Discussion Paper No.261

Numerical Analysis of Three Time Delays in Monetary Policy: The Case of a Sticky-Price Model

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February 2016

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Abstract

In this study, we develop a New Keynesian model that includes the policy rule with which the nominal interest rate's responses are induced according to fluctuations in three economic variables, namely output, the inflation rate, and asset prices. In this model, we also assume that there is a time lag in the interest rate's response to each variable. The model economy is represented as a "differential equation system with three delays." For the determinacy analysis, we use the numerical method developed by Gu and Naghnaeian [K. Gu and M. Naghnaeian, Stability crossing set for systems with three delays, *IEEE* Transactions on Automatic Control 56 (2011), pp. 11–26] to find the parameter regions that achieve local determinacy in order to examine the effects of the three policy lags on local equilibrium determinacy. This is the first such application of this method to New Keynesian economics. We demonstrate that implementations of monetary policy should be "purposefully" delayed to achieve local equilibrium determinacy. Hence, the central bank should determine its target variables by considering not only the responsiveness of the nominal interest rate to output, the inflation rate, and asset prices but also the lag lengths associated with policy implementations.

JEL Classification Codes: E32; E52

Keywords: New Keynesian model, policy lag, interest rate rule, equilibrium determinacy, delay-differential equation

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1 Introduction

Optimizing models that consider price stickiness are referred to as New Keynesian (NK) models. The standard NK model comprises three equations: an Euler equation, Phillips curve, and monetary policy rule.[1] Benhabib et al. (2003) assume the backward-looking monetary policy rule wherein the nominal interest rate responds to the weighted mean of past inflation rates (i.e., lagged inflation rates).

Generally, time lags can be classified into two types, namely distributed lag models and fixed lag models, of which Benhabib et al.'s (2003) model can be considered to be the former. In this study, we develop an NK model that includes fixed time lags. This is, we assume that the nominal interest rate responds to target variables evaluated at fixed points in time. Conceptually, fixed lag models can be regarded as limiting cases of distributed lag models wherein the weighting factor for each target variable is unity at a certain point in time. Algebraically, however, it is impossible to treat fixed lags as special cases of distributed lags. Accordingly, spatial methods must be used.

In the simplest case where the system includes only one fixed lag, its characteristic function can generally be expressed as follows: $\Delta(\lambda) = p_0(\lambda) + p_1(\lambda)e^{-\lambda\tau}$, where λ is a root, $\tau > 0$ is the lag length, and $p_0(\lambda)$ and $p_1(\lambda)$ are polynomials of λ. Owing to the existence of the exponential function $e^{-\lambda\tau}$, the equation $\Delta(\lambda) = 0$ includes an infinite number of roots. Bellman and Cooke (1963), among others, have characterized the solutions of this type of function.

Recently, more complex configurations of characteristic functions have been examined:

- Beretta and Kuang (2002): lag-dependent parameters; $\Delta(\lambda) = p_0(\lambda, \tau) + p_1(\lambda, \tau)e^{-\lambda\tau}$.
- Gu et al. (2005): two lags; $\Delta(\lambda) = p_0(\lambda) + p_1(\lambda)e^{-\lambda\tau_1} + p_2(\lambda)e^{-\lambda\tau_2}$.
- Deng et al. (2006): time-dependent lag; $\Delta(\lambda) = p_0(\lambda) + p_1(\lambda)e^{-\lambda\tau(t)}$.
- Gu and Naghnaeian (2011): three lags; $\Delta(\lambda) = p_0(\lambda) + p_1(\lambda)e^{-\lambda\tau_1} + p_2(\lambda)e^{-\lambda\tau_2} + p_3(\lambda)e^{-\lambda\tau_3}$.
- Lin and Wang (2012): two lags (where one of the exponential functions includes the sum of lags); $\Delta(\lambda) = p_0(\lambda) + p_1(\lambda)e^{-\lambda\tau_1} + p_2(\lambda)e^{-\lambda\tau_2} + p_3(\lambda)e^{-\lambda(\tau_1+\tau_2)}$.

[1] Introductory textbooks for NK economics are presented by Woodford (2003), Walsh (2010), and Galí (2015).

In economics (especially in the context of NK economics), however, few works have examined the effects of fixed time lags,[2] especially compared with the many applications of these mathematical studies in biology.

Tsuzuki (2014, 2015) develops NK models that include only one fixed lag in the monetary policy rule. Here, the nominal interest rate is assumed to change in response to fluctuations in the inflation rate with a time lag, which represents an inflation targeting policy with delay. Tsuzuki's models can thus be considered to be fixed lag versions of Benhabib et al.'s (2003) model. According to Tsuzuki (2014, 2015), an increase in the lag time necessarily increases the number of roots with positive real parts. Accordingly, a policy lag may resolve the problem of equilibrium indeterminacy. However, in a system with two fixed lags, as in the model of Tsuzuki et al. (2015) that includes monetary and fiscal policy lags, an increase in the lag may raise the number of roots with negative real parts. This finding implies that a policy lag may resolve the problem of instability rather than that of indeterminacy.

In the present study, we consider the coexistence of three fixed lags in monetary policy responses. The model economic system is developed based on the model of Carlstrom and Fuerst (2007). In their model, the central bank was assumed to manipulate the nominal interest rate according to variations in both the inflation rate and asset prices. We add output as the third variable of the policy rule in accordance with the formulation in Bullard and Mitra (2002).

In the simplest NK model that does not include a time lag and in which the nominal interest rate responds only to the inflation rate, the necessary and sufficient condition for achieving local equilibrium determinacy is that the nominal interest rate increases by more than one unit when a one-unit increase in the inflation rate occurs; in other words, a monetary policy must be "active." This is the well-known policy norm referred to as the "Taylor principle." However, some notable studies have demonstrated that even if the Taylor principle is not satisfied, equilibrium determinacy can be achieved. Meng and Yip (2004), Bilbiie (2008), and Gliksberg (2009) show that equilibrium determinacy can be achieved when endogenous investment, limited asset market participation, and the

[2] In economics, Guerrini and Sodini (2013) and Tsuzuki and Shinagawa (2015) apply Beretta and Kuang's (2002) method, while Matsumoto and Szidarovszky (2015b) apply Lin and Wang's (2012) method. In addition, Matsumoto and Szidarovszky (2015a) present an application of Gu et al.'s (2005) method.

existence of capital adjustment costs, respectively, are assumed.[3] Furthermore, Bullard and Mitra (2002) and Carlstrom and Fuerst (2007) demonstrate that in discrete-time sticky-price models, an increase in the responsiveness of the nominal interest rate to both the inflation rate and output increases the possibility of the Taylor principle holding, whereas an increase in the responsiveness of the nominal interest rate to asset prices decreases this possibility.

In this study, we investigate the effects of monetary policy lags (i.e., the interaction between multiple policy lags) on the condition for local equilibrium determinacy. First, we consider the case where no policy lags are present. We demonstrate that the responses of the nominal interest rate to both the inflation rate and output increase the possibility of equilibrium determinacy, whereas responses to asset prices decrease that possibility, in line with the results shown by Carlstrom and Fuerst (2007) and Bullard and Mitra (2002). Next, we introduce three time lags to the monetary policy rule. For the determinacy analysis, we use the numerical method developed by Gu and Naghnaeian (2011) to find the parameter regions that achieve local determinacy. This is the first such application of this method to NK economics.

An extensional part of the discussion about the rights and wrongs of the central bank's response to fluctuations in asset prices pertains to the difference between the views of the Federal Reserve and the Bank for International Settlements. The Fed's view, which derives from a monetary policy perspective, is that the central bank should be dedicated to the stabilization of commodity prices and that it does not need to respond to fluctuations in asset prices. Bernanke and Gertler (2001) provide a theoretical ground with this view. Conversely, the Bank for International Settlements' view advocates a need for vigorous austerity measures when confronted with excessive increases in asset prices. Carlstrom and Fuerst (2007) establish the validity of the Fed's view in the presence of sticky prices. This is also confirmed in our model without a lag. However, if positive policy lags exist, how would this result change, if at all? The present study provides an answer to this question.

The remainder of this paper is organized as follows. Section 2 presents a differential equation system that describes the dynamics of the model economy. Section 3 examines the case with no policy lags. Section 4 examines the case with three policy lags. Section

[3]Buffie (2013) demonstrates that Bilbiie's (2008) result strongly depends on the assumption of real wage flexibility. If the real wage rate is sufficiently sticky, the Taylor principle reasserts itself as the condition for determinacy.

5 concludes.

2 The model

In this section, we present the standard NK model. The model economy is constructed by firms, households, and the public sector. Each firm produces differentiated goods by using workers under monopolistic competition. Likewise, each household supplies differentiated workers to firms under monopolistic competition and consumes goods.

2.1 Intratemporal optimization of firms

Various types of differentiated workers indexed by j ($j \in [0, 1]$) exist. Firms first aggregate the differentiated workers l_j via the Dixit–Stiglitz function as follows:

$$l = \left[\int_0^1 l_j^{\frac{\eta-1}{\eta}} dj \right]^{\frac{\eta}{\eta-1}}, \tag{1}$$

where l is the composite labor and $\eta > 1$ is the elasticity of substitution among workers.

The first-order condition for cost minimization yields the demand function for worker l_j as follows:[4]

$$l_j = \left(\frac{W_j}{W} \right)^{-\eta} l, \tag{2}$$

where W_j is the nominal wage rate of worker l_j and W is the nominal wage rate of the whole economy, defined as $W = [\int_0^1 W_j^{1-\eta} dj]^{\frac{1}{1-\eta}}$.

2.2 Intratemporal optimization of households

Various types of differentiated consumption goods indexed by i ($i \in [0, 1]$) exist. Households first aggregate their differentiated goods and then consume them as a composite good. As in the previous section, we express the aggregation of goods as the Dixit–Stiglitz function: $y = [\int_0^1 y_i^{\frac{\phi-1}{\phi}} di]^{\frac{\phi}{\phi-1}}$, where y is the amount of the composite good and $\phi > 1$ is the elasticity of substitution among goods. The first-order condition for cost minimization yields the demand function for good i as follows:

$$y_i = \left(\frac{p_i}{p} \right)^{-\phi} y, \tag{3}$$

[4]See Blanchard and Kiyotaki (1987).

where p_i is the price of good i and p is the price level, represented as $p = [\int_0^1 p_i^{1-\phi} di]^{\frac{1}{1-\phi}}$.

2.3 Intertemporal optimization of firms

We assume a linear technology and specify the production function of good i as follows:

$$y_i = l_i, \tag{4}$$

where y_i is the output of good i and l_i is the amount of composite labor used to produce good i.

Considering the constraints expressed in Equations (3) and (4), firm i solves the profit maximization problem as follows:

$$Maximize_{\pi_i} \int_0^\infty \left[\frac{p_i y_i - W l_i}{p} - \frac{\gamma}{2}(\pi_i - \pi^*)^2 y \right] e^{-\int_0^t r(s)ds} dt,$$
$$subject\ to\ \dot{p}_i = \pi_i p_i,$$

where $\pi_i = \dot{p}_i / p_i$ is the price change rate of good i, r is the real interest rate, and π^* is the steady-state value of the inflation rate. In addition, $\frac{\gamma}{2}(\pi_i - \pi^*)^2 y$ represents a price revision cost. Owing to the existence of this cost, the price becomes sticky. Hence, $\gamma > 0$ can be interpreted as a parameter that reflects the price stickiness measure; the larger the value of γ, the greater the stickiness. Here, we formulate the price revision cost in a quadratic function consistent with that outlined by Rotemberg (1982). Moreover, for simplicity, we assume that the price revision cost is a firm's payment to households that do not spend on goods. For example, it is considered to be a lump-sum payment to workers who handle price replacement tasks.

In the following discussion, we examine a "symmetric equilibrium" in which all firms' behavior is based on the same equations. In this case, we can drop subscript i from all the variables. Furthermore, as the number of goods is normalized to unity (see the Dixit–Stiglitz function), the following expressions hold: $p_i = p$, $\pi_i = \pi$, and $y_i = y$. By using these expressions along with the solutions to the optimization problem above, we can obtain

$$\dot{\pi} + (\pi - \pi^*)\frac{\dot{y}}{y} = r(\pi - \pi^*) - \frac{1-\phi}{\gamma} - \frac{\phi}{\gamma}\frac{W}{p}. \tag{5}$$

This equation is the NK Phillips curve. Furthermore, economically significant solutions would also require satisfying the transversality condition expressed as $\lim_{t\to\infty} p(t)e^{-\int_0^t r(s)ds}dt = 0$.

2.4 Intertemporal optimization of households

In every period, household j obtains utility from consumption c_j and real money holding m_j, and disutility from labor supply l_j. We specify the utility function as follows:

$$u(c_j, m_j, l_j) \equiv \ln c_j + \ln m_j - \frac{l_j^{1+\psi}}{1+\psi}, \tag{6}$$

where $\psi > 0$ is the disutility elasticity of labor supply.

Households possess assets that comprise money $M_j \equiv pm_j$, bond B_j, and a constant volume of stock. We standardize the measure of stock to unity. The stock price relative to the commodity price is denoted as Q and the stock yields dividend D in every period. Nominal asset level A_j can be expressed as follows: $A_j = M_j + B_j + pQ$. Assets can be increased based on income $W_j l_j$, bond interest RB_j (where $R \equiv r + \pi$ is the nominal interest rate), dividend D, capital gain $\dot{Q}$, lump-sum income from firms $T \equiv \frac{\gamma}{2}(\pi_i - \pi^*)^2 y$, and benefits from the government X, whereas assets can be decreased based on consumption c_j. Thus, we obtain

$$\dot{A}_j = W_j l_j + RB_j + pD + p\dot{Q} + pT + pX - pc_j. \tag{7}$$

By using the non-arbitrage condition, the nominal yields on stock represented as $\frac{\dot{Q}+D}{Q} + \pi$ must equal the nominal interest rate for bonds R, i.e.,

$$\dot{Q} = RQ - D - \pi Q. \tag{8}$$

By using this expression, Equation (7) can be rewritten as follows:

$$\dot{a}_j = w_j l_j + ra_j + T + X - c_j - Rm_j - \pi Q, \tag{9}$$

where $a_j \equiv A_j/p$ is the real asset level of household j and $w_j \equiv W_j/p$ is the real wage rate of worker l_j.

Considering Equations (2) and (9), household j solves the utility maximization problem as follows:

$$Maximize_{c_j, m_j, W_j} \int_0^\infty \left[\ln c_j + \ln m_j - \frac{l_j^{1+\psi}}{1+\psi} \right] e^{-\rho t} dt,$$
$$subject\ to\ \dot{a}_j = w_j l_j + ra_j + T + X - c_j - Rm_j - \pi Q,$$

where $\rho > 0$ is the subjective discount rate of households.

Under the assumption of the symmetry condition, the following equations hold: $c_j = c$, $W_j = W$, and $l_j = l$. By using these expressions along with the solutions to the optimization problem above,[5] we obtain

$$\frac{\dot{c}}{c} = r - \rho, \tag{10}$$

$$w = \frac{\eta}{\eta - 1} c l^{\psi}. \tag{11}$$

Equation (10) is an Euler equation, which is one of the optimal conditions for a Ramsey-type utility maximization problem. Equation (11) is a labor supply function. If the labor market is perfectly competitive, the real wage rate w should equal the marginal disutility of labor measured in terms of goods cl^{ψ}.[6] However, in the present model, the labor market is monopolistically competitive. Hence, the real wage rate becomes equal to the marginal disutility of labor multiplied by the markup $\eta/(\eta - 1) > 1$, as in Equation (11). Furthermore, economically significant solutions would require satisfying the transversality condition expressed as $\lim_{t\to\infty} a(t)e^{-\rho t}dt = 0$.

Finally, the profits of firms are paid to households in the form of a dividend; therefore,

$$D = y - \frac{W}{p} l - \frac{\gamma}{2}(\pi - \pi^*)^2 y. \tag{12}$$

2.5 Central bank

The central bank manipulates nominal interest rate R according to fluctuations in output y, inflation rate π, and asset price Q, which implies that it targets output, inflation, and asset prices simultaneously. In this case, the monetary policy rule is expressed as follows:

$$R = R(y, \pi, Q);\ \alpha_y \equiv \frac{\partial R}{\partial y} > 0;\ \alpha_\pi \equiv \frac{\partial R}{\partial \pi} > 0;\ \alpha_q \equiv \frac{\partial R}{\partial Q} > 0;\ \bar{R} = R(y^*, \pi^*, Q^*), \tag{13}$$

where α_y, α_π, and α_q are the responsiveness of the nominal interest rate to output, the inflation rate, and asset prices, respectively. In addition, $\bar{R} > 0$ is the target level of the

[5] One of the conditions for optimality is given by $\frac{1}{m_j} - \mu R = 0$. The dynamics of the model economic system can be examined without using this condition. Therefore, we ignore it from the simultaneous equation system of the model economy.

[6] The inverse of the marginal utility of consumption is expressed as $\partial c/\partial u = c$. In addition, the marginal disutility of labor is expressed as $\partial u/\partial l = -l^{\psi}$. Therefore, the marginal disutility of labor measured in terms of goods is given by $\frac{dc}{dl} = -\frac{\partial u/\partial l}{\partial u/\partial c} = cl^{\psi}$.

nominal interest rate that corresponds to the target levels of these three variables (we regard their steady-state values here).

When delays are present in the central bank's responses to economic fluctuations, Equation (13) can be rewritten as follows:

$$R(t) = R(y(t-\tau_1), \pi(t-\tau_2), Q(t-\tau_3)), \tag{14}$$

where τ_1, τ_2, and τ_3 are delays in the output, inflation rate, and asset-price targeting policies, respectively.

3 Case with no policy lags

To emphasize the effects of policy lags on equilibrium determinacy, we first consider the case with no policy lags (i.e., $\tau_1 = \tau_2 = \tau_3 = 0$).

In this case, the model economic system comprises Equations (4), (5), (8), and (10)–(13). By using the goods market equilibrium condition $y = c$, the system can be summarized in the following three equations:

$$\begin{aligned} \dot{y} &= [R(y, \pi, Q) - \pi - \rho]y, \\ \dot{\pi} &= \rho(\pi - \pi^*) - \frac{1-\phi}{\gamma} - \frac{\phi\eta}{\gamma(\eta-1)} y^{1+\psi}, \\ \dot{Q} &= R(y, \pi, Q)Q - \left(1 - \frac{\eta}{\eta-1} y^{1+\psi} - \frac{\gamma}{2}(\pi - \pi^*)^2\right) y - \pi Q. \end{aligned} \tag{15}$$

The non-trivial solutions to this system are expressed as follows:

$$\begin{aligned} y^* &= \left[\frac{(\phi-1)(\eta-1)}{\phi\eta}\right]^{\frac{1}{1+\psi}}, \\ \pi^* &= \bar{R} - \rho, \\ Q^* &= \left(1 - \frac{\eta}{\eta-1} y^{*1+\psi}\right) \frac{y^*}{\rho}. \end{aligned} \tag{16}$$

The Jacobian matrix of System (15) evaluated at the steady state can be given by

$$J \equiv \begin{bmatrix} \alpha_y y^* & (\alpha_\pi - 1)y^* & \alpha_q y^* \\ -P_1 & \rho & 0 \\ \alpha_y Q^* + P_2 & (\alpha_\pi - 1)Q^* & \alpha_q Q^* + \rho \end{bmatrix},$$

where

$$
\begin{aligned}
P_1 &\equiv \frac{\phi\eta(1+\psi)}{\gamma(\eta-1)} y^{*\psi} > 0, \\
P_2 &\equiv -\left(1 - \frac{\eta}{\eta-1} y^{*1+\psi}\right) + \frac{\eta(1+\psi)}{\eta-1} y^{*1+\psi} \\
&= -\left(1 - \frac{\eta(2+\psi)}{\eta-1} y^{*1+\psi}\right) \\
&= -\left(1 - \frac{(2+\psi)(\phi-1)}{\phi}\right).
\end{aligned}
$$

For plausible parameter values, $P_2 > 0$ holds.[7] Hence, in the following discussion, we assume that this inequality holds.

Thus, the characteristic equation of the above system can be represented as

$$
\Delta_1(x) \equiv |xI - J| = x^3 + b_1 x^2 + b_2 x + b_3 = 0, \tag{17}
$$

where x is a characteristic root, I is an identity matrix,

$$
\begin{aligned}
b_1 &\equiv -traceJ \\
&= -(\alpha_y y^* + \alpha_q Q^* + 2\rho) < 0, \\
b_2 &\equiv \text{sum of the second-order principal minors of } J \\
&= -(P_2 y^* - \rho Q^*)\alpha_q + \rho^2 + 2\alpha_y y^* \rho + P_1(\alpha_\pi - 1)y^*, \\
b_3 &\equiv -detJ \\
&= \alpha_q y^* \rho P_2 - \alpha_y y^* \rho^2 - P_1(\alpha_\pi - 1)y^* \rho.
\end{aligned}
$$

If $\alpha_q = 0$, Jacobian matrix J becomes decomposable. In other words, the dynamics of y and π are unaffected by those of Q (whereas the dynamics of Q are affected by y and π). Therefore, in this case, the dynamic structure of the system is consistent with that of a simple NK model. Hence, the Taylor principle would hold as the necessary and sufficient condition for determinacy. However, as demonstrated by Bullard and Mitra (2002), when the nominal interest rate responds not only to the inflation rate but also to output, the condition for determinacy becomes more complicated; in this case, the condition is expressed as $\alpha_y \rho + P_1(\alpha_\pi - 1) > 0$. If $\alpha_y = 0$, then the well-known determinacy condition, $\alpha_\pi > 1$, can be obtained. When $\alpha_q > 0$, the following lemma holds:

[7] If $\psi > \phi/(\phi-1) - 2$, $P_2 > 0$; therefore, if specifically $\phi > 2$, $P_2 > 0$ holds for all $\psi > 0$.

Lemma 1 The equilibrium is locally determinate if and only if

$$\alpha_q < \alpha_{q3} \equiv \frac{\alpha_y \rho + P_1(\alpha_\pi - 1)}{P_2}. \tag{18}$$

Proof. Matrix J includes three roots. For equilibrium determinacy, all these roots must have positive real parts. The necessary and sufficient conditions for all roots to have positive real parts (Inverse Routh–Hurwitz theorem) are given by $b_1 < 0$, $b_2 > 0$, and $b_3 < 0$.[8] Irrespective of the value of α_q, $b_1 < 0$ holds. Furthermore, $b_3 < 0$ if and only if the condition in Equation (18) is satisfied. A necessary condition for Equation (18) to hold is that $\alpha_y \rho + P_1(\alpha_\pi - 1) > 0$. Under this condition, b_2 is ensured to become positive if $P_2 y^* < \rho Q^*$; however, if $P_2 y^* > \rho Q^*$, it becomes positive only when

$$\alpha_q < \alpha_{q2} \equiv \frac{\rho^2 + 2\alpha_y y^* \rho + P_1(\alpha_\pi - 1) y^*}{P_2 y^* - \rho Q^*}.$$

As $\alpha_{q2} > \alpha_{q3}$, if Equation (18) is satisfied, $\alpha_q < \alpha_{q2}$ is also satisfied. Thus, $b_2 > 0$ and $b_3 < 0$ if and only if $\alpha_q < \alpha_{q3}$. □

The economic implications of this result can be explained as follows. In our model, an increase in the inflation rate increases the real wage rate, which is a marginal cost of firms. This change further results in decreases in profits and asset prices (by $P_2 > 0$). The central bank responds not only to the variation in the inflation rate but also to the variation in asset prices. Therefore, the response of the nominal interest rate to the inflation rate is partly offset by the response to asset prices. This fact implies that the Taylor principle becomes less likely to be satisfied. Thus, the larger the value of α_q, the more likely is the occurrence of indeterminacy. Carlstrom and Fuerst (2007) use a discrete-time sticky-price model to demonstrate the equivalent result.[9]

[8] See Asada et al. (2007) and Tsuzuki (2013).

[9] See Proposition 1 in Carlstrom and Fuerst (2007). They also consider the cases where (i) nominal wages are sticky and (ii) both prices and nominal wages are sticky. In this study, we examine the standard case where only prices are sticky.

4 Case with positive policy lags

The model economic system when Equation (14) is used as the policy rule is expressed as follows:

$$\begin{aligned}
\dot{y}(t) &= [R(y(t-\tau_1), \pi(t-\tau_2), Q(t-\tau_3)) - \pi(t) - \rho]y(t), \\
\dot{\pi}(t) &= \rho(\pi(t) - \pi^*) - \frac{1-\phi}{\gamma} - \frac{\phi\eta}{\gamma(\eta-1)}y(t)^{1+\psi}, \\
\dot{Q}(t) &= R(y(t-\tau_1), \pi(t-\tau_2), Q(t-\tau_3))Q(t) \\
&\quad - \left(1 - \frac{\eta}{\eta-1}y(t)^{1+\psi} - \frac{\gamma}{2}(\pi(t)-\pi^*)^2\right)y(t) - \pi(t)Q(t).
\end{aligned} \tag{19}$$

This is a differential equation system with three delays.

The steady-state values of System (19) are given by Equation (16). We linearize this system at the steady state to obtain

$$\begin{aligned}
\dot{\hat{y}}(t) &= [\alpha_y \hat{y}(t-\tau_1) + \alpha_\pi \hat{\pi}(t-\tau_2) + \alpha_q \hat{Q}(t-\tau_3) - \hat{\pi}(t)]y^*, \\
\dot{\hat{\pi}}(t) &= \rho\hat{\pi}(t) - P_1\hat{y}(t), \\
\dot{\hat{Q}}(t) &= \alpha_y Q^* \hat{y}(t-\tau_1) + \alpha_\pi Q^* \hat{\pi}(t-\tau_2) + \alpha_q Q^* \hat{Q}(t-\tau_3) + P_2\hat{y}(t) - Q^*\hat{\pi}(t) + \rho\hat{Q}(t),
\end{aligned} \tag{20}$$

where $\hat{y}(t) \equiv y(t) - y^*$, $\hat{\pi}(t) \equiv \pi(t) - \pi^*$, and $\hat{Q}(t) \equiv Q(t) - Q^*$. Assuming that the exponential functions $\hat{y}(t) = C_1 e^{xt}$, $\hat{\pi}(t) = C_2 e^{xt}$, and $\hat{Q}(t) = C_3 e^{xt}$ are the solutions to this system, where C_1, C_2, and C_3 are arbitrary constants, and plugging these functions into System (20), we get the following:

$$\begin{bmatrix} x - \alpha_y y^* e^{-\tau_1 x} & y^* - \alpha_\pi y^* e^{-\tau_2 x} & -\alpha_q y^* e^{-\tau_3 x} \\ P_1 & x - \rho & 0 \\ -P_2 - \alpha_y Q^* e^{-\tau_1 x} & Q^* - \alpha_\pi Q^* e^{-\tau_2 x} & x - \rho - \alpha_q Q^* e^{-\tau_3 x} \end{bmatrix} \begin{bmatrix} \hat{y}(t) \\ \hat{\pi}(t) \\ \hat{Q}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of the left-hand side matrix must equal zero for non-trivial solutions to exist:

$$\Delta_2(x) \equiv s_0(x) + s_1(x)e^{-\tau_1 x} + s_2(x)e^{-\tau_2 x} + s_3(x)e^{-\tau_3 x} = 0, \tag{21}$$

where

$$s_0(x) \equiv x^3 - 2\rho x^2 + (\rho^2 - P_1 y^*)x + P_1 y^* \rho,$$
$$s_1(x) \equiv -\alpha_y y^* x^2 + 2\alpha_y y^* \rho x - \alpha_y y^* \rho^2,$$
$$s_2(x) \equiv P_1 \alpha_\pi y^* x - P_1 \alpha_\pi y^* \rho,$$
$$s_3(x) \equiv -\alpha_q Q^* x^2 + (\rho \alpha_q Q^* - P_2 \alpha_q y^*)x + \alpha_q y^* \rho P_2.$$

Equation (21) is the characteristic equation of System (19).

Owing to the existence of the terms that contain the exponential functions $e^{-\tau_1 x}$, $e^{-\tau_2 x}$, and $e^{-\tau_3 x}$, Equation (21) has an infinite number of roots.[10] Furthermore, unlike in the case of ordinary differential equations, delay differential equations require the initial values of $y(t)$, $\pi(t)$, and $Q(t)$ evaluated not only at time zero (t_0; present time) but also at $t_0 - \tau_1 \leq t < t_0$, $t_0 - \tau_2 \leq t < t_0$, and $t_0 - \tau_3 \leq t < t_0$. However, the only values that economic agents can determine at time t_0 are $(y(t_0), \pi(t_0), Q(t_0))$ because "past values" must be considered as given. Therefore, equilibrium determinacy can be achieved only when there are exactly three roots with positive real parts among the infinite number of roots. The equilibrium is indeterminate if fewer than three roots have positive real parts and it is unstable if more than three roots have positive real parts (i.e., an equilibrium path does not exist).

4.1 Method of analysis

The following analysis is performed based on the numerical method developed by Gu and Naghnaeian (2011). The procedure is given as follows:

1. The set of the imaginary part of the roots that generates pure imaginary roots (crossing frequency set: Ω) is characterized. The crossing frequency set is broadly grouped into two types: Grashof and Non-Grashof sets.

2. The set of lags corresponding to the crossing frequency set (stability crossing set: $\mathcal{T} \in (\tau_1, \tau_2, \tau_3)$) is characterized.

3. By assuming plausible parameter values, we depict the stability crossing set as a surface in a 3D space.

[10] See Chapter 3 in Bellman and Cooke (1963).

The notations used in the following sections are defined as follows: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{Z}_3^+ = \{1, 2, 3\}$; i = imaginary unit. Furthermore, let $u' = (u \bmod 3) + 1$ for $u \in \mathbb{Z}_3^+$; that is, $1' = 2$, $2' = 3$, and $3' = 1$. In addition, let $u'' = (u')' = [(u + 1) \bmod 3] + 1$. Then, $\{u, u', u''\} = \mathbb{Z}_3^+$.

4.2 Preliminaries

First, to apply Gu and Naghnaeian's (2011) method, some preconditions should be examined. The polynomials $s_u(x)$, $u = 0, 1, 2, 3$, must satisfy the following conditions (nontriviality assumptions):

Assumption 1 $\deg(s_0(x)) \geq \max\{\deg(s_u(x))|u \in \mathbb{Z}_3^+\}$,

Assumption 2 $s_0(0) + s_1(0) + s_2(0) + s_3(0) \neq 0$,

Assumption 3 $\lim\limits_{x\to\infty} \dfrac{|s_1(x)| + |s_2(x)| + |s_3(x)|}{|s_0(x)|} < 1$.

Assumption 1 ensures the existence of a set $(\tau_1, \tau_2, \tau_3) \in \mathbb{R}_+^3$ that establishes perfect stability, wherein a root with a positive real part does not exist. However, this study examines equilibrium determinacy, which is different from the concept of stability used in mathematics. Hence, the existence of a perfectly stable state is not necessarily required; nevertheless, we adopt this assumption only to apply Gu and Naghnaeian's (2011) method. Because $\deg(s_0(x)) = 3$ and $\max\{\deg(s_u(x))|u \in \mathbb{Z}_3^+\} = 2$, this assumption is satisfied. Assumption 2 implies that $\Delta_2(0) \neq 0$; i.e., zero is not a root. For the same reason as for Assumption 1, we adopt this assumption, while it also plays the only role of a necessary condition to ensure the existence of a perfectly stable state. We should assume $detJ \neq 0$ because $\Delta_2(0) = -detJ$. Finally, Assumption 3 is required to ensure the continuity of roots over (τ_1, τ_2, τ_3). Because $\lim_{x\to\infty} \frac{|s_1(x)|+|s_2(x)|+|s_3(x)|}{|s_0(x)|} = 0$, Assumption 3 is satisfied.

Under these assumptions, we can state the following: as the lags (τ_1, τ_2, τ_3) continuously vary within $\mathbb{R}_+^3$, the number of roots of $\Delta_2(x) = 0$ lying on $\mathbb{C}_+$ can change if a root appears on or crosses the imaginary axis (Lemma 1 in Gu and Naghnaeian (2011)).

4.3 Crossing frequency set

Denote a pure imaginary root as $x = i\omega$, where $\omega > 0$ is the imaginary part.[11] In this section, we characterize the values of $\omega > 0$ that satisfy $\Delta_2(i\omega) = 0$ (crossing frequency). Let

$$a_u(x) = \frac{s_u(x)}{s_0(x)}, \quad u = 0, 1, 2, 3. \tag{22}$$

Obviously, $a_0(x) = 1$. Moreover, we also define that

$$f_0(x) = |a_1(x)| + |a_2(x)| + |a_3(x)|, \tag{23}$$

$$f_u(x) = |a_{u'}(x)| + |a_{u''}(x)| - |a_u(x)|, \quad u = 1, 2, 3. \tag{24}$$

By using Equation (22), $\Delta_2(i\omega) = 0$ can be rewritten as follows:

$$\Delta_3(i\omega) = 1 + a_1(i\omega)e^{-i\omega\tau_1} + a_2(i\omega)e^{-i\omega\tau_2} + a_3(i\omega)e^{-i\omega\tau_3} = 0. \tag{25}$$

Furthermore, by considering each term of Equation (25) as a vector depicted on the complex plane, the four terms form a rectangle, as shown in Figure 1.[12]

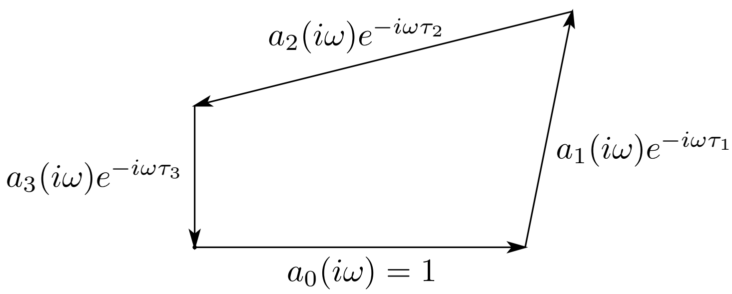

Figure 1: Rectangle formed by the four vectors

From the geometric fact that a rectangle can be formed only when the length of one side does not exceed the sum of the other lengths, we can postulate that a crossing frequency set comprises $\omega > 0$ that satisfy the following four inequalities:

$$f_0(i\omega) \geq 1$$
$$f_u(i\omega) \geq -1, \quad u = 1, 2, 3.$$

[11]We can assume that $\omega > 0$ without loss of generality because the complex roots will always be conjugated.

[12]Note that the length of each vector is independent of (τ_1, τ_2, τ_3). ((τ_1, τ_2, τ_3) determine the direction of the vector.)

Thus, when we depict $f_u(i\omega)$, $u = 0, 1, 2, 3$, against ω, the crossing frequency set Ω comprises all $\omega > 0$ that the curve $f_0(i\omega)$ is above 1 and the curves $f_1(i\omega)$, $f_2(i\omega)$, and $f_3(i\omega)$ are above -1.

Moreover, if the sum of the longest side and the shortest side of a rectangle is smaller than that of the other two sides, the rectangle is recognized as a Grashof rectangle. By using this definition, we can classify $\omega \in \Omega$ into the following four types:

Type 0 Grashof set: Ω_G^0

Set of $\omega \in \Omega$ that satisfies

$$f_u(i\omega) > 1, \quad u = 1, 2, 3. \tag{26}$$

Type u Grashof set: Ω_G^u

Set of $\omega \in \Omega$ that satisfies

$$f_u(i\omega) > 1, \quad u = 1, 2, 3, \tag{27}$$

$$f_{u'}(i\omega) < 1, \tag{28}$$

$$f_{u''}(i\omega) < 1. \tag{29}$$

Type 0 Non-Grashof set: Ω_N^0

Set of $\omega \in \Omega$ that satisfies

$$f_u(i\omega) \leq 1, \quad u = 1, 2, 3. \tag{30}$$

Type u Non-Grashof set: Ω_N^u

Set of $\omega \in \Omega$ that satisfies

$$f_u(i\omega) \leq 1, \quad u = 1, 2, 3, \tag{31}$$

$$f_{u'}(i\omega) \geq 1, \tag{32}$$

$$f_{u''}(i\omega) \geq 1. \tag{33}$$

Ω_G^u, $u = 0, 1, 2, 3$, are open intervals and Ω_N^u, $u = 0, 1, 2, 3$, are closed intervals. (However, Ω_N^u, $u = 0, 1, 2, 3$, become semi-open intervals when $\omega = 0$ is a solution of Equations 31–33 or Equation 30. See Proposition 3 in Gu and Naghnaeian (2011) for details.)

Following Fujiwara (2008), we set the structural parameter values as shown in Table 1. Further, we set the target inflation rate at a realistic value of 2%, which implies that

$\bar{R} = 0.03$. Finally, the monetary policy reaction parameters are set at their typical values, as shown in Table 2, where the value of τ_q is set so that Equation 18 is satisfied (i.e., if a policy lag is not present, the equilibrium is determinate).

η	ϕ	γ	ψ	ρ
21	6.0	27.454	1.0	0.01

Table 1: Structural parameters

α_y	α_π	α_q
0.5	1.5	0.01

Table 2: Policy reaction parameters

Under these assumptions, we investigate the effects of policy lags on equilibrium determinacy. The functions $f_u(i\omega)$, $u = 0, 1, 2, 3$, are depicted as shown in Figure 2. This figure indicates that

$$\Omega = [\omega_1, \omega_4] = [0.2203, 0.8176].$$

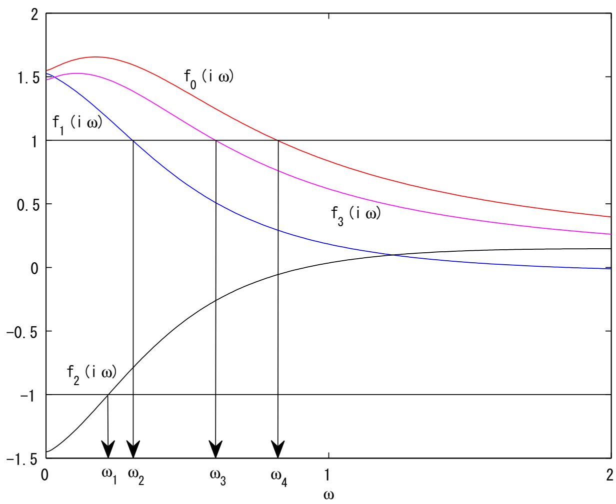

Figure 2: $f_u(i\omega)$, $u = 0, 1, 2, 3$

Moreover, the crossing frequency set Ω can be resolved into the following three components:

$$\Omega = \bigcup_{h=1}^{3} \Omega^h$$

$$\Omega^1 = [\omega_1, \omega_2] = [0.2203, 0.3054] \subset \Omega_N^2,$$
$$\Omega^2 = (\omega_2, \omega_3) = (0.3054, 0.5922) \subset \Omega_G^3,$$
$$\Omega^3 = [\omega_3, \omega_4] = [0.5922, 0.8176] \subset \Omega_N^0.$$

Gu and Naghnaeian (2011) show the geometric configurations of the stability crossing sets ($\mathcal{T}^h$, $h = 1, 2, 3$) that correspond to all possible types of crossing frequency sets (Ω^h, $h = 1, 2, 3$) (see Theorem 1 in Gu and Naghnaeian (2011)). According to their study, $\mathcal{T}^1$ and $\mathcal{T}^3$ shape caps and $\mathcal{T}^2$ shapes wavy sheets in the (τ_1, τ_2, τ_3) space.

4.4 Concretization of the stability crossing sets

To illustrate the stability crossing sets $\mathcal{T}^h$, $h = 1, 2, 3$, in the (τ_1, τ_2, τ_3) space, we present their specific expressions.

Let $\{u, v, w\} = \mathbb{Z}_3^+$; then, Equation (25) can be rewritten as follows:

$$\Delta_3(i\omega) = a_v(i\omega)e^{-i\omega\tau_v} + a_w(i\omega)e^{-i\omega\tau_w} + a_d(i\omega, \tau_u) = 0, \tag{34}$$

where

$$a_d(i\omega, \tau_u) = 1 + a_u(i\omega)e^{-i\omega\tau_u}. \tag{35}$$

Considering the three terms in Equation (34) as vectors on the complex plane and depicting them, a triangle can be formed, as shown in Figure 3.[13]

This figure indicates that θ_v and θ_w can be expressed as follows:

$$\mp\theta_v = \arg\left(a_v(i\omega)\right) - \omega\tau_v - \arg\left(a_d(i\omega, \tau_u)\right) + 2r_v\pi,$$
$$\pm\theta_w = \arg\left(a_w(i\omega)\right) - \omega\tau_w - \arg\left(a_d(i\omega, \tau_u)\right) + 2r_w\pi,$$

[13] For details of the discussion here, see Section 3 in Gu et al. (2005).

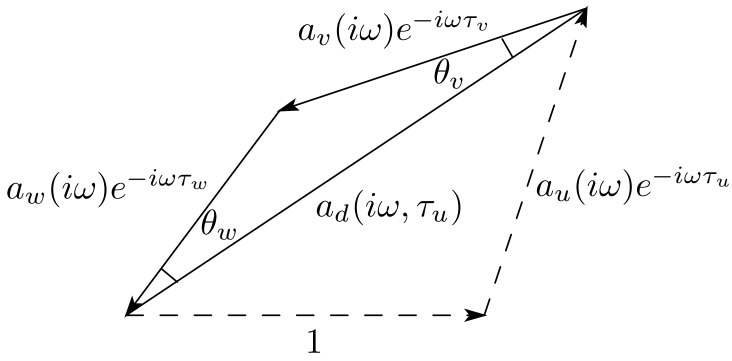

Figure 3: Triangle formed by $|a_d(i\omega, \tau_u)|$, $|a_v(i\omega)|$, and $|a_w(i\omega)|$

where $r_v, r_w = 0, 1, 2, 3, \cdots$. Thus, we obtain

$$\begin{aligned} \tau_v &= \tau_v^{\pm}(\omega, \tau_u, r_v) \\ &= \frac{\arg\left(a_v(i\omega)\right) - \arg\left(a_d(i\omega, \tau_u)\right) \pm \theta_v + 2r_v\pi}{\omega}, \end{aligned} \tag{36}$$

$$\begin{aligned} \tau_w &= \tau_w^{\mp}(\omega, \tau_u, r_w) \\ &= \frac{\arg\left(a_w(i\omega)\right) - \arg\left(a_d(i\omega, \tau_u)\right) \mp \theta_w + 2r_w\pi}{\omega}. \end{aligned} \tag{37}$$

By using the cosine theorem, θ_v and θ_w can also be expressed as

$$\begin{aligned} \theta_v &= \cos^{-1}\left(\frac{|a_d(i\omega, \tau_u)|^2 + |a_v(i\omega)|^2 - |a_w(i\omega)|^2}{2|a_d(i\omega, \tau_u)| \cdot |a_v(i\omega)|}\right), \\ \theta_w &= \cos^{-1}\left(\frac{|a_d(i\omega, \tau_u)|^2 + |a_w(i\omega)|^2 - |a_v(i\omega)|^2}{2|a_d(i\omega, \tau_u)| \cdot |a_w(i\omega)|}\right), \end{aligned}$$

which are substituted into Equations (36) and (37), respectively.

A similar procedure derives the following expression from Equation (35):

$$\begin{aligned} \tau_u &= \tau_u(\omega, r_u) \\ &= \frac{\arg\left(a_u(i\omega)\right) + (2r_u - 1)\pi}{\omega}, \quad r_u = 0, 1, 2, 3, \cdots. \end{aligned} \tag{38}$$

Equations (36)–(38) are used to express the stability crossing sets.

Incidentally, for the triangle shown in Figure 3 to exist, the following condition must hold:

$$||a_v(i\omega)| - |a_w(i\omega)|| \le |a_d(i\omega, \tau_u)| \le |a_v(i\omega)| + |a_w(i\omega)|. \tag{39}$$

These inequalities define the motion range of τ_u for the given values of ω. The motion range depends on the type of Ω^h. Hence, the representation of $\mathcal{T}^h$ also differs accordingly.

When Ω^h is a Grashof set ($\Omega_G = \bigcup_{u=0}^{3} \Omega_G^u$), the motion range of τ_u can be defined as follows (see Appendix A.1):

$$\mathcal{T}_u(\omega, r_u) = [\tau_u(\omega, r_u), \tau_u(\omega, r_u + 1)].$$

On the contrary, when Ω^h is a Non-Grashof set ($\Omega_N = \bigcup_{u=0}^{3} \Omega_N^u$), the motion range of τ_u can be defined as follows (see Appendix A.2):

$$\mathcal{T}_u(\omega, r_u) = [\tau_{um}(\omega, r_u), \tau_{uM}(\omega, r_u)],$$

where

$$\theta_{um} = \cos^{-1}\left(\frac{1 + |a_u(i\omega)|^2 - (|a_v(i\omega)| + |a_w(i\omega)|)^2}{2|a_u(i\omega)|}\right), \tag{40}$$

$$\tau_{um}(\omega, r_u) = \frac{\arg(a_u(i\omega)) + 2r_u\pi - \theta_{um}}{\omega},$$
$$\tau_{uM}(\omega, r_u) = \frac{\arg(a_u(i\omega)) + 2r_u\pi + \theta_{um}}{\omega}.$$

Thus, by defining the set $\mathcal{T}^{h\pm}$ as

$$\mathcal{T}^{h\pm} = \left\{ (\tau_1, \tau_2, \tau_3) \;\middle|\; \begin{array}{c} \tau_u \in \mathcal{T}_u(\omega, r_u), \\ \tau_v = \tau_v^{\pm}(\omega, \tau_u, r_v), \\ \tau_w = \tau_w^{\mp}(\omega, \tau_u, r_w) \end{array} \right\}, \tag{41}$$

the stability crossing sets $\mathcal{T}^h$, $h = 1, 2, 3$, can be represented for the given (r_u, r_v, r_w) and $\{(\omega, \tau_u) | \omega \in \Omega^h, \tau_u \in \mathcal{T}_u(\omega, r_u)\}$, as follows:

$$\mathcal{T}^h = (\mathcal{T}^{h+} \bigcup \mathcal{T}^{h-}) \bigcap \mathbb{R}_+^3. \tag{42}$$

4.5 Drawing the stability crossing sets

Under the assumptions in Tables 1 and 2, $\mathcal{T}^1$, $\mathcal{T}^2$, and $\mathcal{T}^3$ are drawn, as shown in Figures 4–6, respectively.[14]

[14] In the case of $\Omega^1 \subset \Omega_N^2$, $u = 2$, $v = 1$, and $w = 3$. Figure 4 shows the case of $(r_u, r_v, r_w) = (0, 0, 0)$. In the case of $\Omega^2 \subset \Omega_G^3$, $u = 3$, $v = 2$, and $w = 1$. Figure 5 shows the case of $r_u = 0, 1, 2$; $r_v = 1$; and $r_w = 1$. Finally, in the case of $\Omega^3 \subset \Omega_N^0$, $u = 1$, $v = 3$, and $w = 2$. Figure 6 shows the case of $(r_u, r_v, r_w) = (1, 1, 1)$.

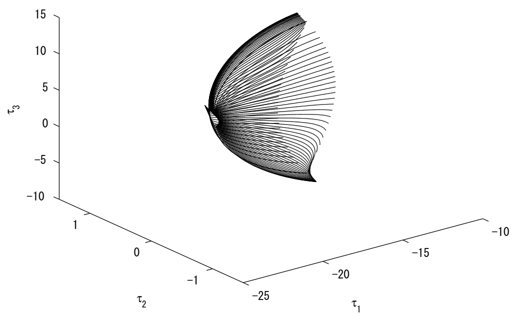

Figure 4: $\mathcal{T}^1$: Cap

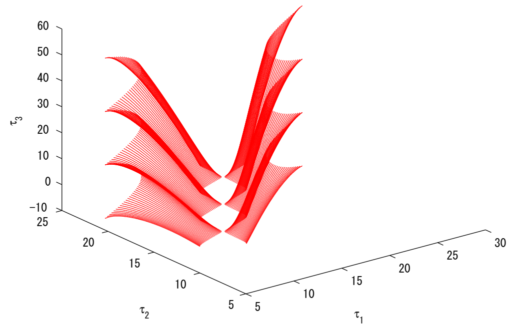

Figure 5: $\mathcal{T}^2$: Wavy sheets

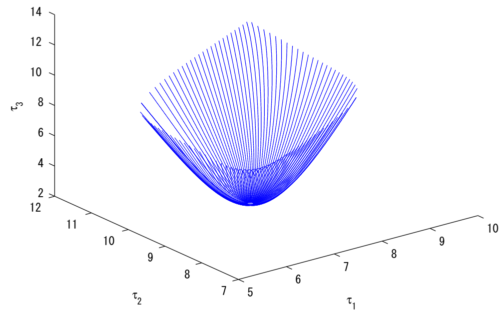

Figure 6: $\mathcal{T}^3$: Cap

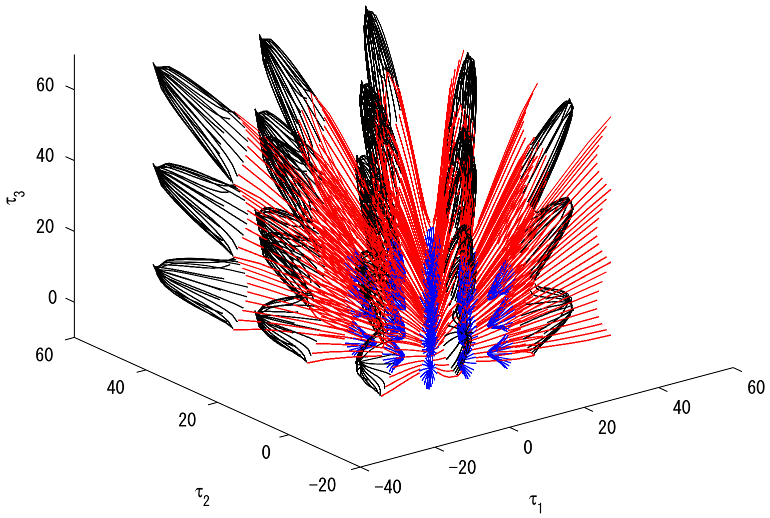

Figure 7: Stability crossing sets

The complete figure of $\mathcal{T}^h$, $h = 1, 2, 3$, corresponding to all combinations of $r_u = 0, 1, 2$; $r_v = 0, 1, 2$; and $r_w = 0, 1, 2$ (27 patterns) can be drawn as shown in Figure 7. *At least* for positive values of (τ_1, τ_2, τ_3) lying inside the surface of $\mathcal{T}^h$, $h = 1, 2, 3$, that enclose the origin, the equilibrium is determinate. To capture the size and shape of the determinacy region more clearly, we show some cross-section diagrams when τ_1, τ_2, or τ_3 is fixed at certain values (0, 10, and 20), as in Figures 8–16. Whenever τ_1, τ_2, or τ_3 crosses these curves (which we call the stability crossing curves), the sign of the real part of the complex roots changes.

The direction of the change (i.e., whether it runs from positive to negative or vice versa) can be examined as follows. In the case of a change in τ_1, the direction is confirmed by determining the sign of the following expression:

$$\mathrm{Re}\left[\frac{\partial x}{\partial \tau_1}\right]_{(\tau_1,\tau_2,\tau_3)=(\tau_1^*,\tau_2^*,\tau_3^*),x=i\omega^*}, \tag{43}$$

where $(\tau_1^*, \tau_2^*, \tau_3^*)$ is a point lying on the surface of a stability crossing set and $x = i\omega^*$ is a pure imaginary root that corresponds to the point. In addition, $\partial x/\partial \tau_1$ can be derived from the implicit function $\Delta_2(x, \tau_1, \tau_2, \tau_3) = 0$ as follows:

$$\frac{\partial x}{\partial \tau_1} = \frac{s_1(x)xe^{-\tau_1 x}}{s_0'(x) + \sum_{j=1}^{3}\{s_j'(x) - s_j(x)\tau_j\}e^{-\tau_j x}}. \tag{44}$$

If $\mathrm{Re}\left[\frac{\partial x}{\partial \tau_1}\right] > 0$, a pair of complex roots moves from left to right on the complex plane when τ_1 crosses τ_1^*; therefore, the real part of the complex roots changes from negative to positive. Conversely, if the inequality runs in the opposite direction, a pair of complex roots moves from right to left on the complex plane; in this case, the real part of the complex roots changes from positive to negative.

For example, at the point $(\tau_1^*, \tau_2^*, \tau_3^*) = (6.9689, 12.0174, 0)$ (see Figure 8), $\omega^* = 0.5395$ and $\mathrm{Re}[\partial x/\partial \tau_1] = 0.0158 > 0$. Accordingly, when τ_1 crosses this point from left to right, the number of roots with positive real parts increases by two. We have already shown that there are exactly three roots with positive real parts in the left-hand side of this point (the equilibrium is determinate when policy lags are not present). Therefore, in the right-hand side region of that point, there are five roots with positive real parts, implying that the equilibrium is unstable. As for the other areas, we can reveal the number of roots with positive real parts by using the same procedure. The values written in each area of Figures 8–16 indicate the number of roots with positive real parts.