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Love Affairs Dynamics with One Delay in Losing Memory or Gaining Affection

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Abstract

A dynamic model of a love affair between two people is examined under different conditions. First the two-dimensional model is analyzed without time delays in the interaction of the lovers. Conditions are derived for the existence of a unique as well as for multiple steady states. The nonzero steady states are always stable and the stability of the zero steady state depends on model parameters. Then a delay is assumed in the mutual-reaction process called the Gaining-affection process. Similarly to the no-delay case, the nonzero steady states are always stable. The zero steady state is either always stable or always unstable or it is stable for small delays and at a certain threshold stability is lost in which case the steady state bifurcates to a limit cycle. When delay is introduced to the self-reaction process called the Losing-memory process, then the asymptotic behavior of the steady state becomes more complex. The stability of the nonzero steady state is lost at a certain value of the delay and bifurcates to a limit cycle, while the stability of the zero steady state depends on model parameters and there is the possibility of multiple stability switches with stability losses and regains. All stability conditions and stability switches are derived analytically, which are also verified and illustrated by using computer simulation.

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1 Introduction

The dynamics of love affairs has been modeled in various ways since Strogatz (1988) has proposed a 2D system of linear differential equations to describe the time evolution of a love affair between two individuals called Romeo and Juliet. Strogatz's purpose was to teach harmonic oscillations by applying a topic that is already on the minds of many college students: the time evolution of a love affair between two people. The study on the love affair dynamics after Strogatz aims to explain dynamic processes of love stories in our life in a formal theoretical framework. On the one hand, real-life observations tell us that love-stories frequently develop very regularly and stay at a plateau of love affair for a long time. Reconstructing the Strogatz model with linear or nonlinear behavioral functions and secure individuals, Rinaldi (1998a, 1998b) shows that one of the model's properties concerning the dynamics of the love affair is a smoothly increasing feeling tending toward a positive stationary point. On the other hand, another real-life observations indicate that love stories often arrive at a fluctuating regime including chaotic motions. Rinaldi (1998c) models the dynamics of the real love affair between Petrarch, a poet of the 14th century, and Laura, a beautiful married lady, with three differential equations and shows the appearance of cyclical pattern ranging from ecstasy to despair. Sprott (2004) applies a 4D system of nonlinear differential equations involving Romeo, Juliet and Romeo's mistress, Guinevere and derive chaotic love regime. Introducing information delays into the Strogatz model, Liao and Ran (2007) find that the stable steady state is destabilized for a delay larger than a threshold value and then bifurcates to a limit cycle via a Hopf bifurcation when Romeo is secure and Juliet is non-secure. Son and Park (2011) investigate the effect of delay on the love dynamics and confirm a cascade of period-doubling bifurcations to chaos analytically as well as numerically. Usually a delay is believed to possess a destabilizing effect in a sense that a longer delay destabilizes a system which is otherwise stable. Bielczyk et al. (2012) reveal the stabilizing effect of the delay by showing that a unstable steady state without time delay can gain stability for certain range of delays.

In this study we follow the Liao-Ran version of the Romeo-Juliet model to investigate how the delay and nonlinearities affect love dynamics. One important issue that Liao and Ran (2011) do not examine is to investigate time evolution in the case of multiple steady state. As is seen shortly, nonlinear behavioral functions can be a source of multiple steady state. However only the unique steady state case has been considered. Our first goal is to investigate dynamics in the multiple case. The second issue we take up concerns the romantic style of Rome and Juliet. There are four specifications of the romantic style for each individual, "eager beaver", "narcissistic nerd", "cautious (or secure) lover" and "hermit."[1] The majority of the population is represented by a cautious or secure lover who loves to be loved (alternatively, hates to be hated) and gradually loses the emotion to the partner when the partner leaves or dies. In

[1] See Strogatz (1994) for more precise specification.

spite of this, most studies confine attention to the case where Romeo and Juliet are heterogeneous, one is secure and the other is non-secure. Furthermore, it is demonstrated that the Romeo-Juliet model without delays does not exhibit cyclic dynamics when both are secure lovers. Our second goal is to investigate how the delay affects love dynamics between secure Romeo and Juliet. We have one more goal. The existing studies mainly focus on the delay that exists in love stimuli sent between Romeo and Juliet. We give a detailed analysis when there is a delay in Romeo's reaction to his own emotional state, referring to the basic study conducted by Bielczky et al. (2013).

This paper is organized as follows. Section 2 presents the basic love dynamic model that has no delays. Section 3 introduces one delay as in the Liao-Ran model and studies the dynamics of multiple steady states. Section 4 considers the case where Romeo loses the feeling for Juliet with a delay and Juliet without any delay. Section 5 concludes the paper.

2 Basic Model

Strogatz (1988) proposes a linear model of love affairs dynamics and Rinaldi (1998a, 1998b) extends it to a more general model in which three aspects of love dynamics, *oblivion*, *return* and *instinct*, are taken into account. If $x(t)$ denotes Romeo's emotions for Juliet at time t while $y(t)$ denotes Juliet's feeling to Romeo at time t, then the rates of change of Romeo's love and Juliet's love are assumed to be composed of three terms,

$$\dot{x}(t) = O_x(x(t)) + R_x(y(t)) + I_x$$

$$\dot{y}(t) = O_y(y(t)) + R_y(x(t)) + I_y$$

where O_z, R_z and I_z for $z = x, y$ are specified as follows. First, O_z gives rise to a loss of interest in the partner and describes the losing-memory process that characterizes decay of love at disappearance of the partner. Second, R_z is a source of interest and describes the reaction of individual z to the partner's love in the gaining-affection process. Lastly, I_z is also a source of interest and describes the reaction of individual z to the partner's appeal reflecting physical, financial, educational, intellectual properties. We adopt the following forms of these reaction functions:

Assumption 1: $O_x(x) = -\alpha_x x,\ \alpha_x > 0$ and $O_y(y) = -\alpha_y y,\ \alpha_y > 0$.

Assumption 2: $R_x(y) = \beta_x \tanh(y)$ and $R_y(x) = \beta_y \tanh(x)$.

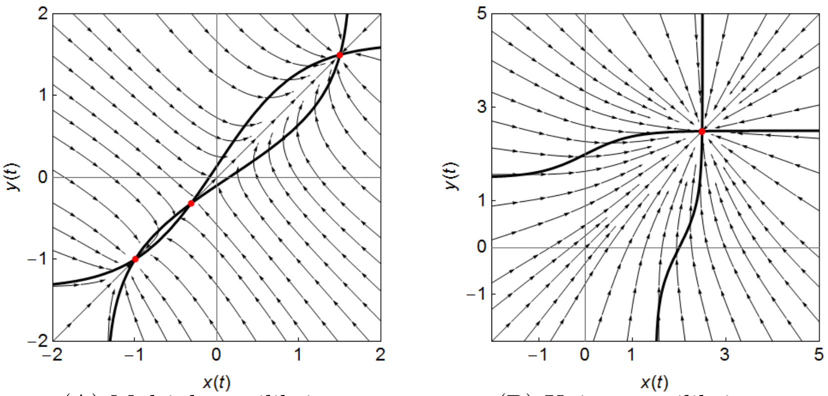
Assumption 3: $I_x = \gamma_x A_y,\ A_y > 0$ and $I_y = \gamma_y A_x,\ A_x > 0$.

Assumption 1 confines attention to the case where the memory vanishes exponentially. In Assumption 2, the hyperbolic function is positive, increasing, concave and bounded from above for positive values and is negative, increasing, convex and bounded from below for negative values. If $\beta_z > 0$. then the feeling of individual z is encouraged by the partner and such an individual is called *secure*. On the other hand, if $\beta_z < 0$, it is discouraged and the individual is though to be *non-secure*. Assumption 3 implies that individuals have time-invariant positive appeal. α_z is called the forgetting parameter while β_z and γ_z are the reaction coefficients of the love and appeal.

Under these assumptions, our basic model is
++

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t) + \beta_x \tanh[y(t)] + \gamma_x A_y, \\ \dot{y}(t) &= \beta_y \tanh[x(t)] - \alpha_y y(t) + \gamma_y A_x. \end{aligned} \tag{1}$$

Two numerical examples are given and the directions of the trajectories are indicated by arrows. In Figure 1(A) with $\alpha_x = \alpha_y = 1$, $\beta_x = \beta_y = 3/2$, $\gamma_x = \gamma_y = 1$ and $A_x = A_y = 1/7$, the isoclines, $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$, intersect at three points denoted by red dots. The middle one is unstable (a saddle) while the one with positive coordinates and the other with negative coordinates are stable nodes. In Figure 1(B) with $\alpha_x = \alpha_y = 1$, $\beta_x = \beta_y = 1/2$, $\gamma_x = \gamma_y = 1$ and $A_x = A_y = 2$, the steady state is unique and stable. As will be seen below, stability of system (1) is rather robust.



(A) Multiple equilibria (B) Unique equilibrium

Figure 1. Orbits of system (1)

Assumption 3 affects the location of a steady state but does not affect dynamic properties. Since we confine our attention to dynamics of the state variables in this study, we, only for a sake of analytical simplicity, replace Assumption 3 with the following:

Assumption 3': $A_x = A_y = 0$.

The steady state of (1) satisfies $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$. Solving $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$ for y yields two functions,

$$y = \tanh^{-1}\left(\frac{\alpha_x}{\beta_x}x\right) \text{ and } y = \frac{\beta_y}{\alpha_y}\tanh(x). \tag{2}$$

Let us denote the right hand side of two equations as $u(x)$ and $v(x)$, respectively. The steady state value of x, denoted as x^*, solves

$$u(x) = v(x) \tag{3}$$

and the steady state value of y, denoted ass y^*, is determined as

$$y^* = u(x^*) \text{ or } y^* = v(x^*). \tag{4}$$

We then have the following result where the proofs of this and further results are given in the Appendix:

Theorem 1 *A zero solution (x_0^*, y_0^*) of system (1) is a unique steady state if $\alpha_x\alpha_y \geq \beta_x\beta_y$ and there are three steady states (x_i^*, y_i^*) for $i = 0, 1, 2$ if $\beta_x\beta_y > \alpha_x\alpha_y$.*

Our next problem is to find out whether a solution of system (1) converges to the steady state or not. First the linearized version of system (1) is obtained by differentiating it in the neighborhood of the steady state,

$$\dot{x}(t) = -\alpha_x x(t) + \beta_x d_y^k y(t),$$

$$\dot{y}(t) = \beta_y d_x^k x(t) - \alpha_y y(t)$$

where

$$d_x^k = \left.\frac{d\tanh(x)}{dx}\right|_{x=x_k^*} \text{ and } d_y^k = \left.\frac{d\tanh(y)}{dy}\right|_{y=y_k^*}.$$

Notice that $d_x^0 = d_y^0 = 1$ at the zero steady state (x_0^*, y_0^*) and $d_x^k = d_y^k < 1$ at the nonzero steady states (x_k^*, y_k^*) for $k = 1, 2$.[2] The steady state is locally asymptotically stable if the roots of the characteristic equation

$$\det\begin{pmatrix} \lambda + \alpha_x & -\beta_x d_y^k \\ -\beta_y d_x^k & \lambda + \alpha_y \end{pmatrix} = 0$$

[2] By definition,

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and its derivative is

$$\frac{d}{dx}\tanh(x) = \left(\frac{2}{e^x + e^{-x}}\right)^2 \leq 1.$$

It is clear that equality holds if $x = 0$. If $e^x = a$ for $x \neq 0$, then

$$e^x + e^{-x} = a + \frac{1}{a} > 2$$

implying

$$\frac{2}{e^x + e^{-x}} < 1$$

Hence the strict inequalty holds if $x \neq 0$.

or

$$\lambda^2 + (\alpha_x + \alpha_y)\lambda + (\alpha_x\alpha_y - \beta_x\beta_y d_x^k d_y^k) = 0$$

have negative real parts. It is now well-known, as a special case of the Routh-Hurwitz stability criterion, that the roots have negative real parts if the following inequality conditions hold,

$$\alpha_x + \alpha_y > 0 \text{ and } \alpha_x\alpha_y - \beta_x\beta_y d_x^k d_y^k > 0. \tag{5}$$

The first inequality always holds by assumption. Thus for the stability of the steady state, we need to check only the second inequality. The local stability results are summarized as follows:

Theorem 2 *The zero steady state (x_0^*, y_0^*) is*

(1) *a saddle point if $\beta_x\beta_y > \alpha_x\alpha_y$,*

(2) *a stable node if $\alpha_x\alpha_y > \beta_x\beta_y > 0$*

and in the case of $\alpha_x\alpha_y > 0 > \beta_x\beta_y$, it is

(3) *a stable node if $(\alpha_x - \alpha_y)^2 + 4\beta_x\beta_y \geq 0$,*

(4) *a stable focus if $(\alpha_x - \alpha_y)^2 + 4\beta_x\beta_y < 0$*

whereas the non-zero steady state (x_k^*, y_k^*) for $k = 1, 2$, is always a stable node.

3 Delay in the Gaining-Affection Process

Son and Park (2011) rise an important question on how an individual know the partner's romantic feeling. Observing a real situation in which the romantic interaction is communicated through various ways such as a talk, a phone call, an email, a letter and a rumor that "she loves you", they find that time is required for the romantic feelings of an individual to transfer to his/her partner. One delay $\tau_x > 0$ is introduced into the gaining-affection process of Juliet in system (1),[3]

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t) + \beta_x \tanh[y(t)], \\ \dot{y}(t) &= \beta_y \tanh[x(t-\tau_x)] - \alpha_y y(t). \end{aligned} \tag{6}$$

Notice that the steady states (x_k^*, y_k^*) for $k = 0, 1, 2$ of the non-delay model are also the steady states of the delay model. The characteristic equation is obtained from the linearized version of system (6)

$$\lambda^2 + (\alpha_x + \alpha_y)\lambda + \alpha_x\alpha_y - \beta_x\beta_y d_x^k d_y^k e^{-\lambda\tau_x} = 0. \tag{7}$$

[3] Liao and Ran (2007) further assume that Romeo also reacts to the delayed Juliet feeling $y(t-\tau_y)$ with $\tau_x \neq \tau_y$. Son and Park (2011) consider the special case where both individuals have the same delay $\tau_x = \tau_y$ in the gaining-affection processes. The dynamic results obtained in those studies are essentially the same as the one to be obtained in the following one delay model.

First the following result is shown:

Theorem 3 *All pure complex eigenvalues of equation (7) are simple.*

Suppose $\lambda = i\omega,\ \omega > 0$ is a root of (7) for some τ_x. Substituting it separates the characteristic equation into the real and imaginary parts,

$$-\omega^2 + \alpha_x\alpha_y - \beta_x\beta_y d_x^k d_y^k \cos\omega\tau_x = 0 \tag{8}$$

and

$$(\alpha_x + \alpha_y)\,\omega + \beta_x\beta_y d_x^k d_y^k \sin\omega\tau_x = 0. \tag{9}$$

Moving the constant terms to the right hand side and then adding the squares of the resultant equations yield a quartic equation

$$\omega^4 + \left(\alpha_x^2 + \alpha_y^2\right)\omega^2 + (\alpha_x\alpha_y)^2 - (\beta_x\beta_y d_x^k d_y^k)^2 = 0. \tag{10}$$

We first consider the stability of the nonzero steady states at which β_x and β_y have identical sign. In the proof of Theorem 1, it is shown that the second inequality condition in (5) holds. Thus all coefficients of equation (10) are positive, so there is no positive solution for ω^2. Therefore there is no stability switch and since they are stable at $\tau_x = 0$, they remain stable for all $\tau_x > 0$. We summarize the result:

Theorem 4 *The nonzero steady states of system (6) are stable for any* $\tau_x \geq 0$.

In Figure 2, we illustrate the basin of attraction of the nonzero steady states, (x_1^*, y_1^*) and (x_2^*, y_2^*), taking $\alpha_x = \alpha_y = 1,\ \beta_x = 3/2,\ \beta_y = 3/2$ and $\tau_x = 2$. Any trajectory starting at an initial point in the light red region coverages to the positive steady state (x_1^*, y_1^*) denoted by the yellow dot and the one starting in the light blue region converges to the negative steady state denoted by the red dot. The downward-slopoing dotted line is the boundary between the two basins when there is no delay, $\tau_x = 0$. Increasing the value of the delay clockwisely rotates the boundary line. Thus the stability region of (x_1^*, y_1^*) in the fourth quadrant is enlarged and the one in the second quadrant is contracted and the same changes, but in opposite direction, occur for the stability region of the steady state (x_2^*, y_2^*). Even if the delay exists in the gaining-affection process, any trajectory converges to the positive equilibrium as far as an initial point is in the first quadrant.

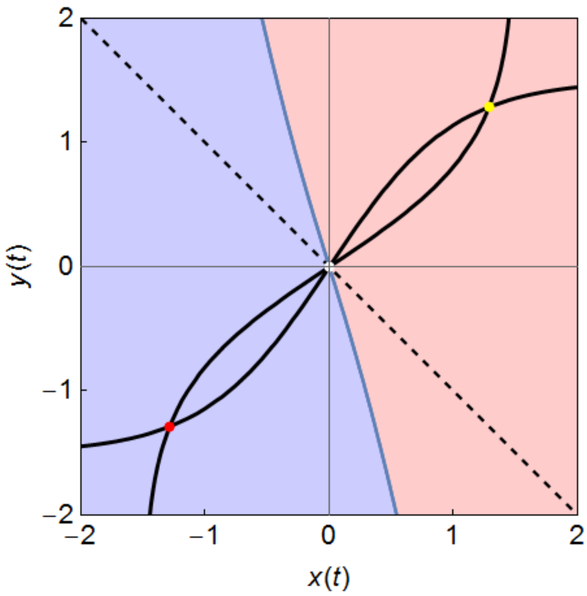


Figure 2. Basin of attraction for system (6)

Consider next the stability of the zero steady state. Solving (10) for ω^2 gives two solutions

$$(\omega_{\pm})^2 = \frac{-\left(\alpha_x^2 + \alpha_y^2\right) \pm \sqrt{D}}{2}$$

with

$$D = \left(\alpha_x^2 - \alpha_y^2\right)^2 + 4(\beta_x\beta_y)^2 > 0.$$

It is clear that $(\omega_-)^2 < 0$ and that $(\omega_+)^2$ is positive if $D > \left(\alpha_x^2 + \alpha_y^2\right)^2$ or

$$(\alpha_x\alpha_y)^2 < (\beta_x\beta_y)^2. \tag{11}$$

If there is no nonzero steady state with $\alpha_x\alpha_y > \beta_x\beta_y > 0$ or $0 > \beta_x\beta_y > -\alpha_x\alpha_y$, then inequality (11) is violated, so there is no positive solution for ω^2, and there is no stability switch in the case of $|\alpha_x\alpha_y| > \left|\beta_x\beta_y\right|$. Notice therefore that equation (11) might hold if, in addition to zero steady state, there are nonzero steady states or $0 > -\alpha_x\alpha_y > \beta_x\beta_y$. Substituting ω_+ into equation (8) and (9) and then looking for τ_x that satisfies both equation, we have from (8)

$$\tau_x^m = \frac{1}{\omega_+}\left[\cos^{-1}\left(\frac{\alpha_x\alpha_y - \omega_+^2}{\beta_x\beta_y}\right) + 2m\pi\right] \text{ for } m = 0, 1, 2, ... \tag{12}$$

and from and (9),

$$\tau_x^n = \frac{1}{\omega_+}\left[\sin^{-1}\left(-\frac{(\alpha_x + \alpha_y)\omega_+}{\beta_x\beta_y}\right) + 2n\pi\right] \text{ for } n = 0, 1, 2, ... \tag{13}$$

Needless to say, these two solutions are different expressions for the same value when $m = n$.

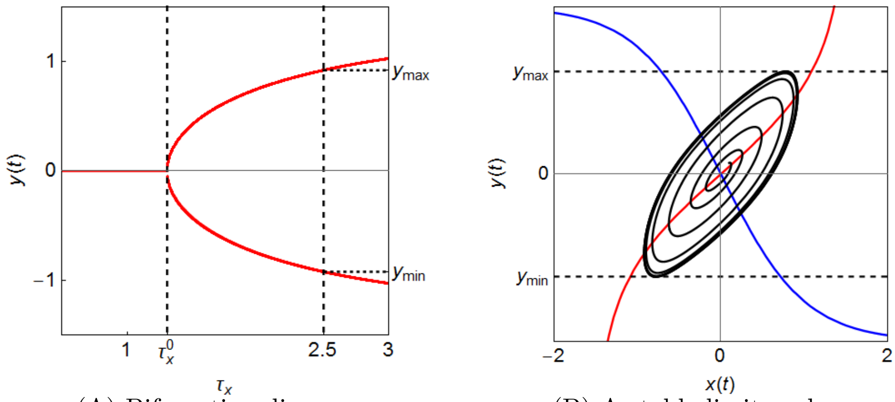
To confirm direction of stability switch, we let $\lambda = \lambda(\tau_x)$ and then determine the sign of the derivative of $\mathrm{Re}[\lambda(\tau_x)]$ at the point where $\lambda(\tau_x)$ is purely imaginary. Simple calculation shows that

$$sign\left[\mathrm{Re}\left(\left.\frac{d\lambda(\tau_x)}{d\tau_x}\right|_{\lambda=i\omega}\right)\right] = sign\left[\omega_+^2\left(2\omega_+^2 + \alpha_x^2 + \alpha_y^2\right)\right].$$

The sign of the right hand side is apparently positive, which implies that crossing of the imaginary axis is from left to right as τ_x increases. Thus, at smallest stability switch (i.e., τ_x^m with $m = 0$), stability is lost and cannot be regained later if steady state is stable without delay. If it is unstable without delay, then it remains unstable for all $\tau_x > 0$. Concerning the stability of the zero steady state, we summarize the following results:

Theorem 5 (1) *If* $|\alpha_x\alpha_y| \geq |\beta_x\beta_y|$, *then the zero steady state is stable regardless of the values of the delay;* (2) *If* $|\alpha_x\alpha_y| < |\beta_x\beta_y|$ *and it is unstable for* $\tau_x = 0$, *then the zero steady state is unstable for any* $\tau_x > 0$; (3) *If* $|\alpha_x\alpha_y| < |\beta_x\beta_y|$ *and it is stable for* $\tau_x = 0$, *then the zero steady state is stable for* $\tau_x < \tau_x^0$, *loses stability for* $\tau_x = \tau_x^0$ *and bifurcates to a limit cycle for* $\tau_x > \tau_x^0$ *where the threshold value* τ_x^0 *is obtained from* (*12*) *with* $m = 0$.

In Figure 3, parameter values are specified as $\alpha_x = \alpha_y = 1$, $\beta_x = 3/2$ and $\beta_y = -3/2$. Result (3) of Theorem 5 is numerically confirmed in Figure 3(A) in which the bifurcation diagram with respect to τ_x is illustrated. Bifurcation parameter τ_x increases from 1/2 to 3 with an increment 1/200. Against each value of τ_x, the local maximum and local maximum values of $y(t)$ for $t \in [750, 800]$ are plotted. The red line starting at $y_0^* = 0$ bifurcates to two branches at $\tau_x = \tau_x^0 (\simeq 1.305)$. If the bifurcation diagram has only one point against the value of τ_x, then the system is stable and converges to the steady state. If it has two points, then one maximum and one minimum of a trajectory is plotted, that is, a limit cycle emerges. The shape of the diagram indicates that the limit cycle become larger as τ_x increases. In Figure 3(A) the dotted vertical line at $\tau_x = 2.5$ intersects the diagram twice. In Figure 3(B) a trajectory starting in the neighborhood of the steady state is oscillatory and converges to a limit cycle that has the maximum and minimum points corresponding to the crossing points in Figure 3(A).



(A) Bifurcation diagram (B) A stable limit cycle

Figure 3. Stability switch and the birth of limit cycles

4 Delay in the Losing-Memory Process

There are millions of people who can't stop loving their partners and live their life in dream of yesterday since they have been left alone. The love motions of those people may be described by a simple one delay differential equation,

$$\dot{x}(t) = -\alpha_x x(t - \tau_x), \ \alpha_x > 0. \tag{14}$$

Taking an exponential solution $x(t) = e^{\lambda t}$ and substituting it into the above equation yield a characteristic equation

$$\lambda = -\alpha_x e^{-\lambda \tau}.$$

Substituting an pure imaginary solution $\lambda = i\omega$ and then separating the resultant equation into the real and imaginary parts, we have

$$\alpha_x \cos \omega\tau = 0 \text{ and } \sin\omega\tau = \frac{\omega}{\alpha_x}.$$

Solving these equations simultaneously determines the threshold value of the delay as

$$\tau_x^0 = \frac{\pi}{2\alpha_x}.$$

If equation (14) is thought to be a linear approximation of the nonlinear equation preventing the possibilities of unbounded passion

$$\dot{x}(t) = -\alpha_x \tanh[x(t - \tau_x)] + A_x$$

where a positive appeal (i.e., $A_x > 0$) leads to a positive steady state. Then the steady state is stable for $\tau < \tau_x^0$ and bifurcates to a cyclic orbit for $\tau > \tau_x^0$.

The memory does not vanish but keeps to oscillate around the steady state that approximates those happy hours. In this section we consider love dynamics of a Romeo who can live in memory and a Juliet who responds instantaneously. We replace Assumption 1 with the following Assumption 1',

Assumption 1': $O_x(x(t-\tau_x)) = -\alpha_x x(t-\tau_x),\ \alpha_x > 0$ and $O_y(y(t)) = -\alpha_y y(t),\ \alpha_y > 0$.

Dynamic system (6) is transformed to the following system with one delay in the losing-memory process,

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t-\tau_x) + \beta_x \tanh[y(t)], \\ \dot{y}(t) &= \beta_y \tanh[x(t)] - \alpha_y y(t). \end{aligned} \tag{15}$$

The characteristic equation is obtained from the linearized version of system (15)

$$\lambda^2 + \alpha_y \lambda - \beta_x \beta_y d_x^k d_y^k + \alpha_x(\lambda + \alpha_y) e^{-\lambda \tau_x} = 0. \tag{16}$$

Suppose again that the equation has a pure imaginary solution, $\lambda = i\omega,\ \omega > 0$. The characteristic equation can be broken down to the real and imaginary parts,

$$\alpha_x \alpha_y \cos \omega\tau + \alpha_x \omega \sin \omega\tau = \omega^2 + \beta_x \beta_y d_x^k d_y^k \tag{17}$$

and

$$-\alpha_x \alpha_y \sin \omega\tau + \alpha_x \omega \cos \omega\tau = -\alpha_y \omega. \tag{18}$$

Squaring both sides of each equation and adding them together yield a fourth-order equation with respect to ω,

$$\omega^4 + \left[(\alpha_y^2 - \alpha_x^2) + 2\beta_x\beta_y d_x^k d_y^k\right]\omega^2 + \left[(\beta_x\beta_y d_x^k d_y^k)^2 - (\alpha_x\alpha_y)^2\right] = 0.$$

Solving the equation with respect to ω^2 yields two solutions

$$(\omega_\pm)^2 = \frac{-\left[(\alpha_y^2 - \alpha_x^2) + 2\beta_x\beta_y d_x^k d_y^k\right] \pm \sqrt{D}}{2}$$

with

$$D = \left[(\alpha_y^2 - \alpha_x^2) + 2\beta_x\beta_y d_x^k d_y^k\right]^2 - 4\left[(\beta_x\beta_y d_x^k d_y^k)^2 - (\alpha_x\alpha_y)^2\right].$$

To simplify the analysis, we assume the following henceforth:

Assumption 4. $\alpha_x = \alpha_y = \alpha$

Then the solutions are simplified as

$$\omega_+^2 = \alpha^2 - \beta_x\beta_y d_x^k d_y^k \tag{19}$$

and

$$\omega_-^2 = -\left(\alpha^2 + \beta_x\beta_y d_x^k d_y^k\right). \tag{20}$$

Solving equations (17) and (18) simultaneously presents two solutions,

$$\cos\omega\tau = \frac{\beta_x\beta_y d_x^k d_y^k}{\alpha^2+\omega^2} \tag{21}$$

and

$$\sin\omega\tau = \frac{\omega\left(\omega^2+\beta_x\beta_y d_x^k d_y^k+\alpha^2\right)}{\alpha(\alpha^2+\omega^2)} \tag{22}$$

Before proceeding, we show the following:

Theorem 6 *If $\lambda = i\omega$ is a solution of equation (16), then it is simple.*

Concerning the direction of motion of the state variable $x(t)$ and $y(t)$ as τ is varied, we have the following result:

Theorem 7 *The stability of the steady state is lost and gained according to whether the following sign is positive or negative,*

$$sign\left[\mathrm{Re}\left(\frac{d\lambda(\tau_x)}{d\tau_x}\bigg|_{\lambda=i\omega}\right)\right] = \begin{cases} sign\left[2\alpha^2-\beta_x\beta_y d_x^k d_y^k\right] & if\ \omega=\omega_+, \\ sign\left[\beta_x\beta_y d_x^k d_y^k\right] & if\ \omega=\omega_-. \end{cases}$$

4.1 Stability of Nonzero Steady State

At any nonzero steady state it is already shown that $\alpha^2 > \beta_x\beta_y d_x^k d_y^k$. So $\omega_+^2 > 0$ while $\omega_-^2 < 0$ since $\beta_x\beta_y > 0$. Then both $\cos\omega\tau$ and $\sin\omega\tau$ are positive so two threshold values of τ_x are obtained, one from equation (21)

$$\tau_x^m = \frac{1}{\omega_1}\left[\cos^{-1}\left(\frac{\beta_x\beta_y d_x^k d_y^k}{\alpha^2+\omega_+^2}\right)+2m\pi\right] \text{ for } m=0,1,2,...$$

and the other from (22)

$$\tau_x^n = \frac{1}{\omega_1}\left[\sin^{-1}\left(\frac{2\alpha\omega_1}{\alpha^2+\omega_+^2}\right)+2n\pi\right] \text{ for } n=0,1,2,...$$

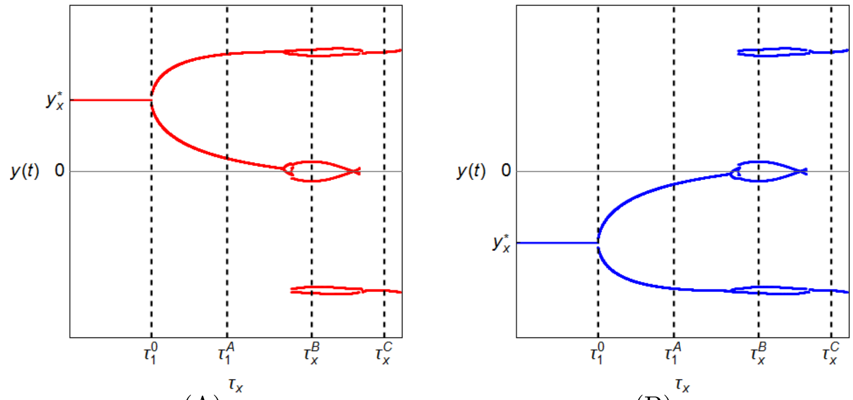
where, as pointed out above, $\tau_x^m = \tau_x^n$ for $m=n$ since these describe the same relation between the delay and the parameters. Due to Theorem 7, we have

$$\mathrm{Re}\left(\frac{d\lambda(\tau_x)}{d\tau_x}\bigg|_{\lambda=i\omega_+}\right) > 0.$$

Then we have the following results concerning the stability switch on the nonzero steady state

Theorem 8 *The nonzero steady state of system (15) is stable for $\tau_x < \tau_x^0$, loses stability for $\tau_x = \tau_x^0$ and bifurcates to a limit cycle for $\tau_x > \tau_x^0$.*

Figures 4 (A) and (B) illustrate bifurcation diagrams with respect to τ_x. The only difference between these diagrams is the selection of the initial functions for system (15) while any other values of the parameters are the same. Simulations for the red curve is performed in the following way. The value of τ_x is increased from 1.5 to 1.825. For each value of τ_x, the delay dynamics system (15) with initial functions $x_0(t) = 0.1\cos(t)$ and $y_0(t) = 0.2\cos(t)$ runs for $0 \leq t \leq 5000$ and data obtained for $t \leq 4950$ are discarded to get rid of transients. The local maximum and minimum from the remaining data of $y(t)$ are plotted against selected values of τ_x. The value of τ_x is increased with 1/500 and then the same procedure is repeated until the value of τ_x arrives at 1.825. The blue curve has initial functions $x_0(t) = -0.1\cos(t)$ and $y_0(t) = -0.2\cos(t)$. Simulation has been done in the same way.



(A) (B)

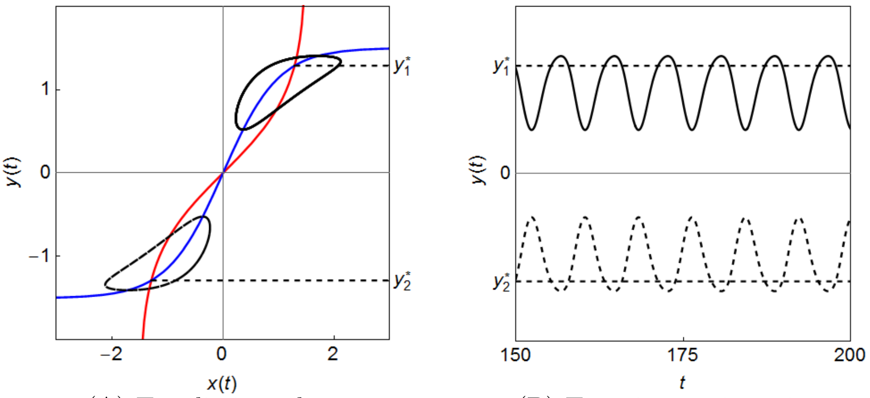
Figure 4. Bifurcation diagrams with different initial functions

Observing the bifurcation diagrams, we find that each diagram has four phases according to which different dynamics arises. To see what dynamics is born in each phase, we select three values of τ_x,

$$\tau_x^A = 1.68, \ \tau_x^B = 1.75 \text{ and } \tau_x^C = 1.81.$$

and then perform simulations to find dynamics in the (x, y) plane and in the $(t, y(t))$ plane. In the first phase where $\tau_x < \tau_x^0$ ($\simeq 1.617$), any trajectory converges to either $y_1^* > 0$ or $y_2^* < 0$ depending on the selection of the initial functions as each steady state is asymptotically stable. In the second phase where the diagrams have two brances and the vertical dotted line at $\tau_x = \tau_x^A$ intersects the blue diagram and the red diagram twice each. The steady state is destabilized as $\tau_x^A > \tau_x^0$. A trajectory starting in the neighborhood of the positive steady state converges to a small limit cycle surrounding the steady state. The same holds for a trajectory starting in the neighborhood of the negative steady state. The simulation results are plotted in Figures 5(A) and

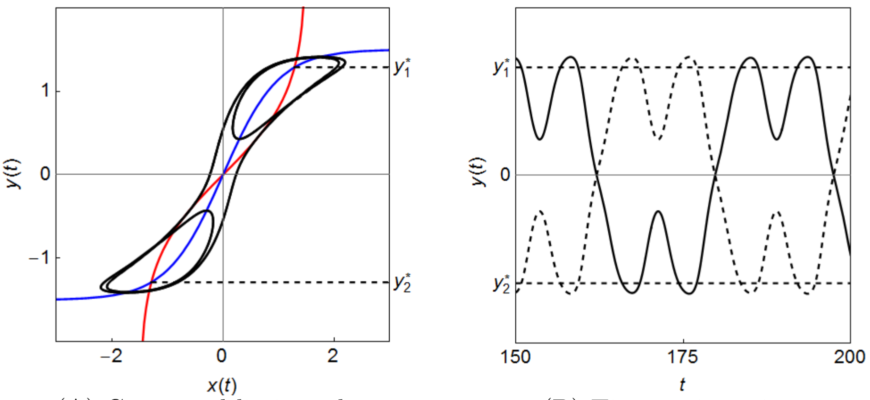
5(B).



(A) Two limit cycles (B) Two time trajectories

Figure 5. Dynamics with $\tau_x = \tau_x^A$

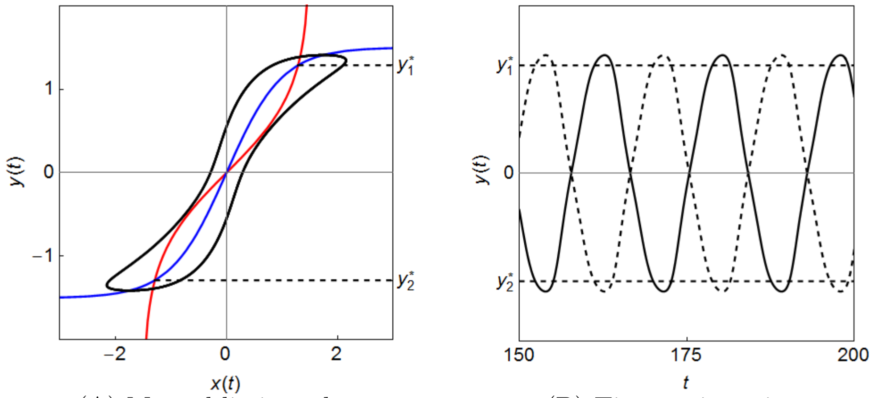
In the third phase, the diagram has six branches and the vertical dotted line at $\tau_x = \tau_x^B$ intersects the diagram six times. This implies two issues. One is that the two independent cycles are connected to form a large one cycle. Two cycles are included in the big one and each cycle has two extreme values leading to six extreme values. The other is that any trajectory converges to the same cyclic attractor regardless of the selection of the initial functions. Figure 6(B) indicates that a trajectory makes two small ups and downs around the positive steady state and moves down in the neighbourhood of the negative steady state within a large cycle. The real curve and dotted curve in Figure 6(B) behave exactly in the same way with some phase shift.



(A) Connected limit cycle (B) Time trajectories

Figure 6. Dynamics with $\tau_x = \tau_x^B$

In the fourth phase, the diagram has two brances and thus the number of intersection of the dotted vertical line at $\tau_x = \tau_x^C$ with the bifurcation diagram decreases to two. As seen in Figure 7(A), the two small cycles are completely merged with the big cycle having one maximum and one minimum. The big limit cycle surrounds the two nonzero steady states, y_1^* and y_2^*.



(A) Merged limit cycle (B) Time trajectories

Figure 7. Dynamics with $\tau_x = \tau_x^C$

4.2 Stability of Zero Steady State

To examine the stability switch of the zero steady state, we consider the three cases depending on the relative magnitude between $\alpha^2 = \alpha_x \alpha_y$ and $\beta_x \beta_y$.

(I) $\beta_x \beta_y \geq \alpha^2$

Under this inequality condition, equations (19) and (20) indicate $\omega_+^2 \leq 0$ and $\omega_-^2 < 0$. The characteristic equation does not have a solution such as $\lambda = i\omega,\ \omega > 0$ and thus the real parts of the eigenvalues do not change their signs if τ_x increases. Hence no stability switch occurs and the stability of the zero steady state is the same as without delay. Due to (1) of Theorem 2, the zero steady state is unstable (i.e., a saddle point) for $\tau_x = 0$, it remains unstable for any $\tau_x > 0$.

(II) $\alpha^2 > \beta_x \beta_y > -\alpha^2$

Due to (3) and (4) of Theorem 2, the zero steady state is stable for $\tau_x = 0$. Equations (19) and (20) with the inequality conditions leads to $\omega_+^2 > 0$ and $\omega_-^2 < 0$, meaning that $\lambda = i\omega_+, \omega_+ > 0$ can be a solution of the characteristic equation under Assumption 4. Due to Theorem 7, we have

$$\mathrm{Re}\left(\left. \frac{d\lambda(\tau_x)}{d\tau_x} \right|_{\lambda = i\omega_+} \right) > 0$$

This implies that the solution crosses the imaginary axis from left to right as τ_x increases. We now determine the threshold value of τ_x at which the real parts of the solutions change their signs. Returning to two equations in (21), we check that the right hand side of both equations are positive. There is a unique $\omega_+\tau_x$, $0 < \omega_+\tau_x < \pi/2$ for which both equations hold,

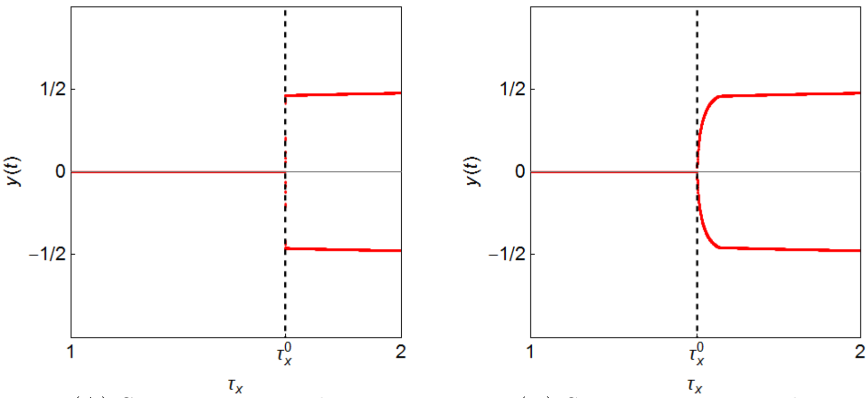
$$\tau_x^m = \frac{1}{\omega_+}\left[\cos^{-1}\left(\frac{\beta_x\beta_y}{\alpha^2+\omega_+^2}\right) + 2m\pi\right] \text{ for } m = 0, 1, 2, ...$$

and

$$\tau_x^n = \frac{1}{\omega_+}\left[\sin^{-1}\left(\frac{\omega_+(\omega_+^2+\beta_x\beta_y+\alpha^2)}{\alpha(\alpha^2+\omega_+^2)}\right) + 2n\pi\right] \text{ for } n = 0, 1, 2, ...$$

It is apparent that $\tau_x^m = \tau_x^n$ for $m = n$. It can be noticed that the zero steady state is asymptotically stable for $\tau_x < \tau_x^0$ and unstable for $\tau_x > \tau_x^0$. Thus τ_x^0 is the threshold value at which the stability switch occurs.

Numerical examples are given to confirm the analytical results. In Figure 8(A) $\alpha_x = \alpha_y = 1$ and $\beta_x = \beta_y = 1/2$ are assumed and both Romeo and Juliet are secure. Stability is lost at $\tau_x = \tau_x^0 \simeq 1.648$ and a limit cycle emerges for $\tau_x > \tau_x^0$. In Figure 8(B), Romeo is still secure but Juliet is non-secure as $\beta_x = 1/2$ and $\beta_y = -1/2$. Stability is lost at $\tau_x = \tau_x^0 \simeq 1.505$ and a limit cycle emerges for $\tau_x > \tau_x^0$. It is to be noticed that the romantic syle in these examples are different, however, evolution of the emotion exhibit essentially the same..



(A) Secure-secure style (B) Secure-nosecure style

Figure 8. Bifurcation diagrams with respect to τ_x

(III) $-\beta_x\beta_y > \alpha^2 > \beta_x\beta_y$

Multiple stability switches occur in this case. equations (19) and (20) indicate $\omega_+^2 \geq 0$ and $\omega_-^2 > 0$. It is to be noticed that (21) with (19) can be written as

$$\cos\omega_+\tau = \frac{\beta_x\beta_y}{\alpha^2+\omega_+^2} \text{ and } \sin\omega_+\tau = \frac{2\alpha\omega_+}{\alpha^2+\omega_+^2}$$

and (21) with (20) as

$$\cos\omega_{-}\tau = -1 \text{ and } \sin\omega_{-}\tau = 0.$$

So we have two different threshold values,

$$\tau_x^m = \frac{1}{\omega_+}\left[\cos^{-1}\left(\frac{\beta_x\beta_y}{\alpha^2+\omega_+^2}\right) + 2m\pi\right] \text{ for } m = 0, 1, 2, ...$$

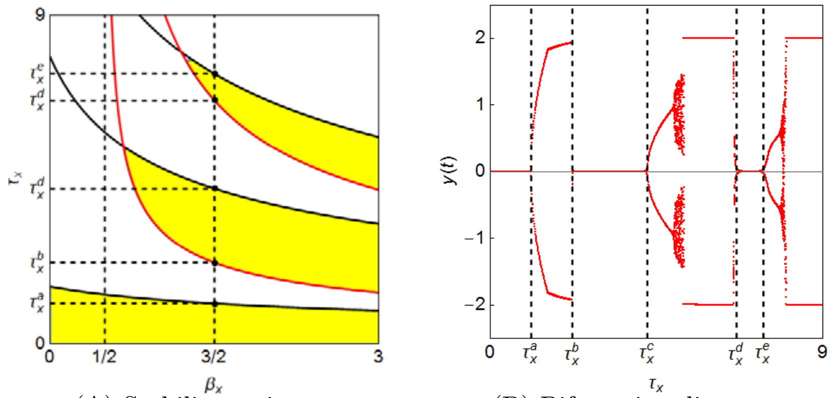
and

$$\tau_x^n = \frac{1}{\omega_-}(\pi + 2n\pi) \text{ for } n = 0, 1, 2, ...$$

Taking $\alpha_x = \alpha_y = 1$ and $\beta_y = -2$, we illustrate three τ_x^m curves for $m = 0, 1, 2$ in black and two τ_x^n curves for $n = 0, 1$ in red against values of $\beta_x \in [0, 3]$. All curves are downward-sloping and increasing the value of m (resp. n) shifts the black (resp. red) curve upward. The red curve is asymptotic to the dotted vertical line at $\beta_x = 1/2$ in Figure 9(A) since ω_- goes to infinity as β_x approaches $1/2$ from above. The steady state is asymptotically stable for (β_x, τ_x) in the yellow regions and unstable otherwise. If we fix the value of β_x at $3/2$ and increases the value of τ_x, then the dotted vertical line at $\beta_x = 3/2$ intersects the downward-sloping curves five times at

$$\tau_x^a \simeq 1.107,\ \tau_x^b \simeq 2.221,\ \tau_x^c \simeq 4.249,\ \tau_x^d \simeq 6.664 \text{ and } \tau_x^e \simeq 7.390.$$

The corresponding bifurcation diagram with respect to τ_x is illustrated in Figure 9(B). These figures illustrate multiple stability switching phenomenon from different points of view. Figure 9(B) indicates three Hopf bifurcation values in τ_x, $\tau_x^a < \tau_x^c < \tau_x^d$. The steady state is stable for $\tau_x = 0$ and remains stable for $\tau_x < \tau_x^a$. It loses stability at $\tau_x = \tau_x^a$ and bifurcates to a limit cycle for for $\tau_x > \tau_x^a$. As the value of τ_x increases further, the steady state repeatedly passes through stability loss and gain and then eventually stays to be unstable. So as far as Figure 9 concerns, the stability loss occurs three time and the stability gain twice for $\tau_x < 9$. Theorem 6 shows that the pure imaginary solutions are simple. Therefore at the crossing points with the stability switching curve only a pair of eigenvalues change the sign of their real part. Without delay the system is stable, all eigenvalues have negative real parts. So at the first crossing when stability is lost one pair of eigenvalues will have positive real part. If at the next crossing point stability might be regained, then the same pair of eigenvalues should change back the sign of their real part to negative, since there is no other pair with positive real parts. So all eigenvalues will have negative real parts again. In case if more than one pairs have positive real parts and the next crossing is when stability might regain, then only one pair changes back the sign of their real part to negative, the others will be still positive, so no stability regain occurs.



(A) Stability regions (B) Bifurcation diagram

Figure 9. Delay effect of τ_x

5 Concluding Remarks

In this paper the dynamic love affair model of Strogartz (1988) was reconsidered. First its nonlinear extension was introduced, the number of steady states was determined and the asymptotic behavior of its steady state was examined under different conditions. Conditions were derived for the existence of a unique and also for multiple steady states. First no time delay was assumed in the interaction of the lovers. In this no-delay case the nonzero steady states were always stable and conditions were derived for the stability of the zero steady state. Next a delay was assumed in the Gaining-affection process. The delay did not alter the stability of the nonzero steady states, the stability of the zero steady state was more complex. Depending on model parameter values it was either stable for all values of the delay, or always unstable, or stable for small values of the delay with stability loss at a certain threshold value of the delay. At this point the steady state bifurcated to a limit cycle. Then a delay was introduced into the Losing-memory process. The nonzero steady state was stable for small values of the delay, then stability was lost and the steady state bifurcated to a limit cycle. So this kind of delay had a destabilizing effect on the nonzero steady states. In examining the stability of the zero steady state we considered three cases depending on the relative magnitude of model parameters. In the first case the zero steady state was always unstable. In the second case stability was lost at a threshold value of the delay, and in the third case multiple stability switches could occur with repeated stability losses and regains. The stability of the steady states was analytically studied and the results were illustrated and verified by using computer simulation. In this paper we considered the cases of no or a single delay. It is a very interesting problem to see how the results of this paper change in the presence of multiple delays. This issue will be the subject of our next research project.

Appendix

Proof of Theorem 1

Proof. The zero steady state, $x_0^* = 0$ and $y_0^* = 0$, is clearly a solution of (3) and (4). Thus the two isoclines intersect at least once at the origin. We investigate whether such an intersection happens only once or not. To this end, we differentiate $u(x)$ and $v(x)$,

$$u'(x) = \frac{\dfrac{\alpha_x}{\beta_x}}{1 - \left(\dfrac{\alpha_x}{\beta_x}x\right)^2},\ u''(x) = \frac{2x\left(\dfrac{\alpha_x}{\beta_x}\right)^3}{\left[1 - \left(\dfrac{\alpha_x}{\beta_x}x\right)^2\right]^2}$$

and

$$v'(x) = \frac{\beta_y}{\alpha_y}\left(\frac{2}{e^x + e^{-x}}\right)^2,\ v''(x) = -\frac{8\beta_y}{\alpha_y}\frac{e^x - e^{-x}}{(e^x + e^{-x})^3}.$$

Although $\alpha_x > 0$ and $\alpha_y > 0$ by assumption, the signs of β_x and β_y are undetermined. We consider three cases, depending on the signs of β_x and β_y.

(i) Assume first that β_x and β_y have different signs. Then $u'(x)$ and $v'(x)$ also have different signs, so one is strictly increasing and the other is strictly decreasing. So $x_0^* = 0$ and $y_0^* = 0$ are the only steady state if $\alpha_x\alpha_y > 0 > \beta_x\beta_y$.

(ii) Assume next that β_x and β_y are both positive. Then

$$u(0) = 0,\ u\left(\frac{\beta_x}{\alpha_x}\right) = \infty,\ u\left(-\frac{\beta_x}{\alpha_x}\right) = -\infty,\ u'(x) > 0,\ u''(x)\begin{cases} > 0 \text{ if } x > 0, \\ < 0 \text{ if } x < 0 \end{cases}$$

and

$$v(0) = 0,\ v(\infty) = \frac{\beta_y}{\alpha_y},\ v(-\infty) = -\frac{\beta_y}{\alpha_y},\ v'(x) > 0,\ v''(x)\begin{cases} < 0 \text{ if } x > 0, \\ > 0 \text{ if } x < 0. \end{cases}$$

Furthermore

$$u'(0) = \frac{\alpha_x}{\beta_x} \text{ and } v'(0) = \frac{\beta_y}{\alpha_y}.$$

Only zero solution is possible if $u'(0) \geq v'(0)$, that is, if

$$\frac{\alpha_x}{\beta_x} \geq \frac{\beta_y}{\alpha_y} \text{ or } \alpha_x\alpha_y \geq \beta_x\beta_y.$$

If $\alpha_x\alpha_y < \beta_x\beta_y$, then there are two nonzero solutions in addition to the zero steady state: one in the positive region $(x_1^*, y_1^*) > 0$ due to the convexity of $u(x)$ and the concavity of $v(x)$ for positive x and the other in the negative region $(x_2^*, y_2^*) < 0$ due to the concavity of $u(x)$ and the convexity of $v(x)$ for negative x.

(iii) Assume finally that $\beta_x < 0$ and $\beta_y < 0$. Equation (3) remains same if β_x and β_y are replaced by $-\beta_x$ and $-\beta_y$, so previous case may apply for existence of nonzero solutions. ■

Proof of Theorem 2

Proof. We omit to prove the first four cases, (1), (2), (3) and (4). For the last case in which $\beta_x\beta_y > \alpha_x\alpha_y$, we consider two sub-cases depending of the signs of β_x and β_y.

(i) We first assume $\beta_x > 0$ and $\beta_y > 0$. At a non-zero solution $v'(x_k^*) < u'(x_k^*)$, that is,

$$\frac{\beta_y}{\alpha_y} d_x < \frac{\frac{\alpha_x}{\beta_x}}{1 - \left(\frac{\alpha_x}{\beta_x} x\right)^2}. \tag{A-1}$$

Since from the first equation in (2),

$$\frac{\alpha_x}{\beta_x} x = \tanh(y),$$

the right hand side of (A-1) is

$$\frac{\frac{\alpha_x}{\beta_x}}{1 - \left(\frac{e^y - e^{-y}}{e^y + e^{-y}}\right)^2} = \frac{\frac{\alpha_x}{\beta_x}}{\left(\frac{2}{e^y + e^{-y}}\right)^2} = \frac{\frac{\alpha_x}{\beta_x}}{d_y}.$$

So we have

$$\frac{\beta_y}{\alpha_y} d_x < \frac{\frac{\alpha_x}{\beta_x}}{d_y} \tag{A-2}$$

or

$$\alpha_x\alpha_y > \beta_x\beta_y d_x d_y. \tag{A-3}$$

(ii) If $\beta_x < 0$ and $\beta_y < 0$, then $v'(x_k^*) > u'(x_k^*)$ for $k = 1, 2$ at any nonzero solution, so inequality (A-1) has opposite direction, as well as inequality (A-2) has opposite direction and by multiplying it by $\alpha_y\beta_x d_y < 0$, equation (A-3) remains valid. ■

Proof of Theorem 3

Proof. If any eigenvalue is multiple, then it also solves the following equation obtained by differentiating the left hand side of equation (7),

$$2\lambda + (\alpha_x + \alpha_y) + \beta_x \beta_y d_x^k d_y^k e^{-\lambda \tau_x} \tau_x = 0. \tag{A-4}$$

From equation (7),

$$\beta_x \beta_y d_x^k d_y^k e^{-\lambda \tau_x} = \lambda^2 + (\alpha_x + \alpha_y)\lambda + \alpha_x \alpha_y$$

that is substituted into equation (A-4),

$$2\lambda + (\alpha_x + \alpha_y) + \lambda^2 \tau_x + (\alpha_x + \alpha_y)\lambda \tau_x + \alpha_x \alpha_y \tau_x = 0$$

or

$$\lambda^2 \tau_x + (2 + \alpha_x \tau_x + \alpha_y \tau_x)\lambda + (\alpha_x + \alpha_y + \alpha_x \alpha_y \tau_x) = 0.$$

This equation cannot have pure complex root since multiplier of λ is positive. ■

Proof of Theorem 6

Proof. The characteristic equation for $\alpha_x = \alpha_y = \alpha$ is simplified as

$$\lambda^2 + \alpha\lambda - \beta_x \beta_y d_x^k d_y^k + \alpha(\lambda + \alpha) e^{-\lambda \tau_x} = 0.$$

If λ is a multiple root, then it also satisfies equation,

$$2\lambda + \alpha + \alpha e^{-\lambda \tau_x} - \tau_x \alpha(\lambda + \alpha) e^{-\lambda \tau_x} = 0.$$

From the first equation

$$e^{-\lambda \tau_x} = -\lambda + \frac{\beta_x \beta_y d_x^k d_y^k}{\lambda + \alpha}$$

and by substituting it into the second equation, we have

$$2\lambda + \alpha + \left(-\lambda + \frac{\beta_x \beta_y d_x^k d_y^k}{\lambda + \alpha} \right) - \tau_x \left(-\lambda(\lambda + \alpha) + \beta_x \beta_y d_x^k d_y^k \right) = 0$$

which can be written as

$$\lambda^3 \tau_x + \lambda^2 (1 + 2\alpha\tau_x) + \lambda \left(2\alpha + \alpha^2 \tau_x - \beta_x \beta_y d_x^k d_y^k \tau_x \right) + \left(\alpha^2 + \beta_x \beta_y d_x^k d_y^k (1 - \alpha\tau_x) \right) = 0.$$

If $\lambda = i\omega$, then

$$\omega^2 = \frac{2\alpha + \alpha^2 \tau_x - \beta_x \beta_y d_x^k d_y^k \tau_x}{\tau_x} = \frac{\alpha^2 + \beta_x \beta_y d_x^k d_y^k (1 - \alpha\tau_x)}{1 + 2\alpha\tau_x}$$

This equation can be simplified as follows:

$$2\alpha + 2\tau_x (2\alpha^2 - \beta_x \beta_y d_x^k d_y^k) + \alpha \tau_x^2 \left(2\alpha^2 - \beta_x \beta_y d_x^k d_y^k \right) = 0.$$

If $\beta_x \beta_y \leq 0$, then the left hand side is positive, so no solution exists. If $\beta_x \beta_y > 0$, then $\omega_+^2 > 0$ if and only if $\alpha^2 > \beta_x \beta_y d_x^k d_y^k$. In this case the left hand side is positive again showing that no solution exists. ■

Proof of Theorem 7

Proof. Select τ_x as the bifurcation parameter and consider λ as the function of $\tau_x,\ \lambda = \lambda(\tau_x)$. Implicitly differentiating the characteristic equation with respect to τ_x gives

$$\left[2\lambda + \alpha + \alpha e^{-\lambda\tau_x} - \alpha\tau_x(\lambda+\alpha)e^{-\lambda\tau_x}\right]\frac{d\lambda}{d\tau_x} - \alpha\lambda(\lambda+\alpha)e^{-\lambda\tau_x} = 0$$

implying that

$$\begin{aligned}\frac{d\lambda}{d\tau_x} &= \frac{\alpha\lambda(\lambda+\alpha)e^{-\lambda\tau_x}}{2\lambda+\alpha+\alpha e^{-\lambda\tau_x}-\alpha\tau_x(\lambda+\alpha)e^{-\lambda\tau_x}}\\ &= \frac{-\lambda^4-2\lambda^3\alpha-\lambda^2\alpha^2+\beta_x\beta_y d_x^k d_y^k\lambda(\lambda+\alpha)}{2\lambda^2+\alpha\lambda+2\lambda\alpha+\alpha^2+(1-\tau_x\lambda-\tau_x\alpha)\left(-\lambda^2-\alpha\lambda+\beta_x\beta_y d_x^k d_y^k\right)}.\end{aligned}$$

Assume that $\lambda = i\omega$, then the numerator becomes

$$\left(-\omega^4+\omega^2\left(\alpha^2-\beta_x\beta_y d_x^k d_y^k\right)\right)+i\omega\left(2\omega^2\alpha+\beta_x\beta_y d_x^k d_y^k\alpha\right)$$

and the denominator is simplified as

$$-\omega^2(1+2\alpha\tau_x)+\left(\alpha^2+\beta_x\beta_y d_x^k d_y^k(1-\alpha\tau_x)\right)+i\left(-\tau_x\omega^3+\omega\left(2\alpha+\alpha^2\tau_x-\tau_x\beta_x\beta_y d_x^k d_y^k\right)\right).$$

Multiplying the numerator and the denominator by the complex conjugate of the denominator shows that $\mathrm{Re}[d\lambda/d\tau_x]$ has the same sign as

$$\omega^4+\omega^2\left(2\alpha^2\right)+\left[\alpha^4+2\alpha^2\beta_x\beta_y d_x^k d_y^k-\left(\beta_x\beta_y d_x^k d_y^k\right)^2\right].$$

At $\omega^2=\omega_+^2=\alpha^2-\beta_x\beta_y d_x^k d_y^k$, this expression becomes

$$2\alpha^2\left(2\alpha^2-\beta_x\beta_y d_x^k d_y^k\right)>0$$

showing that at the stability switch stability is lost or instability is retained. At $\omega^2=\omega_-^2=-\left(\alpha^2+\beta_x\beta_y d_x^k d_y^k\right)$, $\mathrm{Re}[d\lambda/d\tau_x]$ has the same sign as

$$2\alpha^2\beta_x\beta_y d_x^k d_y^k$$

which is positive if $\beta_x\beta_y>0$ and negative if $\beta_x\beta_y<0$. In the first case stability is lost or instability is retained and in the second case stability is regained or stability is retained. ■

References

Bielczyk, N., Forys, U. and Platkowski, T., Dynamical Models of Dyadic Interactions with Delay, *Journal of Mathematical Sociology*, 37, 223-249, 2013.

Bielczyk, N., Bondnar, M. and Forys, U., Delay can Stabilize: Love Affairs Dynamics, *Applied Mathematics and Computation* 219, 3923-3937, 2012.

Liao, X., and Ran, J., Hopf Bifurcation in Love Dynamical Models with Nonlinear Couples and Time Delays, *Chaos Solitions and Fractals* 31, 853-865, 2007.

Rinaldi, S. and Gragnani, A., Love Dynamics Between Secure Individuals: A Modeling Approach, *Nonlinear Dynamics, Psychology, and Life Science* 2, 283-301, 1998a.

Rinaldi, S., Love Dynamics: The Case of Linear Couples, *Applied Mathematics and Computation* 95, 181-192, 1998b.

Rinaldi, S., Laura and Petrach: An Intriguing Case of Cyclical Love Dynamics, *SIAM Journal on Applied Mathematics*, 58, 1205-1221, 1998c.

Son, W-S and Park, Y-J., Time Delay Effect on the Love Dynamic Model, *arXiv preprint arXiv:1108.5786*, 1-8, 2011.

Sprott, J., Dynamical Models of Love, *Nonlinear Dynamics, Psychology, and Life Sciences*, 8, 303-314, 2004.

Strogatz, S., Love Affairs and Differential Equations, *Mathematics Magazine* 61, p.35, 1988.

Strogatz, S., *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering*, Reading, MA, Addison-Wesley, 1994.