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Extended Oligopolies with Pollution Penalties and Rewards*

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Abstract

An extended n-firm oligopoly with product differentiation is considered. It is assumed that the government selects an emission standard for the industry and based on the output and technology of each firm it selects a maximum allowed amount of emission for each firm. If the actual amount is higher than the allowed maximum, then the firm has to pay a constant multiple of the excess to the government, otherwise it is rewarded similarly based on the saved emission amount. The existence of the unique interior equilibrium is first proved, and then time delay is introduced into the penalties the firms have to pay and into the rewards the firms receive. In analyzing the stability of the equilibrium both continuous and discrete time scales are considered. For mathematical simplicity the case of symmetric firms is analyzed. In the continuous case the equilibrium is either always stable or stable if the delay is sufficiently small and at the critical value Hopf bifurcation occurs. In the discrete case the delay is assumed to be unity. The equilibrium is stable if either the total industry output is sufficiently large or the common speed of adjustment of the firms is sufficiently small. The effect of the level of penalty or reward and that of the emission standard on the industry output and therefore on the total emission level is also examined.

Keywords. Pollution penalties, Nonpoint source pollution, Policy lag, Cournot competition

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1 Introduction

The effect of different environmental regulation policies has been investigated by many researchers (Downing and White, 1986; Jung et al., 1996; Montero, 2002 among others). Mostly single firms were considered in relation to environmental R&D, and very few works were devoted to the extension of oligopoly models in this direction. Montero (2002) examined the effect of R&D investment for pollution abatement technology with different environmental policies in duopolistic product markets. Static models were investigated in earlier stages, the existence of equilibrium in Cournot oligopoly with pollution treatment cost sharing was proved by Okuguchi and Szidarovszky (2002), which work was further extended by including emission standard and R&D in the oligopoly model (Okuguchi and Szidarovszky, 2007). If the government is unable to assess the individual emission levels of the different firms, then it can measure only the total level of pollution, and ambient charges are introduces. In this policy (Segerson, 1988) the government defines a cut-off for the total pollution level and regardless of the specific emission level of each firm, all are equally punished or rewarded. Ganguli and Raju (2012) demonstrated that in a Bertrand duopoly increasing ambient charges could lead to greater pollution, however Raju and Ganguli (2013) showed the opposite effect in a Cournot duopoly framework when the increase of ambient charges reduce pollution. This result was further generalized by Matsumoto et al. (2017) for n-firm oligopolies where the stability of the dynamic model with naive expectation was also examined. If the government is familiar with the used technology and production output of each firm, then it is able to assess the proportion of each firm from the total pollution level. Therefore each firm can be punished or rewarded according to its assessed individual emission level compared to its allowed proportional maximum from the government defined cut-off threshold. In this paper this idea will be elaborated. After the formulation of the mathematical model the existence of the static equilibrium will be proved. Assuming gradient adjustment of the firms, dynamic models will be developed with both continuous and discrete time scales, and conditions will be derived for the stability of the equilibrium. The stability conditions will be then analyzed and compared. And finally the effect of penalty and reward parameters on the industry output and the total pollution level will be investigated.

2 Model and Cournot-Nash Equilibrium

Consider n firms in an oligopoly with differentiated products. Let q_k be output of firm k. The price of the product of firm k is seen as

$$p_k = \alpha_k - q_k - \gamma_k \sum_{i \neq k}^{n} q_i \tag{1}$$

with $n \geq 2$ and $\alpha_k > 0$ and $0 < \gamma_k < 1$. Firm k emits pollution $e_k q_k$ in connection to its production with $e_k > 0$. The government can measure the total emission quantity and has an exogenously selected environmental standard $\bar{E}$. So the maximum allowed emission of firm k is clearly

$$\frac{e_k q_k}{\sum_{i=1}^n e_i q_i} \bar{E}. \tag{2}$$

If a firm exceeds this amount then it has to pay a penalty of m times the exceeded amount, and if its emission amount is below the maximum allowed amount, then the firm is rewarded by m times the saved emission amount. So the payoff of firm k becomes

$$\pi_k = (p_k - K_k)\, q_k - m \left(e_k q_k - \frac{e_k q_k}{\sum_{i=1}^n e_i q_i} \bar{E} \right) \tag{3}$$

where K_k is the marginal cost of firm k. Using equation (1),

$$\pi_k = \left(\alpha_k - q_k - \gamma_k \sum_{i \neq k}^n q_i - K_k \right) q_k - \left(m e_k - \frac{m e_k}{\sum_{i=1}^n e_i q_i} \bar{E} \right) q_k. \tag{4}$$

The first term corresponds to revenue and production cost and the second term refers to emission penalty or reward. Assuming interior optimum, the first order condition implies that

$$\frac{\partial \pi_k}{\partial q_k} = \left(\alpha_k - 2q_k - \gamma_k \sum_{i \neq k}^n q_i - K_k \right) - \left(m e_k - \frac{m e_k \sum_{i \neq k}^n e_i q_i}{\left(\sum_{i=1}^n e_i q_i \right)^2} \bar{E} \right) = 0. \tag{5}$$

Notice that (5) strictly decreases in q_k with fixed values of q_i $(i \neq k)$. At $q_k = 0$, its value is

$$\alpha_k - \gamma_k \sum_{i \neq k}^n q_i - K_k - m e_k + \frac{m e_k}{\sum_{i \neq k}^n e_i q_i} \bar{E}.$$

If this value is nonpositive, then $q_k = 0$ is optimum, which is not interior. As $q_k \to \infty$, the value of (5) tends to $-\infty$, so there is always a unique best response.

For mathematical simplicity, let us assume symmetric firms in the sense that $\gamma_1 = \gamma_2 = ... = \gamma_n = \gamma$ and $e_1 = e_2 = ... = e_n = e$, that is, firms have identical substitutability and technology in emission production. Then equation (5) becomes

$$\alpha_k - 2q_k - \gamma \sum_{i=1}^n q_i + \gamma q_k - K_k - me + \frac{m \left(\sum_{i=1}^n q_i - q_k \right)}{\left(\sum_{i=1}^n q_i \right)^2} \bar{E} = 0$$

and with notation $Q = \sum_{i=1}^n q_i$, we have

$$\alpha_k - \gamma Q - K_k - me + \frac{m\bar{E}}{Q} = q_k \left(2 - \gamma + \frac{m\bar{E}}{Q^2} \right).$$

So

$$q_k = \frac{(\alpha_k - K_k - me)\,Q^2 - \gamma Q^3 + m\bar{E}Q}{(2-\gamma)\,Q^2 + m\bar{E}}. \tag{6}$$

By adding these equations for $k = 1, 2, ..., n$ and dividing by Q,

$$(2-\gamma)\,Q^2 + m\bar{E} = \left(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme\right) Q - n\gamma Q^2 + nm\bar{E}$$

which is a quadratic equation for Q,

$$[(2-\gamma) + n\gamma]\,Q^2 - \left(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme\right) Q + (1-n)m\bar{E} = 0. \tag{7}$$

At $Q = 0$, the left hand side is negative and as $Q \to \infty$, it converges to $+\infty$, so there is real root. Since the constant term is negative, one root is positive and the other is negative. So only the positive root has economic meaning. Then the corresponding equilibrium levels of the firms are given by (6).

3 Dynamic Extensions and Stability Analysis

Assume the government has a time delay $\tau > 0$ in posing penalty or giving reward to the firms, and the firms use gradient adjustments in continuous time scales. Then the resulting dynamic system becomes

$$\dot{q}_k(t) = S_k \left[(\alpha_k - 2q_k(t) - \gamma Q(t) + \gamma q_k(t) - K_k) - \left(me - \frac{m\bar{E}\,(Q(t-\tau) - q_k(t-\tau))}{Q(t-\tau)^2} \right) \right]. \tag{8}$$

It is a nonlinear delay differential equation. Let g_k denote the right hand side, then

$$\frac{\partial g_k}{\partial q_k(t)} = -S_k\,(2-\gamma) < 0,$$

$$\frac{\partial g_k}{\partial q_k(t-\tau)} = -\frac{S_k m\bar{E}}{Q(t-\tau)^2} < 0,$$

$$\frac{\partial g_k}{\partial Q(t)} = -S_k\gamma < 0$$

and

$$\frac{\partial g_k}{\partial Q(t-\tau)} = \frac{S_k m\bar{E}\,[2q_k(t-\tau) - Q(t-\tau)]}{Q(t-\tau)^3} < 0.$$

if there is no dominant term. Let these derivatives be denoted by A_k, B_k, C_k and D_k, respectively, then the linearized equation has the form, where q_k and Q_k are now their distances from equilibrium levels:

$$\dot{q}_k(t) = A_k q_k(t) + B_k q_k(t-\tau) + C_k Q(t) + D_k Q(t-\tau). \tag{9}$$

By looking for the solution as usual,

$$q_k(t) = e^{\lambda t} u_k,$$

we have

$$u_k \left(\lambda - A_k - B_k e^{-\lambda\tau} \right) - \sum_{i=1}^{n} u_i \left(C_k + D_k e^{-\lambda\tau} \right) = 0. \tag{10}$$

For the sake of simplicity, we assume in addition that $S_1 = S_2 = ... = S_n = S$, the equilibrium is symmetric and the initial output levels are identical. Then the coefficients A_k, B_k, C_k and D_k are also identical. Then the u_k coefficients must be also the same, therefore we have a delay equation:

$$\lambda - A - Be^{-\lambda\tau} - nC - nDe^{-\lambda\tau} = 0$$

or

$$\lambda - (A + nC) - (B + nD)e^{-\lambda\tau} = 0. \tag{11}$$

At the equilibrium

$$A + nC = -S\left[(n-1)\gamma + 2\right] < 0$$

and

$$B + nD = \frac{Sm\bar{E}(1-n)}{Q^2} < 0.$$

As $\tau = 0$, when the system is without delay,

$$\lambda = A + nC + B + nD < 0,$$

implying that the system is asymptotically stable. Stability switch occurs if $\lambda = i\omega$ with some $\omega > 0$. With the notation

$$U = A + nC < 0 \text{ and } V = B + nD < 0,$$

we have

$$i\omega - U - V(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts,

$$-U - V\cos\omega\tau = 0 \tag{12}$$

and

$$\omega + V\sin\omega\tau = 0. \tag{13}$$

So

$$\omega^2 = V^2 - U^2 = (V - U)(V + U).$$

In our case

$$V - U \quad = \frac{Sm\bar{E}(1-n)}{Q^2} + S\left[(n-1)\gamma + 2\right] \tag{14}$$

The first term is negative and the second positive. Since $V + U < 0$, there is no solution for ω if (14) is nonnegative, and therefore there is no stability switch. If (14) is negative, then there is a unique value of ω,

$$\omega = \sqrt{V^2 - U^2}.$$

Hopf bifurcation is used to find the direction of the stability switch. Let τ be the bifurcation parameter and consider λ as function of τ: $\lambda = \lambda(\tau)$. By implicitly differentiating equation (11) with respect to τ, we have

$$\frac{d\lambda}{d\tau} - Ve^{-\lambda\tau}\left(-\lambda - \frac{d\lambda}{d\tau}\tau\right) = 0$$

showing that

$$\begin{aligned} \frac{d\lambda}{d\tau} &= \frac{-Ve^{-\lambda\tau}\lambda}{1 + Ve^{-\lambda\tau}\tau}, \\ &= \frac{-\lambda(\lambda - U)}{1 + \tau(\lambda - U)}, \end{aligned}$$

where equation (11) is used. When $\lambda = i\omega$,

$$\begin{aligned} \frac{d\lambda}{d\tau} &= \frac{\omega^2 + i\omega U}{1 - \tau U + i\omega\tau}, \\ &= \frac{(\omega^2 + i\omega U)(1 - \tau U - i\omega\tau)}{(1 - \tau U)^2 + (\tau\omega)^2} \end{aligned}$$

with real part having the same sign as

$$\omega^2(1 - \tau U) + \omega U(\tau\omega) = \omega^2 > 0$$

showing that the sign of the real part changes from negative to positive, so stability is lost. That is, at the smallest stability switching point, stability is lost and it cannot be regained later. If there is positive solution for ω, then $|V| > |U|$, and since from (12) and (13) we know that $\sin\omega\tau > 0$ and $\cos\omega\tau < 0$, the smallest (critical) value of τ is the following:

$$\tau^* = \frac{1}{\sqrt{V^2 - U^2}} \cos^{-1}\left(-\frac{U}{V}\right).$$

At the critical value of τ there is the possibility of the birth of limit cycles. From (14), we know that the system is always asymptotically stable if

$$m\bar{E}(1 - n) + [(n - 1)\gamma + 2]\, Q^2 \geq 0. \tag{15}$$

From (7), the left hand side equals

$$\left(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme\right) Q \tag{16}$$

so the system is stable if multiplier of Q is nonnegative. Notice that the multiplier of Q is the total marginal profit of the firms at zero environmental standard when all firms have zero production levels.

If discrete time scales are assumed and the government delay is unity, then the continuous time scales differential equation (8) is modified to a second order difference equation,

$$q_k(t+1) = q_k(t) + S_k \left[(\alpha_k - 2q_k(t) - \gamma Q(t) + \gamma q_k(t) - K_k) - \left(me - \frac{m\bar{E}\left[Q(t-1) - q_k(t-1)\right]}{Q(t-1)^2} \right) \right] \tag{17}$$

and after linearization it becomes

$$q(t+1) = q(t) + [Aq(t) + Bq(t-1) + nCq(t) + nDq(t-1)] \tag{18}$$

where we again assume symmetric firms. By introducing the new variables

$$a(t) = q(t-1) \text{ and } b(t) = q(t),$$

we have

$$\begin{aligned} a(t+1) &= b(t) \\ b(t+1) &= (B+nD)a(t) + (A+nC+1)b(t) \end{aligned} \tag{19}$$

with coefficient matrix

$$\begin{pmatrix} 0 & 1 \\ V & U+1 \end{pmatrix}$$

leading to a quadratic characteristic polynomial

$$-\lambda(U+1-\lambda) - V = \lambda^2 - (U+1)\lambda - V. \tag{20}$$

Asymptotic stability is guaranteed by

$$\begin{aligned} U+1-V+1 &> 0, \\ -U-1-V+1 &> 0, \\ -V &< 1. \end{aligned}$$

The second condition is clearly satisfied. The first inequality can be rewritten as

$$V - U = \frac{S}{Q}\left(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme \right) < 2 \tag{21}$$

and the third condition has the form

$$-V = \frac{Sm\bar{E}(n-1)}{Q^2} < 1. \tag{22}$$

In comparing the stability conditions in the continuous and discrete case we notice the following.

(a) If $\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme < 0$, then (21) holds and (22) is satisfied if Q is sufficiently large or the speed of adjustment S is sufficiently small. The continuous system loses stability if τ is sufficiently large.

(b) If $\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme \geq 0$, then the continuous system is stable. The discrete system is stable if Q is sufficiently large or the value of S is sufficiently small.

An equivalent condition can be given by rewriting (21) and (22) as

$$Q > \frac{S}{2}\left(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme\right)$$

and

$$Q > \sqrt{Sm\bar{E}(n-1)}.$$

Letting

$$Q^* = \max\left[\frac{S}{2}\left(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme\right), \sqrt{Sm\bar{E}(n-1)}\right], \tag{23}$$

the stability condition becomes

$$P(Q^*) < 0 \tag{24}$$

where $P(Q)$ denotes the left hand side of equation (7).

4 Effect of Penalty or Reward on Pollution Levels

We can analyze how the total production level (and therefore the total emission level) depends on the penalty factor m. Considering Q as a function of m, and implicitly differentiating equation (7) with respect to m, we have

$$[(2-\gamma)+n\gamma]\, 2Q\frac{dQ}{dm} + neQ - \left(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme\right)\frac{dQ}{dm} + (1-n)\bar{E} = 0$$

implying that

$$\frac{dQ}{dm} = \frac{-neQ + (n-1)\bar{E}}{2Q\left[(2-\gamma)+n\gamma\right] - \left(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} K_k - nme\right)}. \tag{25}$$

Based on equation (7), the denominator can be rewritten as

$$Q\left[(2-\gamma)+n\gamma\right] + \frac{(n-1)m\bar{E}}{Q}$$

which is positive. So the sign of dQ/dm depends on the sign of the numerator. The first term is negative, the second term is positive. Using again equation (7), we see that the m-multiple of the numerator equals

$$-\left(\sum_{k=1}^{n}\alpha_k - \sum_{k=1}^{n}K_k\right)Q + \left[(2-\gamma)+n\gamma\right]Q^2.$$

It is reasonable to assume that the first term is negative. Therefore $dQ/dm > 0$ if and only if

$$Q > \frac{\sum_{k=1}^{n}\alpha_k - \sum_{k=1}^{n}K_k}{(2-\gamma)+n\gamma}. \tag{26}$$

Let $\bar{Q}$ denote the right hand side of this inequality and $P(Q)$ again the left hand side of equation (7), then this is the case when $P(\bar{Q}) < 0$ meaning that

$$\bar{E} > \frac{\left[(2-\gamma)+n\gamma\right]\bar{Q}^2 - \left(\sum_{k=1}^{n}\alpha_k - \sum_{k=1}^{n}K_k - nme\right)\bar{Q}}{(n-1)m}. \tag{27}$$

It is easy to see that the numerator is positive. This inequality means that increase in the value of m has an increasing effect on the total industry output as well as in the total emission level if the emission standard $\bar{E}$ is sufficiently large. Otherwise the opposite effect can be observed.

Next we examine the effect of increasing in the value of $\bar{E}$. Considering now Q as function of $\bar{E}$ and implicitly differentiating equation (7) we have

$$\left[(2-\gamma)+n\gamma\right]2Q\frac{dQ}{d\bar{E}} - \left(\sum_{k=1}^{n}\alpha_k - \sum_{k=1}^{n}K_k - nme\right)\frac{dQ}{d\bar{E}} + (1-n)m = 0$$

showing that

$$\frac{dQ}{d\bar{E}} = \frac{(n-1)m}{2Q\left[(2-\gamma)+n\gamma\right] - \left(\sum_{k=1}^{n}\alpha_k - \sum_{k=1}^{n}K_k - nme\right)}. \tag{28}$$

We already established that the denominator is positive, so $dQ/d\bar{E} > 0$ showing that the increase in the emission standard always has an increasing effect on the industry output as well as on the total emission level.

5 Concluding Remarks

This paper examined n-firm oligopolies with product differentiation when the firms face penalties or rewards depending on the amounts of their pollution levels. The government selects an emission standard for the entire industry, and based on the specific technology and output of each firm the government determines its maximum allowed emission level. The amount of penalty or that of the reward is determined by the difference of the actual emission level and the maximum allowed amount. This simple mathematical model can be considered as the counterpart of models with ambient pollution charges discussed earlier in the literature. The existence of the unique interior equilibrium was first proved, and then dynamic extensions were introduced and the stability of the equilibrium was examined with both discrete and continuous time scales. In the continuous time scales a positive delay was introduced in the penalty and reward terms and in the case of discrete time scales unit delay was assumed. In the case of continuous time scales the equilibrium is always stable if the total marginal profit of the firms at zero output levels is non-negative, otherwise it is stable if the length of the delay is sufficiently small. At the critical value of the delay Hopf bifurcation occurs. In the case of discrete time scales the equilibrium is stable if either the industry output is sufficiently large or the common speed of adjustment of the firms is sufficiently small. We also established that an increase in the value of m has an increasing effect on the total pollution level if the emission standard $\bar{E}$ is sufficiently large, otherwise the opposite effect occurs. An increase in the value of $\bar{E}$ always has an increasing effect on the total pollution level of the industry.

Symmetric firms were assumed for mathematical simplicity, in which case the equilibrium could be given analytically, however in the non-symmetric case it is not possible making the further analysis much more complicated. This issue will be the subject of our next project.

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