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Love Dynamics Between Cautious Romeo and Juliet With Delays in Self and Cross Reactions

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Love Dynamics Between Cautious Romeo and Juliet with Delays in Self and Cross Reactions *

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Abstract

We investigate love dynamics of two individuals in a delay Romeo and Juliet model in which both are assumed to be cautious, the most natural romantic style. The local stability analysis proves first that the steady state is fairly stable when there are no delays and second that solving the characteristic equation generates a set of positive delays for which the steady state loses stability. Through numerical analysis, we confirm the following three main results: (1) cyclic oscillations of love feeling emerge via Hopf bifurcation; (2) multiple delays cause the double edge effect implying that alternation of stability and instability repeatedly occurs; (3) complicated dynamics involving chaotic oscillations emerges and then merges to a limit cycle as the length of one delay increases with fixed values of the other delay.

Keywords: Love dynamics, Delay differential equations, Cautious lovers, Hopf bifurcation, Double edge effect

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1 Introduction

This paper considers love affair dynamics with two distinct delays and extends the study of Matsumoto and Szidarovszky (2016) that deals with love dynamics with one delay. It presents a new characterization of love evolution by which a wide spectrum of love dynamics can be constructed, ranging from monotonic convergence to complicated oscillations including chaotic phenomena.

Differential equations have been used to describe dynamic phenomena in various fields of science; Navier-Stockes equations in physics, Schrödinger equation in quantum chemistry, Lotka-Voltera equations in biology, price dynamic system following the law of supply and demand in economics, to name only a few. In social psychology, Strogatz (1988) could be the first to model the dynamics of romantic feelings of two individuals named Romeo and Juliet with two differential equations. It is designed to evolve a love affair between Romeo and Juliet through two routes. It may be very natural that one's love grows more strongly when the other expresses his/her love more passionately. It is, on the contrary, also possible that one's feeling gets gradually down when this individual is a fickle lover. So the first route describes a change in one's own level of love feeling affected by the partner's level. We say that time evolution of the love affair through this route is nurtured in a *cross-reaction* process. Second, one's own level can change even in absence of the partner. If Juliet disappears by some reasons, Romeo gradually loses his affections as time goes on or he could live in old days memories, preserving his affections to Juliet. We say that love affection through this route evolves in a *self-reaction* process. Further, how love grows or decays depends on the romantic styles of Romeo and Juliet. According to Strogatz (1988), the styles are classified into four specifications, "eager beaver" if both of the self and cross reactions are positive, "cautious lover" if the self reaction is negative and the cross-reaction is positive, "narcissistic nerd" if the self-reaction is positive and the cross-reaction is negative and "hermit" or "stoic lover" if both are negative.

Taking into account of the real-life fact that the majority of the population is the cautious lover, we confine our main analysis to a case in which Romeo and Juliet are cautious lovers. Further, real life observations often tell us two distinguished evolutions of love after they meet for the first time. One is that love stories develop monotonically and, sooner or later, stay at their plateau for a long time (i.e., convergent dynamics). The other is that love stories arrive at a oscillating regime in which the feelings of the individuals sometimes fall into the bottomless pit, some other times rise up to the peak of happiness and these ups and downs repeat themselves (i.e., cyclic dynamics). In the existing literature, two types of models are known to explain these observations. On one hand, it has been demonstrated that a variant of Romeo and Juliet model can interpret various types of love stories described in "Pride and Prejudice," (Rinaldi et al. (2014)) "Gone with the Wind" (Rinaldi at al. (2013a)) and "Beauty and the Beast" (Rinaldi at al. (2013b)) by focusing on the appeals of the individuals. On the other hand, following natural phenomena that interactions between the individuals could be delayed, Liao and Ran (2007), Son and Park (2011) and

Bielczyk et al. (2013) focus on a delay in a nonlinear cross-reaction process and show the birth of cyclic dynamics through Hopf bifurcation. This finding indicates that the nonlinear delay model of Romeo and Juliet explains various oscillatory dynamics of two individuals' romantic feelings. In the existing literature, however, not much has been revealed with respect to multiple delays in reactions of the individuals to stimuli. In this paper we construct a class of love dynamic models of cautious individuals in which there are delays in both self-reaction and cross-reaction of love accumulation process and study the delay effects on love dynamics analytically as well as numerically.

In what follows, Section 2 presents the basic love dynamic model with no delays. In the first half of Section 3, we review Liao and Ran (2007) in which two delays are assumed in the cross-reaction process. Then in its latter half, we additionally introduce the self-reaction delay to their model and analytically derive a *stability switching curve* on which stability is switched to instability or vice versa. Section 4 conducts numerical simulations and demonstrates that the two delay model can generate rich dynamics to describe "many couples, many ways to express their love and affection". Section 5 concludes the paper and provides directions of future research.

2 Basic Model

We now construct a Romeo and Juliet model without delays, which is called a basic model. Let $x(t)$ and $y(t)$ denote levels of the romantic feelings of Romeo to Juliet and that of Juliet to Romeo at time t, respectively, if $x(t) > 0$ and $y(t) > 0$. The negativity of these state variables represents a level of the non-romantic or dislike feelings such as antagonism and disdain. Then the rates of change describe the feeling accumulation processes of the individuals and are assumed to have the following forms according to Rinaldi (1998):

$$\begin{aligned}\dot{x}(t) &= O_x(x(t)) + R_x(y(t)) + I_x \\ \dot{y}(t) &= O_y(y(t)) + R_y(x(t)) + I_y\end{aligned} \tag{1}$$

each of which is composed of three terms, *oblivion* denoted as O_z, for $z = x, y$, *return* by R_z and *instinct* by I_z. First, O_z gives rise to a loss of interest that describes the self-reaction process and depends on his/her own feeling level. It characterizes decay of love at disappearance of the partner in the self-reaction process. Second, R_z is a source of interest and describes the reaction of individual z to the partner's love in the cross-reaction process. Lastly, I_z is also a source of interest and describes the reaction of individual z to the partner's appeal reflecting physical, financial, educational, intellectual, well-born properties, the family background, etc. We adopt the following forms of these reaction functions:

Assumption 1: $O_x(x) = -\alpha_x x,\ \alpha_x > 0$ and $O_y(y) = -\alpha_y y,\ \alpha_y > 0$.

Assumption 2: $R_x(y) = \beta_x \tanh(y)$ and $R_y(x) = \beta_y \tanh(x)$.

Assumption 3: $I_x = \gamma_x A_y,\ A_y \geq 0$ and $I_y = \gamma_y A_x,\ A_x \geq 0$.

Assumption 1 confines attention to the case where the affection vanishes exponentially in the absence of the partner. Assumption 2 implies that the reaction is determined by a product of two terms, the reaction coefficient and the hyperbolic reaction that is positive, increasing, concave and bounded from above for positive values and is negative, increasing, convex and bounded from below for negative values. The love affection of individual z is encouraged or discouraged by the partner according to whether $\beta_z > 0$ or $\beta_z < 0$. Assumption 3 implies that individuals have time-invariant appeal. Non-zero A_z affects not only the location of the steady state but also love dynamics as mentioned in the Introduction. In this study, since we confine attention to the delay effects on love evolution, $A_z = 0$ is assumed to simplify the analysis.[1] Under these assumptions, equations (1) are reduced to a more specific system,

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t) + \beta_x \tanh[y(t)], \\ \dot{y}(t) &= -\alpha_y y(t) + \beta_y \tanh[x(t)], \end{aligned} \tag{2}$$

where α_z is the self-reaction or forgetting coefficient and β_z is the cross-reaction or return coefficient. The basic structure of system (2) is the same as that of the model Strogatz (1988) proposes. The minor difference is that a linear return function of Strogatz is replaced with a nonlinear hyperbolic function. We call it a basic Romeo and Juliet model. Matsumoto and Szidarovszky (2016) have already demonstrate that it has a unique zero steady state if $\alpha_x\alpha_y \geq \beta_x\beta_y$ and two more non- zero steady states if $\beta_x\beta_y > \alpha_x\alpha_y$. Accordingly, let (x_0^*, y_0^*) be the zero steady state and (x_k^*, y_k^*) the positive steady state if $k = 1$ and the negative steady state if $k = 2$.

To examine stability of the steady states, system (2) is linearized,

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t) + \beta_x d_y^k y(t), \\ \dot{y}(t) &= \beta_y d_x^k x(t) - \alpha_y y(t) \end{aligned} \tag{3}$$

where

$$d_x^k = \left.\frac{d\tanh(x)}{dx}\right|_{x=x_k^*} \text{ and } d_y^k = \left.\frac{d\tanh(y)}{dy}\right|_{y=y_k^*}.$$

Substituting exponential solutions $x(t) = e^{\lambda t}u$ and $y(t) = e^{\lambda t}v$ into system (3) yields the corresponding characteristic equation

$$\det\begin{pmatrix} \lambda + \alpha_x & -\beta_x d_y^k \\ -\beta_y d_x^k & \lambda + \alpha_y \end{pmatrix} = 0$$

[1] On the other hand a number of papers of Rinaldi treats the appeal as an ingredient factor for love evolution. Since our approach could complement Rinaldi's approach, a delay model with the appeal could bring about more fruitful results.

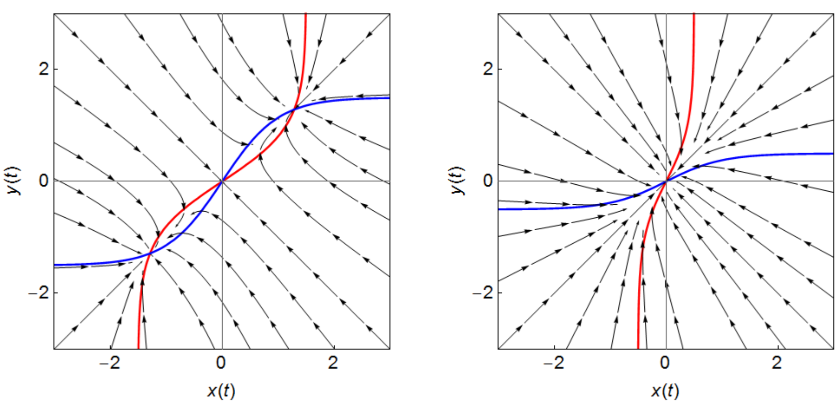
or

$$\lambda^2 + (\alpha_x + \alpha_y)\lambda + \alpha_x\alpha_y - \beta_x\beta_y d_x^k d_y^k = 0, \tag{4}$$

where $\alpha_x + \alpha_y > 0$ always by Assumption 1. It is also demonstrated in Matsumoto and Szidarovszky (2016) that $\alpha_x\alpha_y - \beta_x\beta_y d_x^k d_y^k > 0$ for $k = 1, 2$ and $d_x^k = d_y^k = 1$ for $k = 0$. Hence we have the following result, which is a summary of Theorem 2 of Matsumoto and Szidarovszky (2016);

Theorem 1 *The non-zero steady states are stable nodes while the zero-steady state is a saddle if $\beta_x\beta_y > \alpha_x\alpha_y$ and stable otherwise.*

The following numerical simulations confirm Theorem 1 and are done with $\alpha_x = \alpha_y = 1$, $\beta_x = \beta_y = 3/2$ in Figure 1(A) and $\beta_x = \beta_y = 1/2$ in Figure 1(B). It is clearly seen that (x_0^*, y_0^*) is a saddle and (x_k^*, y_k^*) for $k = 1, 2$ are stable in Figure 1(A) whereas the unique steady state (x_0^*, y_0^*) in Figure 1(B) is a stable node.[2]



(A) $\beta_x\beta_y > \alpha_x\alpha_y > 0$ (B) $\alpha_x\alpha_y > \beta_x\beta_y > 0$

Figure 1. Stability properties of the basic model

3 Two Delay Model

We introduce time delays in the interactions of the individuals in the basic model and consider how the delays affect love evolutions just described above by the nonlinear Romeo and Juliet model without any delays. Matsumoto and Szidarovszky (2016) have already confirmed the appearance of cyclic love dynamics in the model with one delay. Henceforth we investigate the effects caused by two distinct delays in this study.

[2] In the same way, it can be checked that the zero-steady state is stable in the cases of $0 > \beta_x\beta_y > -\alpha_x\alpha_x$ and $0 > -\alpha_x\alpha_x > \beta_x\beta_y$.

3.1 Liao and Ran Model

We first review the dynamic results provided by Liao and Ran (2007) that introduce two delays τ_x and τ_y into the cross-reaction processes of the basic model,

$$\dot{x}(t) = -\alpha_x x(t) + \beta_x \tanh[y(t-\tau_y)],$$

$$\dot{y}(t) = \beta_y \tanh[x(t-\tau_x)] - \alpha_y y(t).$$

The steady states of this model are the same as the ones of the basic model. To examine the stability of the steady states, the model is linearized in the neighborhood of the steady state (x_k^*, y_k^*)

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t) + \beta_x d_y^k y(t-\tau_y), \\ \dot{y}(t) &= \beta_y d_x^k x(t-\tau_x)] - \alpha_y y(t). \end{aligned} \tag{5}$$

Substituting exponential solutions $x(t) = e^{\lambda t}u$ and $y(t) = e^{\lambda t}v$ into system (5) yields, after arranging the terms, the corresponding characteristic equation

$$\lambda^2 + (\alpha_x + \alpha_y)\lambda + \alpha_x\alpha_y - \beta_x\beta_y d_x^k d_y^k e^{-\lambda\tau} = 0 \tag{6}$$

with $\tau = \tau_x + \tau_y$. Although Liao and Ran (2007) introduce two delays into the cross-reaction processes, dynamics generated by their model is essentially the same as the one by the model with one delay in the cross-reaction process since only the sum of two delays plays a crucial role in determining dynamic behavior in their model. Hence, following Matsumoto and Szidarovszky (2016), it can be shown first that all pure complex eigenvalues of equation (6) are simple and second that the non-zero steady states are locally asymptotical stable. Liao and Rao (2007), on the other hand, concern with dynamics of the zero-steady state and show the following result, which restates their Theorem 1:

Theorem 2 *Concerning the zero-steady state, (x_0^*, y_0^*), no stability switch occurs if $\beta_x\beta_y \geq \alpha_x\alpha_y$ and $\alpha_x\alpha_y > \left|\beta_x\beta_y\right|$ while a stability switch occurs only once at $\tau = \tau_0$ if $0 > -\alpha_x\alpha_y > \beta_x\beta_y$, implying that the zero stationary state is locally asymptotically stable if $\tau < \tau_0$, loses stability if $\tau = \tau_0$ and bifurcates to a limit cycle if $\tau > \tau_0$ where the threshold value of τ is defined as*

$$\tau_0 = \frac{1}{\omega_0}\sin^{-1}\left(\frac{(\alpha_x+\alpha_y)\omega_0}{\beta_x\beta_y}\right)$$

with

$$\omega_0 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{(\alpha_x^2-\alpha_y^2)^2 + 4\left(\beta_x\beta_y\right)^2} - \left(\alpha_x^2+\alpha_y^2\right)}.$$

A Hopf bifurcation resulting in cyclic dynamics occurs under two conditions, (i) the product of the cross reaction coefficients is larger than the product of the self reaction coefficients and (ii) one of the two individuals is a hermit and the other is cautious. The first one could be possible, however, the second might be unusual, as already been pointed out by Son and Park (2011). To remedy this deficit, we will limit the styles of Romeo and Juliet to cautious lovers henceforth and then pursue possible emergencies of cyclic oscillations:

Assumption 2'. $\beta_x > 0$ and $\beta_y > 0$.

3.2 Delay Romeo and Juliet Model

We introduce multiple delays into the self- and cross-reaction processes in such a way that each delay has an independent role:

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t-\tau_x^s) + \beta_x \tanh[y(t-\tau_y^c)], \\ \dot{y}(t) &= \beta_y \tanh[x(t-\tau_x^c)] - \alpha_y y(t) \end{aligned} \tag{7}$$

where $\tau_x^s > 0$ is a self-reaction delay of Romeo[3] while $\tau_x^c \geq 0$ and $\tau_y^c \geq 0$ cross-reaction delays of Romeo and Juliet satisfying $\tau_x^c + \tau_y^c > 0$. The linearized version of model (7) is

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t-\tau_x^s) + \beta_x d_y^k y(t-\tau_y^c), \\ \dot{y}(t) &= \beta_y d_x^k x(t-\tau_x^c) - \alpha_y y(t), \end{aligned}$$

and the analysis of the Jacobian as before reveals that the characteristic equation is

$$\lambda(\lambda+\alpha_y) + \alpha_x(\lambda+\alpha_y)e^{-\lambda\tau_1} - \beta_x\beta_y d_x^k d_y^k e^{-\lambda\tau_2} = 0 \tag{8}$$

where $\tau_1 = \tau_x^s$ and $\tau_2 = \tau_x^c + \tau_y^c$. Following the method provided by Gu et al. (2005), we detect the location of the eigenvalues of this characteristic equation in the same way as in Matsumoto and Szidarovszky (2015).

Since $\alpha_x\alpha_y - \beta_x\beta_y d_x^k d_y^k > 0$ for $k = 1, 2$ is shown in Matsumoto and Szidarovszky (2016), $\lambda = 0$ is not a solution of equation (8). Then dividing both sides of the characteristic equation by $\lambda(\lambda+\alpha_y)(\neq 0)$ reduces the left hand side to

$$a(\lambda) = 1 + a_1(\lambda)e^{-\lambda\tau_1} + a_2(\lambda)e^{-\lambda\tau_2}$$

where

$$a_1(\lambda) = \frac{\alpha_x}{\lambda} \text{ and } a_2(\lambda) = -\frac{\beta_x\beta_y d_x^k d_y^k}{\lambda(\lambda+\alpha_y)}.$$

Theorem 1 ensures the stability of the non-zero steady state when there are no delays (i.e., $\tau_1 = \tau_2 = 0$). We now examine whether the switching from stability to instability or vice verse can occur at some positive values of the delays. To this end, we check if the characteristic equation (8) has a pair of pure imaginary solutions at the threshold values of the delays. Let us suppose

[3] The qualitatively same results will be obtained even if the self-reaction delay of Romeo is replaced with a self-reaction delay of Juliet.

that $\lambda = i\omega,\ \omega > 0$ where we have the same results under $\omega < 0$ as the solutions are conjugate. We then have

$$a_1(i\omega) = -i\frac{\alpha_x}{\omega}$$

and

$$a_2(i\omega) = \frac{\beta_x\beta_y d_x^k d_y^k \omega}{\omega(\omega^2 + \alpha_y^2)} + i\frac{\alpha_y\beta_x\beta_y d_x^k d_y^k}{\omega(\omega^2 + \alpha_y)}.$$

Their absolute values are

$$|a_1(i\omega)| = \frac{\alpha_x}{\omega}$$

and

$$|a_2(i\omega)| = \frac{\left|\beta_x\beta_y d_x^k d_y^k\right|}{\omega\sqrt{\omega^2 + \alpha_y^2}}.$$

Furthermore,

$$\arg\left[a_1(i\omega)\right] = \frac{3\pi}{2} \text{ and } \arg\left[a_2(i\omega)\right] = \tan^{-1}\left[\frac{\alpha_y}{\omega}\right].$$

Solving $a(i\omega) = 0$ directly is not an easy job. However, if the three terms in $a(i\omega)$ are considered to be three vectors in the complex plane, then solving it is equivalent to constructing a triangle from the three terms satisfying the following (triangle) conditions,

$$(1)\ 1 \leq |\alpha_1(i\omega)| + |\alpha_2(i\omega)|$$

$$(2)\ \ |\alpha_1(i\omega)| \leq 1 + |\alpha_2(i\omega)|$$

$$(3)\ \ |\alpha_2(i\omega)| \leq 1 + |\alpha_1(i\omega)|$$

Conditions (1) and (2) are written as

$$f(\omega) = (\omega - \alpha_x)^2\left(\omega^2 + \alpha_y^2\right) \leq \left(\beta_x\beta_y d_x^k d_y^k\right)^2$$

and condition (3) as

$$g(\omega) = (\omega + \alpha_x)^2\left(\omega^2 + \alpha_y^2\right) \geq \left(\beta_x\beta_y d_x^k d_y^k\right)^2.$$

Hence, the triangle conditions are satisfied if

$$f(\omega) \leq \left(\beta_x\beta_y d_x^k d_y^k\right)^2 \leq g(\omega).$$

It is to be noticed that

$$f(0) = (\alpha_x\alpha_y)^2,\ f(\alpha_x) = 0,\ f(\pm\infty) = \infty \text{ and } f(\omega) < g(\omega).$$

Differentiating $f(\omega)$ yields

$$f'(\omega) = (\omega - \alpha_x)(4\omega^2 - 2\omega\alpha_x + 2\alpha_y^2).$$

The first factor is negative if $\omega < \alpha_x$ and positive if $\omega > \alpha_x$ while the sign of the second factor seems to be ambiguous. Let

$$\varphi(\omega) = 4\omega^2 - 2\omega\alpha_x + 2\alpha_y^2$$

and the roots of $\varphi(\omega) = 0$ are

$$\bar{\omega}_{1,2} = \frac{\alpha_x \pm \sqrt{\alpha_x^2 - 8\alpha_y^2}}{4}.$$

For simplicity, we make the following[4]:

Assumption 4: $\alpha_x \simeq \alpha_y$ such that $\alpha_x^2 < 8\alpha_y^2$ always.

The negative discriminant implies that $\varphi(\omega) \geq 0$ for all ω. Hence we have

$$f'(\omega) \begin{cases} < 0 & \text{if } \omega < \alpha_x, \\ \\ > 0 & \text{if } \omega > \alpha_x. \end{cases}$$

So the domain for ω in which the triangle conditions are satisfied is defined as follows.

Theorem 3 *Let $B = \left(\beta_x\beta_y d_x^k d_y^k\right)^2$. Then the domain is between roots of $f(\omega) = B$ if $B \leq (\alpha_x\alpha_y)^2$ and between roots of $g(\omega) = B$ and $f(\omega) = B$ if $B > (\alpha_x\alpha_y)^2$.*

Let the three terms in $a(i\omega)$ be three vectors forming a triangle. We suppose that $|1|$, the absolute value of vector 1, is its base and denote the angle between $|1|$ and $|\alpha_1(i\omega)|$ by θ_1 and the angle between $|1|$ and $|\alpha_2(i\omega)|$ by θ_2. Then by the law of cosine, we have

$$\cos\theta_1 = \frac{(\omega^2 + \alpha_x^2)(\omega^2 + \alpha_y^2) - (\beta_x\beta_y d_x^k d_y^k)^2}{2\alpha_x\omega(\omega^2 + \alpha_y^2)}$$

and

$$\cos\theta_2 = \frac{(\omega^2 - \alpha_x^2)(\omega^2 + \alpha_y^2) + (\beta_x\beta_y d_x^k d_y^k)^2}{2\beta_x\beta_2\omega\sqrt{\omega^2 + \alpha_y^2}}.$$

[4] If $\alpha_x^2 \geq 8\alpha_y^2$ holds, then the discriminant is nonnegative. Both roots $\bar{\omega}_1$ and $\bar{\omega}_2$ are real, positive and less than α_x. So we have

$$\varphi(\omega) \begin{cases} > 0 & \text{if } \omega < \bar{\omega}_1 \text{ or } \omega > \bar{\omega}_2, \\ \\ < 0 & \text{if } \bar{\omega}_1 < \omega < \bar{\omega}_2. \end{cases}$$

Hence the sign of $f'(\omega)$ is determined such as

$$f'(\omega) \begin{cases} < 0 & \text{if } \omega < \bar{\omega}_1 \text{ or } \bar{\omega}_2 < \omega < \alpha_x, \\ \\ > 0 & \text{if } \bar{\omega}_1 < \omega < \bar{\omega}_2 \text{ or } \omega > \alpha_x. \end{cases}$$

So domain of ω can be defined in various ways. However, it is numerically checked that dynamics obtained under $\alpha_x^2 \geq 8\alpha_y^2$ is essentially the same as the one under $\alpha_x^2 < 8\alpha_y^2$.

Solving these two equations for θ_1 and θ_2 gives

$$\theta_1(\omega) = \cos^{-1}\left[\frac{(\omega^2+\alpha_x^2)(\omega^2+\alpha_y^2)-(\beta_x\beta_y d_x^k d_y^k)^2}{2\alpha_1\omega(\omega^2+\alpha_y^2)}\right]$$

and

$$\theta_2(\omega) = \cos^{-1}\left[\frac{(\omega^2-\alpha_x^2)(\omega^2+\alpha_y^2)-(\beta_x\beta_y d_x^k d_y^k)^2}{2\beta_1\beta_2\omega\sqrt{\omega^2+\alpha_y^2}}\right].$$

Since the triangle may be located above and below the horizontal axis in the complex plane, we have two possibilities,

$$\left\{\arg\left[\alpha_1(i\omega)e^{-i\omega\tau_1}\right]+2k\pi\right\}\pm\theta_1(\omega)=\pi$$

and

$$\left\{\arg\left[\alpha_2(i\omega)e^{-i\omega\tau_2}\right]+2n\pi\right\}\mp\theta_2(\omega)=\pi.$$

Using the formula $\arg[\alpha_k(i\omega)e^{-i\omega\tau_k}] = \arg[\alpha_k(i\omega)] + \arg[e^{-i\omega\tau_k}]$ and solving these equations for τ_1 and τ_2 yield the threshold values of the delays,

$$\tau_1^{\pm}(\omega,k) = \frac{1}{\omega}\left\{\arg\left[\alpha_1(i\omega)\right]+2(k-1)\pi\pm\theta_1(\omega)\right\}$$

and

$$\tau_2^{\mp}(\omega,n) = \frac{1}{\omega}\left\{\arg\left[\alpha_2(i\omega)\right]+2(n-1)\pi\mp\theta_2(\omega)\right\}$$

for $(k,n)=0,1,2,...$ We can find the pairs of (τ_1,τ_2) constructing the *partition curves* consisting of two sets of parametric segments for $k,n\geq 0$,

$$L_1(k,n)=\left\{\tau_1^+(\omega,k),\tau_2^-(\omega,n)\right\}\text{ for }\omega\in[\omega_s,\omega_e]$$

and

$$L_2(k,n)=\left\{\tau_1^-(\omega,k),\tau_2^+(\omega,n)\right\}\text{ for }\omega\in[\omega_s,\omega_e].$$

Here $[\omega_s,\omega_e]$ denotes the domain of ω in which the triangle conditions hold. According to Theorem 2, the left hand extreme value or the starting value ω_s solves $f(\omega)=B$ if $B\leq(\alpha_x\alpha_y)^2$ and $g(\omega)=B$ if $B>(\alpha_x\alpha_y)^2$ while the right hand extreme value or the ending value ω_e always solves $f(\omega)=B$. The next result confirms that the segments of $L_1(k,n)$ and $L_2(k,n)$ with fixed k and varying n smoothly connected to form one continuous curve.[5]

Theorem 4 *With a fixed value of k, the segments of $L_1(k,n)$ and $L_2(k,n)$ form a continuous curve as n increases.*

Notice two issues, one is that characteristic equation (8) has a pair of pure imaginary roots on these partition curves and the other is that given k and n, the partition curve divides the (τ_1,τ_2) region into subregions according to the number of the eigenvalues whose real parts are positive. In consequence, since characteristic equation (8) is reduced to equation (4) as $\tau_1=\tau_2=0$, the steady state is locally stable for any pair of (τ_1,τ_2) in the separated region including the origin.

[5]Applying Theorem 1 of Matsumoto and Szidarovszky (2015), we can demonstrate that these segments from a continuous curve

4 Numerical Simulations

In this section we specify the parameter values and perform numerical simulations first to confirm the conditions under which stability switching occurs and then to detect what kind of dynamics emerges when stability is lost. Under Assumptions 1, 2 and 2', we can identify two cases, $\beta_x\beta_y > \alpha_x\alpha_y > 0$ and $\alpha_x\alpha_y > \beta_x\beta_y > 0$. These cases are successively considered in the sequel.

4.1 $\beta_x\beta_y > \alpha_x\alpha_y > 0$

Under this parametric conditions, the following two issues have been already shown in the basic or non-delay model:

(1) three steady states exist,

(2) the non-zero steady states are locally asymptotically stable and the zero-steady state is a saddle.

In this section we are concerned with the delay effects caused by changing the values of τ_1 and τ_2 on dynamics. More precisely, specifying the parameter values, we numerically examine whether the delays can destabilize the non-zero steady states and what dynamics can emerge when the stability is lost. As in Figure 1(A), we make the following:

Assumption 4: $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 3/2$

Under Assumption 4, Figure 2 illustrates the segments of $L_1(k, n)$ in red and $L_2(k, n)$ in blue with the values of ω varying from $\omega_s \simeq 0.883$ to $\omega_e \simeq 1.104$ for $k = 0$ and $n = 0, 1, 2$. The green and purple dots are connecting points of the red and blue segments, the red segments start at the green points and end at the purple points while the blue segments end at the green points and start at the purple points.[6] It is to be noticed that the red and blue segments shift upward when n increases and rightward when k increases. The $L_1(0, 0)$ segment is located below the horizontal axis so it is not depicted. In order to keep τ_2 non-negative, the lower most blue segment of $L_2(0, 0)$ is illustrated only for $\tau_1 \geq \tau_1^0 \simeq 1.617$.[7] Increasing the value of τ_2 along the vertical dotted line at $\tau_1 = \tau_1^0$ intersects the partition curve three times at

$$\tau_2^a \simeq 2.618,\ \tau_2^b \simeq 6.837 \text{ and } \tau_2^c \simeq 8.697$$

which are denoted by the black dots.

Concerning stability, we first examine dynamics in the following two regions of Figure 2, the LHS (left hand side) region shaded by positive-sloping lines in

[6] In the case of $n = 1$, segment $L_1(0, 1)$ starts at the end point (i.e., the lowest green point) of $L_2(0, 0)$ and ends at the starting point (i.e., the lower purple point) of $L_2(0, 1)$.The same connection is repeated as k increases.

[7] Taking $k = n = 0$ and solving $\tau_2^+(\omega, 0) = 0$ yields $\omega = \bar{\omega}$. Substituting into $\tau_1^-(\omega, 0)$ presents approximately this threshold value.

which $0 \leq \tau_1 < \tau_1^m$ and $\tau_2 \geq 0$ and the RHS (right hand side) region shaded by the negative-sloping lines in which $\tau_1 \geq \tau_1^M$ and $\tau_2 \geq 0$. Here τ_1^m is the minimum τ_1-value of the segments $L_2(0,n)$ and τ_1^M is the maximum τ_1-value of the segments $L_1(0,n)$.[8] Since the origin, $\tau_1 = \tau_2 = 0$ is in the LHS region, any combination of the delays in the LHS region do not affect stability of the non-zero steady states and such a delay is called *harmless.* In order to arrive at the RHS region for any value of τ_2, increasing τ_1 from zero must cross the stability switching curve and then the vertical dotted line at $\tau_1 = \tau_1^M$. This implies that at least one of the eigenvalues must have a positive real part for any pair (τ_1, τ_2) in the RHS region. In other words, the stability is lost in the RHS region. Roughly speaking, for any value of τ_2, the stability of the non-zero steady states is preserved for smaller values of τ_1 and lost for larger values of τ_1. These results are summarized as follows:

Proposition 5 *The non-zero steady states of system (7) are locally asymptotically stable if* $0 \leq \tau_1 < \tau_1^m$ *and* $\tau_2 \geq 0$ *and always unstable if* $\tau_1 \geq \tau_1^M$ *and* $\tau_2 \geq 0$.

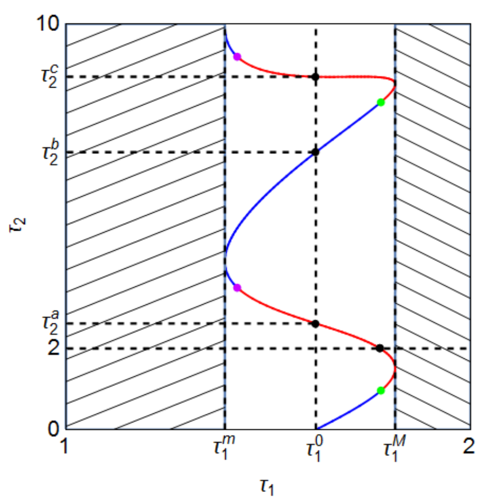


Figure 2. The stability switching curve for $k = 0$ and $n = 0, 1, 2$.

We now turn attention to the remaining area in which $\tau_1^m \leq \tau_1 \leq \tau_1^M$ and $\tau_2 \geq 0$. The red-blue connected curve in Figure 2 is called a *stability switching curve* since dynamics is switched from stability to instability (i.e., stability loss) or from instability to stability (i.e., stability gain) on this curve. We numerically

[8] Solving $d\tau_1^-/d\omega = 0$ for ω to obtain the minimizer $\omega_m \simeq 1.084$ and then substituting it into $\tau_1^-(\omega, 0)$ yields $\tau_1^m \simeq 1.393$. The other value $\tau_1^M \simeq 1.813$ is determined in the similar way.

confirm how the stability switch takes place on the stability switching curve and what dynamics arises when a stability loss takes place.[9] We first consider the effects caused by the change in τ_1 with a fixed value of τ_2 and then proceed to the effect caused by the change in τ_2, with a fixed value of τ_1.

Simulation 1: $1 \leq \tau_1 \leq 3.5,\ \tau_2 = 2$.

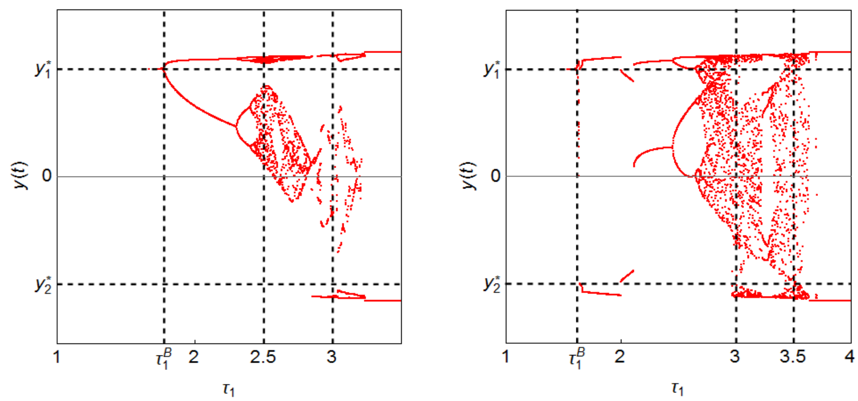
The dotted horizontal line at $\tau_1 = 2$ crosses the red segment $L_1(0,1)$ at the dotted point in Figure 2. Although it is not labelled, the τ_1-value of the intersection will be denoted by $\tau_1^B \simeq 1.776$ in which B implies bifurcation. Figure 3 illustrates two bifurcation diagrams with respect to τ_1 in which delay system (7) runs for $0 \leq t \leq 1000$ and initial functions defined for $t \in [-\tau_1, 0]$ are selected to be constant, $x_0(t) = x_1^* + 0.1$ and $y_0(t) = y_1^* + 0.2$. In Figure 3(A), τ_1 is increased from 1 to 3.5 with an increment of 1/400 along the dotted line at $\tau_2 = 2$. The local maximum and minimum values of $y(t)$ for $950 \leq t \leq 1000$ are plotted again each value of τ_1 to take away initial disturbances. The horizontal part of the diagram for $y = y_1^*$ and $1 \leq \tau_1 < \tau_1^B$ implies that the positive steady state is locally asymptotically stable. The stability is lost at $\tau_1 = \tau_1^B$ and a limit cycle oscillating around y_1^* emerges for $\tau_1 > \tau_1^B$ when the diagram has two branches. It then gets complicated more through a period-doubling-like cascade as τ_1 further increases. As τ_1 becomes closer to 3.5, the complicated dynamics converges to a big limit cycle including two steady states inside. If we take the values of the constant initial functions in the neighborhood of the negative steady state, y_2^*, we then have exactly the same dynamics whose bifurcation diagram is symmetric to the one in Figure 3(A) with respect to the horizontal axis, which is not illustrated to avoid confusion.

Simulation 2: $1 \leq \tau_1 \leq 4.5,\ \tau_2 = \tau_2^a$.

In Figure 3(B), the fixed value of τ_2 is increased to τ_2^a and the same procedure generates a different shape of the bifurcation diagram showing larger and more complicated oscillations when stability is lost at $\tau_1 = \tau_1^B (= \tau_1^0)$ at which the horizontal line at $\tau_2 = \tau_2^a$ crosses the $L_1(0,1)$ segment in Figure 2. It is to be noticed that $\tau_1^B = \tau_1^0 \simeq 1.617$ in Figure 3(A) and $\tau_1^B \simeq 1.776$ in Figure 3(B)

[9] It is possible to analytically confirm the direction of the stabililty switch. See Theorems 3 and 4 in Matsumoto and Szidarovszky (2015) and Proposition 6.1 in Gu et al. (2005).

though the same notation is used.



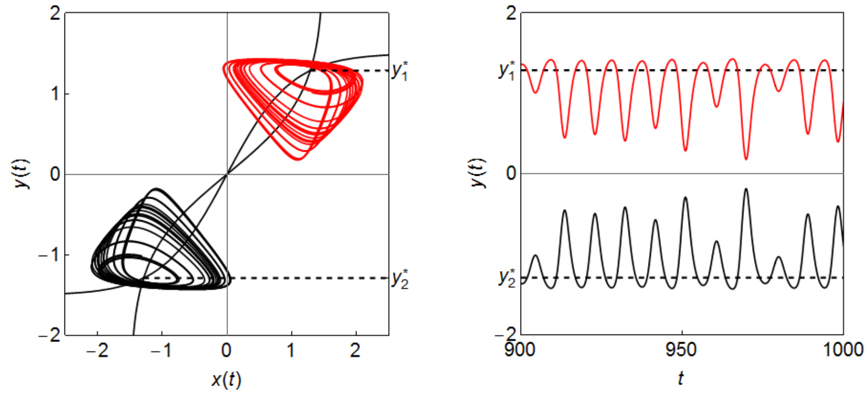
(A) $\tau_2 = 2$ (B) $\tau_2 = \tau_2^a$

Figure 3. Bifurcation diagrams with respect to τ_1

Simulation 3: $\tau_1 = 2.5$ and $\tau_2 = 2$

In the next simulation, we keep $\tau_2 = 2$ and fix the values of τ_1 at 2.5. By doing so, we will see dynamics scattered along the dotted vertical line at $\tau_1 = 2.5$ seen in Figure 3(A) from a different point of view. With these parameter values, the dynamic system runs for $0 \leq t \leq 1000$ with two different initial functions defined for $t \leq 0$, one with $x_0(t) = x_1^* + 0.1$ and $y_0(t) = y_1^* + 0.2$ as before and the other with $x_0(t) = x_2^* - 0.1$ and $y_0(t) = y_2^* - 0.2$. Two phase diagrams are plotted in the (x, y) plane in Figure 4(A) in which a trajectory with the positive initial functions converges to an upper attractor surrounding the positive steady state y_1^* and a trajectory with the negative initial functions approaches the lower attractor surrounding the lower steady state y_2^*. The attractors are point symmetric with respect to the origin. The corresponding time trajectories of $y(t)$ for $900 \leq t \leq 1000$ are depicted in Figure 4(B) in which both are symmetric with respect to the horizontal axis.

These simulations explain well our two typical love experiences. The first one is that love developments strongly depends on the first feelings that the individuals obtain at their first meeting. When trajectories start in the first quadrant of the (x, y) plane, both individuals are thought to have "good feeling" and can develop their romantic feelings to more favorable situation. On the contrary when both have "not good feelings," the developments grow into the opposite direction. The second one is that love feelings are often fluctuating in a wavering manner around their "true love" levels even if they fell in love. As is seen in Figure 4(B), the romantic feeling of Rome and Juliet are alternating between staying and going.



(A) Phase diagrams

(B) Time trajectories

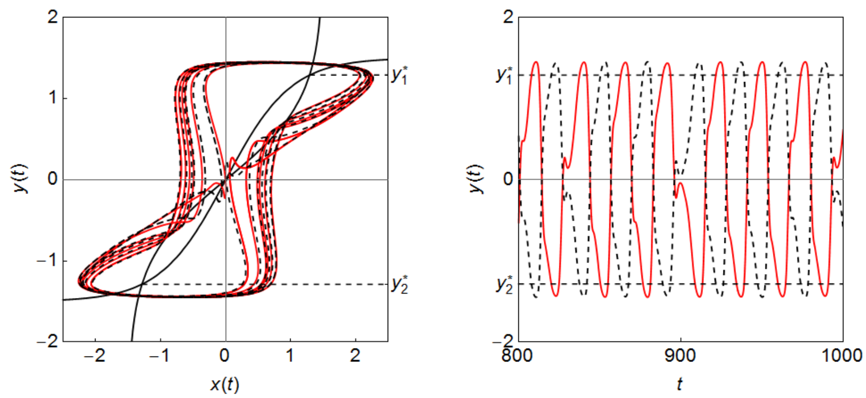
Figure 4. Dynamics with $\tau_1 = 2.5$ and $\tau_2 = 2$

Simulation 4: $\tau_1 = 3$ and $\tau_2 = 2$

We increase τ_1 to 3 from 2.5, taking $\tau_2 = 2$ in the next example. We again observe love dynamics on the vertical line at $\tau_1 = 3$ in Figure 3(A). Resultant dynamics with the positive initial functions is depicted in red and dynamics with the negative initial functions in black.[10] A parametric difference seems to be minor, however, major different dynamics emerges. Comparing Figure 5(A) with Figure 4(A), we can see that increasing the value of τ_1 makes two attractors merge into one large attractor including the two steady states inside by enlarging the size of each attractor. In Figure 5(A) two large attractors are depicted in red and black. Due to the point symmetry, the attractors are not identical, however, generated dynamics are essentially the same and line-symmetric with respect to the horizontal axis as seen in Figure 5(B). Although love and hate are opposite emotions, our daily-life experiences indicate that one can often turn into the other and vice versa. These examples describe these alternative feelings between love and hate. These results remind us of cyclical emotional feelings of Petrach, an Italian poet of the 14th century to Laura, a beautiful but married woman described by Jones (1995). Stimulated by this work, Rinaldi (1998) has successfully described the poet's regular cyclic behavior ranging from ecstasy to despair with a dynamic system of three ordinary differential equations without delays but with instincts (i.e., appeals) which are assumed away in our model

[10] The initial functions $x_0(t) = 0.1$ and $y_0(t) = 0.2$ for $t < 0$ are selected for the red curve and $x_0(t) = -0.1$ and $y_0(t) = -0.2$ for the black curve.

by Assumption 3.



(A) Phase diagrarams (B) Time trajectories

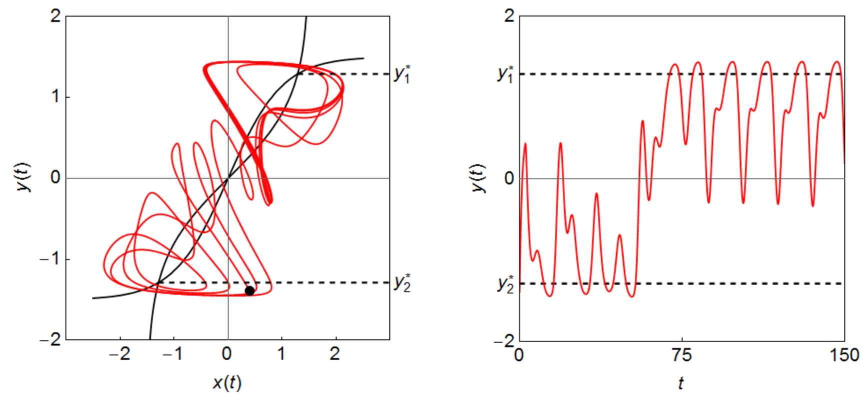
Figure 5. Dynamics with $\tau_1 = 3$ and $\tau_2 = 2$

Simulation 5: $\tau_1 = 3$ and $\tau_2 = \tau_1^a$

We now turn attention to the bifurcation diagram in Figure 3(B) with $\tau_2 = \tau_2^a$. Taking $\tau_1 = 3$, Figure 6(A) depicts a phase diagram with the initial functions $x_0(t) = 0.4$ and $y_0(t) = -y_2^* - 0.1$ for $t \leq 0$. Notice that dynamics starts with the point $(x_0(0), y_0(0))$ denoted by the black dot in the fourth quadrant of the (x, y) plane, oscillates around the negative steady state for a while and then converges to a limit cycle surrounding the positive steady state. This numerical example provides a green light to the Beast, a witched young handsome prince, in "Beauty and the Beast." Even starting with a unfavorable point where the Beast has good feelings to Beauty named Belle but Beauty has negative feelings due to the terrifying and ugly appearance of the Beast, their love story described by delay system (7) with suitable parametric values arriving at a regime where Belle is happy and the Beast can be transformed to a prince.[11] It is also numerically confirmed that a similar cyclic oscillation around the negative steady state is obtained when the initial point is selected in the second quadrant where Romeo has negative feelings and Juliet has positive feeling when

[11] Rinaldi et al.(2013b) interprete this love story with non-delay model (1) without Assumption 3. Positivity of the appeal A_z is a key factor that brings Beauty and the Beast to their plateau.

they first meet.

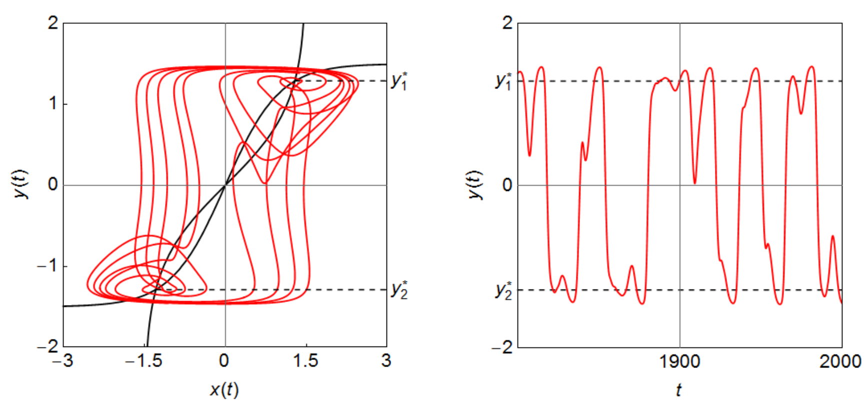


(A) Phase diagram (B) Time trajectory

Figure 6. Dynamics with $\tau_1 = 3$ and $\tau_2 = \tau_2^a$

Simulation 6: $\tau_1 = 3.5$ and $\tau_2 = \tau_2^a$

In the last simulation with Figure 3(B), the value of τ_1 is increased to 3.5. More realistic love-hate alternations are described in Figures 7(A) and 7(B). Emotions oscillate around the positive steady state (i.e., love regime) for a while and then suddenly change their directions, going toward the negative steady state, around which emotions oscillate in the neighborhood sooner or latter, a sudden change occurs again to bring back the negative oscillations to the positive oscillations. This alternative oscillation continues. Contrary to a regular cyclic pattern in Figure 5(A), a quite irregular pattern appears ranging from one extreme to the other.

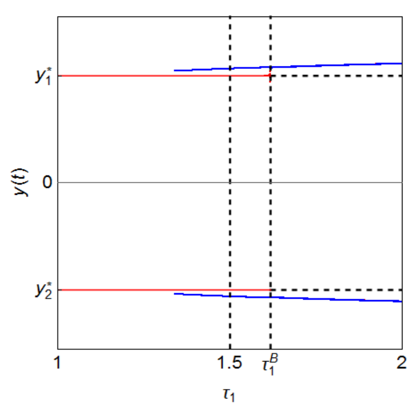


(A) Phase diagram (B) Time trajectory

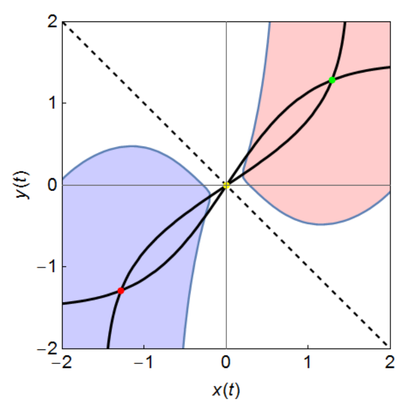
Figure 7. Dynamics with $\tau_1 = 3.5$ and $\tau_2 = \tau_2^a$

Simulation 7: $1 \leq \tau_1 \leq 2,\ \tau_2 = \tau_2^b$.

Now the value of τ_2 is increased to τ_2^b from τ_2^a. In Figure 8(A) two blue curves, one located slightly above the $y = y_1^*$ locus and the other below the $y = y_2^*$ locus, present bifurcation diagrams with respect to τ_1. With this, it is first observed that no more complicated dynamics appears but only a simple limit cycle emerges when the stability of the non-zero steady states is lost for $\tau_1 > \tau_1^B (= \tau_1^0)$.[12] This result implies two issues. One is that given τ_1, a larger value of τ_2 generates simplified dynamics. The other is that multi-stability can occur as two branches are illustrated for a small deviation of τ_1 from τ_1^B. A basin of attraction is presented in Figure 8(B) with $\tau_1 = 3/2$ and $\tau_2 = \tau_2^b$. The region of the initial points, $(x_0(0), y_0(0))$, is divided into three subregions, the light-red region including the positive steady state denoted by the green dot, the light-blue region including the negative steady state by the red dot and the white region including the zero-steady state denoted by the yellow dot. This division indicates an initial point dependency of dynamics, namely, dynamics converges to the positive steady state, to the negative steady state or to a limit cycle according to whether the point $(x_0(0), y_0(0))$ is selected from the red region, the blue region or the white region of Figure 8(B). This dependency also explains something that happens often in a love story, that is, even though Romeo and Juliet are into something goods for example, a small misunderstanding described by a small change in the initial functions can lead them to a different worlds in which one's love feeling fluctuates from one extrem to the other.



(A) Bifurcation diagrams



(B) Basin of attraction

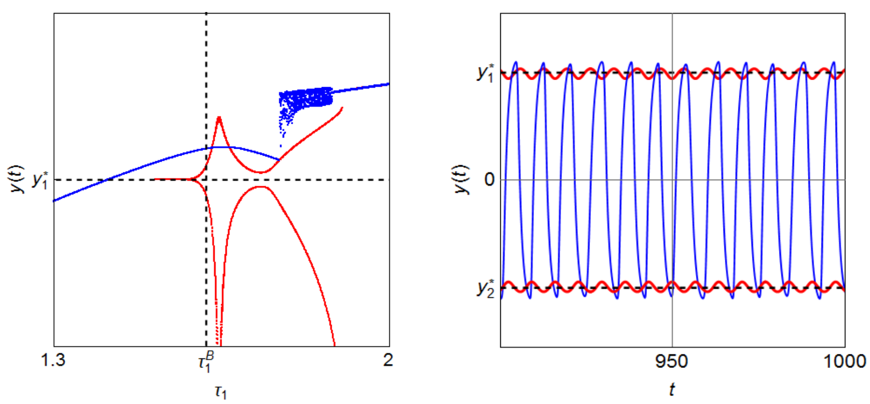
Figure 8. Dynamics with $\tau_1 = 3/2$ and $\tau_2 = \tau_2^a$

Simulation 8: $1 \leq \tau_1 \leq 2,\ \tau_2 = \tau_2^c$.

In the last simulation in this section, the value of τ_2 is further increased to τ_2^c. It is confirmed that multi-stability occurs in the neighborhood of the

[12] Although it is not illustrated in Figure 8(A), the simple shape of the bifurcation diagrams is numerically confirmed for larger values of τ_1.

bifurcation value of $\tau_1^B(=\tau_1^0)$. As can be seen, small red limit cycles surrounding the non-zero steady state coexist with a large blue limit cycle surrounding the zero steady state and including the non-zero steady states inside. Concerning the red cycle, we notice the followings: the positive steady state y_1^* is stable for $\tau_1 < \tau_1^B$ and bifurcates to a limit cycle for $\tau_1 > \tau_1^B$; a radius of the limit cycle is sensitive to the value of τ_1; the red cycle suddenly disappears when the value of τ_1 is larger than about roughly 1.9. Concerning the blue cycle, we also notice the followings: the limit cycle can emerge for $\tau_1 < \tau_1^B$; for τ_1 in the vicinity of 1.8, the bifurcation diagram has a collection of the local maximum value, indicating that the cycle repeats small fluctuations in the local maximum and minimum, although fludtuations in the local minimum are not seen in Figure 9(A). These numerical simulations imply the inital point dependency with which a small change in the initial point might result in largely different dynamics.



(A) Bifurcation diagrams (B) Time trajectories

Figure 9. Dynamics with $\tau_1 = 1.8$ and τ_2^c

4.2 $\alpha_x\alpha_y > \beta_x\beta_y > 0$

We now focus on the case in which the self reaction dominates the cross reaction. Keeping the values of β_x and β_y fixed, we increase the values of α_x and α_y to the extent that the direction of the inequality is reversed:

Assumption 5: $\alpha_x = \alpha_y = 8/5$.

Under these parametric conditions, two issues can be confirmed: (1) the basic model has the zero solution as the unique steady state and (2) the eigenvalues of the characteristic equation (4) are real and negative indicating that the zero steady state is a stable node, as shown in Figure 1(B). Our concerns are, as before, upon the delay effects on dynamics, in particular we are interested in whether larger delays can destabilize the zero steady state and what dynamics could emerge when the stability is lost.

Figure 10 illustrates the parts of the segments of $L_1(k,n)$ in red and $L_2(k,n)$ in blue for $k=0$ and $n=0,1,2$ under Assumption 5 and $\beta_x=\beta_y=3/2$. The stability switching curve is the outer envelope of the partition curves for various values of n and its shape differs from the one in Figure 2.[13] Since the steady state is stable without delays (i.e., $\tau_1=\tau_2=0$) and the origin of Figure 10 is located left to the stability switching curve, the steady state with positive delays is stable in the left part to the curve and unstable in the right. The value of $\tau_1^B(=0.6)$ is selected in such a way that the dotted vertical line at $\tau_1=\tau_1^B$ intersects the stability switching curve three times at

$$\tau_2^a \simeq 1.771,\ \tau_2^b \simeq 3.536 \text{ and } \tau_2^c \simeq 4.478.$$

Multiple intersections suggest that delay τ_2 has the double edge effect for $\tau_1=\tau_1^B$, which is numerically confirmed below.

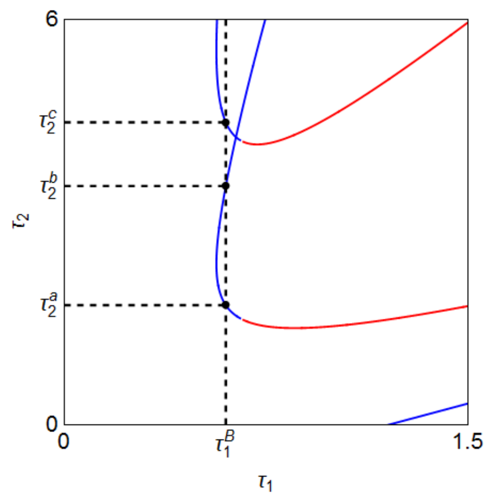


Figure 10. Stability switching curves

Simulation 9: $\tau_1=\tau_1^B,\ 0\le\tau_2\le 6.$

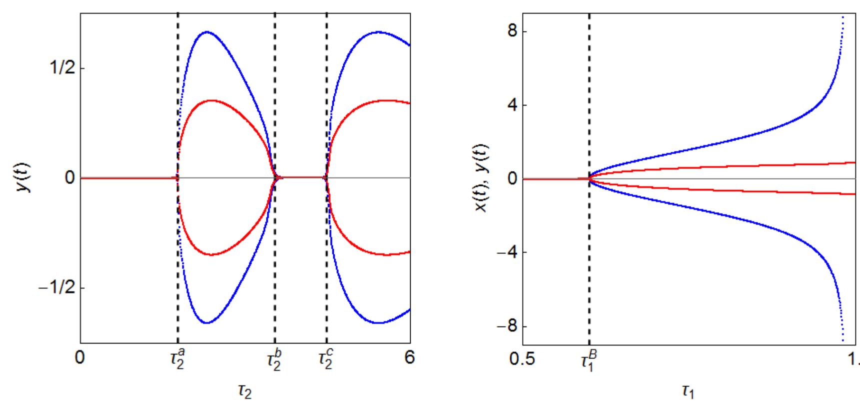
Figure 11(A) presents a bifurcation diagram with respect to τ_2 in which a value of τ_2 is increased along the dotted vertical line at $\tau_1=\tau_1^B$ in Figure 10. As is seen both in Figure 10 and Figure 11(A), the steady state loses stability at $\tau_2=\tau_2^a$ and $\tau_2=\tau_2^c$ and regains it at $\tau_2=\tau_2^b$. Further, Figure 11(A) depicts time trajectories of $x(t)$ in blue and $y(t)$ in red. It is seen that the steady state bifurcates to a limit cycle surrounding the zero steady state whenever the stability is lost and the cycle merges to the zero steady state when the stability is regained. Provided a positive delay in τ_1, some positive value of τ_2 keeps both

[13] Under these specifications, the end points of the domain for the stability switching curves are $\omega_s \simeq 0.805$ and $\omega_e \simeq 1.163$.

Romeo and Juliet in suspense resulting in a cyclic behavior and some other value makes them stay at the zero steady state.

Simulation 10: $0.5 \leq \tau_1 \leq 1,\ \tau_2 = \tau_2^a$

Figure 11(B) gives bifurcation diagrams of $x(t)$ in blue and $y(t)$ in red with respect to τ_1. Two issues are observed: (i) Given the fixed value of τ_2, different values of τ_1 do not qualitatively affect dynamics as the time trajectories in Figure 11(B) exhibit similar behavior (i.e., limit cycles) and (ii) time trajectories of $x(t)$ exhibit huge oscillations for τ_1-values close to unity, indicating that the model may be inappropriate to describe love evolutions of Romeo and Juliet.



(A) With respect to τ_2 (B) With respect to τ_1

Figure 11. Bifurcation diagrams

5 Concluding Remarks

We have investigated a delay Romeo and Juliet model in which both are supposed to be cautious lovers, the most natural romantic style between two individuals. After checking that the no-delay version of the model exhibits fairly stable dynamics, we theoretically derive the instability conditions of the delay version by analyzing the characteristic equations and then use them to illustrate a stability switching curve on which stability of the delay version is lost. Concerning global behavior, we perform numerical simulations to examine that the unstable steady state turns to cyclic oscillations through Hopf bifurcation. In comparison with a one-delay model, the stability switching curve has more complicated shape and dynamics become more complicated in the two-delay model. In consequence, it is confirmed that the multiple delays have the double edge effect, implying the occurrence of multiple alternations of stability and instability with increasing value of one delay and fixing the value of the other delay at some level. Furthermore, various love evolutions ranging from simple to complex dynamics involving chaos can explain various types of love stories between two individuals.

For further investigation, we first plan to introduce delays in the two self-reaction processes, with which the delay model could be more realistic. Secondly, it may be interesting to add one more individual to the model and consider a triangle relation whose dynamics could be much more complicated and therefore much more interesting.

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