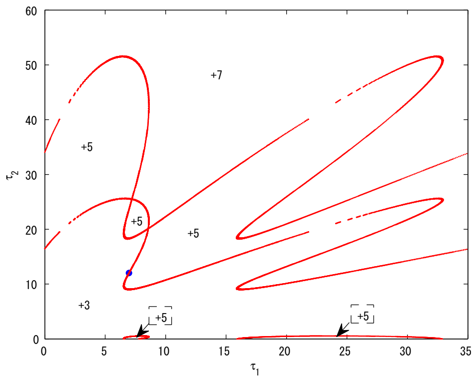

Figure 8: $\tau_3 = 0$

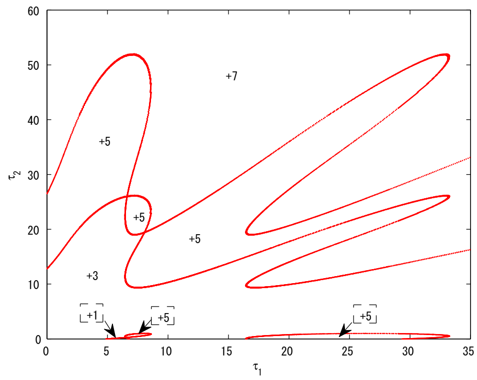

Figure 9: $\tau_3 = 10$

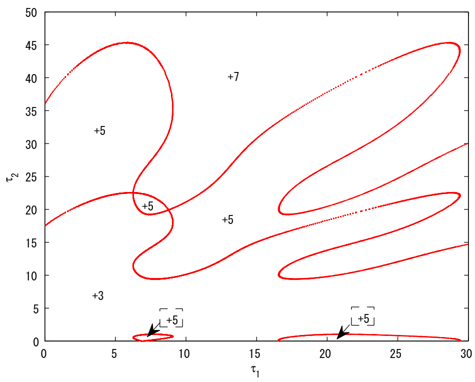

Figure 10: $\tau_3 = 20$

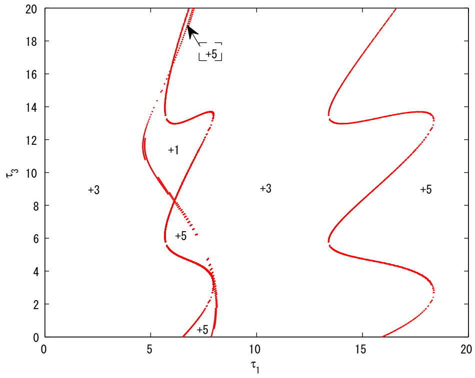

Figure 11: $\tau_2 = 0$

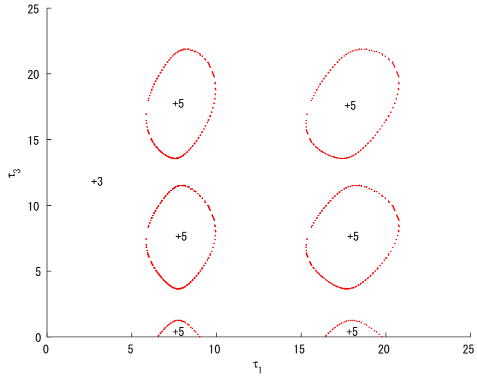

Figure 12: $\tau_2 = 10$

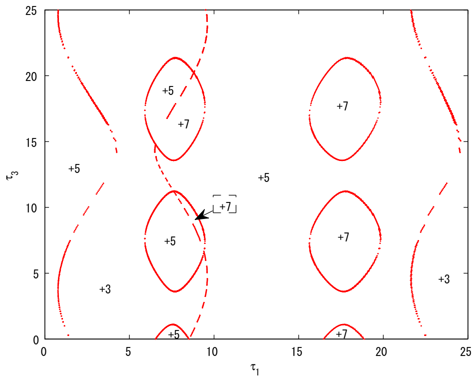

Figure 13: $\tau_2 = 20$

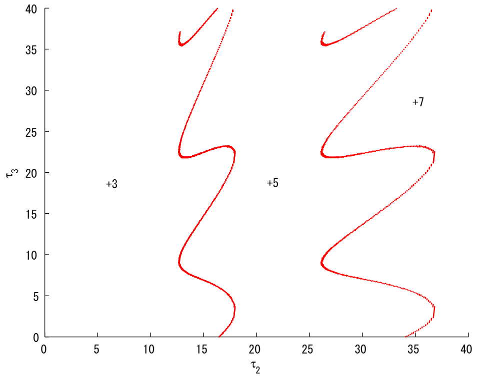

Figure 14: $\tau_1 = 0$

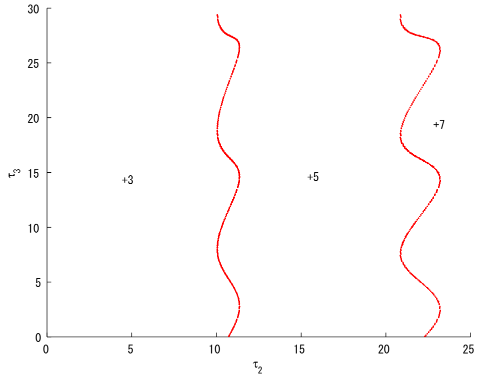

Figure 15: $\tau_1 = 10$

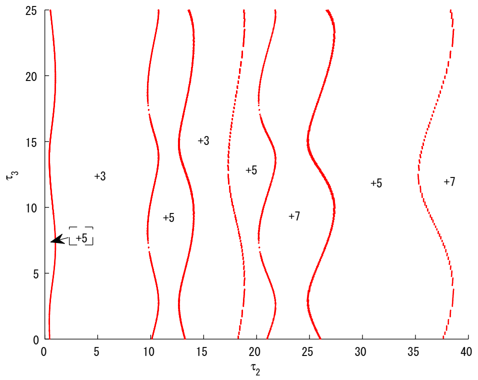

Figure 16: $\tau_1 = 20$

In each of these figures, an increase in the value of some lag parameter may cause any one of the following: determinacy, indeterminacy, or instability. For example, in Figure 11, if $\tau_3 = 10$, the dynamic property of the system changes with an increase in τ_1 as follows: determinate—indeterminate—determinate—unstable. Specifically, the regions of τ_1 that achieve determinacy are given by $\tau_1 \in [0, 5.13]$ and $\tau_1 \in [6.73, 16.4]$, which implies that if an implementation lag does not present in inflation targeting and a lag of approximately five years exists in asset-price targeting, then lags in output targeting must be approximately 0–2.5 or 3.4–8.2 years.

Furthermore, moving on the vertical axis of Figures 11–16 does not change the number of roots with positive real parts. This finding implies that a lag in asset-price targeting does not affect equilibrium determinacy, at least when τ_1 or τ_2 is sufficiently small.

In addition, as shown in Section 3, in the case where a lag is not present, if the condition in Equation (18) does not hold, the equilibrium is indeterminate. In this case, the signs of the real parts of the three roots are necessarily $+ + -$ (the case of $+ - -$ cannot occur). Accordingly, in the above case, there is no possibility that equilibrium determinacy is achieved by introducing a lag in a policy response because the number of roots with positive real parts necessarily changes by two in any case.

Moreover, cycles or other complex fluctuations may exist around the steady state because on the stability crossing curves and thus all conditions for a Hopf bifurcation are satisfied.

5 Conclusion

In this study, we used the NK model to analyze the effects of three policy lags on local equilibrium determinacy. Unlike Tsuzuki (2014, 2015), who only studies the effect of inflation targeting, the existence of multiple target variables in monetary policy (i.e., not only the inflation rate but also output and asset prices) provides a new possibility for a lag. In other words, even if the system includes only one lag, the lag can resolve the problem of instability. For example, when τ_1, which represents a lag in output targeting, moves on the horizontal axis of Figures 8 and 11, the equilibrium changes as follows: determinate—unstable—determinate. However, increases in τ_2 and τ_3, which denote lags in inflation and asset-price targeting, respectively, do not have such an effect. In this case, an increase in τ_2 necessarily causes instability, whereas that in τ_3 does not affect the dynamic property.

In cases where multiple policy lags coexist, the analysis becomes considerably more complicated. All lags can have a stabilizing effect on the equilibrium. Specifically, in Figures 13 and 16, τ_1 or τ_3 must be positive to achieve determinacy. If the values of these lags are zero, the equilibrium is unstable. This finding suggests that the central bank may be required to "purposefully" delay its policy implementation.

Moreover, the above results are valid only for the plausible parameter values assumed in Section 4. They do not have anything like generality. Depending on the parameter values, the type of Grashof set may change. Accordingly, the configuration of a stability crossing set may also dramatically alter.

A more theoretical investigation of the effects of policy lags requires an algebraic approach to differential equation systems with multiple delays. Unfortunately, such a method has not thus far been established. Nonetheless, the analysis performed in this study is helpful for policymakers. Indeed, the presented findings suggest that the central bank should determine its target variables by considering not only the responsiveness of the nominal interest rate to these variables but also the lag lengths associated with policy implementations.

In addition, the present study argues that (i) if a delay exists only in asset-price targeting, the Fed's view would assert its validity at least for a slight change in policy responsiveness, as in the case with no policy lags; and (ii) if multiple policy lags coexist, the Bank of International Settlements' view can become valid policy, depending on the lag parameter set.

A Appendix

A.1 Motion range of τ_u: the case of Grashof sets

In the case of $\Omega^h \subset \Omega_G^u$, $u = 0, 1, 2, 3$, the expression in Equation (39) holds with strict inequalities for all $\tau_u \in \mathbb{R}_+$ (see Equations 26 and 27–29). Therefore, $\mathcal{T}^h$ can be defined for $(\omega, \tau_u) \in \Omega^h \times \mathbb{R}_+$. Thus, the motion range of τ_u can be defined as

$$\mathcal{T}_u(\omega, r_u) = [\tau_u(\omega, r_u), \tau_u(\omega, r_u + 1)].$$

A.2 Motion range of τ_u: the case of Non-Grashof sets

In the case of $\Omega^h \subset \Omega_N^u$, $u = 0, 1, 2, 3$, the first inequality in Equation (39) holds for all $\tau_u \in \mathbb{R}_+$ (see Equations 30 and 31–33). However, the second inequality, $|a_d(i\omega, \tau_u)| \leq |a_v(i\omega)| + |a_w(i\omega)|$, is not ensured to hold. This inequality is equivalent to the following expression: $\pi - \theta_{um} \leq \arg\left(a_u(i\omega)e^{-i\omega\tau_u}\right) + 2r_u\pi \leq \pi + \theta_{um}$, where

$$\theta_{um} = \cos^{-1}\left(\frac{1 + |a_u(i\omega)|^2 - (|a_v(i\omega)| + |a_w(i\omega)|)^2}{2|a_u(i\omega)|}\right).$$

Therefore, we obtain

$$\tau_{um}(\omega, r_u) = \frac{\arg\left(a_u(i\omega)\right) + 2r_u\pi - \theta_{um}}{\omega},$$
$$\tau_{uM}(\omega, r_u) = \frac{\arg\left(a_u(i\omega)\right) + 2r_u\pi + \theta_{um}}{\omega}.$$

By using these expressions, the motion range of τ_u can be represented as follows:

$$\mathcal{T}_u(\omega, r_u) = [\tau_{um}(\omega, r_u), \tau_{uM}(\omega, r_u)].$$

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