing set Ω ($\beta = 4.1$)

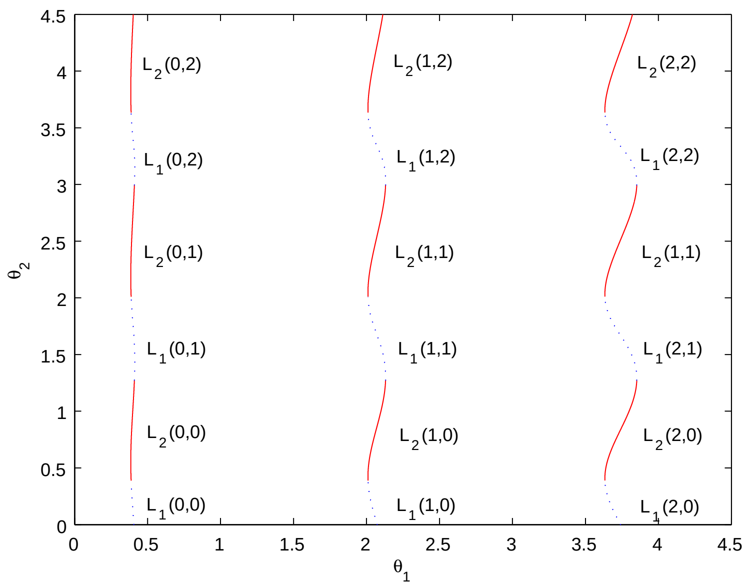

Figure 3: Crossing curves ($\beta = 4.1$)

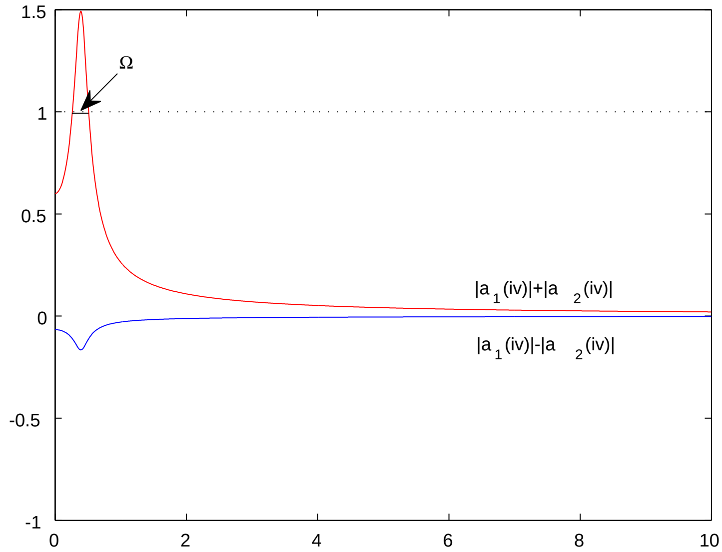

Figure 4: Crossing set Ω ($\beta = 0.1$)

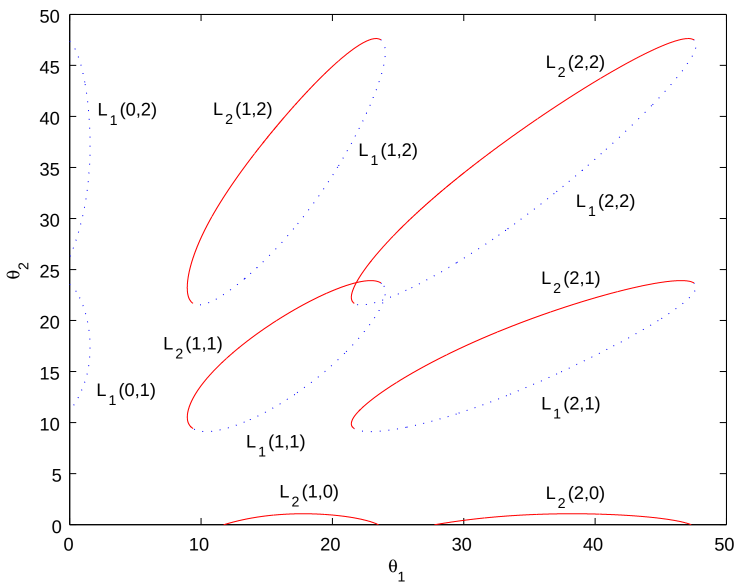

Figure 5: Crossing curves ($\beta = 0.1$)

from right to left with an increase in θ_1 (which indicates stabilization). For convenience of calculation, we observe the sign of $\text{Re}(d\lambda/d\theta_1)^{-1}|_{\lambda=iv}$ instead of that of $d\text{Re}\lambda/d\theta_1|_{\lambda=iv}$.

Differentiating Equation (17) with respect to θ_1, we obtain

$$\left[a_1'(\lambda)e^{-\theta_1\lambda} - a_1(\lambda)e^{-\theta_1\lambda}\theta_1 + a_2'(\lambda)e^{-\theta_2\lambda} - a_2(\lambda)e^{-\theta_2}\theta_2\right]\frac{d\lambda}{d\theta_1} = a_1(\lambda)e^{-\theta_1\lambda}\lambda,$$

or equivalently

$$\left(\frac{d\lambda}{d\theta_1}\right)^{-1} = \frac{a_1'(\lambda)e^{-\theta_1\lambda} + a_2'(\lambda)e^{-\theta_2\lambda} - a_2(\lambda)e^{-\theta_2\lambda}\theta_2}{a_1(\lambda)e^{-\theta_1\lambda}\lambda} - \frac{\theta_1}{\lambda}, \tag{31}$$

where

$$a_1'(\lambda) = \frac{\alpha\beta p_0(\lambda) - \alpha\beta(\lambda - I_K)(2\lambda + b_1)}{p_0(\lambda)^2},$$
$$a_2'(\lambda) = \frac{\alpha c\tau p_0(\lambda) - \alpha c\tau(\lambda - I_K)(2\lambda + b_1)}{p_0(\lambda)^2}.$$

3.4.1 Example 1

Suppose that $\beta = 4.1$. In this case, describing the real part of Equation (31) as a function of $v \in \Omega$, we can derive Figure 6, where the dotted curves are the functions evaluated on curve $L_1(m, n)$, and the solid curves are the functions evaluated on curve $L_2(m, n)$.

Figure 6 shows that $\text{Re}(d\lambda/d\theta_1)^{-1}|_{\lambda=iv} > 0$ holds for all cases in Figure 3. Therefore, at least two imaginary roots with positive real parts emerge when θ_1 crosses the crossing curves from left to right.

Now, a curve formed by connecting curves $L_j(0, n)$ (where $j = 1, 2$; $n = 0, 1, 2$) is termed as $m0$ (an enlarged representation of this curve is proposed in Figure 7). Then, we can make the following proposition:

Proposition 3.1 For lags (θ_1, θ_2) lying to the left of curve $m0$, the steady state is locally stable. However, for lags (θ_1, θ_2) lying to the right of curve $m0$, the steady state is unstable.

Based on this proposition, we can state the following: In the case of $\theta_1 < 0.384$, the steady state is locally stable irrespective of the value of θ_2, i.e., if a government expenditure lag is sufficiently small, a tax collection lag does not affect economic stability. Moreover, in the case of $\theta_1 \in (0.384, 0.412)$, the steady state fluctuates between stability and instability as θ_2 increases. Thus, a tax collection lag can contribute toward stabilizing an economy.

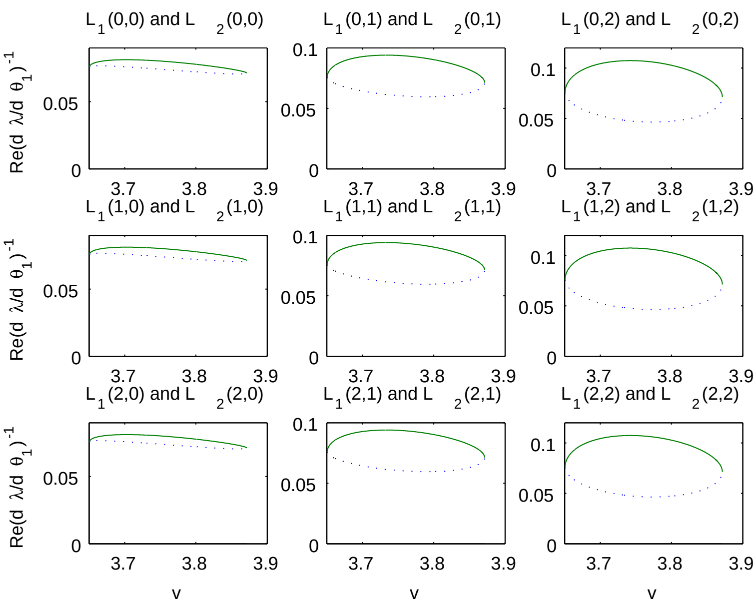

Figure 6: Direction of crossing ($\beta = 4.1$)

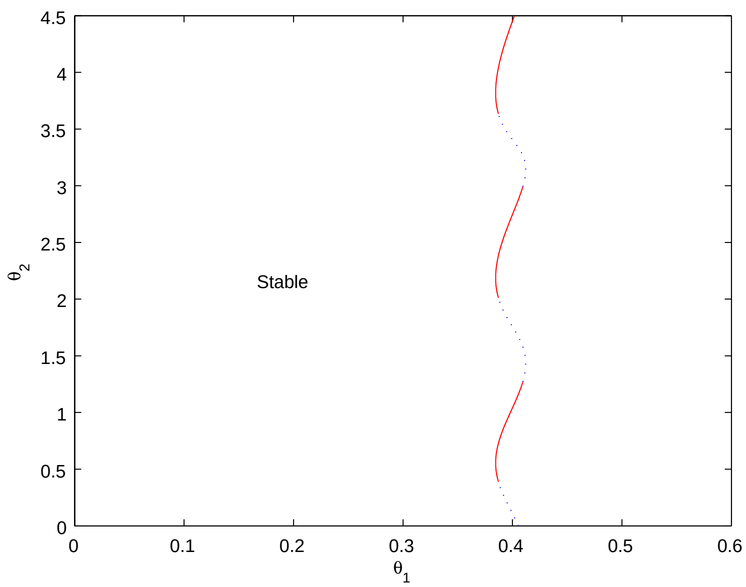

Figure 7: Curve $m0$

3.4.2 Example 2

When $\beta = 0.1$, the direction of crossing is determined by Figure 8. Figures 8 and 5 demonstrate the following proposition:

Proposition 3.2 In Figure 5, the regions enclosed within curves $L_1(m, n)$ and $L_2(m, n)$ (i.e., regions inside the circles) are unstable, whereas the others are stable.

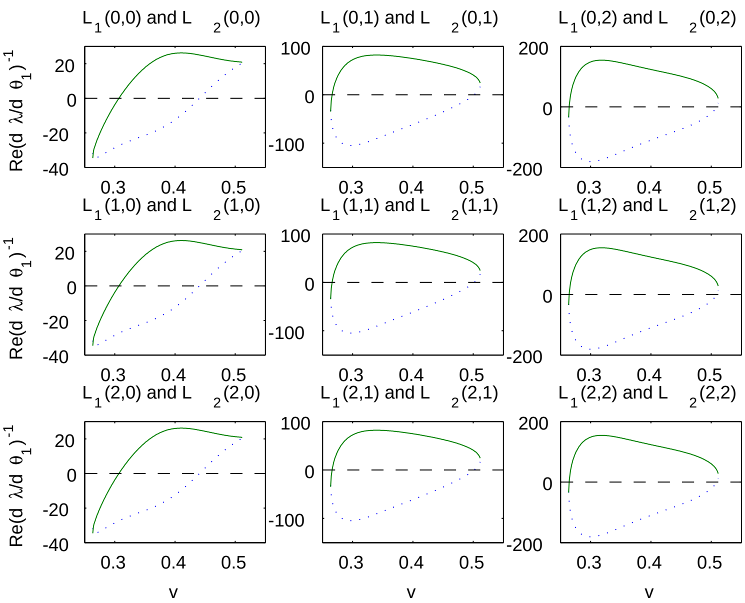

Figure 8: Direction of crossing ($\beta = 0.1$)

Comparing the case of a passive policy ($\beta = 0.1$) with that of an active policy ($\beta = 4.1$) within an economically meaningful region of (θ_1, θ_2) (i.e., θ_1 and θ_2 take values between 0 and 3), the former achieves a larger stable region. This suggests that an active policy stance may increase economic instability. This result cannot be derived from a model without a time lag. Furthermore, as indicated by Figure 5, in the case of $\beta = 0.1$, the steady state fluctuates between stability and instability with increases in not only θ_2 but also θ_1. Therefore, not only tax collection but also government expenditure lags can contribute towards stabilizing an economy.

4 Global dynamics

Thus far, we analyzed the local dynamics of System (13) with regard to the steady state. In this section, we illustrate phase diagrams to visually confirm the result established in the previous section and provide an example of global dynamics of the system.

We set the same parameter values as those in the previous section and assume that $\beta = 4.1$ (This section only examines the case with an active fiscal policy.). Further, we assume $\theta_1 = 0.4$. As indicated by Figure 7, if θ_2 is sufficiently small (i.e., $\theta_2 \leq 0.038$), the steady state is locally stable. However, if $\theta_2 > 0.038$, then the dynamics of the solutions change depending on the value of θ_2 (Figure 9). When $\theta_2 = 0.7$, a stable cycle exists and the solutions starting from the initial values of $(Y(0), K(0)) = (390, 55)$ converge to the cycle. When $\theta_2 = 1.7$, the steady state becomes locally stable again, and the solutions converge to the steady state. Moreover, when $\theta_2 = 3.6$, a strange-shaped attractor emerges, and the solutions exhibit chaotic behaviors.

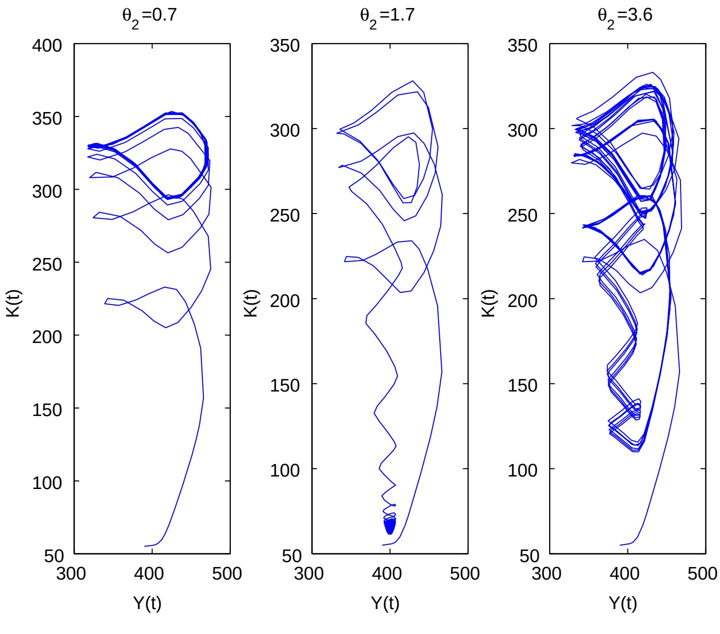

Figure 9: $\theta_1 = 0.4$

This example demonstrates that while an increase in a tax collection lag contributes toward local stability, it can cause globally complex behaviors.

5 Conclusion

In this study, we developed the Kaldorian model with government expenditure and tax collection lags and examined the effects of these lags on local stability by using numerical simulations. In addition, we also examined global dynamics.

As shown by Asada (1987), under a fiscal policy without a lag, the steady state is locally stable as long as the government is sufficiently active. However, Asada and Yoshida (2001) show that under a policy with a sufficiently large expenditure lag, the steady state becomes unstable even if the government is sufficiently active. This study showed that under a policy with government expenditure and tax collection lags, a policy lag can have a stabilizing effect on the steady state.

Under an active policy stance, if a government expenditure lag exceeds a certain threshold level, then the steady state becomes unstable. This result is similar to that in Asada and Yoshida's (2001) study. However, we further demonstrated that in the neighborhood of the threshold, certain positive values of a tax collection lag can achieve local stability. Therefore, a tax collection lag can contribute toward economic stability.

Similarly, under a passive policy stance, both tax collection as well as government expenditure lags can contribute to stabilizing an economy.

We also demonstrated that in an unstable parameter region, limit cycles and complex behaviors can emerge. Therefore, while an increase in a tax collection lag contributes toward local stability, it can cause globally complex behaviors.

According to Friedman (1948), policy lags are classified into three types: recognition, implementation, and diffusion lag. Unlike recognition and diffusion lag, implementation lag can be considered as adjustable to some extent. Therefore, this study suggests that an adjustment of the timing of policy implementation can be a means to achieve stabilization.

References

[1] Asada, Toichiro (1987). "Government finance and wealth effect in a Kaldorian cycle model," *Journal of Economics* 47(2), 143–166.

[2] Asada, Toichiro (2010). "Central banking and deflationary depression: a Japanese perspective," in M. Cappello and C. Rizzo (eds.) *Central Banking and Globalization*, Nova Science Publishers, New York, 91–114.

[3] Asada, Toichiro, and Akio Matsumoto (2014). "Monetary policy in a delay dynamic Keynesian model," *Annual of the Institute of Economic Research* 45, Chuo University, 143–160 (in Japanese).

[4] Asada, Toichiro, and Hiroyuki Yoshida (2001). "Stability, instability and complex behavior in macrodynamic models with policy lag," *Discrete Dynamics in Nature and Society* 5(4), 281–295.

[5] Bellman, Richard Ernest, and Kenneth L. Cooke (1963). *Differential-Difference Equations*, Academic Press, New York.

[6] Cai, Jianping (2005). "Hopf bifurcation in the IS-LM business cycle model with time delay," *Electronic Journal of Differential Equations* 2005(15), 1–6.

[7] Chang, Winston W., and David J. Smyth (1971). "The existence and persistence of cycles in a non-linear model: Kaldor's 1940 model re-examined," *Review of Economic Studies* 38(1), 37–44.

[8] Fanti, Luciano, and Piero Manfredi (2007). "Chaotic business cycles and fiscal policy: an IS-LM model with distributed tax collection lags," *Chaos, Solitons & Fractals* 32(2), 736–744.

[9] Friedman, Milton (1948). "A monetary and fiscal framework for economic stability," *American Economic Review* 38(3), 245–264.

[10] Gabisch, G., and H. W. Lorenz (1989). *Business Cycle Theory: A Survey of Methods and Concepts*, Springer-Verlag, New York.

[11] Gandolfo, Giancarlo (2010). *Economic Dynamics*, Fourth Edition, Springer-Verlag, Berlin Heidelberg.

[12] Gu, Keqin, Silviu-Iulian Niculescu, and Jie Chen (2005). "On stability crossing curves for general systems with two delays," *Journal of Mathematical Analysis and Applications* 311(1), 231–253.

[13] Kaldor, Nicholas (1940). "A model of the trade cycle," *Economic Journal* 50(197), 78–92.

[14] Kalecki, Michal (1935). "A macrodynamic theory of business cycles," *Econometrica, Journal of the Econometric Society* 3(3), 327–344.

[15] Matsumoto, Akio, and Ferenc Szidarovszky (2013). "Dynamics in delay IS-LM model with tax collections," *IERCU Discussion Paper* No.208, Chuo University.

[16] Matsumoto, Akio, and Ferenc Szidarovszky (2014). "Nonlinear multiplier-accelerator model with investment and consumption delays," *IERCU Discussion Paper* No.227, Chuo University.

[17] Neamţu, Mihaela, Dumitru Opriş, and Constantin Chilarescu (2007). "Hopf bifurcation in a dynamic IS-LM model with time delay," *Chaos, Solitons & Fractals* 34(2), 519–530.

[18] Sasakura, Kazuyuki (1994). "On the dynamic behavior of Schinasi's business cycle model," *Journal of Macroeconomics* 16(3), 423–444.

[19] Schinasi, Garry J. (1981). "A nonlinear dynamic model of short run fluctuations," *Review of Economic Studies* 48(4), 649–656.

[20] Sportelli, Mario, and Luigi De Cesare (2005). "A dynamic IS-LM model with delayed taxation revenues," *Chaos, Solitons & Fractals* 25(1), 233–244.

[21] Sportelli, Mario, Luigi De Cesare, and Maria T. Binetti (2014). "A dynamic IS-LM model with two time delays in the public sector," *Applied Mathematics and Computation* 243, 728–739.

[22] Szydlowski, Marek (2002). "Time-to-build in dynamics of economic models I: Kalecki's model," *Chaos, Solitons & Fractals* 14(5), 697–703.

[23] Szydlowski, Marek (2003). "Time to build in dynamics of economic models II: models of economic growth," *Chaos, Solitons & Fractals* 18(2), 355–364.

[24] Tsuduki, Eiji (2015). "Stability explanations in macrodynamic models with a policy lag: Old and New Keynesian economics," mimeo.

[25] Yoshida, Hiroyuki, and Toichiro Asada (2007). "Dynamic analysis of policy lag in a Keynes–Goodwin model: stability, instability, cycles and chaos," *Journal of Economic Behavior & Organization* 62(3), 441–469.

[26] Zhou, Lujun, and Yaqiong Li (2009). "A dynamic IS-LM business cycle model with two time delays in capital accumulation equation," *Journal of Computational and Applied Mathematics* 228(1), 182–187.